Particle Physics

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Books

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1. Kane, G: *The Particle Garden*, Addison Wesley, 1995
   [Not a textbook, just a good read without any maths]
2. Williams, W S C: *Nuclear and Particle Physics*, Clarendon Press, 1991
   [Good level, good coverage, but certain things missing (accelerators, experimental methods, some theoretical/mathematical details)]
3. Griffiths, D: *Introduction to Elementary Particles*, John Wiley & Sons, 1987
   [Again, good level, good coverage, but accelerators and experimental methods are not covered]
4. Martin, B R and Shaw, G: *Particle Physics*, John Wiley &
   [Slightly more theoretically-biased than previous ones]
5. Perkins, D H: *Introduction to High Energy Physics*, Cambridge University Press
   [Most comprehensive coverage of topics, including accelerators and experimental methods, but sometimes requires some previous knowledge]
6. Seiden, A: *Particle Physics, A Comprehensive Introduction*, Addison Wesley,
   [The best coverage of underlying maths, including basics of QFT and Group Theory]
7. Thomson, M: *Modern Particle Physics*, Cambridge University Press [The most up-to-date in the list]

Web Resources

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1. Lancaster Particle Physics Package for A-level students:
   http://www.hep.lancs.ac.uk/package/
   Some basic stuff - worth a look or two (feedback welcome)

2. Paricle Physics in the UK website, plenty of info and links:
   http://hepweb.rl.ac.uk/ppUK/

3. FNAL (Fermi National Accelerator Laboratory), home of the Tevatron:
   http://www.fnal.gov/

4. CERN (European Centre for Nuclear Research), home of LEP and LHC:
   http://public.web.cern.ch/public/

5. The ultimate resource: Particle Data Group website
   http://pdg.lbl.gov
   The official reference for all particle data. Many useful review articles, too

Particle Physics — What's This About?

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'Elementary' Particles — $e, p, n, \nu, \mu, \tau, \gamma, W, Z \ldots$ and their interactions with each other.

You should already know a few things about them.

Would you expect Particle Physics to be a hard subject?

Compared to other areas of physics (nuclear, solid state, bio-. . .) and other sciences (botany, chemistry, zoology, medicine) I believe PP is actually **very simple**:

- ✦ Particles have (relatively) few properties ('quantum numbers').
- ✦ These properties usually have few discrete values.
- ✦ Particles obey very simple, relatively few, well-defined laws.
- ✦ All elementary particles of the same type are **absolutely identical**.

Why does PP Seem So Hard Then?

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- The world of particles is so far from our everyday experience, that all these simple properties and simple laws may look and seem unnatural and weird;

  What can we do?

  'Friendly' names: strangeness, charm, colour, top, bottom... Find analogies and simple rules

- Many mathematical methods used to describe the world of particles are quite advanced (Group Theory, Quantum Field Theory, Advanced Statistical Methods ...)

  What can we do?

  Use simplified maths, skip derivations...

- Your intuition fails to work

  What can we do?

  Help you build your intuition by giving you lots of various problems to solve!

What's the Scale?

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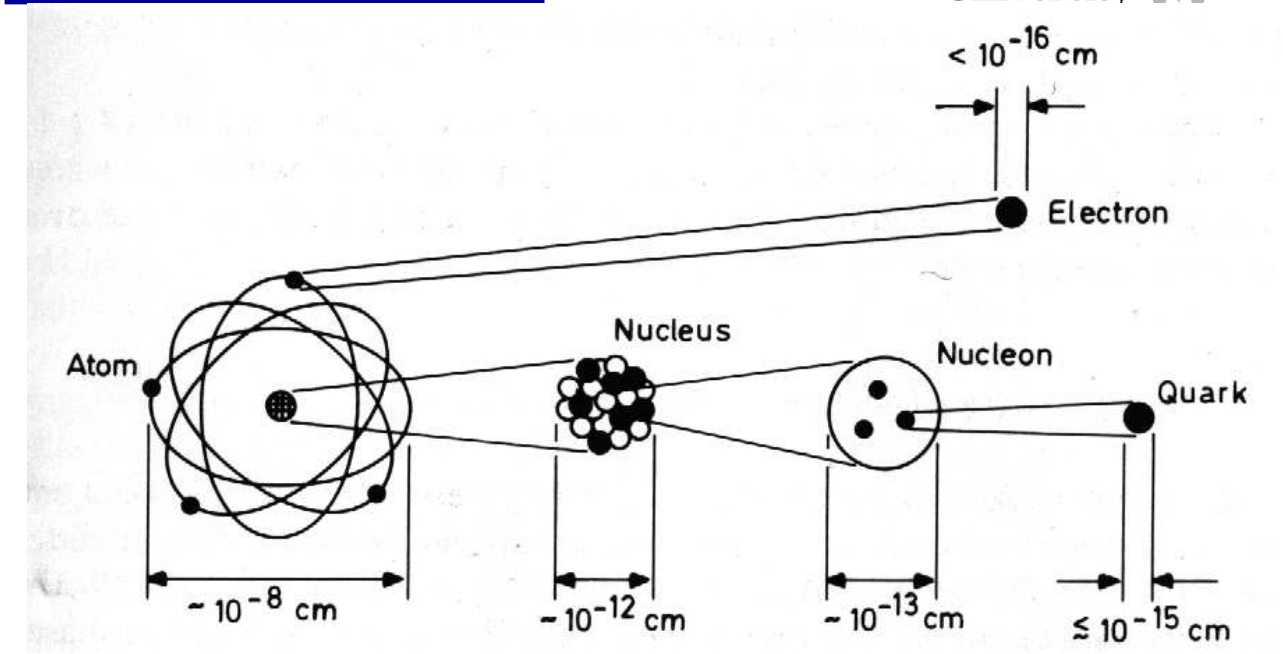
'Elementary' Particles:
the smallest constituents
of matter (known so far):
leptons and quarks, and also
the interaction carriers:
photons γ, gluons g,
$W^{\pm}$ and Z^0 bosons.



Well-established models and theories at present exclude gravitational interactions:

1. quantum theory of gravity has not been built yet;
2. may (should!) be tied to properties of space-time at tiny scales;
3. too weak to matter for particles under 'usual' circumstances.

However, **weak, electromagnetic** and **strong** interactions are understood and described reasonably well. In this course we will concentrate on these three.

Is SI System Useful in Particle Physics?

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Main properties of particles: mass m, charge e, spin s.

For an electron in SI system:

$$\begin{aligned} m_e &= 9.109 \times 10^{-31}\ \mathrm{kg} \\ e &= -1.602 \times 10^{-19}\ \mathrm{C} \\ s_z &= \pm\hbar/2 = \pm(1/2) \times 1.055 \times 10^{-34}\ \mathrm{J \cdot s} \end{aligned}$$

Particle physicists **do not** use SI system. Instead, a particle physicist would write:

$$\begin{aligned} m_e &= 0.51\ \mathrm{MeV}/c^2 \\ e &= -1\ \text{proton charge} \\ s_z &= \pm 1/2 \end{aligned}$$

The last equation suggests: in particle physics

$$\hbar = 1.055 \times 10^{-34}\ \mathrm{J \cdot s} = 1$$

which, for one thing, states that in particle physics the product of units of [energy] and [time] is dimensionless.

Can we Make it Even Simpler?

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So, it's natural to choose units such that $\hbar = 1$. This means that

[energy] $\times$ [time] $=1$ and also [momentum] $\times$ [distance] $=1$

Now, remember the relativistic relation between Energy E, momentum $\mathbf{p}$ and mass m:

$$E^2 = \mathbf{p}^2\, c^2 + m^2\, c^4$$

Relativistic particles move with speeds close to speed of light. Carrying all these huge factors like (300000000 m/s)^2 around will be avoided in a system of units where $c = 1$, which simply means that [new unit of time] is [old unit of time]$/c$.

The choice $\hbar = 1$ and $c = 1$ would mean that

- Energy, momentum and mass are measured in the same units
- Angular momentum is dimensionless
- Time and distance are measured in the same units
- Energy is inverse of time
- One needs just **one** dimesional unit, which is usually chosen as the unit of energy
- In Particle Physics this is 1 GeV

Natural System of Units

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The system of units with $\hbar = 1$ and $c = 1$ is called the Natural system:

$$\begin{aligned} 1 \text{ unit of length} &= 1 \text{ GeV}^{-1} \simeq 0.1978 \text{ fm} \\ 1 \text{ unit of time} &= 1 \text{ GeV}^{-1} \simeq 0.6588 \cdot 10^{-24} \text{ s} \\ 1 \text{ unit of energy} &= 1 \text{ GeV} \\ 1 \text{ unit of momentum} &= 1 \text{ GeV} \quad \text{sometimes GeV}/c \\ 1 \text{ unit of mass} &= 1 \text{ GeV} \quad \text{sometimes GeV}/c^2 \end{aligned}$$

Note: 1 GeV = 1000 MeV and $(1 \text{ GeV})^{-1} = (1000 \text{ MeV})^{-1}$, but $1000 \text{ GeV}^{-1} = 1 \text{ MeV}^{-1}$

One more unit: **barn b** for cross section: $1 \text{ b} = 10^{-24} \text{ cm}^2$.

One barn is far too big a unit for particle physics:

$$1 \text{ b} = 10^3 \text{ mb} = 10^6 \ \mu\text{b} = 10^9 \text{ nb} = 10^{12} \text{ pb} = 10^{15} \text{ fb}$$

The cross sections of most interesting processes in particle physics are usually measured in femtobarns fb.

Here is a useful relation:

$$1 \text{ GeV}^{-2} = 0.389 \text{ mbarn}$$

How About Electric Charge?

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The fine structure constant α is a dimensionless constant. In SI system:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.036} \simeq \frac{1}{137}$$

In natural units it still has the same numerical value:

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036} \simeq \frac{1}{137}$$

This relation can be used to determine the electron charge e in the natural system, as electric charge here is dimensionless.

The numerical value of e is not used much. However, α is one of the most well-measured constants in physics.

There are two things which are very interesting here:

- the fact that α is a very small number, $\alpha \ll 1$ (allows use of perturbation theory)
- the fact that the charge is quantised. The smallest non-zero charge we know of is exactly $e/3$ (more about this later).

A Bit of Quantum Mechanics

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Remember the Heisenberg Uncertainty Principle?

Uncertainty in momentum times uncertainty in coordinate cannot be smaller than $\hbar$:

$$\Delta p_x \, \Delta x \gtrsim \hbar/2$$

and similarly for y and z directions.

Uncertainty in energy times uncertainty in time cannot be smaller than $\hbar$:

$$\Delta E \, \Delta t \gtrsim \hbar/2$$

Think about it:

the energy of a system can be measured with high precision **only if** it lives long enough.

If the system is **unstable**, its energy (and hence its mass) is **not** defined precisely.

Of course, in natural units $\hbar = 1$ and the uncertainty relations above, when used for order-of-manitude estimates, usually read

$$\Delta p \, \Delta x \sim 1 \qquad \Delta E \, \Delta t \sim 1$$

Relativistic Notation

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For a free particle of mass m, energy E and momentum $\mathbf{p}$

$$E^2 - \mathbf{p}^2 = m^2$$

Introducing 4-vectors: $p_1 = (E_1, \mathbf{p}_1), p_2 = (E_2, \mathbf{p}_2)$, with the **dot product** defined as:

$$p_1 \cdot p_2 \equiv E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \equiv p_{1\mu} p_{2\nu} g^{\mu\nu} \equiv p_{1\mu} p_2^{\mu}$$

Important:

- 4-vectors are usually written as p_μ or simply p (no arrows, no bold fonts)
- The 4-indices $\mu, \nu \ldots$ have values 0,1,2,3 or t, x, y, z.
- $g^{\mu\nu}$ is the "metric tensor", $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
- $p_0 \equiv E$ is energy, $p_{1,2,3} \equiv p_{x,y,z}$ are the components of the 'good old' 3-momentum
- Hence, in the dot product, the product of energies enters with a plus sign, while the dot product of 3-momenta enters with a minus sign

So, in this notation, for a free particle of mass m and 4-momentum $p = (E, \mathbf{p})$ one has

$$p^2 = m^2 \qquad \text{where } p^2 \equiv E^2 - |\mathbf{p}|^2$$

Of course, $|\mathbf{p}|$ is the length (module) of the 3-momentum.

Lorentz Transformation

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As you know, energies and momenta of particles depend on the system of reference.

Lorentz transformation governs the way energies and momenta are transformed from one reference system to another.

- Particle in system one has 4-momentum $p = (E, p_x, p_y, p_z)$
- Same particle in system two has 4-momentum $p' = (E', p'_x, p'_y, p'_z)$
- System two is moving with velocity v m/s with respect to system one in the direction of their common z axis
- Calculate $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$

$$\begin{aligned} E' &= \gamma E + \beta\gamma p_z \\ p'_x &= p_x \\ p'_y &= p_y \\ p'_z &= \beta\gamma E + \gamma p_z \end{aligned}$$

Particle mass m remains **invariant** under Lorentz transformation:

$$E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 = E'^2 - p_x'^2 - p_y'^2 - p_z'^2$$

Useful relations

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Let the particle of mass m be at rest in frame one. If we know the modulus of particle's momentum $p \equiv |\mathbf{p}|$ in frame two (and hence its energy $E = \sqrt{\mathbf{p}^2 + m^2}$):

- The easiest way of calculating the speed β is:

$$\beta = p/E$$

- Similarly, the easiest way of calculating the γ-factor is:

$$\gamma = E/m$$

- Once again, the easiest way of calculating the product $\beta\gamma$ is:

$$\beta\gamma = p/m$$

It's worth checking, that when calculated this way, we still have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Usually this is the easiest way of determining the parameters of Lorentz transform: you have to find (or "create") a particle which is at rest in that frame!

Decay Length

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Decay length l is the distance travelled by an unstable particle during its lifetime t.

Obviously, if the particle is at rest, $l = 0$ no matter how long it lives. For a particle moving with velocity v in the lab frame, the observed lifetime t is different from the 'proper' lifetime in the particle's own rest frame t^*, following the Lorentz transformation (in the rest frame $z^* = 0$):

$$\begin{aligned} t &= \gamma t^* + \beta\gamma z^* = \gamma t^* \\ l &= vt = c\beta t = c\beta\gamma t^* \end{aligned}$$

Hence, the mean decay length in the lab frame is $\langle l \rangle = \beta\gamma c\tau$.

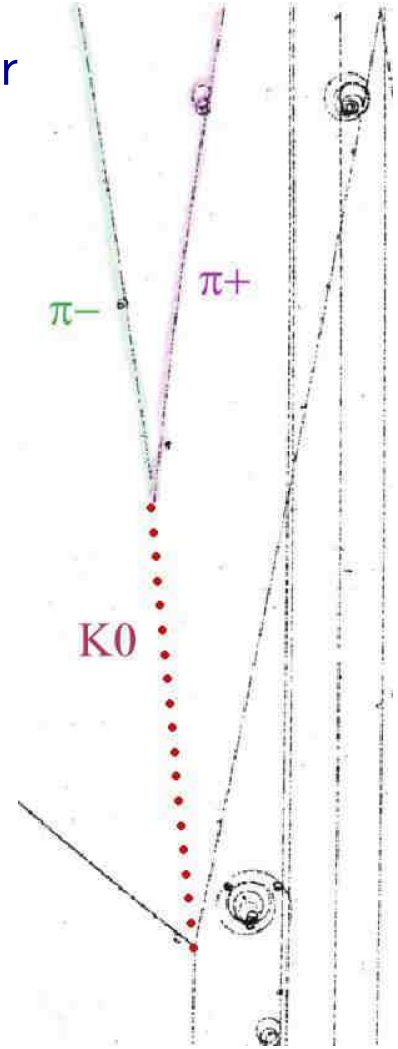
For a particle with 3-momentum modulus $p \equiv |\mathbf{p}|$ its energy is $E = \sqrt{p^2 + m^2}$. Hence:

$$\beta = \frac{p}{E}, \quad \gamma = \frac{E}{m}, \quad \beta\gamma = \frac{p}{m}$$

With τ in seconds and c in m/s, $\langle l \rangle$ will be in meters. Example:

$K^0_S \to \pi^+\pi^-$ decay has mean life $\sim 10^{-10}$ s with $c\tau = 2.7$ cm.

However, 10 GeV kaons will travel on average $\sim 2.7 \cdot 10/0.5 = 54$ cm.



Invariant Mass

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The dot product of **any** two 4-vectors is **invariant** under Lorentz transformations.

Consider a system of two particles: $p_1 = (E_1, \mathbf{p}_1)$ with mass m_1 and $p_2 = (E_2, \mathbf{p}_2)$ with mass m_2.

Calculate the sum of the two 4-vectors: $P = (E_1 + E_2, \mathbf{p}_1 + \mathbf{p}_2)$.

The quantity M is called the **invariant mass** (sometimes **effective mass)** of our two particles, if

$$M^2 \equiv P^2 = P \cdot P = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2) \cdot (\mathbf{p}_1 + \mathbf{p}_2)$$

Let's do the maths:

$$\begin{aligned} M^2 &= (p_1 + p_2) \cdot (p_1 + p_2) = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ &= m_1^2 + m_2^2 + 2E_1E_2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \\ &= m_1^2 + m_2^2 + 2E_1E_2 - 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta \end{aligned}$$

where θ is the angle between the two 3-vectors $\mathbf{p}_1$ and $\mathbf{p}_2$.

The invariant mass is — surprise, surprise! — invariant under Lorentz transformations: the result of this calculation will be the same in any reference frame.

But remember: all involved quatities (energies, momenta, angles) must be calculated **in the same frame**!

More Useful Relations

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Let's to solve a little problem:

We have a system of two particles (produced, say, as a result of another particle's decay, e.g. $\Delta^0 \to p\pi^-$). We have identified the two particles (i.e. know which is p and which is π^-) and we have measured their momenta, $\mathbf{p}_1$ and $\mathbf{p}_2$. We know the masses, so we can calculate the energies E_1 and E_2.

The question is: can we figure out the speed β (and hence the γ-factor) of the the (p, π) system in the lab frame?

The answer is very simple:

$$\beta = \frac{|\mathbf{p}_1 + \mathbf{p}_2|}{E_1 + E_2}$$

$$\gamma = \frac{E_1 + E_2}{M}$$

where M is the invariant mass of the (p, π) system (which should be equal to the mass of the "parent" Δ - see later).

This calculation will be valid for any system of particles, even if they are not the decay products of a single particle.

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Remember how Shrödinger's equation was obtained? Started from the classical relation for energy, and replaced physical quantities with corresponding operators, acting upon the wave function ψ

$$E = \frac{\mathbf{p}^2}{2m} + V \qquad \mathbf{p} \Rightarrow -i\boldsymbol{\nabla}$$

$$-\frac{1}{2m}\boldsymbol{\nabla}^2\psi + V\psi = E\psi$$

For a free particle $V = 0$, and the solution of the remaining equation is the plane wave

$$\psi = C\, e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad E = \mathbf{p}^2/2m$$

Time-dependence of the free particle wave function is also easy to obtain:

$$i\frac{\partial}{\partial t}\psi = E\psi, \qquad \psi \sim e^{-iEt}$$

So the full wave function of a free particle is $\psi(\mathbf{x}, t) \sim e^{-iEt+i\mathbf{p}\cdot\mathbf{x}}$.

Relativistic Quantum Mechanics

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In relativistic quantum theory, the base equation for a free particle is the familiar

$$E^2 - \mathbf{p}^2 = m^2 \qquad \text{or} \qquad p^2 - m^2 = 0$$

Rewritten with quantum-mechanical operators, it's called Klein-Gordon equation:

$$\left\{ -\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2 \right\} \phi = m^2 \phi \qquad \text{or} \qquad \left\{ -\frac{\partial^2}{\partial x_\mu \partial x^\mu} - m^2 \right\} \phi = 0$$

which is satisfied by the wave function

$$\phi(x) \equiv \phi(t, \mathbf{x}) \sim e^{-ip\cdot x}, \qquad p \cdot x = Et - \mathbf{p} \cdot \mathbf{x}$$

That is, the wave function of a free particle in the relativistic quantum theory is still a plane wave.

A free particle is clearly an idealisation. It fills the whole of space-time with equal probability and thus its $\mathbf{x}$ and t are undetermined; hence, its energy-momentum is well defined and can be measured. However, a free particle is free as long as it does not interact with anything, hence it is unobservable...

What Do We Observe?

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Observing a stable system is not much fun: nothing happens $\Rightarrow$ nothing to observe.

1) An unstable system may undergo some kind of **decay** process, with stable decay products that can be observed one way or another.

- Radioactive decay of an unstable nucleus; decay products are a lighter nucleus, α, β and γ 'rays'.
- A charged pion from a cosmic ray shower decays into a muon and a neutrino: $\pi^{\pm} \rightarrow \mu^{\pm} + \nu$.

2) Something interesting may happen if two (stable) objects are made to interact with each other in some kind of a **collision**. This collision may occur naturally, or could be organised artificially as a result of a huge design and engineering effort.

- Light radiation by a hot object — mechanical collisions between atoms cause some electrons to move to higher energy orbits; subsequently they return to the stable orbit by radiating photons. The energy level structure of the atoms can be studied by measuring the energies (frequencies) of these photons.
- Two protons of 7000 GeV energy each are collided head-to-head; each collision produces on average a few hundred particles, and particle physicists spend their time trying to make sense of this.

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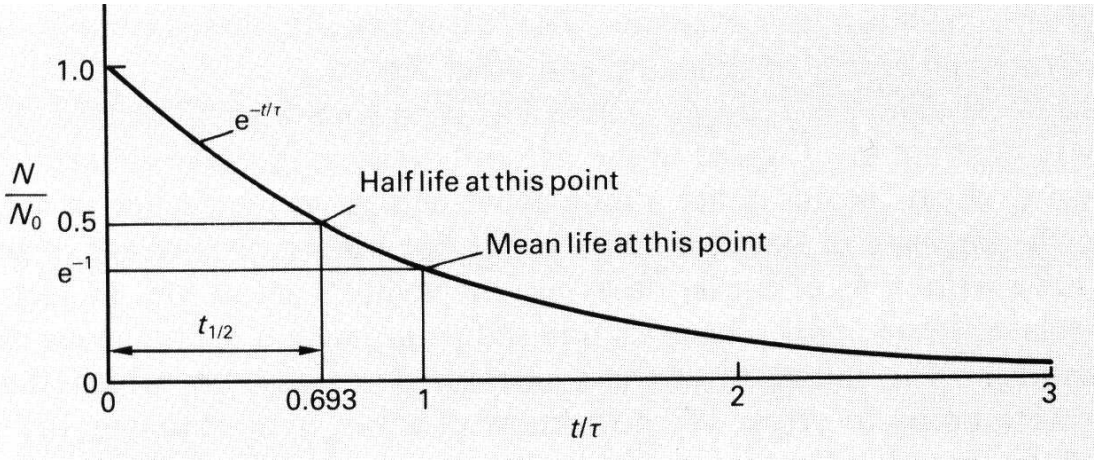
$N(t)$ — number of (unstable) objects of the same type, still intact at moment t.

If the objects decay spontaneously and independently of each other, the number of objects decayed, dN, during the small time interval dt must be proportional to the number of objects still present at the start of the interval, $N(t)$:

$$dN = -\Gamma\, N\, dt \qquad \Rightarrow \qquad N(t) = N_0 e^{-\Gamma t}$$

where Γ is a constant, which determines the rate of the decay. It has dimension [1/time] and hence [energy] or [mass], and is equal to the decay probability per unit time.



Quantity $\tau = 1/\Gamma$ has the dimension of [time], and is called 'mean life'. It is closely related to another measure of decay rate, half-life $t_{1/2}$.

After $t = \tau$ only the fraction $1/e$ of the initial sample is left undecayed.

After $t = t_{1/2}$ half of the initial sample has decayed; $t_{1/2} = \tau \log 2 = 0.693\,\tau$.

Decays in Quantum Theory

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For a free particle with mass m, at rest, one has $\mathbf{p} = 0 \;\Rightarrow E = m$ and the plane wave wavefunction reads

$$\phi(t, \mathbf{x}) \sim e^{-iEt + i\mathbf{p}\cdot\mathbf{x}} = e^{-imt}$$

So that the probability density of finding the particle, $|\phi|^2 = \phi^*\phi = \text{const}$ is independent of time. Hence, plane waves only describe stable free particles which populate all space-time uniformly.

For an unstable particle with mean life $\tau = 1/\Gamma$, the mass is complex:

$$m \;\Rightarrow\; m - i\Gamma/2 \quad \text{and} \quad |\phi|^2 = \phi^*\phi \sim e^{imt - \Gamma t/2} e^{-imt - \Gamma t/2} = e^{-\Gamma t}$$

We see that the introduction of a (negative) imaginary part $\Gamma/2$ to the mass of a particle causes its probability density to decay with time, with mean life $\tau = 1/\Gamma$.

So, the life time of an unstable particle is finite, $\Delta t \simeq \tau$. The uncertainty relation now tells us that the energy of this particle can only be determined with some finite accuracy:

$$\Delta E\, \Delta t \sim 1 \quad \text{hence} \quad \Delta E \sim 1/\tau = \Gamma$$

Hence, whenever the mass of an unstable particle is being measured, the result will be a distribution of masses with the width of order Γ, rather than a fixed number every time.

Mean Life and Width

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So, an unstable particle is characterised by its mean life τ, or its inverse $\Gamma = 1/\tau$, decay probability per unit time.

Usually, for 'long-lived' particles with $\tau \gtrsim 10^{-20}$ s, the numbers quoted in the tables are for τ in seconds and/or $c\tau$ in centimetres.

For particles with shorter lifetimes, the number used to describe the decay rate is $\Gamma = 1/\tau$, measured in units of energy:

- $\tau = 0.66 \cdot 10^{-21}$ seconds corresponds to $\Gamma = 1$ MeV
- $\tau = 0.66 \cdot 10^{-22}$ seconds corresponds to $\Gamma = 10$ MeV
- $\tau = 0.66 \cdot 10^{-23}$ seconds corresponds to $\Gamma = 100$ MeV
- $\tau = 0.66 \cdot 10^{-24}$ seconds corresponds to $\Gamma = 1$ GeV

Often Γ is referred to as the width of a particle (we sill soon see why).

Several Decay Channels?

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If a particle has more than one decay channel, partial decay probabilities Γ_i should be added up to calculate the total decay probability $\Gamma_{\rm tot}$:

$$\Gamma_{\rm tot} = \Gamma_1 + \Gamma_2 + \ldots, \qquad \tau = 1/\Gamma_{\rm tot}$$

Mean life is calculated as the inverse of the total decay probability. It is **not** correct to add up 'partial lifetimes'!

The Branching Fraction $BR(i)$ for a particular decay channel with partial width Γ_i is

$$BR(i) = \frac{\Gamma_i}{\Gamma_{\rm tot}}$$

π^+ DECAY MODES		Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\mu^+\nu_\mu$	[b]	(99.98770 ± 0.00004) %		30
$\mu^+\nu_\mu\gamma$	[c]	(2.00 ± 0.25) × 10^{-4}		30
$e^+\nu_e$	[b]	(1.230 ± 0.004) × 10^{-4}		70
$e^+\nu_e\gamma$	[c]	(1.61 ± 0.23) × 10^{-7}		70
$e^+\nu_e\pi^0$		(1.025 ± 0.034) × 10^{-8}		4
$e^+\nu_e e^+e^-$		(3.2 ± 0.5) × 10^{-9}		70
$e^+\nu_e\nu\overline{\nu}$		< 5 × 10^{-6}	90%	70

The lifetimes, full widths and branching fractions of various decay modes are listed in the Tables of Particle Properties. Some particles have many tens of different decay modes.

A charged pion has mass $m_\pi = 140$ MeV, mean life $\tau = 2.6 \cdot 10^{-8}$ s, and $c\tau = 7.8$ m.

What's That p in the Last Column?

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p is the 3-momentum available for the decay products, in the initial particle rest frame.

In two-particle decays, p is the momentum of **each** or the two decay products. In multi-particle decays, p shows the maximum momentum for each particle.

For a two-particle decay $M \to m_1 + m_2$, the value of $p \equiv |\mathbf{p}|$ can be easily calulated using Lorentz-invariance and energy-momentum conservation:

In its own rest frame, the 4-momentum of the decaying particle is $q = (M, \mathbf{0})$.

The decay products have 4-momenta $p_1 = (E_1, \mathbf{p}_1)$ and $p_2 = (E_2, \mathbf{p}_2)$, with $E_1 = \sqrt{m_1^2 + \mathbf{p}_1^2}$, $E_2 = \sqrt{m_2^2 + \mathbf{p}_2^2}$; energy conservation gives $E_1 + E_2 = M$.

Momentum conservation yields $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0} \;\; \Rightarrow \;\; |\mathbf{p}_1| = |\mathbf{p}_2| \equiv p$. Let's calculate p_2^2:

$$\begin{aligned} m_2^2 \;=\; p_2^2 \;=\; (q - p_1)^2 \;&=\; q^2 + p_1^2 - 2qp_1 = M^2 + m_1^2 - 2ME_1 + 2 \times \mathbf{0} \cdot \mathbf{p}_1 \\ m_2^2 \;&=\; M^2 + m_1^2 - 2ME_1 \\ E_1 \;&=\; \frac{M^2 + m_1^2 - m_2^2}{2M} \qquad p = \sqrt{E_1^2 - m_1^2} = |\mathbf{p}_1| = |\mathbf{p}_2| \end{aligned}$$

It should be easy to calculate E_2, either from energy conservation or simply by symmetry.

Describing Collisions

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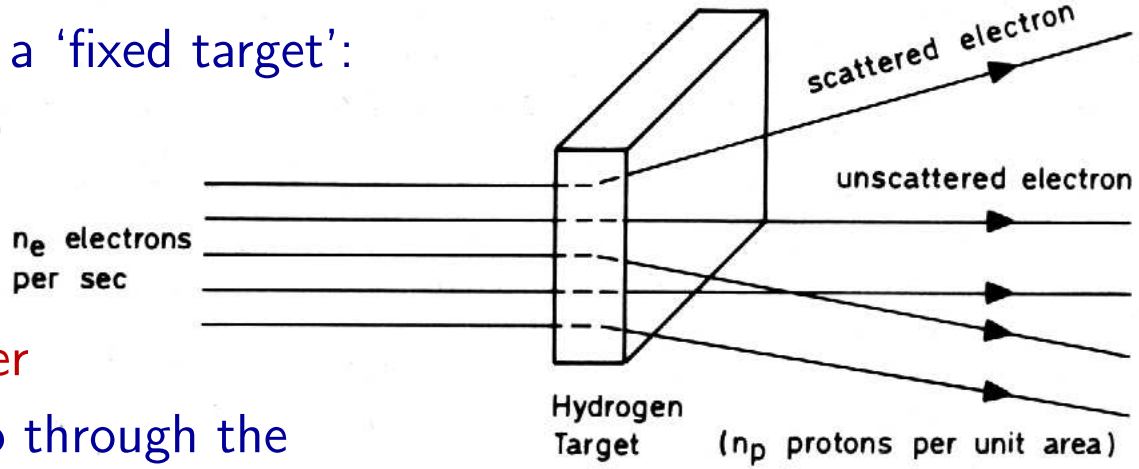
Decays provide lots of useful information, but you have to find a way of **producing** the unstable particles first. Those which are produced 'naturally' (Big Bang, cosmic rays) tend to decay without being detected by physicists, for various reasons).

First experiments used natural radioactive sources. Now giant accelerator complexes are being built, thanks to huge international efforts.

Until 1970's, experiments usually had a 'fixed target': a beam of particles (n_e per unit time) hits a (solid, liquid or gaseous) target (n_p protons per unit area). Some of the projectile particles (n_s per unit time) get scattered, some just go through the target unperturbed. The **cross section**



$$\sigma = \frac{n_s}{n_e \cdot n_p} \qquad \left[\text{dimension } \frac{T^{-1}}{T^{-1}L^{-2}} = L^2\right]$$

represents an effective area per target particle such that an incident particle hitting that area will be scattered.

More Cross Sections

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The cross section describes the probability of scattering: the cross section times the number of target particles per unit area gives the fraction of incident particles which will be scattered.

The cross section depends on the enrgy of the collision and on the types of colliding particles.

A beam of point-like particles falling on a target full of a gas consisting of small hard spheres of radius R will have a cross section $\sigma = \pi R^2$.

Particles do not behave like hard spheres, maybe more like slightly opaque objects with soft edges.

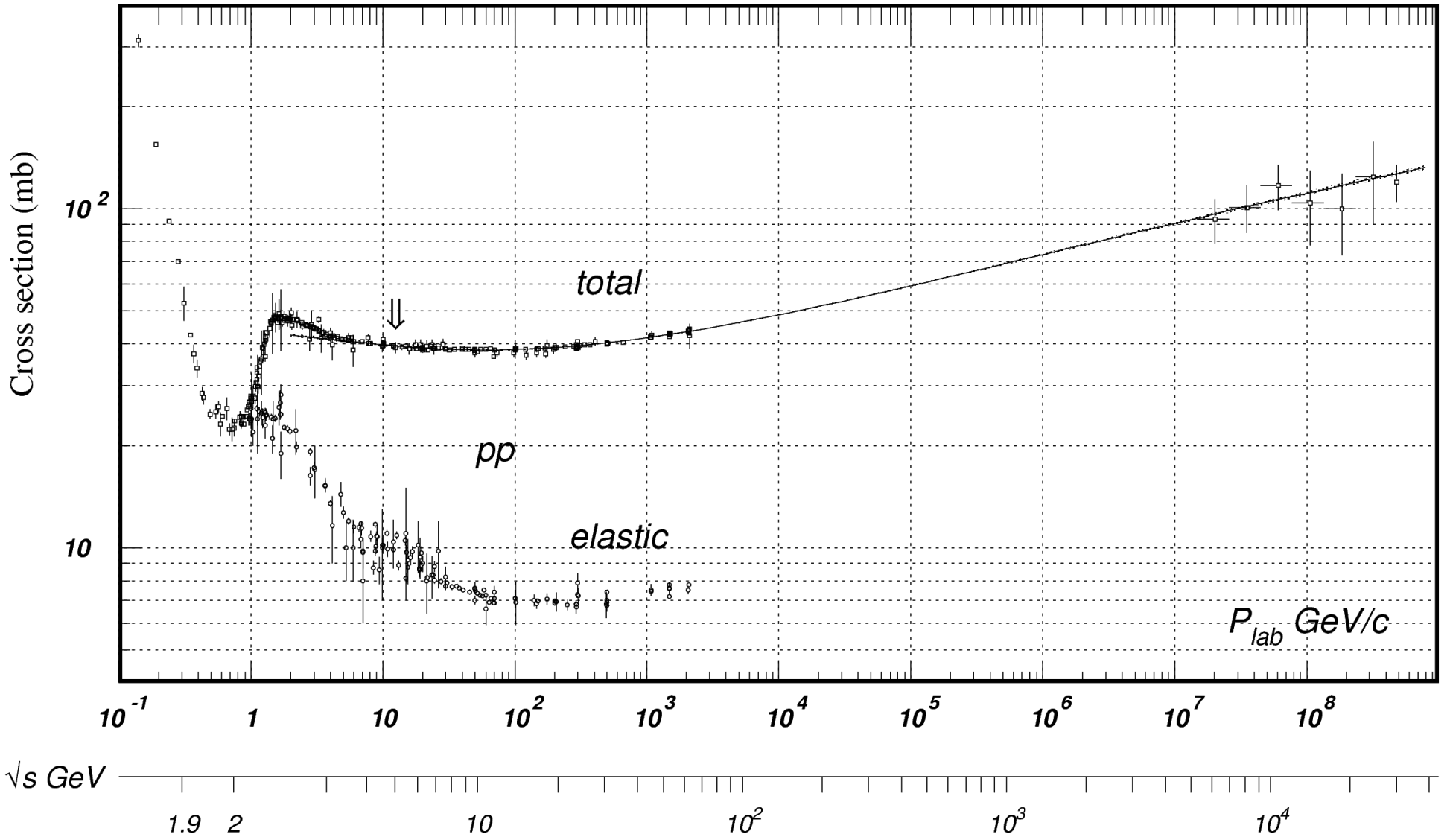
In general, the weaker the interaction between the beam and the target particles, and/or the shorter the range of the interaction, the smaller the cross section.

If the detector is capable of measuring the scattering angles θ, φ one can talk about **differential cross section**

$$\frac{d\sigma}{d\Omega} \equiv \frac{d\sigma}{\sin\theta d\theta d\varphi}$$

which represents the fraction of scattered particles per solid angle interval $d\Omega = \sin\theta d\theta d\varphi$, per target particle.

Proton-Proton Total and Elastic Cross Sections from PDG



Log-log? Elastic? Total? Tevatron? LHC? Highest energy points? $P_{\rm lab}$? $\sqrt{s}$? Threshold?

Q & A

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- Log-log: hard to plot otherwise: y range ~ 100, x range is 10^{10}.
- Elastic: no change in (kinetic) energy, i.e. $p + p \to p + p$
- Total: all possible processes summed over: elastic, $p + p \to p/n + p/n + N\pi$, resonance production, jet production...
- Tevatron: ~ 2 TeV — but not on the plot, since it's a $\bar{p}p$ collider
- LHC ~ 14 TeV: it's a pp collider – not on the plot either, sorry...
- Highest energy points: not accelerator, but cosmic ray measurement. Huge energies, unaccessible in man-made accelerators... Where do they come from?
- $P_{\rm lab}$: beam momentum in the target frame. In colliders, target frame $\neq$ lab frame.
- $\sqrt{s} \equiv E_{\rm cm}$: collision energy in the centre-of-mass frame of the colliding particles.
- Threshold: at small $P_{\rm lab}$ values, only elastic scattering is possible. The two curves start separating at the point where the processes $p + p \to p + p + \pi^0$ and $p + p \to p + n + \pi^+$ become possible.

$E_{\rm cm}$ versus $P_{\rm lab}$

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The kinematic variable s is one of three Mandesltam variables (see later). s in nothing else but the invariant mass of the colliding particles squared.

$$s \equiv (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2$$

In c.m. frame $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and hence $s = (E_1 + E_2)^2 = E_{\rm cm}^2$.

In symmetric high energy colliders (LEP, Tevatron, LHC) c.m.frame **is** the lab frame, $\mathbf{P}_{\rm beam1} = -\mathbf{P}_{\rm beam2}$, $E_{\rm beam1} = E_{\rm beam2} \equiv E_{\rm beam}$ and $\sqrt{s} = E_{\rm cm} = 2E_{\rm beam}$.

In a fixed target experiment,

$$\begin{aligned} P_{\rm lab} &\equiv |\mathbf{p}_1|, \quad E_1 = \sqrt{\mathbf{p}_1^2 + m_1^2}, \quad \mathbf{p}_2 = \mathbf{0}, \quad E_2 = m_2, \\ E_{\rm cm}^2 &= s = (p_1 + p_2)^2 = (E_1 + m_2)^2 - (\mathbf{p}_1 + \mathbf{0})^2 = m_1^2 + m_2^2 + 2E_1 m_2 \end{aligned}$$

At high energies one can neglect masses and $E_1 \approx P_{\rm lab}$, hence

$$\sqrt{s} = E_{\rm cm} \simeq \sqrt{2m_2 P_{\rm lab}}$$

Now you know why particle physicists prefer to build colliders!

Luminosity

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Let's define the cross section in terms appropriate for both fixed target and collider experiments.

Luminosity $\mathcal{L}$ is the number of colliding particles per unit time per unit area, and is measured in $\text{cm}^{-2}\text{s}^{-1}$.

Modern accelerators may have luminosity as high as 10^{34} $\text{cm}^{-2}\text{s}^{-1}$.

The number of particles dN per unit time, passing through area $d\sigma$, is

$$dN = \mathcal{L} d\sigma \qquad \Rightarrow \qquad \frac{d\sigma}{d\Omega} = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

The differential cross section is the number of particles per unit time scattered into solid angle $d\Omega$, divided by $d\Omega$ and by the luminosity.

$$\text{event rate} \; = \; \text{cross section} \; \times \; \text{luminosity}$$

The luminosity characterises the frequency of collisions (i.e. the capabilities of the accelerator), while the cross section describes the nature and properties of the collisons.

Fermi's Golden Rules

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So, there are two kinds of processes to be studied and described:

- ✦ Decays, described by the decay rate (width) Γ
- ✦ Scattering, described by the cross section σ

Quantum Field Theory provides us with recipes to calculate both of these, in terms of two ingredients:

1. The quantum-mechanical amplitude ('matrix element') for the process $\mathcal{M}$.
2. The phase space available for the final state of the process.

The Golden Rule for decays states:

$$\text{Decay Rate } d\Gamma \sim |\mathcal{M}(1 \to N)|^2 \times (\text{Phase Space of N})$$

while the Golden Rule for scattering says:

$$\text{Cross Section } d\sigma \sim \frac{1}{\text{Flux Factor}} \; |\mathcal{M}(1+2 \to N)|^2 \times (\text{Phase Space of N})$$

where I have ignored some constant factors for the moment.

Quantum Field Theory

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Here come several slides just to show you how a physical theory is built from scratch.

The aim is to convince you that a few basic principles may be enough to build a theory capable of describing and explaining most known phenomena in the world.

In non-relativistic quantum mechanics, physical quantities become operators acting upon the wave functions, which describe the particles.

In quantum field theory, wave functions themselves become operators, which can create or annihilate quanta of fields which represent particles: $\psi = \psi^+ + \psi^-$.

For example, an electron-creation operator $\psi^+(p)$, acting on vacuum $|0\rangle$, will create an electron with 4-momentum p:

$$\psi(p)^+|0\rangle = |u(p)\rangle$$

A physical theory in QFT is defined by a **Lagrangian density** L, which is constructed from those creation and annihilation operators.

Crash course on Quantum Field Theory

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Modern view of the world (in Physics!) relies on the assumption that all fundamental interactions are described by a Quantum Filed Theory (QFT).

A physical system is described by a Lagrangian (density) L, with is a function of various "generalised coordinates" and their first-order derivatives. In QFT, these generalised coordinates are in fact the field operators, representing various particles (which, in turn, depend on space-time coordinates).

Action S is defined as a space-time integral of L:

$$S = \int L\, dtdxdydz$$

Action must be a Lorentz invariant, which makes L Lorentz invariant too.

Clearly, S depends on L, the configuration of the fields and their space-time evolution.

The Action principle states, that the space-time evolution of the fileds, which actually takes place, must correspond to a **minimum** of S.

So, the condition $\delta S = 0$ can be used to find out the **equations of motion** which the fields must satisfy. This is in fact a generalisation of the classical Euler-Lagrange formalism.

QED Lagrangian

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The Lagrangian of Quantum Electrodynamics (QED), which describes a world consisting of charged fermions of one type (say, electrons, operator ψ) and bosons carrying the interaction (photons, operator A) looks like this:

$$L = L_{\text{fermion}} + L_{\text{boson}} + L_{\text{interaction}}$$

$$\begin{aligned} L_{\text{fermion}} &= i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi \\ L_{\text{boson}} &= -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \\ L_{\text{interaction}} &= -e\bar{\psi}\gamma_\mu\psi A^\mu \end{aligned}$$

where m and e are electron mass and charge, γ_μ are Dirac's matrices, and the summation over repeated indices $\mu, \nu = 0, 1, 2, 3$ is understood.

Starting from this Lagrangian, by applying the well known (Euler-Lagrange) formalism from classical mechanics, one can **derive** the equations which describe much of the dynamics of this world: Dirac's equation and Maxwell's equations.

This ambitious programme is beyond the Particle Physics course, but it is not as hard as it may seem. Finding solutions to those equations is a lot harder.

Noether's Theorem

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The Lagrangian is said to be invariant against some transformation, if it remains unchanged when this transformation is performed.

If the Lagrangian is invariant, so are the laws of physics derived from it.

Noether's theorem:

Every symmetry of nature yields a conservation law

$$\Updownarrow$$

Every conservation law reveals an underlying symmetry

Assuming that the nature is described by a Lagrangian, this theorem can be proven mathematically (again, not as hard as it seems).

Example:

Laws of nature are the same today as they were yesterday $\Rightarrow$

$\Rightarrow$ Lagrangian of the system is invariant under translations in time $\Rightarrow$

$\Rightarrow$ the energy of the system is conserved

Symmetries of the Lagrangian

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Several invariances and corresponding conservation laws are especially important:

Symmetry		Conserved Quantity
Translations in time	$\Leftrightarrow$	Energy
Translations in space	$\Leftrightarrow$	Momentum
Rotations in space	$\Leftrightarrow$	Angular momentum
Gauge transformations	$\Leftrightarrow$	Electric charge (current)
Spatial reflections	$\Leftrightarrow$	Parity
Charge conjugation	$\Leftrightarrow$	C-parity

Kinetic terms in the Lagrangian, describing free fermionic and bosonic fields L_{fermion}, L_{boson} are as symmetric and hence as invariant as they can possibly be, hence

the symmetry properties of the Lagrangian are determined by the interaction term.

Example: in weak interactions parity is **not** conserved, because the weak interaction Lagrangian term is not invariant under spatial reflections.

Phase Space

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Phase space is another name for momentum space. All phase space matters are referred to as **particle kinematics**, as opposed to **dynamics**, which is governed by interactions and is decribed by the amplitude $\mathcal{M}$.

The differential volume element of the phase space of a particle with 4-momentum p is

$$d^4p \equiv dE dp_x dp_y dp_z$$

However, the four components are not independent: for any real particle $E^2 = m^2 + |\mathbf{p}|^2$. After applying this condition, the phase space element per each final particle i becomes

$$\frac{dp_{ix} dp_{iy} dp_{iz}}{2E_i} \equiv \frac{d^3\mathbf{p}_i}{2E_i}, \qquad E_i = \sqrt{m_i^2 + |\mathbf{p}_i|^2}$$

In addition, each component of the final 4-momenta must satisfy the energy-momentum conservation:

$$(P_{\text{initial}})_\mu = \sum_{i=\text{final}}^{N} (p_i)_\mu \qquad \mu = 0, 1, 2, 3$$

where the total initial 4-momentum P_{initial} is usually well-known: for decays, it's the 4-momentum of the decaying particle; for collisions it's the sum of the 4-momenta of the two colliding particles.

Threshold

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One interesting consequence of energy conservation is immediately obvious. For a decay of a particle of mass M at rest, into two particles with masses m_1 and m_2, we have

$$M = E_1 + E_2$$

However, $E_1 = \sqrt{m_1^2 + |\mathbf{p}_1|^2} \geq m_1$ and $E_1 = \sqrt{m_2^2 + |\mathbf{p}_2|^2} \geq m_2$, hence this condition will never be met if $m_1 + m_2 > M$. In order for the decay to proceed, some (whatever small) amount of energy should be left over to create some 3-momentum: a particle cannot decay if the sum of final masses exceeds or is equal to the initial mass.

A similar analysis can be performed for collisions, but M should be replaced with $E_{cm} = \sqrt{s}$. This quantity should exceed the sum of the masses of final particles, otherwise the process cannot go ahead.

The minimum c.m. energy at which a particular reaction is kinematically allowed is called the **threshold** of this reaction:

$$(E_{\rm cm})_{\rm threshold} = \text{sum of masses of final particles}$$

At energies just above threshold the final particles will be almost at rest in c.m.s.

Generic Process $2 \to 2$

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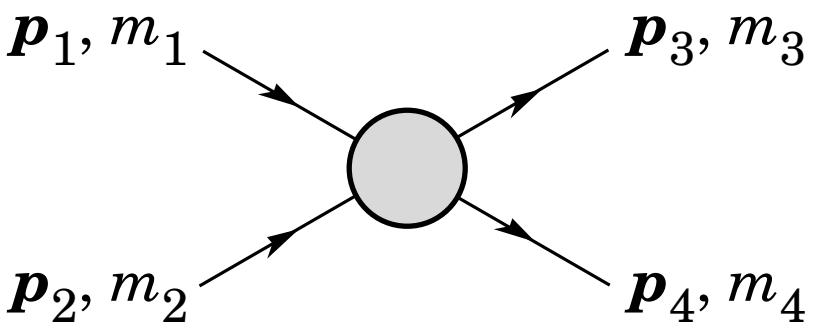
Two particles with masses m_1, m_2 and momenta $\mathbf{p}_1, \mathbf{p}_2$ collide.

Two particles with masses m_3, m_4 and momenta $\mathbf{p}_3, \mathbf{p}_4$ are produced.



In general, all masses can be different. Calculate energies:

$$E_1 = \sqrt{\mathbf{p}_1^2 + m_1^2}, \quad E_2 = \sqrt{\mathbf{p}_2^2 + m_2^2}, \quad E_3 = \sqrt{\mathbf{p}_3^2 + m_3^2}, \quad E_4 = \sqrt{\mathbf{p}_4^2 + m_4^2}$$

Energy and momentum conservation states:

$$E_1 + E_2 = E_3 + E_4 \qquad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$$

Let's see how far we can go with this information alone (and a bit of common sense).

How Many Variables?

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So, we have $4 \times 4 = 16$ quantities describing this collision: 4 energies and 3×4 components of momenta. How many free, interesting and non-trivial variables do we actually have?

In $1 \to 2$ decay, $M \to m_1 + m_2$, we ended up with **no** free variables at all: after fixing the reference frame, all components of both final momenta, as well as both final energies, were uniquely calculated through the three masses.

Start with 16 variables.

- 4 energies are calculated from momenta and masses.
- two initial momenta (i.e. 6 components) are fixed: we know what we are colliding!
- the sum of the two final energies is fixed by the energy conservation (one equation).
- the sum of the two final momenta is fixed by the momentum conservation (three equations).
- If the initial particles are spinless (or in fact are unpolarised) there must be azimuthal symmetry around the collision axis, i.e. one variable, the azimuthal angle ϕ, is trivial $\Rightarrow$ easily "integrated out".

How many "true" variables are we left with? Do the maths...

Mandelstam Variables s, t, u

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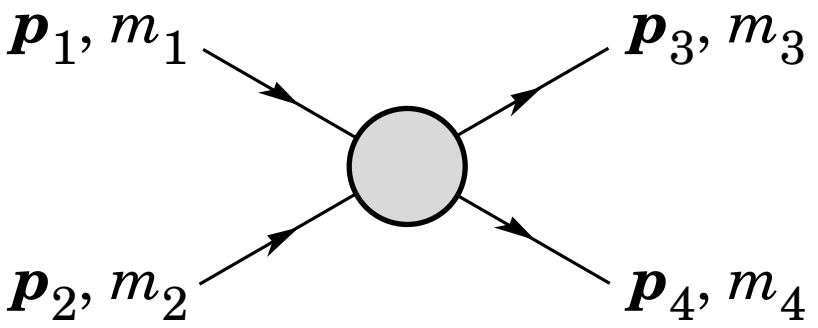
Define three **invariant** kinematic variables:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_2 - p_3)^2 = (p_1 - p_4)^2 \end{aligned}$$



Note: p_1, p_2, p_3, p_4 are all 4-vectors here!

Using 4-momentum conservation, one can immediately show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

So, one of the three (usually u) is obsolete: knowing masses, s and t one can calculate u.

s is the invariant mass squared of the two initial (or two final) particles. Show that $\sqrt{s}$ is equal to the centre-of-mass energy of the collision:

$$\sqrt{s} \equiv E_{\rm cm}$$

Momenta of initial and final particles can be calculated with the formulae used in the $M \to m_1 + m_2$ decay, by replacing M with $\sqrt{s}$.

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t is the square of the 4-momentum transfer from particle 1 in the initial state to particle 3 in the final state.

If all masses are equal, $m_1 = m_2 = m_3 = m_4 \equiv m$, then

$$|\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}_3| = |\mathbf{p}_4| \quad \equiv \quad P$$

$$E_1 = E_2 = E_3 = E_4 \quad \equiv \quad E = \sqrt{P^2 + m^2} = \frac{\sqrt{s}}{2}$$

It's easy to show that in this case

$$t = -2P^2(1 - \cos\theta) = -4P^2 \sin^2\frac{\theta}{2}, \qquad u = -2P^2(1 + \cos\theta) = -4P^2 \cos^2\frac{\theta}{2},$$

where θ is the scattering angle of particle 3. Note: both t and u are (usually) negative!

At very high energies, $s \gg m_1^2 + m_2^2 + m_3^3 + m_4^2$, with $\mathbf{p}_1$ along $+z$ axis, one has:

$$t \approx -\frac{s}{2}(1 - \cos\theta), \qquad u \approx -\frac{s}{2}(1 + \cos\theta),$$

Elastic Cross Section

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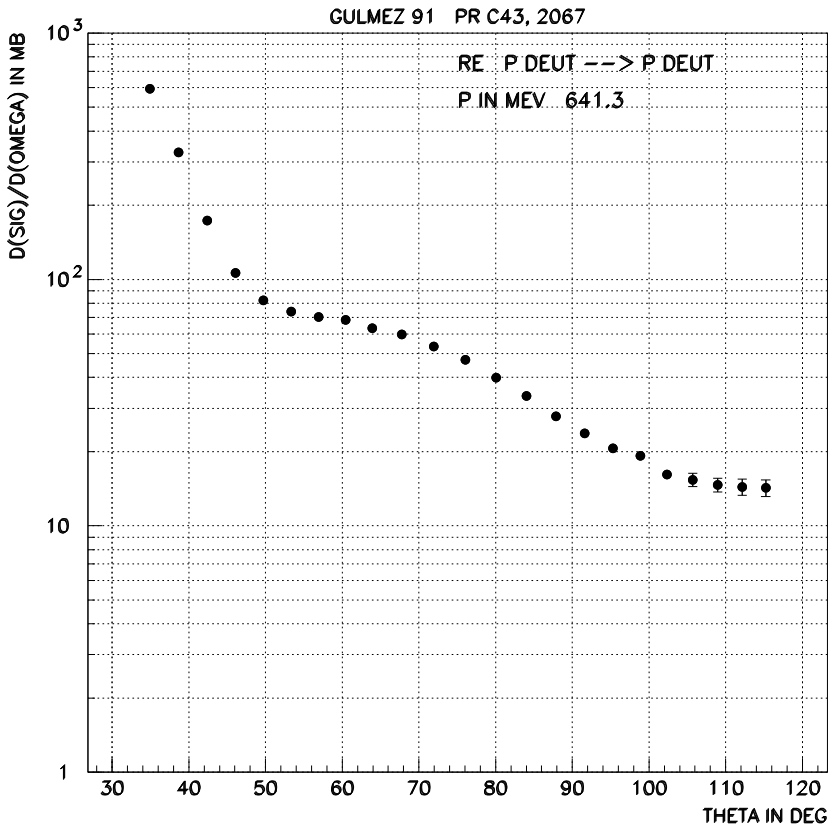
Either θ, the c.m.s. scattering angle of particle 3, or t, the 4-momentum transfer squared, may be chosen as our only independent variable used to describe the collision $2 \to 2$.

One can derive simple relations between various differential cross sections:

$$\frac{d\sigma}{d\cos\theta} = 2\pi \frac{d\sigma}{d\Omega} = \frac{s}{2}\frac{d\sigma}{d|t|}$$

Of course, in a symmetric collider, θ is the angle measured directly. But the above formula is only true if θ is the scattering angle in c.m.s.

However, t is **invariant**, and is thus a convenient way of comparing various experiments.



Differential elastic cross sections usually have a huge peak near $\theta = 0$ which also corresponds to $t = 0$.

Electron-Muon Scattering

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Consider elastic electron-muon scattering, i.e. the reaction

$$e^- + \mu^- \to e^- + \mu^-$$

To keep things simple, let's neglect all masses, i.e. consider the high energy limit: in c.m.s. all four particles have momenta with magnitudes P, and θ is the scattering angle.

The differential cross section describing electron-muon scattering:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8P^2} \frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}}$$

where α is the fine structure constant, $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$.

Show that it can be re-written in terms of s, t, u like this:

$$\frac{d\sigma}{dt}(e^-\mu^- \to e^-\mu^-) = \frac{2\pi\alpha^2}{s^2} \frac{s^2 + u^2}{t^2}$$

Do charges matter? What's the difference between $e^-\mu^-$ and, say, $e^-\mu^+$ cross sections?

Golden Rules Re-visited

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Armed with all the knowledge about true free variables, one can show that Fermi Golden Rules for processes involving just two final particles look actually quite simple:

$$\begin{aligned}
\Gamma(1 \to 2) &= \frac{p_f}{8\pi M^2} \left|\mathcal{M}_{1\to 2}\right|^2 \\
\frac{d\sigma(2 \to 2)}{d\Omega} &= \frac{1}{64\pi^2 s} \frac{p_f}{p_i} \left|\mathcal{M}_{2\to 2}(s, \cos\theta)\right|^2 \\
\frac{d\sigma(2 \to 2)}{dt} &= \frac{1}{64\pi s} \frac{1}{p_i^2} \left|\mathcal{M}_{2\to 2}(s, t)\right|^2
\end{aligned}$$

Quantities p_i, p_f are the moduli of the 3-momenta of initial and final particles, respectively, in the c.m.s.:

$$\begin{aligned}
p_i &= \frac{1}{2\sqrt{s}} \sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]} \\
p_f &= \frac{1}{2\sqrt{s}} \sqrt{[s - (m_3 + m_4)^2][s - (m_3 - m_4)^2]}
\end{aligned}$$

These are obtained from E_1 and E_2 calculated before. For decays, s should be replaced by M^2. Obviously, at high energies one neglects masses and $p_i = p_f = \sqrt{s}/2$.

Electron-Muon scattering: the Matrix Element

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Comparing the Golden Rule with the expression for the cross section

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{s^2}\frac{s^2+u^2}{t^2}$$

we can separate the kinematical factor and the matrix element squared to obtain

$$|\mathcal{M}_{e\mu\to e\mu}(s,t)|^2 = 32\pi^2\,\alpha^2\frac{s^2+u^2}{t^2}$$

Given this matrix element, one can immediately find the matrix element describing another related process:

$$e^+ + e^- \to \mu^+ + \mu^-$$

This is accomplished using cross-symmetry.

Cross-symmetry

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Here is the recipe:

- Take the reaction $e^- + \mu^- \to e^- + \mu^-$.
- Replace initial μ^- with its antiparticle μ^+ and change the sign of its 4-momentum p_2. Now the μ^- in the in itial state is transformed into a μ^+ in the final state.
- Replace final e^- with its antiparticle e^+ and change the sign of its 4-momentum p_3. Now the e^- in the final state is transformed into a e^+ in the initial state.
- Now we have the reaction $e^+ + e^- \to \mu^+ + \mu^-$.

The interesting thing is that this new process is still described by the same matrix element, with the momenta changed accordingly:

$$\mathcal{M}_{e^-\mu^-\to e^-\mu^-}(p_1, p_2, p_3, p_4) = \mathcal{M}_{e^+e^-\to\mu^+\mu^-}(p_1, -p_3, -p_2, p_4)$$

But this change of momenta is equivalent to swapping s and t. Hence

$$\begin{aligned} |\mathcal{M}_{e^+e^-\to\mu^+\mu^-}(s,t)|^2 &= 32\pi^2\,\alpha^2\,\frac{t^2+u^2}{s^2} \\ \frac{d\sigma}{dt}(e^+e^-\to\mu^+\mu^-) &= \frac{2\pi\alpha^2}{s^2}\,\frac{t^2+u^2}{s^2} \end{aligned}$$

Physical theories: how things work

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The development cycle of a physical theory:

1. Find (or create!) a mathematical concept/model/theory which has the appropriate structure and properties relevant to your area of physics.

2. Formulate your problem in terms of this mathematical theory.

3. Solve the mathematical problem (nothing to do with physics whatsoever!)

4. Try to understand and interpret the solution.

5. If/when this solution becomes unsatisfactory (new data, higher precision), go to 1.

History of physics knows many examples illustrating this cycle (Newton, Shrödinger, Bohr, Heisenberg, Einstein, Dirac, Feynman. . .)

Symmetries and Groups

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- Symmetry is our main (if not the only!) weapon in the quest for understanding fundamental laws of Nature.
- This is especially true for areas of physics which are so far away from our every-day experience, that our intuition is often misleading.
- Relies on the existence of mathematical concepts and methods (some already existed, some needed to be deveopled by physicists!)
- Most of the symmetries in Particle Physics are described by various kinds of **groups** of matrices.
- Still 'not entirely clear' why these methods have anything to do with nature — but this is not our main worry right now...

Symmetries and transformations

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Transformation $\Rightarrow$ invariance (or covariance) $\equiv$ symmetry

Invariance: no change!

Covariance: there are changes which happen in a very particular way

Clearly, invariance is a special case of covariance

Examples:

- Mirror (reflection, left-right)
- Rotations in 2D and 3D spaces – scalars, vectors etc.
- Car $\rightarrow$ traffic $\rightarrow$ spontaneous breaking (Britain vs Europe)
- Isospin ($p \leftrightarrow n$, pions, hadronic amplitudes)
- Quark model (hadron classification, magnetic moments)
- Special theory of relativity ($c =$ const with boosts, relativistic kinematics)
- Dirac's equation (Lorentz covariance $\rightarrow$ bi-spinors)
- ...

Definition of a group

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A collection of elements T_i is a group if the following four **group axioms** are satisfied:

1. Every pair of elements T_i, T_j is associated to an element T_k from the same collection, by an operation (called 'group multiplication'):
$$T_i T_j = T_k$$
2. This "group multiplication operation" is associative, i.e. for any three elements
$$(T_k T_m) T_n = T_k (T_m T_n)$$
3. There exists an identity element E such that for any element T_k
$$E T_k = T_k E = T_k$$
4. For each T_k there is an inverse T_k^{-1} such that
$$T_k T_k^{-1} = T_k^{-1} T_k = E$$

Some general remarks

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The definition looks simple enough — but has far-reaching concequenses. . .

Group theory is the basis of a big area of modern maths called 'algebra'.

Mathematicians (and some theoretical physicists) are very good at operating with abstract concepts.

More complex mathematical structures exist: sets with *two* group operations ('rings') or *three* group operations ('fields').

Fortunately, these are well outside our course. We particle physicists try to remain in touch with experimental reality. So far no burning need to go into these.

On the other hand: **Group theory proved itself to be very useful in many areas of (particle) physics.**

Please make sure you understand the difference between **associative** and **commutative** operations:

Associativity $(AB)C = A(BC)$ is about the sequence the operation is applied to the operands (you need at least three of them). A group operation *must be associative!*

Commutativity $AB = BA$ is about the sequence of the operands (two at a time). A group operation may or may not be commutative – more about this later.

Examples

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- Two numbers: 1 and -1, under regular multiplication
- Two numbers: 0 and 1, under 'modulo-2 addition'
- Rotations of a regular hexagon by $n \times 60^\circ, n = 0, 1, 2, 3, \ldots$
- Rotations of a circle by an arbitrary angle ϕ
- Others?...

THINK! For each of these:

What are the elements of the group? How many elements does the group have?

What is the group multiplication operation? Is it associative?

What is the identity element?

Can you find the inverse of each element?

Multiplication table of a discrete group

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Example: Cyclic group with 3 elements, C_3:

Elements: $0, 1, 2$; group multiplication: addition modulo 3.

Group multiplication table:

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Check that all products belong to the original set $0, 1, 2$

Check that each row (or column) contains identity element "0" (i.e. there is an inverse).
Find the inverse for each element!

Addition is associative, so all four group axioms are satisfied, and hence this is indeed a group.

Cyclic group C_n

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Cyclic group C_n consists of n elements $a_1, a_2, \ldots, a_n$, which have the following properties:

$$\begin{aligned} a_1 &\equiv b \\ a_2 &= a_1 b = b^2 \\ a_3 &= a_2 b = b^3 \\ &\ldots \\ a_{n-1} &= a_{n-2} b = b^{n-1} \\ a_n &= a_{n-1} b = b^n = E \quad \text{(Identity element)} \end{aligned}$$

I.e. all elements of the group are "powers" of one element, whose n-th power is the identity element.

In general, from the group theory point of view, we do not have (or need) to know what kind of object this b is, or what "real-world" mathematical operation the group multiplication corresponds to, **as long as the above propereties are satisfied**.

E.g. it could be a set of unimodular complex numbers under multiplication, or integers under addition-modulo-n, or something else.

All these would be **equivalent** from group-theoretical point of view.

Isomorphism

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If there is a **one-to-one correspondence** between the elements of two groups G_A and G_B (in both directions), then the two groups are **isomorphic** (i.e. equivalent) to each other.

This would imply that:

- the identity elements are mapped onto each other;
- if elements A_1 and A_2 are mapped onto B_1 and B_2, respectively, then the product $A_3 = A_1 A_2$ is mapped onto $B_3 = B_1 B_2$, and vice versa.

So, the group of n integers under addition-modulo-n is isomoprphic to the cyclic group C_n.

Similarly, the grouop of dicrete rotations of unimodular numbers on complex plane, which consists of n elements

$$\exp\left(\frac{2\pi i k}{n}\right), \qquad k = 0, 1 \ldots n-1$$

under normal multiplication, is also isomorphic to C_n — in fact, here b simply is $b = \exp\left(\frac{2\pi i}{n}\right)$.

For discrete groups, isomorphism can be established by comparing that the multiplication tables are equivalent.

From group-theoretical point of view, all isomorphic groups are equivalent. Any group G may have many different explicit realisations ("representations"), but these will be equally applicable to any other group which is isomorphic to G.

Types of groups. Subgroups

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- If a group has a finite number of elements, it is a *finite* group; otherwise it is *infinite*.
- The trasformations — and hence the groups — can be *discrete* or *continuous*.

  All continuous groups are infinite.
- If the group operation is commutative, i.e. $T_i T_j = T_j T_i$ for **all** elements of the group, the group is *Abelian*, otherwise it is *non-Abelian*.

  Abelian groups usually have a fairly simple structure.

  Non-Abelian groups are much more interesting/useful, but can be very complicated.
- A subset of a group which itself is a group, with the same group multiplication operation and the same identity element, is called a *subgroup* of the original group.
- Identity alone and the group itself are the two *trivial* subgroups, others are *non-trivial*.
- A continuous group can have discrete subgroups. In fact, any cyclic group is a subgroup of the group of continuous rotations on a plane.

Continuous groups

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In continuous groups, group elements can be labelled by a number r of continuous real parameters, denoted collectively by, say, $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_r)$:

$$T_i \to T(\alpha)$$

A very simple example: group of complex numbers with unit modulus, $e^{i\phi}$ w.r.t. multiplication:

$$e^{i\phi_1} e^{i\phi_2} = e^{i\phi_3} \qquad \text{where} \qquad \phi_3 = \phi_1 + \phi_2$$

Here, the "set" of parameters α is just a single parameter ϕ (which can be restricted to the range $-\pi < \phi \leq \pi$).

Elements of this group represent rotations of the unit circle by angle ϕ on a complex plane.

Note that the inverse of each element is its complex conjugate:

$$(e^{i\phi})^{-1} = e^{-i\phi} = (e^{i\phi})^*$$

which makes $e^{i\phi}$ a *unitary* matrix of dimension 1×1. Hence, this group is called $U(1)$.

$U(1)$ looks almost trivial, but it forms the basis of Quantum Electrodynamics (among other things), so treat it with respect!

Lie groups – I

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For a continuous group, by convention, $T(0) = E$, i.e. the identity element corresponds to all parameters set to zero.

identity element $\Rightarrow$ no transformation $\Rightarrow$ all parameters$= 0$

Group multiplication now implies that

$$T(\alpha)T(\beta) = T(\gamma(\alpha, \beta))$$

i.e. the product of $T(\alpha)$ and $T(\beta)$ is another element of the same group $T(\gamma)$, where parameters $\gamma = \gamma(\alpha, \beta)$ are continuous functions of parameters α and β.

Now, **if** $\gamma(\alpha, \beta)$ are not just continuous, but **analytic** functions of α and β, at and around the "origin" $\alpha = 0, \beta = 0$ then the group is a Lie group.

A function is analytic at some point if it is infinitely differentiable at that point, and hence can be locally represented by a convergent infinite power series within a certain area around that point.

Lie groups – II

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So analyticity means that you can move from one group element to a 'neighbouring' element by analytic continuation in the parameter space, hence we can reach any element within the 'continuously connected' area of the parameter space, starting from the original element.

In particular, Lie proved that **the properties of all the elements of a Lie group — which are continuously reachable from the identity element — are determined by the immediate neighbourhood of the identity element.**

This means that the **generators** of a Lie group — the derivatives of a general group element with respect to a parameter, near the identity element — acquire a very important, if not definitive, role in the properties of the group. More about them later.

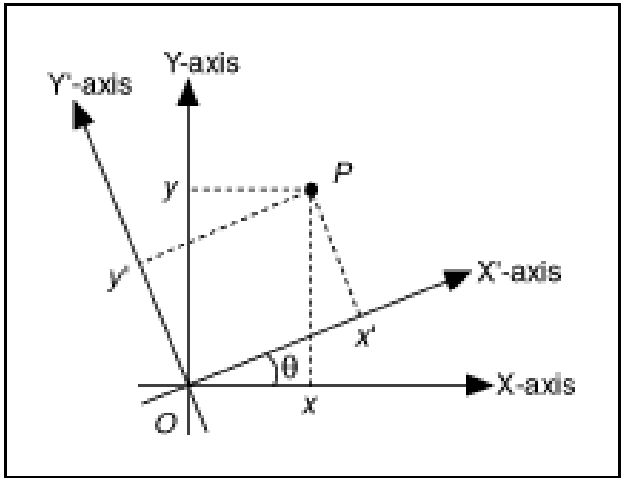
All continuous groups considered in this course will be Lie groups.

In particular, $U(1)$ is a Lie group, as

$$\phi_3 = \phi_3(\phi_1, \phi_2) = \phi_1 + \phi_2$$

is an analytic function of ϕ_1 and ϕ_2.

2D rotations



Another example of a Lie group: 2D rotations, one parameter ϕ:

$$\begin{aligned} x' &= \cos\phi\, x + \sin\phi\, y \\ y' &= -\sin\phi\, x + \cos\phi\, y \end{aligned}$$

or, in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Or, equivalently

$$\mathbf{x}' = R(\phi)\mathbf{x} \qquad \mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

It's not too hard to check that matrices $R(\phi)$ form a group under matrix multiplication.

In particular, $R(\phi_1)R(\phi_2) = R(\phi_3)$ with $\phi_3 = \phi_1 + \phi_2$.

This group is called $SO(2)$ - a "special" (with $\det R = +1$) group of real orthogonal 2×2 matrices. More about these later.

Isomorphism of $SO(2)$ and $U(1)$

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There is, quite obviously, a bi-directional one-to-one correspondence between the elements of the group $U(1)$, given by $e^{i\phi}$, and the elements of the group $SO(2)$, as represented by the matrix $R(\phi)$:

$$\begin{aligned} e^{i\phi} \quad &\Leftrightarrow \quad R(\phi) \\ e^{i\phi_1} e^{i\phi_2} = e^{i(\phi_1+\phi_2)} \quad &\Leftrightarrow \quad R(\phi_1)R(\phi_2) = R(\phi_1+\phi_2) \end{aligned}$$

This is indeed a one-to-one mapping – i.e. isomorphism – of two groups.

From the group-theoretical point of view, isomorphic groups are identical

It's the group structure that matters, **not** the particular implementation.

Either set of objects –

– complex numbers of unit modulus $e^{i\phi}$ or 2×2 matrices $R(\phi)$ –

– can be used to **represent either of the groups** $U(1)$ and $SO(2)$.

Group representations

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In general, **any group G can be mapped**, element by element, **onto a group of matrices** which has the identical group structure as the group G, with respect to the matrix multiplication as the group operation.

This group of matrices is said to be a **representation** of the original group G.

So, 1×1 unitary matrices form a **representation** of group $U(1)$. Of course, we used 1×1 unitary matrices with respect to (matrix) multiplication to **define** the group $U(1)$, so this is trivial.

Similarly, 2×2 orthogonal matrices $R(\phi)$ obviously form a 2×2 representation of the group $SO(2)$.

However, it's much more interesting that, because the groups $U(1)$ and $SO(2)$ are isomorphic to each other, **both sets represent both groups**:

- 1×1 unitary matrices form a 1×1 complex representation of group $SO(2)$, while
- matrices $R(\phi)$ form a 2×2 real representation of the group $U(1)$.

In fact, **any group has an infinite number of different matrix representations.**

Also: **the same group can be represented by matrices of different dimensions.**

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So, matrices $R(\phi)$ form a 2-dimensional representation of $SO(2)$.

We have seen that these describe the transformation of 2D vectors $\{x, y\}$ under the rotation of the coordinate system by the angle ϕ.

However, there are other types of objects, which may require larger or smaller matrices to describe **("represent")** their transformation under $SO(2)$.

In particular, a **scalar** object is invariant under rotations, and hence the corresponding transformation is

$$s' = s$$

This, in fact, is the definition of a scalar. Hence, the matrices from the scalar representation of $SO(2)$ are rather trivial: they have dimension 1×1 and are equal to 1.

So, there are (at least) two 1×1 representations of $SO(2)$:

- scalar, 1
- unitary, $e^{i\phi}$

There are, of course, many other (and larger) representations as well.

Isomorphism and Homomorphism

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Apart from isomorphism (one-to-one mapping), there may be another type of relation between two groups:

one element in one group is mapped to 2 or more elements in the other. This is called **homomorphism**.

The relationship between the two 1×1 representations of $SU(2)$, scalar (1) and unitary $(e^{i\phi})$ is a rather extreme example of homomorphism: **All** elements from the unitary representation are mapped onto the only element, identity, in the scalar representation.

Can we find other 1×1 representations of $SO(2)$ (or, equivalently, of $U(1)$)?

Of course: $e^{-i\phi}$ is a wonderful example. $e^{2i\phi}$ is another.

How about 2×2 representations other than the familiar $R(\phi)$ from page 62? Here are a few:

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \qquad \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{2i\phi} \end{pmatrix} \qquad \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

Any of these form a full isomorphic representation: as the matrices are diagonal, different rows and columns do not mix. So it's easy to spot the 1×1 "fundamental" representations inside these "composite" ones.

But are these new 2×2 representations related to the "good old" 2×2 representation $R(\phi)$?

Equivalent representations

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So, in general, the same group can be represented by matrices of different sizes, but they all must be square matrices — the type of the object under transformation must remain the same! I.e. for an object $\mathbf{x}$ with n components, the matrix R_n needs to be $n \times n$:

$$\mathbf{x}' = R_n \mathbf{x} \qquad \Rightarrow \qquad \begin{pmatrix} x'_1 \\ \dots \\ x'_n \end{pmatrix} = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \dots & \dots & \dots \\ r_{n1} & \dots & r_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$$

Let R_n be a representation of group G of dimension $n \times n$, and S be a non-singular constant $n \times n$ matrix (i.e. the inverse S^{-1} exists). Then, the set of matrices

$$\tilde{R}_n = S^{-1} R_n S$$

also forms a representation of the same group G. The sets R_n and $\tilde{R}_n$ are related through the **equivalence transformation**, and are equivalent representations.

So, if, for example, S is not a diagonal matrix, the equivalent representation to any of the three 2×2 representations in page 66 will not, in general, be diagonal.

Reducible and irreducible representations

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Now, here comes the important part: if you can find S such that some particular matrix representation of a group takes a block-diagonal form (with the rest filled with zeroes), then this representation is deemed **reducible**. If there is no such S, then the representation is **irreducible**.

Irreducible representations play the role similar to that of the basis vectors in some space, or of the eigenfunctions of an operator:

you can, in general, expand any representation into a superposition of irreducible representations.

Try applying the equivalence transformation on $R(\phi)$ from p.62 using the following choice of matrix S:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

(need to find S^{-1} first of course...)

More about irreducible representations later...

Generators of a Lie group

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A **generator** J of a Lie group is an operator performing an infinitesimal (infinitely small) transformation near the identity element of the group (which stands for 'no transformation').

E.g. a finite transformation by angle ϕ can be accomplished by applying a large number n of successive transformations by small angles $\delta\phi = \phi/n$, where $n \to \infty$.

$$\begin{aligned} R(\phi) &= R(\delta\phi)R(\delta\phi)R(\delta\phi)\ldots R(\delta\phi) \qquad (n \text{ times}) \\ &= [R(\delta\phi)]^n \mid_{n\to\infty} \\ &= (1 + i\delta\phi J)^n \mid_{n\to\infty} \\ &= \exp(i\phi J) \end{aligned}$$

This is called **exponentiation** of the infinitesimal transformation **generated by** the operator (matrix) J.

Factor i was introduced for convenience. Use familiar techniques to find the only generator of $SO(2)$:

$$J = \frac{1}{i}\frac{dR(\phi)}{d\phi}\Big|_{\phi=0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

This is illustrated here for $SO(2)$ and its 2×2 representation $R(\phi)$, but is rather general.

A Lie group with k continuous parameters will have k generators.

All properties of a Lie group are determined by its generators.

Exponentiation of a Generator

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Look what we have got:

$$R(\phi) = \exp(i\phi J), \qquad J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

But we know that

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

What does this mean?!

In order to understand this better, let's do a harmless exercise:

Calculate J^n for any n:

$$J^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv I \qquad J^3 = J^2 J = J \qquad \text{etc.}$$

Any even power of J is the identity matrix; any odd power of J equals J.

Exponentiation of a Generator - II

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Remember that

$$\exp(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots$$

For our matrix this means:

$$R(\phi) = \exp(i\phi J) = I + i\phi J - \frac{\phi^2}{2}J^2 - i\frac{\phi^3}{6}J^3 \ldots$$

Let's collect even powers of J first:

$$I - \frac{\phi^2}{2}I + \ldots = (1 - \frac{\phi^2}{2} + \ldots)I = I\cos\phi$$

Now let's collect odd powers of J:

$$i\phi J - i\frac{\phi^3}{6}J + \ldots = i(\phi - \frac{\phi^3}{6} + \ldots)J = iJ\sin\phi$$

Finally

$$R(\phi) = I\cos\phi + iJ\sin\phi = \cos\phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

So, indeed, exponentiation of a generator yields a full finite transformation!

Coordinate Transformations: active and passive

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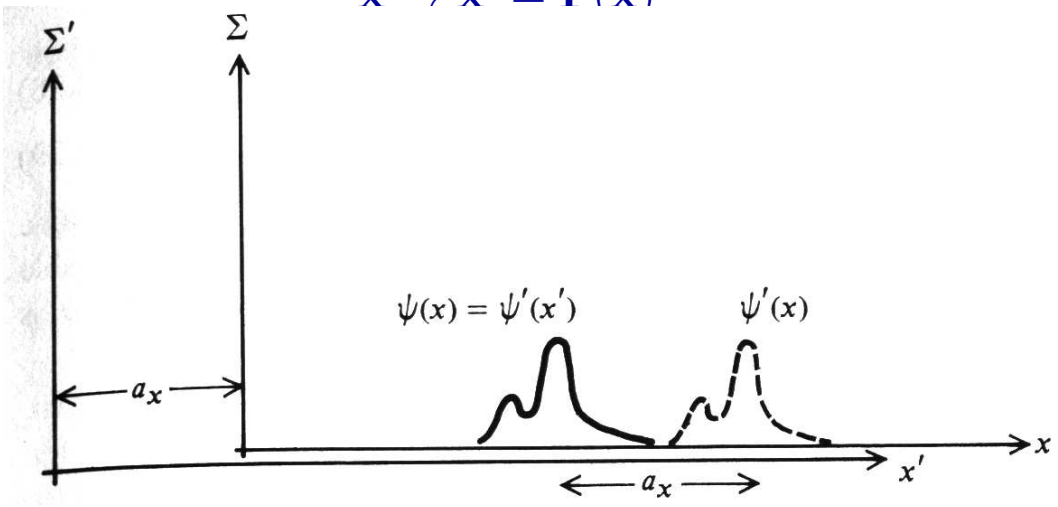
Consider transformation of coordinates (e.g. translation by a_x), from reference system Σ to system Σ':

$$\mathbf{x} \to \mathbf{x}' = \mathbf{F}(\mathbf{x})$$



Solid line: passive interpretation, **same physical wave function in both frames** Σ and Σ':

$$\psi'(\mathbf{x}') = \psi(\mathbf{x})$$

This is more mathematical ("change description").

Dashed line: active interpretation, **a new wave function in the old frame** Σ:

$$\psi'(\mathbf{x}) = \psi(\mathbf{F}^{-1}(\mathbf{x}))$$

This is more physical ("change object").

Coordinate Transformations: active and passive — II

The wavefunction of a QM state at any given physical point should have the same physical value in both Σ and Σ':

$$\psi'(\mathbf{x}') = \psi(\mathbf{x}) \qquad \text{(passive interpretation)} \tag{1}$$

Now, let's solve $\mathbf{F}(\mathbf{x})$ for $\mathbf{x}$:

$$\mathbf{x} = \mathbf{F}^{-1}(\mathbf{x}')$$

Equation (1) can now be re-written like this:

$$\psi'(\mathbf{x}') = \psi(\mathbf{F}^{-1}(\mathbf{x}'))$$

Since $\mathbf{x}$ is nowhere to be seen any more, we can rename $\mathbf{x}'$ as $\mathbf{x}$:

$$\psi'(\mathbf{x}) = \psi(\mathbf{F}^{-1}(\mathbf{x})) \qquad \text{(active interpretation)}$$

So, the two interpretations lead to two different treatments.

At the end of the day, these two treatments are equivalent, but one may be preferred to the other in various circumstances.

In particular, the active picture is very useful when calculating generators etc.

Translations in space

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Consider coordinate transformation such that

$$\mathbf{x}' = \mathbf{x} + \delta\mathbf{a} \qquad \mathbf{x} = \mathbf{x}' - \delta\mathbf{a}$$

This is a translation of the coordinate frame by a (small) vector $\delta\mathbf{a}$.

What happens to the wave function under such transformation?

$$\psi'(\mathbf{x}) = U(\delta\mathbf{a})\psi(\mathbf{x}) = \psi(\mathbf{x} - \delta\mathbf{a})$$

[We have used active interpretation here]

For simplicity, assume $\delta\mathbf{a}_x = \delta a, \delta\mathbf{a}_y = 0, \delta\mathbf{a}_z = 0$.

Then, we only consider x-dependence, and we can expand:

$$\begin{aligned} U(\delta a)\psi(x) = \psi(x - \delta a) &= \psi(x) - \delta a \frac{\partial\psi(x)}{\partial x} \\ &= \psi(x) - i\delta a\left(-i\frac{\partial\psi(x)}{\partial x}\right) = \left[1 - i\delta a\left(-i\frac{\partial}{\partial x}\right)\right]\psi(x) \end{aligned}$$

The operator $P_x \equiv -i\frac{\partial}{\partial x}$ is the generator of translation operation along x-axis.

At the same time, this is the quantum-mechanical operator of the x-component of momentum.

Rotations in 3D: around z axis

Lancaster University

Let's identify the generators of rotations in 3D, starting with rotations around z.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad R_z^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If θ is small, then $\cos\theta = 1$ and $\sin\theta = \theta$, hence

$$\begin{aligned} \psi'(\mathbf{x}) = U(R_z)\psi(\mathbf{x}) = \psi(R_z^{-1}\mathbf{x}) &= \psi(x - \theta y, y + \theta x, z) \\ &= \psi(x,y,z) + \theta\left(x\frac{\partial\psi(x,y,z)}{\partial y} - y\frac{\partial\psi(x,y,z)}{\partial x}\right) \\ &= \left[1 + \theta\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi(x,y,z) \end{aligned}$$

Hence:

$$U(R_z) = (1 + i\theta J_z + \ldots) \quad \Rightarrow \quad J_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

which you may recognise as the z component of the angular momentum operator.

More 3D rotations: around x, y axes

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Derivation for rotations around x and y axes is very similar. Starting from respective rotations (through angle ϕ around x, and angle ψ around y) we have:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \qquad R_y = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

one easily gets expressions for the generators acting upon the wave function:

$$U(R_x) = (1 + i\phi J_x + \ldots) \qquad U(R_y) = (1 + i\psi J_y + \ldots)$$

$$J_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad J_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

which are x and y components of the angular momentum operator.

Groups $SO(3)$ and $O(3)$

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A general 3D rotation is defined by the product of matrices R_x, R_y, R_z defined in pp. 75 and 76:

$$R(\phi, \psi, \theta) = R_x(\phi) R_y(\psi) R_z(\theta)$$

The matrices $R(\phi, \psi, \theta)$ describing 3D rortations form a group:

1. Rotations are matrix multiplications, hence operation is associative.
2. A combination of two rotations is another rotation;
3. There is an identity element (no rotation);
4. Any rotation can be "undone" by rotating back, hence the inverse.

Rotations conserve distance, hence R are orthogonal, $R^T R = I$.

It's easy to show that $\det R = \pm 1$.

Orthogonal matrices of size 3×3 form the group $O(3)$. These, in general, include reflections.

Those matrices that correspond to proper rotations – i.e. no reflections – form the "special" group $SO(3)$ with $\det R = +1$ (similarly to the 2D case).

However, in contrary to 2D case, even $SO(3)$ **is a non-Abelian group**, as its generators do not commute with each other. This is understandable: R_x, R_y, R_z do **not** commute, so e.g. $R_x(\phi) R_y(\psi) R_z(\theta)$ and $R_z(\theta) R_y(\psi) R_x(\phi)$ are different rotations!

Matrix form of generators J_x, J_y, J_z

From the expressions of matrices R_x, R_y, R_z, one can easily derive the matrix form of the generators J_x, J_y, J_z:

$$J_x = \frac{1}{i}\frac{dR_x}{d\phi}|_{\phi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_y = \frac{1}{i}\frac{dR_y}{d\psi}|_{\psi=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$J_z = \frac{1}{i}\frac{dR_z}{d\theta}|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using this form, one can easily prove the commutation relations, and find the corresponding expression for the matrix describing J^2.

A general rotation in terms of J_x, J_y, J_z

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Based on our previous experience in 2D case, and properties of generators, it should be fairly straightforward to obtain a general expression for a matrix describing a rotation through a **vector of angles** $\mathbf{a} = \{a_x, a_y, a_z\}$:

$$R(\mathbf{a}) = \exp\left(i\mathbf{a} \cdot \mathbf{J}\right)$$

This describes a generic element of the group $SO(3)$ (same as the product of three rotations in page 39, but in a different parameterisation).

Note the **dot product** of two 3D vectors in the exponential.

The length a of vector $\mathbf{a}$ is the rotation angle around the direction of vector $\mathbf{a}$

This means that $\mathbf{a} = a\mathbf{n}_a$ where $\mathbf{n}_a$ is a unit vector in direction of vector $\mathbf{a}$.

One can express the angles θ, ϕ, ψ in terms of the components of vector $\mathbf{a}$, but this is not a trivial task due to non-Abelianity of the group.

Algebra of $SO(3)$ generators

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Just to summarize:

- Group $SO(3)$ is defined as a group formed by "special" orthogonal matrices of size 3×3 under matrix multiplication.
- "Special" refers to the property that the determinant of those matrices is $+1$ (rather than -1).
- The group describes transformations of cartesian coordinates of a point in 3D space under various rotations of the coordindate system.
- Although actual parameterisations may vary, a generic group element of $SO(3)$ has three angles as the parameters describing the rotation.
- Hence, the group has three generators, J_x, J_y, J_z. These generators do not commute, hence the group is non-Abelian.

The algebra of a group is defined as the full set of commutation relations of its generators.

For $SO(3)$ the algebra is given by the equations

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y$$

This set of commutators determines all the properties of the group $SO(3)$:
everything else can be derived from here.

Representations and Commutators

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It is the structure of the commutation relations between the generators of a Lie group which determines the structure and properties of the group, **not** the size of matrices representing it for a particular type of objects, or even the size and type of matrices used to define the group!

As mentioned before, all properties of a Lie group are defined by the **algebra of its generators**, i.e. the collection of their commutators.

Commutation relations between generators of the group — i.e. the group algebra — must be the same for all representations of the same group.

A scalar representation is the obvious exception of this rule: being invariant, scalars do not depend on any group parameters, and hence all generators are zero.

So far, all our Lie groups were Abelian — except $SO(3)$, where we only saw one non-trivial representation — so all this talk seems pointless.

But very soon we will see that **irreducible representations of $SO(3)$ come in all sizes.**

More representations of $SO(3)$?

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So far, we have considered two irreducible representations of $SO(3)$: 1-component (scalar) given by 1×1 matrix, and 3-component (vector) given by the "founding fathers" of the group, 3×3 orthogonal matrices.

Is there a 2×2 representation of $SO(3)$?

Yes, there is! But it is **not** by orthogonal 2×2 matrices: these do not have enough variety:

- They are only "good enough" for 2D rotations,
- Just one-parameter freedom,
- Have to be abelian.

How about 2×2 **unitary** matrices, $U^\dagger U = UU^\dagger = I$, are these any good?

Matching with $SO(3)$ rotations

Lancaster University

Basic rotations on 3D vectors in 3D space, around each of the three axes x, y, z, were performed by the three matrices

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \quad R_y = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \quad R_z = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We should try to find such three choices of our complex a and b, that there is a one-to-one correspondence between R_x, R_y, R_z and the 2×2 unitary U_x, U_y, U_z.

Most importantly, **their properties under commutation should be the same**.

Now, only one of these can be diagonal (two diagonal matrices would commute) and only one can be real ("real unitary" matrices are orthogonal matrices and would also commute). So, here is a reasonable choice:

$$U_x = \begin{pmatrix} \cos A_x & i\sin A_x \\ i\sin A_x & \cos A_x \end{pmatrix} \quad U_y = \begin{pmatrix} \cos A_y & \sin A_y \\ -\sin A_y & \cos A_y \end{pmatrix} \quad U_z = \begin{pmatrix} e^{iA_z} & 0 \\ 0 & e^{-iA_z} \end{pmatrix}$$

where A_x, A_y, A_z are three parameters (angles), related to the three angles ϕ, θ, ψ in the R_x, R_y, R_z.

To do the matching, we need to find the generators of matrices U_x, U_y, U_z, calculate their commutators and make sure these are the same as for matching J_x, J_y, J_z.

Matching with $SO(3)$ generators

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From the expressions of U_x, U_y, U_z one can easily calculate the generators of respective infinitesimal transformations, treating A_x, A_y, A_z as parameters:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are, of course, **Pauli matrices**, which have non-trivial commutation relations:

$$[\sigma_x, \sigma_y] = 2i\sigma_z \quad [\sigma_y, \sigma_z] = 2i\sigma_x \quad [\sigma_z, \sigma_x] = 2i\sigma_y$$

These are **similar**, but different from what we need, by a factor of 2. But there is a simple way of making the commutators match exactly: introduce a factor of $1/2$ between the angles A_x, A_y, A_z and ϕ, ψ, θ:

$$A_x = \frac{\phi}{2}, \quad A_y = \frac{\theta}{2}, \quad A_z = \frac{\psi}{2},$$

Then, the generators of U_x, U_y, U_z, calculated with respect to the angles ϕ, ψ, θ, will be $s_j = \frac{\sigma_j}{2}$ $(j = x, y, z)$ which will have the correct commutation relations. Hence, such matrices will form a valid 2×2 complex representation of the group $SO(3)$.

This means that there are 2-component (complex) objects which form a basis for this representation of $SO(3)$. Under arbitrary rotations of 3D space, these 2-component objects are "rotated" by 2×2 unitary matrices U_x, U_y, U_z. The additional factors of $1/2$ mean, that these 2-component objects will be rotated by half-angles, compared to $SO(3)$ vectors. These 2-component objects are called **spinors**.

Matching with $SO(3)$ rotations

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So, going back to the general parameterisation of U in page 83, one can do the following matching.

The choice $A_z = \frac{\theta}{2}$ can be matched to R_z:

$$U_z = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \quad \leftrightarrow \quad R_z = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The choice $A_y = \frac{\psi}{2}$ corresponds to R_y, while $A_z = \frac{\phi}{2}$ corresponds to R_x:

$$U_y = \begin{pmatrix} \cos\frac{\psi}{2} & \sin\frac{\psi}{2} \\ -\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix} \quad \leftrightarrow \quad R_y = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

$$U_x = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \quad \leftrightarrow \quad R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

The resulting matrices U_x, U_y, U_z perform basic rotations around x, y, z axes, by angles ϕ, ψ, θ respectively, on spinors.

Representations of $SO(3)$ so far

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- **Scalars** are the lowest order representation of $SO(3)$. These are objects with 1 component, which remain invariant under rotations.
- **Vectors** have 3 components, transform under rotations according to the matrices $R_x(\phi), R_y(\psi), R_z(\theta)$. The properties of these matrices are fully determined by their respective generators J_x, J_y, J_z.

  In other words, for an arbitrary vector $\mathbf{v}$, the rotated vector is $\mathbf{v}' = R\mathbf{v}$, where R can be parameterised in various ways, e.g.

$$R = R(\phi, \psi, \theta) = R_x(\phi)R_y(\psi)R_z(\theta) \qquad \text{or} \qquad R = R(\mathbf{a}) = \exp(i\mathbf{a} \cdot \mathbf{J})$$

- **Spinors** have 2 (complex) components, and transform under rotations according to the matrices $U_x(\phi), U_y(\psi), U_z(\theta)$. The properties of these matrices are fully determined by their respective generators s_x, s_y, s_z.

  In other words, for an arbitrary 2-component spinor $b \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, the rotated spinor b' is $b' = Ub$, where U can be parameterised in various ways, e.g.

$$U = U(\phi, \psi, \theta) = U_x(\phi)U_y(\psi)U_z(\theta) \qquad \text{or} \qquad U = U(\mathbf{a}) = \exp(i\mathbf{a} \cdot \mathbf{s})$$

Here $\mathbf{a}$ is the same vector as in page 79: its direction defines the axis of rotation, while its magnitude defines the angle of rotation around that axis.

Casimir operator and Schur's Lemma

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The operator J^2 commutes with all generators of $SO(3)$. This can be shown in general, using operators, so should be true in any representation of the group.

An operator which commutes with all generators of a Lie group is called a Casimir operator of the group. J^2 is the only Casimir operator of $SO(3)$.

Schur's lemma: a matrix which commutes with all matrices that form an irreducible representation of a group must be proportional to the unit matrix.

Thus, for any **irreducible** representation of $SO(3)$, the representation of operator $J^2 = \lambda_j I$.

For any such matrix, any state ("object of rotation", e.g. a vector) must be an eigenstate, corresponding to that eigenvalue λ_j:

$$J^2|\text{state}\rangle = \lambda_j|\text{state}\rangle$$

where the eigenvalue λ_j is the same for any element belonging to the same irreducible representation.

For a different irreducible representation, λ_j must be different, otherwise Schur's lemma is violated, i.e. the Casimir operator in a composite (reducible) representation will then be proportional to I.

Hence, **the eigenvalues of Casimir operators can be used to classify irreducible representations.**

Classification for $SO(3)$

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So, irreducible representations of $SO(3)$ are classified by λ_j, the eigenvalue of J^2. Various distinct separate **states** within each irreducible representation should belong to the category of eigenstates of another conserved operator, which commutes with J^2. Any — but only one — of the three J's can be chosen to do this job. Usually it's J_3.

Hence, the states which define an irreducible representation of $SO(3)$ can be classified by two numbers: λ_j, for the representation, and m, for individual states within the representation.

Let's call them

$$|\text{state}\rangle \equiv |j, m\rangle$$

As a starting point, we have

$$J^2|j, m\rangle = \lambda_j|j, m\rangle$$
$$J_3|j, m\rangle = m|j, m\rangle$$

where we really want to know the values of λ_j, m as well as the number of independent states in each representation, i.e. the dimension of the respective 'vector space'.

Classification for $SO(3)$ – II

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Define two new operators $J_\pm$, with some useful properties:

$$J_+ \equiv J_1 + iJ_2, \quad J_- \equiv J_1 - iJ_2$$

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3, \quad J^2 = J_3^2 + J_3 + J_-J_+$$

Let's see what happens, when J_3 acts upon a state given by $J_+|j,m\rangle$:

$$\begin{aligned} J_3J_+|j,m\rangle &= J_+J_3|j,m\rangle + [J_3, J_+]|j,m\rangle \\ &= J_+m|j,m\rangle + J_+|j,m\rangle \\ &= (m+1)J_+|j,m\rangle \end{aligned}$$

Hence, $J_+|j,m\rangle$ — if it exists! — is an eigenstate of J_3, with eigenvalue $(m+1)$.

Similarly, it can be shown that $J_-|j,m\rangle$, when it exists, is an eigenstate of J_3 with eigenvalue $(m-1)$. J_+ and J_- are called **raising** and **lowering** operators, respectively.

Classification for $SO(3)$ – III

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So, $J_+|j,m\rangle = \text{const}\,|j,m+1\rangle$ — or, if that state does not exist, $J_+|j,m\rangle = 0$.

Let's continue the process:

$$J_+|j,m+1\rangle = \text{const}\,|j,m+2\rangle, \quad \text{or 0, if the new state does not exist.}$$

We are after finite-dimensional representations, so let us assume that after k steps like this, we get

$$J_+|j,m+k\rangle = 0 \quad J_+|j,j\rangle = 0 \quad J_3|j,j\rangle = j|j,j\rangle$$

where we assumed $m+k=j$, the largest eigenvalue of J_3 for this representation.

Let's now calculate λ_j for $J^2|j,j\rangle$:

$$J^2|j,j\rangle = (J_3^2 + J_3 + J_-J_+)|j,j\rangle = (j\cdot j + j + 0)|j,j\rangle = j(j+1)|j,j\rangle$$

Hence $\lambda_j = j(j+1)$. As the eigenvalue of J^2 must be the same within an irreducible representation, this holds for any m.

Half-way there...

Classification for $SO(3)$ – IV

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Let's now start repeatedly applying J_- on $|j,j\rangle$. We will get a sequence of states $|j,j-1\rangle, |j,j-2\rangle, \ldots, |j,l\rangle$ until for some l we get $J_-|j,l\rangle = 0$. So, we have

$$J_-|j,l\rangle = 0 \quad \Rightarrow \quad \langle j,l|J_-^\dagger = 0 \quad \Rightarrow \quad \langle j,l|J_-^\dagger J_-|j,l\rangle = 0$$

But $J_-^\dagger = J_+$, and $J_+J_- = J_-J_+ + 2J_3 = J^2 - J_3^2 + J_3$, hence

$$0 = \langle j,l|J_+J_-|j,l\rangle = j(j+1) - l^2 + l = j(j+1) - l(l-1).$$

This equation is only satisfied if $l = -j$. Hence, the whole "spectrum" of eigenstates of J_3 consists of $2j+1$ discrete states, corresponding to eigenvalues

$$m = j, j-1, \ldots, -j+1, -j$$

Hence, $2j+1$ must be a positive integer or zero, meaning that only the following values for j are allowed:

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

Correspondingly, the dimensions of irreducible representations of $SO(3)$ are

$$2j+1 = 1, 2, 3, 4, \ldots$$

Basis for irreducible representations of $SO(3)$

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Let's summarise what we have achieved. Based on commutation relations between the three generators of $SO(3)$, we have found that:

- Irreducible representations of $SO(3)$ can be classified by the eigenvalues of the Casimir operator, the angular-momentum-square J^2, which are equal to $j(j+1)$. Index j can take values $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
- The (orthogonal and normalised) basis of the j-th irreducible representation consists of $2j+1$ states $|j,m\rangle$, which are eigenfunctions of operators J^2 and J_3 with the following eigenvalues:

$$\begin{aligned} J^2|j,m\rangle &= j(j+1)|j,m\rangle \\ J_3|j,m\rangle &= m|j,m\rangle \end{aligned}$$

In addition to the above, one can show that $J_\pm$ operators in any representation have the following properties:

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{j(j+1)-m(m+1)}|j,m+1\rangle \\ J_-|j,m\rangle &= \sqrt{j(j+1)-m(m-1)}|j,m-1\rangle \end{aligned}$$

Physics behind representations

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An irreducible representation of $SO(3)$ labelled j is defined on the space describing quantum-mechanical states with a particular value j of angular momentum.

For single-particle states, in an appropriately chosen coordinate frame, this would mean that j is the particle's own intrinsic angular momentum, i.e. spin:

- $j = 0$ corresponds to scalars, particles/states with spin zero.
- $j = \frac{1}{2}$ corresponds to spinors, particles/states with spin $1/2$, with 2 components.
- $j = 1$ corresponds to vectors, particles/states with spin 1, with 3 components.
- $j = \frac{3}{2}$ corresponds to particles/states with spin $3/2$, with 4 components.
- $j = 2$ corresponds to tensor particles, with spin 2, with 5 components.
- ...

=====

Rotating Spinors

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In particular, the basic space on which the 2×2 representation of $SO(3)$ operates is the space of 2-component complex **spinors**, with the basis

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A general rotation is given by

$$\begin{aligned} U(\mathbf{n}, \theta) &= \exp(i\theta_1\sigma_1/2 + i\theta_2\sigma_2/2 + i\theta_3\sigma_3/2) \\ &= \exp(i\theta(\mathbf{n}\cdot\boldsymbol{\sigma})/2) \\ &= \exp(i(\boldsymbol{\theta}\cdot\boldsymbol{\sigma})/2) \\ &= I\cos(\frac{\theta}{2}) + i(\mathbf{n}\cdot\boldsymbol{\sigma})\sin(\frac{\theta}{2}) \end{aligned}$$

All above expressions are equivalent (just differ by notation and/or parameterisation) and define what happens to spinors under $SO(3)$ rotations. Here $\theta_i = n_i\theta$, hence

$$\boldsymbol{\theta}\cdot\boldsymbol{\sigma} = \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3 = \theta(n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3) = \theta(\mathbf{n}\cdot\boldsymbol{\sigma})$$

and $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector defining the direction of the rotation axis.

Any spinor state can be obtained by rotating the base spinors, with the help of matrix U.

Group $SU(2)$

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Unitary matrices U of size 2×2 with $\det U = 1$ form a group called $SU(2)$.

Strictly speaking, the generators of the group $SU(2)$ in its **fundamental** representation 2×2 are Pauli matrices $\sigma_i, i = 1, 2, 3$ which satisfy the commutation relations

$$[\sigma_k, \sigma_l] = 2\, i\, \epsilon_{klm} \sigma_m$$

This representation was used to **define** the group, and can also be used to build all other representations as well.

As with $SO(3)$, the general element of $SU(2)$ can be obtained by multiplying the three basic "rotations":

$$U(\phi, \psi, \theta) = U_x(\phi) U_y(\psi) U_z(\theta)$$

$$U_x(\phi) = \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix} \qquad U_y(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \qquad U_z(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Note that we wrote full angles here. We only needed to work with half-angles if/when we wanted to have the matching between the commutators of $SU(2)$ and $SO(3)$ generators.

But maybe it is a good idea to keep that matching, and hence keep those 1/2 in the angles?

Relation between $SU(2)$ and $SO(3)$

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Group $SU(2)$ is very similar to $SO(3)$. The only difference is in that factor of 2 in their respective sets of commutators (i.e. **group algebras**):

$$\begin{aligned} SU(2) \text{ with full angles}: &\quad [\sigma_k, \sigma_l] = 2i\epsilon_{klm}\sigma_m \\ SU(2) \text{ with half angles}: &\quad [s_k, s_l] = i\epsilon_{klm}s_m \\ SO(3): &\quad [J_k, J_l] = i\epsilon_{klm}J_m \end{aligned}$$

All the properties of these groups and their representations can be derived from these basic relations (as we indeed have done for $SO(3)$).

With the choice of basic matrices U_x, U_y, U_z from p. 95, one gets the Pauli matrices σ_k as the generators of $SU(2)$, with that extra "2" in the commutators. So we will need to go through the process of determining the eigenvalues, eigenfunctions and representations for $SU(2)$ in a process **similar, but not identical** to what we did with $SO(3)$. That factor of 2 will show up in various places in various powers.

Alternatively, we could keep the $1/2$ in the angles for the three basic 2×2 U-matrices, i.e. have them defined as in p. 85.

Then, the generators of $SU(2)$ would be $s_k = \frac{1}{2}\sigma_k$, whose algebra is exactly the same as for the generators of $SO(3)$. Hence, we can **directly reuse all results from** $SO(3)$ **to** $SU(2)$.

In this case, the two groups are effectively isomorphic for most, if not all, intents and purposes, although some irreducible representations – the unitary ones, with half-integer spin – retain that ambiguity we discussed earlier (for **both** groups).

Relation between $SU(2)$ and $SO(3)$ — II

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So, if that factor of $\frac{1}{2}$ is included into the definitions of the angles in the unitary representations, then all the representations, their classification and properties for $SU(2)$ are identical with those of $SO(3)$: we will have scalars, spinors, vectors etc. in both, classified through their total angular momentum ("spin") j and its projection m.

There are some residual differences between integer-j (hence real orthogonal) and half-integer-j (hence complex unitary) representations:

- If any of the angles ϕ, ψ, θ is increased by 2π, then the real "$SO(3)$" matrices $R_{x,y,z}$ do not change, while the unitary "$SU(2)$" matrices $U_{x,y,z}$ (with the $\frac{1}{2}$ in place) gain an overall **minus** sign.
- Hence, when mapping R to U, both U and $-U$ correspond to the same R, i.e. there is a two-to-one mapping of the elements of $SU(2)$ onto the elements of $SO(3)$.
- You need two full rotations in real representations to complete one full rotation in unitary ones.
- Strictly speaking, the relation between $SU(2)$ and $SO(3)$ is a $2 \to 1$ **homomorphism**, but it becomes an **isomorphism** if we limit ourselves to the compact area around the identity element.
- Despite these differences, all representations of $SU(2)$ are also valid representations of $SO(3)$, and vice-versa. Nature does not seem to care about these differences.
- The differences disappear once one realises that in the unitary representations, usually it's the modulus squared which is the quantity carrying any physical meaning.
- All our results obtained for $SO(3)$ are fully applicable to $SU(2)$, as they only used commutators, which, with $\frac{1}{2}$ in place, are the same for both.

Representation matters

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Why do we care about representations?

The logic is simple:

- Physics is invariant under 3D rotations
- Any physical object has **some** transformation properties under rotations
- Fundamental objects tend to have simple transformation properties, hence belong to separate irreducible representations
- In general, a combination of two irreducible representations forms a **reducible** representation
- The latter can be **decomposed** into a sum of states belonging to various irreducible representations

We will look at these matters next, before considering bigger – and arguably more important – groups.

Spin and particle properties

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Spin is the intrinsic angular momentum of the particle, and is quantised as such. The spin of a particle can be 0, 1/2, 1, 3/2, 2... We will soon see how and why this happens.

Particles with whole spins are called bosons; they obey Bose-Einstein statistics, their number is not conserved, and the wave function of a state describing two identical bosons must be symmetric.

Particles with spins 1/2, 3/2 etc. are called fermions; they obey Fermi-Dirac statistics, they can only be produced in pairs, and the wave function of a state describing two identical fermions must be antisymmetric.

All particles which are usually associated with matter are fermions: electrons, protons and neutrons are all fermions, as are their antiparticles positrons, antiprotons and antineutrons. So are other charged leptons $\mu^{\pm}, \tau^{\pm}$ and all neutrinos. All six quarks and their antiquarks are fermions too.

All fundamental fermions $e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t$ and their antiparticles have spin $1/2$.

On the other side, particles describing classical fields, i.e. the interaction carriers, are bosons: photons γ, gluons g, weak inetraction carriers $W^{\pm}, Z^0$ all have spin 1.

One Standard Model particle — the Higgs boson H^0— has spin zero.

Conservation of Angular Momentum

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A particle with spin S has $2S+1$ different components (projections, polarisations).

Apart from the intrinsic angular momentum — spin $\mathbf{S}$ — particles can have "good old" orbital angular momentum $\mathbf{L}$ relative to each other.

Two-particle system: spins $\mathbf{S}_1$, $\mathbf{S}_2$, orbital momentum $\mathbf{L}$. Total spin $\mathbf{S}$ is the **vector** sum of the two spins, calculated according to quantum-mechanical rules:

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

E.g. the total spin of the hydrogen atom can be either 0 or 1, with the matching number of components:

$$2 \otimes 2 = 1 \oplus 3$$

Total angular momentum $\mathbf{J}$ of the system is the vector sum of the total spin $\mathbf{S}$ and the orbital momentum $\mathbf{L}$:

$$\mathbf{J} = \mathbf{S} + \mathbf{L}$$

- **A combination of two fermions is a boson**, because the orbital momentum must be a whole number.
- **Angular momentum conservation means that the total angular momentum $|\mathbf{J}|$ must be conserved, as well as its projection J_z** — but not spin or orbital momentum separately.

Some "simple" examples

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Hydrogen atom consists of an electron (spin $1/2$) and a proton (spin $1/2$).

Each of these belong to an irreducible 2-component spinor representation of $SO(3)$.

What are the transformation properties of the hydrogen atom, as a whole? I.e. is it a scalar (with total momentum 0) or a vector (total momentum 1) or maybe something else?

The actual answer is fairly complicated, as apart from spins, there is also the orbital momentum involved, and the momentum addition rules can be tricky, especially if you don't know what you are doing...

Very similar rules exist within the quark model, where the isospin symmetry in a two-quark-flavour world is governed by $SU(2)$.

We are now talking about the spectroscopy of mesons (i.e. quark-antiquark systems) and baryons (three quark systems).

Parity

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Parity is a very useful quantum number, characterising a certain property of any system of particles.

Parity operator $\mathcal{P}$ changes the directions of all three spatial coordiante axes (x, y, z) to opposite, thus transforming a right-handed coordinate system into a left-hand coordinate systemed.

This is equivalent to looking at the mirror image of the system.

When operator $\mathcal{P}$ acts on a wave function $\Psi(x, y, z)$, you get

$$\mathcal{P}\Psi(x, y, z) = \Psi(-x, -y, -z)$$

If we do this twice, space returns to its original state, and hence the wave function should remain the same:

$$\mathcal{P}^2\Psi(x, y, z) = \Psi(x, y, z)$$

Parity Eigenvalues and Eigenstates

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Hence, if $\Psi(x,y,z)$ is an eigenfunction of the $\mathcal{P}$ operator with eigenvalue P, one has:

$$\begin{aligned} \mathcal{P}\Psi(x,y,z) &= \Psi(-x,-y,-z) = P\Psi(x,y,z) \\ \mathcal{P}^2\Psi(x,y,z) &= \Psi(x,y,z) = P^2\Psi(x,y,z) \end{aligned}$$

This means that $P^2 = 1$ and hence $P = \pm 1$, i.e there are two types of eigenstates:

- Those that do not change sign under parity transformation are **parity-even**.
- Those that change sign under parity transformation are **parity-odd**.

Even a single particle has some (intrinsic) parity. Parities of bosons are well defined. However, fermions and their antifermions must have opposite parities. By agreement, a proton is assigned parity of $+1$, hence an antiproton has parity -1.

Parity is a multiplicative quantum number: the parity of a two-particle state is the product of the parities of each state, times the parity of the wavefunction describing their relative motion.
The latter is $(-1)^L$ where L is the orbital momentum of the system: $L = 0, 1, 2 \ldots$.

Full wave function of a composite system

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Hadrons are particles that participate in strong interactions.

All hadrons consist of quarks, antiquarks and gluons.

Quarks carry various quantum numbers: position/momentum, spin, colour. The full wave function of a hadron is a product of three wave functions, each defined in its own space. E.g., for a meson one has

$$\Psi(\mathbf{r},\mathbf{S}) = \Psi_{\rm Space}(\mathbf{r}) \times \Psi_{\rm Spin}(\mathbf{S}_1,\mathbf{S}_2) \times \Psi_{\rm Colour}(C_1,\bar{C}_2)$$

where $\mathbf{r}$ is the relative position of the quark and antiquark, $\mathbf{S}_{1,2}$ are their spins and $C_{1,2}$ are their colours.

Colour wave function of a meson is symmetric (see QCD later).

For baryons there will be three constituents, all quarks:

$$\Psi(\mathbf{r},\mathbf{S}) = \Psi_{\rm Space}(\mathbf{r}_{12},\mathbf{r}_{13}) \times \Psi_{\rm Spin}(\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3) \times \Psi_{\rm Colour}(C_1,C_2,C_3)$$

Very similarly, there will be three antiquarks in an antibaryon.

Colour wave function of a baryon or an antibaryon is fully antisymmetric (see QCD later).

Example of the Spin wave Functions

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- Spin-0 state has just one projection, $S_z = 0$.
- Spin-1 state has three projections: $S_z = +1, 0, -1$
- Spin-1/2 state has two projections: $S_z = +1/2, -1/2$, or $\uparrow$ and $\downarrow$.

Two spin-1/2 partcles can be in four different states:

$$\uparrow\uparrow, \quad \uparrow\downarrow, \quad \downarrow\uparrow, \quad \downarrow\downarrow$$

First and last obviously form the two extreme projections of a spin-1 state. They are clearly symmetric, hence the symmetric combination of the two middle ones should form the zero-projection of the spin-1 state. This leaves the antisymmetric combination as the spin-0 state:

$$\Psi_{\rm Spin}(0) = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow); \qquad \Psi_{\rm Spin}(1) = \uparrow\uparrow, \ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \ \downarrow\downarrow .$$

Spin-0 wave function is antisymmetric (i.e. changes sign when swapping 1 and 2), while spin-1 wavefunction is symmetric. Here is a good way of remembering this: the parity of the spin wave function is $(-1)^{S+1}$, where S is the total spin of the two-fermion state.

Space Part of the Wave Function

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- Space part depends on relative coordinate(s) between quarks and antiquarks inside hadrons, and hence their relative orbital angular momenta $\mathbf{L}$.
- The lowest-energy states (ground states) will have spherically symmetric wave functions, corresponding to zero orbital momenta, $L = 0$.
- Thus, lowest-mass mesons and baryons have symmetric space wave functions.
- "Excited" mesons and baryons may have $L = 1$ or higher.
- In general, a wave function corresponding to orbital momentum L has parity $(-1)^L$.
- Total angular momentum of the bound state $\mathbf{J}$ — the spin of the hadron as a whole — is calculated as a **vector** sum of $\mathbf{L}$ and $\mathbf{S}$, $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

Spins of Mesons and Baryons

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- Ground state mesons will have $L = 0$ and hence $J = S$, i.e. 0 or 1, depednig on the value of the total spin:
  - Antisymmetric spin wave function corresponds to total spin zero, hence with $L = 0$ total momentum (i.e. spin of the meson) will be zero.
  - Symmetric spin wave function corresponds to total spin one, hence with $L = 0$ total momentum (i.e. spin of the meson) will be one.

- Ground state baryons will have $J = 1/2$ or $3/2$, depending on the spin wave function.
  - If the first two quarks formed a spin-0 state, adding the third will give a spin-$1/2$ baryon. This spin state has mixed symmetry properties: antisymmetric between 1 and 2, symmetric between 13 and 23.
  - If the first two quarks formed a spin-1 state, adding the third will give a spin-$3/2$ baryon. This spin state is fully symmetric for all three permutations: 12, 13, 23.

Parities of Mesons

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A fermion has no definite intrinsic parity, but the instrinsic parities of fermions and antifermions are opposite. Parity of the proton is taken as $+1$.

When all three spatial dimensions are inverted, $\mathbf{r} \to -\mathbf{r}$, all ground state meson wave functions will change sign. This is because they all consist of a fermion and an antifermion, while there is no additional contribution coming from the space part of the wave function as $L = 0$.

In general, for any meson, parity is $P = (-1)^{L+1}$.

Now let's try charge conjugation, for those mesons which are their own antiparticles.

Changing quark into an antiquark in a meson is the same as swapping them (i.e. space inversion, introducing $(-1)^{L+1}$) and simultaneously swapping their spins (thus introducing $(-1)^{S+1}$).

Colour wave function is symmetric and remains unaffected.

Hence, C-parity of a meson is $C = (-1)^{L+S+2} = (-1)^{L+S}$.

For spin-0 mesons, $J^{PC} = 0^{-+}$. They are C-even pseudoscalars.

For spin-1 mesons, $J^{PC} = 1^{--}$. They are C-odd (true) vectors.

Two-quark World: Mesons

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Two light quarks u and d are usually considered as two states of the same quark. Similarly to two spin states — projections — of a fermion, they are said to have isospin projection $I_z = +1/2$, i.e. up (for u) and $-1/2$, i.e. down (for d) (hence the names of the quarks!)

For antiquarks, it's the other way around: $\bar{u}$ has $I_z = -1/2$, $\bar{d}$ has $I_z = +1/2$.

In a two-quark world, there will be four possible mesonic spin-0 states (pseudoscalars):

$$u\bar{d}, \quad u\bar{u}, \quad d\bar{d}, \quad d\bar{u}$$

Similarly to the spin case, the first and the last have isospin projections $+1$ and -1 respectively, and can be considered as two extremes of an isotriplet state with $I = 1$. These can be identified as π^+ and π^-.

The "middle" projection of this state, π^0, will be the mixture $(u\bar{u} - d\bar{d})/\sqrt{2}$, while the other mixture $(u\bar{u} + d\bar{d})/\sqrt{2}$ is the isosinglet $(I = 0)$. Let's tentatively call it $\tilde{\eta}$.

Four more mesons will exist with identical quark content, but spin 1 (i.e. vector).

The isotriplet states are ρ^+, ρ^0, ρ^-, and the isosinglet state is called ω.

Two-quark World: Baryons

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Let's start with spin 3/2 baryons, which have fully symmetric spin wave function, and fully antisymmetric colour wave function. As lowest-energy states in their sector, they are supposed to have lowest possible angular momentum too, which means the spatial part of the wave function is also fully symmetric. This makes the full wave function **antisymmetric**, and hence will allow all quarks to have the same type (flavour).

Remember — identical fermions must have antisymmetric wave functions!

Hence, we have four possible spin-3/2 baryons:

$$uuu\ (\Delta^{++}),\quad uud\ (\Delta^{+}),\quad udd\ (\Delta^{0}),\quad ddd\ (\Delta^{-})$$

However, the spin wavefunction of the two quarks in a spin-0 state is antisymmetric, and combined with the antisymmetric colour wave function and a symmetric spatial wave function, yield a symmetric state for those two quarks. Hence, in spin-1/2 baryons, those two quarks which have zero combined spin cannot have identical flavour! They must be different, and hence in a two-quark world we only have two possible spin-1/2 baryons:

$$udu\ (\text{proton}),\quad udd\ (\text{neutron})$$

where the first two quarks ud are in a spin-0 state, $(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$.

In fact, these symmetry properties of baryonic wave functions were one of the reasons of introducing the concept of colour in the first place!

Magnetic moments of nucleons

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The magnetic moment $\boldsymbol{\mu}$ of a pointlike particle with charge e, spin $\boldsymbol{s}$ and mass m is

$$\boldsymbol{\mu} = \frac{\boldsymbol{s}e}{m}$$

For electrons this formula works fine, but it does NOT work for protons and neutrons.

In units of **nuclear magneton** $\mu_N = \frac{e}{2m_N}$, the proton and the neutron have magnetic moments

$$\mu_p = +2.793\ldots, \qquad \mu_n = -1.913\ldots$$

instead of $+1$ and 0 respectively.

Charge, mass and spin values are all correct. Where is the problem?

Protons and neutrons are NOT pointlike particles, they have structure, i.e. they consist of other, "more fundamental" particles.

The proton magnetic moment was measured by Otto Stern in 1933 (Nobel Prise in 1943).

It took a couple of decades to develop the quark model, which explained this fact (and many others as well!)

Decomposition of reducible representations

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A composite system formed from two systems A and B with angular momenta j_A and j_B can be described in two ways, in terms of either

- the basis of its constituents, $|j_A, j_B, m_A, m_B\rangle \equiv |j_A, m_A\rangle |j_B, m_B\rangle$
- or of the eigenfunctions $|J, M\rangle$ of the combined angular momentum $J = j_A + j_B$.

These are two alternative descriptions of the same system, hence we should be able to relate them to each other.

States $|j_A, m_A\rangle$ and $|j_B, m_B\rangle$ contain $(2j_A + 1)$ and $(2j_B + 1)$ states respectively, so there are $(2j_A + 1)(2j_B + 1)$ states in the composite "space", which is the **product** of two irreducible representations.

Our aim is to **decompose** this product into a **sum** of irreducible representations of dimensions $(2J + 1)$, where J takes all allowed values. The set of allowed values of J can be easily identified: the eigenvalues M of the operator $J_3 = J_{A3} + J_{B3}$ are

$$M = m_A + m_B, \quad m_A = -j_A, \ldots, j_A, \quad m_B = -j_B, \ldots, j_B,$$

Hence any value of J which has a 3-projection from the above set will be allowed:

$$J = |j_A - j_B|, |j_A - j_B| + 1, \ldots, j_A + j_B - 1, j_A + j_B.$$

Clebsch-Gordan coefficients

Lancaster University

So, in general we have the following decomposition of an irreducible representation into a sum of combinations:

$$|J,M\rangle = \sum_{m_A,m_B} C(m_A,m_B;J,M)|j_A,m_A\rangle\,|j_B,m_B\rangle, \qquad m_B = M - m_A$$

where the sum runs over those m_A, m_B which satisfy the condition $m_A + m_B = M$

(so it's a single sum, not a double sum!)

This answers the question: What are the different ways to construct a state with total angular momentum J (and its projection M) out of two states with angular momenta j_A and j_B ?

Conversely, a combination $|j_A,m_A\rangle\,|j_B,m_B\rangle$ can be decomposed into a sum of irreducible representations:

$$|j_A,m_A\rangle\,|j_B,m_B\rangle = \sum_J G(J,M;m_A,m_B)|J,M\rangle$$

This answers a different question: When two states with angular momenta j_A and j_B are combined together, which irreducible representations (with total angular momentum J) can they end up in?

Here $C(m_A,m_B;J,M)$ and $G(J,M;m_A,m_B)$ are **Clebsch-Gordan coefficients**.

In general, they also depend on j_A, j_B, but I did not show this dependence explicitly.

Clebsch-Gordan coefficients — II

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One can use orthogonality and completeness of the basis "vectors" (wavefunctions) to calculate these coefficients:

$$\begin{aligned} \langle j_A, m_A | \langle j_B, m_B | J, M \rangle &= C(m_A, m_B; J, M) \\ \langle J, M | j_A, m_A \rangle | j_B, m_B \rangle &= G(J, M; m_A, m_B) \end{aligned}$$

from where it is obvious that $C^* = G$.

Even better, under "Condon and Shortley phase convention" both can be made real, so

$$C(m_A, m_B; J, M) = G(J, M; m_A, m_B)$$

which clearly makes our lives easier.

The values of C-G coefficients are tabulated for all important groups (and some other ones too). But they are not that hard to calculate, once you get hold of the procedure.

Clebsch-Gordan coefficients — III

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The C-G coefficients must satisfy lots of various symmetry normalisation conditions.

In particular:

$$\sum_J |C|^2 = 1 \quad \text{for any} \quad j_A, m_A; j_B, m_B$$

$$\sum_{m_A} |C|^2 = 1 \quad \text{for any} \quad j_A, j_B, J, M \quad (m_B = M - m_A)$$

A very simple example: $j_B = 0$, $J = j_A$; i.e. what happens to the angular momentum of a system if we add to it another sytem with total momentum zero?

Clearly, $m_B = 0$ and hence $M = m_A$ and there is only **one** term in the sum. Then, simply out of mormalisation condition, we get that the corresponding $C = 1$.

Hence, adding a zero angular momentum does not change the system's angular momentum (surprise, surprise!)

Clebsch-Gordan coefficients — IV

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So, we have the definition:

$$|J,M\rangle = \sum_{m_A,m_B} C(m_A,m_B;J,M)|j_A,m_A\rangle|j_B,m_B\rangle$$

In order to determine the values for C-G coefficients, once and for all, one can use our old friend, the lowering operator

$$J_- = J_{A-} + J_{B-}$$

We also need a good starting point. This is usually chosen as the **largest possible** $M = J$ **for the state** $J = j_A + j_B$**,** which means $m_A = j_A, m_B = j_B$. Clearly, there is only one term contributing to the sum:

$$|j_A + j_B, j_A + j_B\rangle = C(j_A,j_B;j_A + j_B, j_A + j_B)|j_A,j_A\rangle|j_B,j_B\rangle$$

hence $C(j_A,j_B;j_A + j_B, j_A + j_B) = 1$.

Now remember the general formula:

$$J_-|j,m\rangle \quad = \quad \sqrt{j(j+1) - m(m-1)}|j,m-1\rangle$$

Calculating Clebsch-Gordan coefficients

Lancaster University

Act with $J_- = J_{A-} + J_{B-}$ on both sides of the equation

$$|j_A + j_B, j_A + j_B\rangle = |j_A, j_A\rangle|j_B, j_B\rangle$$

In the l.h.s., we get

$$\sqrt{J(J+1) - J(J-1)}|J, J-1\rangle = \sqrt{2J}|J, J-1\rangle$$

where $J = j_A + j_B$.

In the r.h.s., we should remember that all J operators are group generators, i.e. perform infinitesimally small transformations, and hence behave like differentiation operators. This means that whenever the sum $J_{A-} + J_{B-}$ acts upon a product of states $|j_A, m_A; j_B, m_B\rangle \equiv |j_A, m_A\rangle|j_B, m_B\rangle$, each generator only "sees" its own state, leaving the other one intact:

$$(J_{A-} + J_{B-})|j_A, j_A\rangle|j_B, j_B\rangle = \sqrt{2j_A}|j_A, j_A - 1\rangle|j_B, j_B\rangle + \sqrt{2j_B}|j_A, j_A\rangle|j_B, j_B - 1\rangle$$

Calculating Clebsch-Gordan coefficients — II

Lancaster University

Hence, the respective C-G coefficients are

$$C(j_A - 1, j_B; j_A + j_B, j_A + j_B - 1) = \sqrt{\frac{j_A}{j_A + j_B}},$$

$$C(j_A, j_B - 1; j_A + j_B, j_A + j_B - 1) = \sqrt{\frac{j_B}{j_A + j_B}}.$$

This process can be continued down the multiplet with $J = j_A + j_B$. In general, the number of contributing terms will increase first, and then start decreasing, before reaching the bottom with $M = -J = -j_A - j_B$, where the respective C will be equal to 1 again.

Of course, we could have started from that one, and moved step-by-step up, using J_+.

But this is not the end of the job! There is the next multiplet waiting in the queue, with $J = j_A + j_B - 1$, and for that one, we will need a new starting point!

Calculating Clebsch-Gordan coefficients — III

Lancaster University

The extreme M for the state $J = j_A + j_B - 1$ is $j_A + j_B - 1$, so it should be possible to express it through a superposition of base functions $|j_A, j_A - 1\rangle|j_B, j_B\rangle$ and $|j_A, j_A\rangle|j_B, j_B - 1\rangle$. But this superposition should be **orthogonal** to the one we found for $(M-1)$-th component of $J = j_A + j_B$, hence

$$|j_A + j_B - 1, j_A + j_B - 1\rangle = \sqrt{\frac{j_B}{j_A + j_B}}|j_A, j_A - 1\rangle|j_B, j_B\rangle - \sqrt{\frac{j_A}{j_A + j_B}}|j_A, j_A\rangle|j_B, j_B - 1\rangle$$

Starting from this one, and applying J_-, one can recover values for the second row of C-s.

We will not do this, I just reached this point to show you how you can calculate C-G coefficients if you need to do it while stranded on a deserted island.

If you just need the coefficients for some practical purpose, some are shown on the next page. Also, here is a link (one of many):

`http://pdg.lbl.gov/2009/reviews/rpp2009-rev-clebsch-gordan-coefs.pdf`

But why would you possibly need them?!

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31. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

1/2×1/2

	1	
	$+1$	$1 \quad 0$
$+1/2\ +1/2$	1	$0 \quad 0$

		$1/2 \quad 1/2$	1
$+1/2\ +1/2$	1	$1/2 \quad 1/2$	1
$+1/2\ -1/2$	$1/2 \quad 1/2$	$1/2 \quad -1/2$	-1
$-1/2\ +1/2$	$1/2 \quad -1/2$		
	$-1/2\ -1/2$	1	

1×1/2

	$3/2$	
	$+3/2$	$3/2 \quad 1/2$
$+1\ +1/2$	1	$+1/2\ +1/2$

		$1/3 \quad 2/3$	$3/2 \quad 1/2$
$+1\ -1/2$	$1/3 \quad 2/3$	$-1/2\ -1/2$	$3/2$
$0\ +1/2$	$2/3 \quad -1/3$		$-3/2$
	$0\ -1/2$	$2/3 \quad 1/3$	
	$-1\ +1/2$	$1/3 \quad -2/3$	
		$-1\ -1/2$	1

2×1

	3		
	$+3$	$3 \quad 2$	
$+2\ +1$	1	$+2 \quad +2$	

		$1/3 \quad 2/3$	$3 \quad 2 \quad 1$
$+2\ 0$	$1/3 \quad 2/3$	$+1 \quad +1 \quad +1$	
$+1\ +1$	$2/3 \quad -1/3$		
	$+2\ -1$	$1/15 \quad 1/3 \quad 3/5$	
	$+1\ 0$	$8/15 \quad 1/6 \quad -3/10$	
	$0\ +1$	$2/5 \quad -1/2 \quad 1/10$	

1×1

	2		
	$+2$	$2 \quad 1$	
$+1\ +1$	1	$+1 \quad +1$	

		$1/2 \quad 1/2$	$2 \quad 1 \quad 0$
$+1\ 0$	$1/2 \quad 1/2$	$0 \quad 0 \quad 0$	
$0\ +1$	$1/2 \quad -1/2$		
	$+1\ -1$	$1/6 \quad 1/2 \quad 1/3$	
	$0\ 0$	$2/3 \quad 0 \quad -1/3$	
	$-1\ +1$	$1/6 \quad -1/2 \quad 1/3$	

		$0 \quad -1$	$1/2 \quad 1/2$	2
$0\ -1$	$1/2 \quad 1/2$	$-1 \quad 0$	$1/2 \quad -1/2$	-2
$-1\ 0$	$1/2 \quad -1/2$		$-1\ -1$	1
	$-1\ -1$	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

	$5/2$	
	$+5/2$	$5/2 \quad 3/2$
$+2\ +1/2$	1	$+3/2\ +3/2$

		$1/5 \quad 4/5$	$5/2 \quad 3/2$
$+2\ -1/2$	$1/5 \quad 4/5$	$+1/2\ +1/2$	
$+1\ +1/2$	$4/5 \quad -1/5$		
	$+1\ -1/2$	$2/5 \quad 3/5$	$5/2 \quad 3/2$
	$0\ +1/2$	$3/5 \quad -2/5$	$-1/2\ -1/2$
		$0\ -1/2$	$3/5 \quad 2/5$	$5/2 \quad 3/2$
		$-1\ +1/2$	$2/5 \quad -3/5$	$-3/2\ -3/2$
			$-1\ -1/2$	$4/5 \quad 1/5$	$5/2$
			$-2\ +1/2$	$1/5 \quad -4/5$	$-5/2$
				$-2\ -1/2$	1

3/2×1/2

	2	
	$+2$	$2 \quad 1$
$+3/2\ +1/2$	1	$+1 \quad +1$

		$1/4 \quad 3/4$	$2 \quad 1$
$+3/2\ -1/2$	$1/4 \quad 3/4$	$0 \quad 0$	
$+1/2\ +1/2$	$3/4 \quad -1/4$		
	$+1/2\ -1/2$	$1/2 \quad 1/2$	$2 \quad 1$
	$-1/2\ +1/2$	$1/2 \quad -1/2$	$-1 \quad -1$
		$-1/2\ -1/2$	$3/4 \quad 1/4$	2
		$-3/2\ +1/2$	$1/4 \quad -3/4$	-2
			$-3/2\ -1/2$	1

3/2×1

	$5/2$	
	$+5/2$	$5/2 \quad 3/2$
$+3/2\ +1$	1	$+3/2\ +3/2$

		$2/5 \quad 3/5$	$5/2 \quad 3/2 \quad 1/2$
$+3/2\ 0$	$2/5 \quad 3/5$	$+1/2 \quad +1/2 \quad +1/2$	
$+1/2\ +1$	$3/5 \quad -2/5$		
	$+3/2\ -1$	$1/10 \quad 2/5 \quad 1/2$	
	$+1/2\ 0$	$3/5 \quad 1/15 \quad -1/3$	
	$-1/2\ +1$	$3/10 \quad -8/15 \quad 1/6$	

		$1/5 \quad 1/2 \quad 3/10$	$3 \quad 2 \quad 1$
$+1\ -1$	$1/5 \quad 1/2 \quad 3/10$	$0 \quad 0 \quad 0$	
$0\ 0$	$3/5 \quad 0 \quad -2/5$		
$-1\ +1$	$1/5 \quad -1/2 \quad 3/10$		
	$0\ -1$	$2/5 \quad 1/2 \quad 1/10$	
	$-1\ 0$	$8/15 \quad -1/6 \quad -3/10$	
	$-2\ +1$	$1/15 \quad -1/3 \quad 3/5$	

		$-1 \quad -1 \quad -1$	$3 \quad 2$
$0\ -1$	$2/5 \quad 1/2 \quad 1/10$	$-2 \quad -2$	
$-1\ 0$	$8/15 \quad -1/6 \quad -3/10$		
$-2\ +1$	$1/15 \quad -1/3 \quad 3/5$		
	$-1\ -1$	$2/3 \quad 1/3$	3
	$-2\ 0$	$1/3 \quad -2/3$	-3
		$-2\ -1$	1

		$1/2 \quad 1/2$	$5/2 \quad 3/2 \quad 1/2$
$+1/2\ -1$	$1/2 \quad 1/2$	$-1/2 \quad -1/2 \quad -1/2$	
$-1/2\ 0$	$1/2 \quad -1/2$		
$-3/2\ +1$			
	$+1/2\ -1$	$3/10 \quad 8/15 \quad 1/6$	
	$-1/2\ 0$	$3/5 \quad -1/15 \quad -1/3$	
	$-3/2\ +1$	$1/10 \quad -2/5 \quad 1/2$	

		$3/2 \quad 1/2$	$5/2 \quad 3/2$
$-1/2\ -1$	$3/5 \quad 2/5$	$-3/2 \quad -3/2$	$5/2$
$-3/2\ 0$	$2/5 \quad -3/5$		$-5/2$
	$-3/2\ -1$	1	

Notation:

J	J	$\ldots$
M	M	$\ldots$
m_1	m_2	
m_1	m_2	Coefficients
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Multi-fermion states

Lancaster University

To simplify our formulae, let's use the following notation:

$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \uparrow \qquad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \downarrow$$

Two spin-$\frac{1}{2}$ particles can be in four different states: $\uparrow\uparrow, \quad \uparrow\downarrow, \quad \downarrow\uparrow, \quad \downarrow\downarrow$

First and last obviously form the two extreme projections of a spin-1 state. They are clearly symmetric, hence the symmetric combination of the two middle ones should form the zero-projection of the spin-1 state. This leaves the antisymmetric combination as the spin-0 state:

$$|1,+1\rangle = \uparrow\uparrow, \qquad |1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \qquad |1,-1\rangle = \downarrow\downarrow,$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow);$$

Symbolically, this is written as: $2 \otimes 2 = 1 \oplus 3$

Multi-fermion states — II

Lancaster University

We just did the Clebsch-Gordan decomposition for a combination of two spin-1/2 states.

Here are all the non-trivial coefficients we have found, using notation $C(m_A, m_B; J, M)$:

$$C(+\tfrac{1}{2}, -\tfrac{1}{2}; 1, 0) = \tfrac{1}{\sqrt{2}}, \quad C(-\tfrac{1}{2}, +\tfrac{1}{2}; 1, 0) = \tfrac{1}{\sqrt{2}}, \quad C(+\tfrac{1}{2}, -\tfrac{1}{2}; 0, 0) = \tfrac{1}{\sqrt{2}}, \quad C(-\tfrac{1}{2}, +\tfrac{1}{2}; 0, 0) = -\tfrac{1}{\sqrt{2}}.$$

The above is useful when forming Cooper pairs of electrons.

The spin-1 state is symmetric, hence the Pauli principle will forbid even orbital momenta.

The spin-0 state is anti-symmetric, hence the Pauli principle will forbid odd orbital momenta.

Similarly, this is the basis for forming **mesons** from a quark-antiquark pair in the quark model.

Both quarks and antiquarks are spinors (but are **not** identical spinors).

At ground states, with no orbital momentum between the quarks, one can only have spin-0 and spin-1 meson.

$SU(2)$ and isospin

Lancaster University

Before we can construct hadrons from quarks, we need to figure out what the **flavour symmetry** and **colour symmetry** look like.

All *hadrons* (strongly interactiong particles like pions, Kaons, protons, neutrons, hyperons, etc.) consist of various quarks and antiquarks, following well-defined rules of the quark model and Quantum ChromoDynamics (QCD).

Each quark exists in one of three **colour** states, which form a unitary symmetry described by group $SU(3)$, the gauge group of QCD. More about it later.

Here we only need to know, that the colour part of the wave function is **symmetric** for mesons, and **fully antisymmetric** for baryons.

Quark model and QCD state that all quark flavours are identical from the point of view of strong interactions, i.e. have the same colour "charges". This implies a 6-dimensional unitary flavour symmetry, (cf. 6 quark "flavours").

However, most quark masses are so different from each other, that this 6-dimensional symmetry is badly broken. Only the two lightest quarks have almost equal masses, and hence form the basis of a very useful symmetry group.

Guess what? That group is another $SU(2)$, this time describing rotations in some **isotopic spin**, or simply **isospin** space.

$SU(2)$ and isospin — II

Lancaster University

In a two-quark world of $SU(2)$ isospin, the basis consists of **up**- and **down**- type quarks:

$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv u \qquad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv d$$

A general isospin spinor ξ is a mixture of these two (a "rotation" in isospin space), i.e. under a general isospin rotation U

$$\xi \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow U\xi$$

The hermitian conjugate spinor $\xi^\dagger$ (a row (ξ_1^*, ξ_2^*)) transforms as $\xi^\dagger \equiv (\xi_1^*, \xi_2^*) \rightarrow \xi^\dagger U^\dagger$.

Hence, ξ and $(\xi^\dagger)^T$ do not transform the same way, as $(U^\dagger)^T \neq U$.

But for **unitary** U, it can be shown that the spinor

$$\begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix}$$

transforms as needed, i.e. as the original ξ.

Complex conjugate of a particle's wavefunction descrtibes its **antiparticle**, hence in the antiquark iso-doublet **it's the $\bar{d}$ which has isospin projection** $+1/2$ (and a relative phase given by a minus sign), while $\bar{u}$ has the isospin projection $-1/2$.

Some quark model applications

Lancaster University

Hence, in a two-quark world, there will be an **isovector**, i.e. 3-component state:

$$|1,+1\rangle = u\bar{d}, \quad |1,0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), \quad |1,-1\rangle = d\bar{u},$$

and an **isosinglet** state

$$|0,0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

If the two **spins** of the quarks combine into a singlet spin-0 state, then we have the isotriplet of pions π^+, π^0, π^- and isosinglet η. These have the spin wave function determined earlier: $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

So the full flavour$\times$spin wave function of a π^0 looks like this:

$$|\pi^0\rangle = \frac{1}{2}(u\uparrow\bar{u}\downarrow - u\downarrow\bar{u}\uparrow - d\uparrow\bar{d}\downarrow + d\downarrow\bar{d}\uparrow)$$

If the **spins** of the quarks combine into a triplet (spin 1) state, then we have the isotriplet of vector mesons ρ^+, ρ^0, ρ^- and isosinglet vector meson ω. Here the spin wave function is different, e.g. for $M = 0$ one has: $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ and hence the full wave function for a ρ_0 with spin projection 0 is

$$|\rho^0\rangle_{M=0} = \frac{1}{2}(u\uparrow\bar{u}\downarrow + u\downarrow\bar{u}\uparrow - d\uparrow\bar{d}\downarrow - d\downarrow\bar{d}\uparrow)$$

Leptonic decays of ρ_0 and ω

Lancaster University

Electromagnetic decays of vector mesons happen through a virtual photon, decaying into a lepton pair:

$$\rho, \omega \to \gamma^* \to e^+e^- \quad (\mu^+\mu^-)$$

The probability of such decays is proportional to the (square of the) effective coupling of ρ and ω to the photon, i.e. to "average electric charge" of the respective quark-antiquark state. The latter is determined by the wave functions we just derived. The spin wave function is the same (both ρ^0 and ω are vector particles) so it's down to flavour wave function:

$$\frac{\Gamma(\rho \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} = \left|\frac{u\bar{u} - d\bar{d}}{u\bar{u} + d\bar{d}}\right|^2 = \left(\frac{\frac{2}{3} + \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}}\right)^2 = 9$$

with the experimental value being (11.7 ± 0.4).

Given the fact that the isospin $SU(2)$ symmetry is **not** perfect (electroweak interactions, as well as quark masses, cause violations) this is a very good result!

Clebsch-Gordan in decays

Lancaster University

Vector mesons K^{*+} and K^{*0} have the quark structures

$$K^{*+} = (u\bar{s}), \qquad K^{*0} = (d\bar{s})$$

They have dominant strong decay modes

$$K^* \to K\pi$$

This is exactly what PDG says, without specifying relative intensities of various possible charge combinations:

$$K^{*+} \to K^+\pi^0, \qquad K^{*+} \to K^0\pi^+$$

$$K^{*0} \to K^+\pi^-, \qquad K^{*0} \to K^0\pi^0$$

This is the classic example of the use of Clebsch-Gordan coefficients: Relative probabilities of these decays **are** in fact the values (squared) of respective C-G coefficients!

Clearly, what we have here is an isodoublet on the left (vector kaon) as a combination of an isodoublet (kaon) and an isotriplet (pion):

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = C_1|\tfrac{1}{2}, +\tfrac{1}{2}\rangle|1, 0\rangle + C_2|\tfrac{1}{2}, -\tfrac{1}{2}\rangle|1, 1\rangle$$

Clebsch-Gordan in decays — II

Lancaster University

Use the Tables I gave you to find out the relative branching fractions for the above decays, remembering that C-G coefficients determine the wave functions, i.e. the *amplitudes*, while decay probabilities are determined by $|$ *amplitude* $|^2$.

In both examples, the values of j_A and j_B are 1/2 (final kaon) and 1 (final pion).

$$K^{*+} \rightarrow K^+\pi^0, \quad J=1/2,\ M=1/2,\ m_1=0,\ m_2=+1/2 \quad C=-\sqrt{\frac{1}{3}}$$

$$K^{*+} \rightarrow K^0\pi^+ \quad J=1/2,\ M=1/2,\ m_1=+1,\ m_2=-1/2 \quad C=+\sqrt{\frac{2}{3}}$$

$$K^{*0} \rightarrow K^+\pi^-, \quad J=1/2,\ M=-1/2,\ m_1=-1,\ m_2=+1/2 \quad C=-\sqrt{\frac{2}{3}}$$

$$K^{*0} \rightarrow K^0\pi^0 \quad J=1/2,\ M=-1/2,\ m_1=0,\ m_2=-1/2 \quad C=\sqrt{\frac{1}{3}}$$

So, in both examples here, the decay channel containing a π^0 has branching fraction 33.3%, while the other one has 66.7%

This is all in fact quite simple, once you know what you are doing!

Three-quark states: baryons

Lancaster University

Baryons are composed of three quarks. Relevant decomposition looks like this:

$$2 \otimes 2 \otimes 2 = (1 \oplus 3) \otimes 2 = (1 \otimes 2) \oplus (3 \otimes 2) = 2 \oplus 2 \oplus 4$$

I.e., without any orbital mexcitations, three spin-1/2 particles can combine into a quartet of spin-3/2 and two doublets of spin-1/2. Here are some example wave functions:

$$\begin{aligned}
|\tfrac{3}{2}, +\tfrac{3}{2}\rangle &= \uparrow\uparrow\uparrow \\
|\tfrac{3}{2}, +\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
|\tfrac{1}{2}, +\tfrac{1}{2}\rangle_S &= \tfrac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) \\
|\tfrac{1}{2}, +\tfrac{1}{2}\rangle_A &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)
\end{aligned}$$

The first one is clearly fully symmetric against all permutations of the three initial fermions. So is the second one.

The last two have mixed symmetry against permutations. Indices S and A stand respectively for symmetric and asymmetric wave functions w.r.t. permutations of particles 1 and 2.

Wave function of the proton

Lancaster University

Nucleons (protons and neutrons) are baryons and hence consist of three quarks.

Nucleons have spin 1/2 and isospin 1/2.

In particular, proton with spin up has $s_z = +\frac{1}{2}$ and $I_z = +\frac{1}{2}$.

The spin wave function may belong to either of the bottom two representations from p. 129:

$$|\text{spin}, S\rangle \equiv |\tfrac{1}{2}, +\tfrac{1}{2}\rangle_S = \tfrac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)$$
$$|\text{spin}, A\rangle \equiv |\tfrac{1}{2}, +\tfrac{1}{2}\rangle_A = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

On the isospin side, things are very similar, just replace $\uparrow$ with u and $\downarrow$ with d:

$$|\text{Isospin}, S\rangle \equiv |I = \tfrac{1}{2}, I_3 = +\tfrac{1}{2}\rangle_S = \tfrac{1}{\sqrt{6}}(udu + duu - 2uud)$$
$$|\text{Isospin}, A\rangle \equiv |I = \tfrac{1}{2}, I_3 = +\tfrac{1}{2}\rangle_A = \tfrac{1}{\sqrt{2}}(udu - duu)$$

In order to have correct symmetry properties, these need to be either both S or both A, equally and symmetrically mixed with each other:

$$\tfrac{1}{\sqrt{2}}|\text{spin}, S\rangle|\text{Isospin}, S\rangle + \tfrac{1}{\sqrt{2}}|\text{spin}, A\rangle|\text{Isospin}, A\rangle$$

Want to see what the end result looks like, in terms of $u\uparrow$, $u\downarrow$, $d\uparrow$, $d\downarrow$?

Wave function of the proton – II

Lancaster University

Here it is, in all its glory, for a proton with spin up:

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}} \quad [\quad 2(u\uparrow u\uparrow d\downarrow) - (u\uparrow u\downarrow d\uparrow) - (u\downarrow u\uparrow d\uparrow)$$
$$2(u\uparrow d\downarrow u\uparrow) - (u\uparrow d\uparrow u\downarrow) - (u\downarrow d\uparrow u\uparrow)$$
$$2(d\downarrow u\uparrow u\uparrow) - (d\uparrow u\uparrow u\downarrow) - (d\uparrow u\downarrow u\uparrow) \quad]$$

And here it is for a proton with spin down:

$$|p\downarrow\rangle = \frac{1}{\sqrt{18}} \quad [\quad 2(u\downarrow u\downarrow d\uparrow) - (u\downarrow u\uparrow d\downarrow) - (u\uparrow u\downarrow d\downarrow)$$
$$2(u\downarrow d\uparrow u\downarrow) - (u\downarrow d\downarrow u\uparrow) - (u\uparrow d\downarrow u\downarrow)$$
$$2(d\uparrow u\downarrow u\downarrow) - (d\downarrow u\downarrow u\uparrow) - (d\downarrow u\uparrow u\downarrow) \quad]$$

Now you know the answer to the question about the proton wave function in terms of quark fields.

(Well, almost: the colour part is still to be included. . .)

Can you work out wavefunctions for the neutron, with spin up and down?

Magnetic moments of nucleons

Lancaster University

The magnetic moment of a proton can be defined as

$$\mu_p = \sum_{\text{quark } i=1,2,3} \langle p\uparrow \mid \mu_i \sigma_{3i} \mid p\uparrow \rangle$$

where quark magnetic moments $\mu_i = (\text{quark charge})_i \left(\frac{e}{2m}\right)$, are considered to be those of a point-like (structureless) spin-1/2 particle (e is electric charge of the proton, and m is quark mass).

By carefully using these formulae, and remembering that each σ_{3i} only acts on its own quark wavefunction, one gets

$$\mu_p = \frac{1}{3}(4\mu_u - \mu_d), \qquad \mu_n = \frac{1}{3}(4\mu_d - \mu_u)$$

where, assuming equal masses to u and d quarks, we have $\mu_u = -2\mu_d$.

So, we get a very strong "prediction":

$$\frac{\mu_n}{\mu_p} = -\frac{2}{3}$$

to be compared with the experimental value

$$\frac{\mu_n}{\mu_p} = -0.685\ldots$$

which is **very impressive**, as for point-like nucleons one gets $\mu_n = 0$ as the neutron has no net charge.

Lorentz group

Lancaster University

You have shown in your homework, that the coordinate transformation

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

describes a valid Lorentz boost, with α closely related to the Lorentz transformation parameters β and hence γ.

Some of you may remember that this transformation keeps intact the space-time *interval*

$$s^2 = t^2 - z^2 \quad \{-x^2 - y^2\}$$

which hence is the kinematic invariant.

[To simplify notation, I will sometimes use x_0, x_1, x_2, x_3 instead of t, x, y, z and of course assume $c = 1$.]

If we write explicitly all 4 space-time coordinates, then the boost along the $x_3 \equiv z$ axis with parameter α_3 is described by

$$L_3 = \begin{pmatrix} \cosh\alpha_3 & 0 & 0 & \sinh\alpha_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha_3 & 0 & 0 & \cosh\alpha_3 \end{pmatrix}$$

Lorentz group — II

Lancaster University

Similarly the boosts along the axes 1 and 2, with parameters α_1 and α_2 respectively, are represented by

$$L_1 = \begin{pmatrix} \cosh\alpha_1 & \sinh\alpha_1 & 0 & 0 \\ \sinh\alpha_1 & \cosh\alpha_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad L_2 = \begin{pmatrix} \cosh\alpha_2 & 0 & \sinh\alpha_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\alpha_2 & 0 & \cosh\alpha_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These are 4×4 matrices representing 4-vector transformations, hence our former favourite $SO(3)$ and our current favourite $SU(2)$ can both be represented by the right-bottom 3×3 sub-matrices, describing regular 3D rotations between axes 1,2,3 (i.e. x, y, z), with parameters θ, ϕ, ψ, from pages 40, 41.

By combining 3 rotations and 3 boosts, we get a single 6-parameter **Lorentz group**, a group which covers all coordinate rotations and Lorentz boosts.

Clearly, it contains an $SO(3)$ as a subgroup, which describes 3D rotations.

Lorentz group — III

Lancaster University

Back in "$SO(3)$" times, we have introduced the generators of the 3 rotations in 3D space. Re-written for 4D space-time, these three now gain an extra column and an extra row of zeroes, corresponding to the 0th, time-axis:

$$J_1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad J_2 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad J_3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As for the generators K_1, K_2, K_3 describing the boosts, they are effectively infinitesimal "rotations" in the planes $01, 02, 03$, (otherwise known as tx, ty, tz) but this time both non-zero elements in each generator have the same sign:

$$K_1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

These generators define the 4×4 representation of the Lorentz group, which define the transformation properties of Lorentz 4-vectors under rotations and boosts.

Algebra of the Lorentz group generators

Lancaster University

The algebra of a group is the same for any representation of the group. So we can use the 6 generators above to establish the algebra of the Lorentz group.

First 3 generators of the Lorentz group are in essence 3 generators of $SO(3)$ or $SU(2)$, so we already know that

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

Once we include the other 3, after some matrix multiplication, we find out that

$$[J_i, K_j] = i\epsilon_{ijk} K_k \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

The commutator of a boost along one axis and a rotation ar ound another is equivalent to a boost along the third axis, while the commutator of two boosts along two axes is equivalent to a rotation around the third!

This means that while the three J's form a **closed algebra**, the three K's do **not**, hence the three K's alone do not form a group of their own.

Generators and representations of the Lorentz group

Let's introduce a slightly different view of the generators J and K:

$$A_i = \tfrac{1}{2}(J_i + iK_i), \qquad B_i = \tfrac{1}{2}(J_i - iK_i),$$

Commutators of A and B now look like this:

$$[A_i, A_j] = i\epsilon_{ijk}A_k \qquad [B_i, B_j] = i\epsilon_{ijk}B_k \qquad [A_i, B_j] = 0$$

A's and B's form two separate $SU(2)$ groups, which commute with each other!

So the algebra of the Lorentz group coincides with the algebra of the outside product $SU(2) \otimes SU(2)$.

Each $SU(2)$ subgroup has its own Casimir operator, hence Lorentz group has two such operators, $A^2 = A_1^2 + A_2^2 + A_3^2$ and $B^2 = B_1^2 + B_2^2 + B_3^2$.

According to Schur's lemma, these in each representation are proportional to unit matrices, hence we can use the two eigenvalues to classify the representations of the Lorentz group by two numbers, two "spins" j and j', one for each $SU(2)$ subgroup.

Note that neither of these $SU(2)$ subgroups coincide with the "original" $SO(3)$ which represented 3D rotations.

Generators and representations of the Lorentz group — II

The general matrix M representing the Lorentz group can be expressed through the generators A_i and B_i is the usual way:

$$\begin{aligned} M &= \exp\left(i\mathbf{a}\cdot\mathbf{A}\right)\exp\left(i\mathbf{b}\cdot\mathbf{B}\right) \\ &= \exp\left(i\mathbf{a}\cdot\mathbf{A}+i\mathbf{b}\cdot\mathbf{B}\right) \end{aligned}$$

since $\mathbf{A}$ and $\mathbf{B}$ commute. Substituting their definitions through $\mathbf{J}$ and $\mathbf{K}$, one can relate the parameters a_i and b_i to the original parameters θ_i and α_i:

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{\theta} - i\boldsymbol{\alpha} \\ \boldsymbol{b} &= \boldsymbol{\theta} + i\boldsymbol{\alpha} \end{aligned}$$

This will help to find out transformation properties of various irreducible representations of the Lorentz group in terms of the original generators J and K.

Simplest representations of the Lorentz group

Lancaster University

So, since the Lorentz group is in fact an outside product of two $SU(2)$ groups, its representations can be classified by a pair of numbers (j, j'), either of which can be a zero, an integer, or a half-integer.

Clearly, the simplest one is $(0, 0)$, representing a Lorentz-invariant, a full **scalar**.

The next simplest ones would be two different types of **spinors**, $\xi \equiv (\frac{1}{2}, 0)$ and $\eta \equiv (0, \frac{1}{2})$. Under a general Lorentz transformation, parameterised by 3 angles θ_i and three boosts α_i, they transform as

$$\xi \to \exp\left[\tfrac{i}{2}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\theta} - i\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\right)\right]\xi = \exp\left[\frac{\boldsymbol{\sigma}}{2}\cdot\left(i\boldsymbol{\theta} + \boldsymbol{\alpha}\right)\right]\xi$$

$$\eta \to \exp\left[\tfrac{i}{2}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\theta} + i\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\right)\right]\eta = \exp\left[\frac{\boldsymbol{\sigma}}{2}\cdot\left(i\boldsymbol{\theta} - \boldsymbol{\alpha}\right)\right]\eta$$

Once again, ξ and η are two different, non-equivalent spinor representations of the Lorentz group:

- ξ is the **right-handed spinor**
- η is the **left-handed spinor**

Representations of the Lorentz group — III

Lancaster University

Here comes a secret: if we extend the Lorentz group to include spatial reflection, i.e. **parity operation**, then generators J, which behave like **a pseudo-vector** (they correspond to angular momentum) remain intact, while K's behave like 3-velocity (**a true vector**) and hence change sign.

That is, under parity transformation, A and B swap places, and all representations behave accordingly: $(j, j') \leftrightarrow (j', j)$ under parity transformation

It follows then, that under parity $\xi \leftrightarrow \eta$, and these two are no longer independent! In order to create an irreducible representation of the Lorentz group extended with parity, we need to combine them into a single 4-spinor (sometimes called a Dirac bi-spinor).

$$\psi \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \exp\left[\frac{i}{2}\left(\sigma\cdot\theta - i\sigma\cdot\alpha\right)\right] & 0 \\ 0 & \exp\left[\frac{i}{2}\left(\sigma\cdot\theta + i\sigma\cdot\alpha\right)\right] \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

This ψ is the object that shows up in the Dirac's equation, which describes a free relativistic particle with spin $1/2$. Apparently, it needs to be a 4-component object simply to have meaningful transformation properties under (parity-extended) Lorentz transformation.

In case you are wondering: 4-vectors are represented by $(j = 1/2, j' = 1/2)$, and their transformations are (of course) governed by the generators shown in page 135.

Poincaré group

Lancaster University

There is a way of extending the Lorentz group — add translations: 3 along the three spatial axes, and one translation in time.

We already know, that corresponding infinitesimal transformations can be represented by differential operators:

$$P_i = -i\frac{\partial}{\partial x_i}, \quad i = 1, 2, 3, \qquad P_0 = -i\frac{\partial}{\partial t}$$

With this extension, the Lorentz transformation becomes **inhomogeneous**:

$$x_\mu \to x'_\mu = \sum_{\nu=0}^{3} \Lambda_{\mu\nu} x_\nu + a_\mu, \qquad \mu = 0, 1, 2, 3.$$

where $\Lambda_{\mu\nu}$ stands for the 4×4 matrix representing **homogeneous** Lorentz transformations, boosts and rotations.

The inhomogeneous Lorentz group is called **Poincaré group**. In addition to old 6 parameters and 6 corresponding generators, it has 4 more, one for each translation.

Algebra of the Poincaré group

Lancaster University

Without going into too much detail of this group, let's have a look at the group algebra.

The commutators between boosts and rotations remain the same as before:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad [J_i, K_j] = i\epsilon_{ijk}K_k \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k$$

The new operators P_i commute with each other, but do **not** commute with rotations and boosts:

$$[J_i, P_j] = i\epsilon_{ijk}P_k \qquad [K_i, P_j] = i\delta_{ij}P_0$$

Finally, translation in time commutes with rotations, but gives a spatial translation when commuted with a boost:

$$[J_i, P_0] = 0, \qquad [K_i, P_0] = iP_i$$

For a free particle, P_0 represents its energy and P_i its momentum components; J describes its angular momentum (spin). Non-Hermitian generators K_i have no observable associated to them; they describe boosts from one reference frame to another.

Casimir operators of the Poincaré group

Lancaster University

It can be shown, that the quadratic combination of the new operators

$$C_1 = P_0^2 - P_i^2$$

commutes with all 10 generators, and hence it is one of two Casimir operators of the Poincaré group.

The other Casimir operator is not so straightforward to find. Define a 4-vector

$$\begin{aligned} w_0 &= \mathbf{P}\cdot\mathbf{J} = P_i J_i, \\ w_i &= (P_0\mathbf{J} + \mathbf{P}\times\mathbf{K})_i = P_0 J_i + \epsilon_{ijk} P_j K_k \end{aligned}$$

Then, the combination

$$C_2 = w_0^2 - w_i^2$$

also commutes with all 10 generators, and hence is the other Casimir operator of the Pioncaré group.

Casimir operators of the Poincaré group — II

Lancaster University

Poincaré group is the group representing the most general coordinate transformations, under which every physical state has to have well-defined transformation properties.

Fundamental objects must belong to irreducible representations of the symmetry group.

Irreducible representations are classified according to the eigenvalues of the Casimir operators of the group.

So, what are physical meanings of C_1 and C_2?

They are both invariants, and hence can be calculated in any reference frame of our choice.

Clearly, for a single free particle in its own frame, $\mathbf{P} = 0$ and we have :

$$\begin{aligned} C_1 &= E^2 - \mathbf{P}^2 = m^2, \\ C_2 &= w_0^2 - \mathbf{w}^2 = -m^2 \mathbf{J}^2 = -m^2\, s(s+1), \end{aligned}$$

where s is the particle's spin.

Now we know why particles in PDG book are classified according to their mass and spin!

Fermions and Bosons

Lancaster University

Poincare group of transformations includes Lorentz boosts, space rotations and translations in space and time. Every quantity which claims that it describes something in the real world must belong to some representation of this group.

Representations are classified according to the invariants of the group. Poincare group has two invariants: mass (squared) and spin (squared).

At the moment we do not know much about the origins of particles' masses. They are considered as parameters of the theory and are measured experimentally.

Spin, on the other hand, is the intrinsic angular momentum of the particle, and is quantised as such. The spin of a particle can be 0, 1/2, 1, 3/2, 2...

Particles with whole spins are called bosons; they obey Bose-Einstein statistics, their number is not conserved, and the wave function of a state describing two identical bosons must be symmetric.

Particles with spins 1/2, 3/2 etc. are called fermions; they obey Fermi-Dirac statistics, they can only be produced in pairs, and the wave function of a state describing two identical fermions must be antisymmetric.

Fermions and Bosons – II

Lancaster University

All particles which are usually associated with matter are fermions: electrons, protons and neutrons are all refmions, as are their antiparticles positrons, antiprotons and antineutrons. So are other charged leptons $\mu^{\pm}, \tau^{\pm}$ and all neutrinos. All six quarks and their antiquarks are fermions too.

- All fundamental fermions $e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t$ and their antiparticles have spin $1/2$.
- Mathematically, their wave function is described by Dirac's (bi)spinors.
- Two components describe the particle, the other two describe the antiparticle.

On the other side, particles describing classical fields, i.e. the interaction carriers, are bosons: photons γ, gluons g, weak inetraction carriers $W^{\pm}, Z^0$ all have spin 1. The graviton, carrier of gravity, has spin 2.

One Standard Model particle — the Higgs boson H^0— has spin zero.

- Spin 0 particles are described by a Lorentz-scalar field.
- Spin 1 particles have Lorentz-vector wave functions.
- Spin 2 bosons are described by rank 2 tensors.

Boost for fermions

Lancaster University

Under pure Lorentz boost, $\theta = 0$, the general matrix looks much simpler, especially if we try to boost ξ and η from rest to some momentum $\mathbf{p}$:

$$\begin{pmatrix} \xi(\mathbf{p}) \\ \eta(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \exp(\frac{\sigma\cdot\alpha}{2}) & 0 \\ 0 & \exp(-\frac{\sigma\cdot\alpha}{2}) \end{pmatrix} \begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix}$$

At rest, the spinors ξ and η are indistinguishable, $\xi(0) = \eta(0)$, and hence interchangeable. Swapping them around, and then re-expressing through $\xi(\mathbf{p})$ and $\eta(\mathbf{p})$, we get

$$\begin{pmatrix} \xi(\mathbf{p}) \\ \eta(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \exp(\frac{\sigma\cdot\alpha}{2}) & 0 \\ 0 & \exp(-\frac{\sigma\cdot\alpha}{2}) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \exp(-\frac{\sigma\cdot\alpha}{2}) & 0 \\ 0 & \exp(\frac{\sigma\cdot\alpha}{2}) \end{pmatrix} \begin{pmatrix} \xi(\mathbf{p}) \\ \eta(\mathbf{p}) \end{pmatrix}$$

The product of the three matrices is

$$\begin{pmatrix} 0 & \exp(\sigma\cdot\alpha) \\ \exp(-\sigma\cdot\alpha) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{E+\sigma\cdot\mathbf{p}}{m} \\ \frac{E-\sigma\cdot\mathbf{p}}{m} & 0 \end{pmatrix}$$

where α, the (3-vector) parameter of the boost, is replaced using

$$\cosh|\alpha| = \gamma = \frac{E}{m}, \qquad \sinh|\alpha| = \beta\gamma = \frac{|\mathbf{p}|}{m}$$

as well as the familiar formula (slightly modified for a real argument) with $\mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}$:

$$\exp(\alpha\cdot\sigma) = I\cosh\alpha + (\mathbf{n}\cdot\sigma)\sinh\alpha$$

Dirac's equation

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So, based purely on the transformation properties of Lorentz spinors, we obtain an interesting equation:

$$\begin{pmatrix} -m & E+\sigma\cdot\mathbf{p} \\ E-\sigma\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} \xi(\mathbf{p}) \\ \eta(\mathbf{p}) \end{pmatrix}=0$$

Re-introducing the 4-component spinor ψ, and introducing 4×4 matrices

$$\gamma_0=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\qquad \gamma_{1,2,3}=\begin{pmatrix} 0 & -\sigma_{1,2,3} \\ \sigma_{1,2,3} & 0 \end{pmatrix},$$

the above equation can be rewritten as

$$(\gamma_\mu p^\mu - m)\psi = 0.$$

This is Dirac's equation, describing a free relativistic fermionic field. The matrices $\gamma_\mu, \mu=0,1,2,3$ are called Dirac's matrices.

There is one more useful matrix, called γ_5, which is equal to $\gamma_5 = i\gamma_0\ \gamma_1\ \gamma_2\ \gamma_3$.

We will obtain Dirac's equation again, using the Euler-Lagrange formalism.

Left-handed and right-handed ψ

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Let's study in detail transformation properties of the bi-spinor ψ and its conjugate $\psi^\dagger$.

Our main weapon is the transformation rule of the 4-spinor ψ under a general Lorentz transformation:

$$\psi \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \exp\left[\frac{i}{2}\left(\sigma\cdot\theta - i\sigma\cdot\alpha\right)\right] & 0 \\ 0 & \exp\left[\frac{i}{2}\left(\sigma\cdot\theta + i\sigma\cdot\alpha\right)\right] \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Before going any further, let's remind ourselves that the 2-spinors ξ and η, introduced before, are in fact **right-handed** and **left-handed** components of the 4-spinor ψ respectively:

$$\xi \equiv \psi_R, \qquad \eta \equiv \psi_L$$

Hence

$$\psi \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix} \equiv \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

In the following we will keep this new notation.

Conjugate 4-spinor $\bar{\psi}$

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Re-writing the Lorentz transformation matrix in a more compact form:

$$\psi \equiv \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \rightarrow \psi' \equiv \begin{pmatrix} \psi'_R \\ \psi'_L \end{pmatrix} = \begin{pmatrix} e^{\frac{\sigma}{2}\cdot(i\theta+\alpha)} & 0 \\ 0 & e^{\frac{\sigma}{2}\cdot(i\theta-\alpha)} \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Here θ and α are the usual 3-vectors defining the rotations and boosts, respectively.

For the Hermitian conjugate ("row") 4-spinor $\psi^\dagger$ we then have (as $\sigma^\dagger = \sigma$):

$$\psi^\dagger \equiv \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} \rightarrow (\psi')^\dagger \equiv \begin{pmatrix} \psi_R'^\dagger & \psi_L'^\dagger \end{pmatrix} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} \begin{pmatrix} e^{\frac{\sigma}{2}\cdot(-i\theta+\alpha)} & 0 \\ 0 & e^{\frac{\sigma}{2}\cdot(-i\theta-\alpha)} \end{pmatrix}$$

It's easy to verify, that a construct $\psi^\dagger\psi = \psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L$ is not an invariant, and hence is not a very useful quantity. Instead, a "proper" Dirac-conjugate 4-spinor needs to be defined:

$$\bar{\psi} \equiv \psi^\dagger\gamma_0 \equiv \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix}$$

which now inevitably has different transformation properties:

$$\begin{aligned} \bar{\psi} \rightarrow \bar{\psi}' &\equiv \begin{pmatrix} \psi_R'^\dagger & \psi_L'^\dagger \end{pmatrix} \gamma_0 = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} \begin{pmatrix} e^{\frac{\sigma}{2}\cdot(-i\theta+\alpha)} & 0 \\ 0 & e^{\frac{\sigma}{2}\cdot(-i\theta-\alpha)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{\frac{\sigma}{2}\cdot(-i\theta+\alpha)} \\ e^{\frac{\sigma}{2}\cdot(-i\theta-\alpha)} & 0 \end{pmatrix} = \bar{\psi} \begin{pmatrix} e^{\frac{\sigma}{2}\cdot(-i\theta-\alpha)} & 0 \\ 0 & e^{\frac{\sigma}{2}\cdot(-i\theta+\alpha)} \end{pmatrix} \end{aligned}$$

Invariance of $\bar{\psi}\psi$

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In other words, it's straightforward to show that

$$\begin{pmatrix} e^{\frac{\sigma}{2}\cdot(-i\theta+\alpha)} & 0 \\ 0 & e^{\frac{\sigma}{2}\cdot(-i\theta-\alpha)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{\sigma}{2}\cdot(i\theta+\alpha)} & 0 \\ 0 & e^{\frac{\sigma}{2}\cdot(i\theta-\alpha)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which means that the construct $\bar{\psi}\psi$ is Lorentz invariant:

$$\bar{\psi}\psi \rightarrow \bar{\psi}'\psi' = \psi'^{\dagger}\gamma_0\psi' = \psi^{\dagger}\gamma_0\psi = \bar{\psi}\psi$$

If ψ is a wavefunction, then $\bar{\psi}\psi$ describes the probability density.

From above, we have

$$\bar{\psi}\psi = \psi_R^{\dagger}\psi_L + \psi_L^{\dagger}\psi_R$$

which, in fact, makes more sense. In addition, we know that under parity transformation $\psi_R \leftrightarrow \psi_L$, hence $\bar{\psi}\psi$ is parity-invariant too, i.e. is a true scalar.

Using similar methods, one can show that $\bar{\psi}\gamma_5\psi$ is indeed a pseudoscalar (i.e. a scalar that changes sign under parity transformation), and that $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\gamma_5\gamma_\mu\psi$ form a true- and a pseudo-vector, respectively.

Lagrangian (density) for a scalar field

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Lagrangian (density) defines the world (more about this later though. . .). In Quantum Field Theory, the generalised coordinates are the fields, while the generalised momenta are their derivatives.

The Lagrangian has to be Lorentz-invariant. It is also expected to depend on the fields themselves, and their first derivatives. Terms containing first order (either of the field, or of its derivative) are pointless, sice they will give constant terms in the equations of motion. This strongly restricts the variety of terms that may show up in the Lagrangian.

In particular, for a scalar field, one possible term would be proportional to the field squared, and the other – to the square of the derivative:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2} m^2 \phi^2$$

where the parameter m needs to have the dimension of energy/momentum (or mass).

The Euler-Lagrange equations, in covariant 3+1 D form, look like this:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$$

This leads nicely to the equation of motion

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

This is called Klein-Gordon equation. It describes a free scalar field, with quanta of mass m.

Covariant forms with ψ and $\bar{\psi}$

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Let's try and classify some bi-linear forms with bi-spinors, some of which which may contribute to the Lagrangian of a fermionic field.

- A combination $\bar{\psi}\psi$ is a scalar.
- A combination $\bar{\psi}\gamma_5\psi$ is a pseudoscalar.
- A combination $\bar{\psi}\gamma_\mu\psi$ is a 4-vector, $\mu = 0, 1, 2, 3$.
- A combination $\bar{\psi}\gamma_\mu\gamma_5\psi$ is an axial 4-vector (pseudo-vector), $\mu = 0, 1, 2, 3$.

In order to build the Lagrangian for a free **scalar** field ϕ, we simply used ϕ^2 with some coefficient for the mass term, and $\partial_\mu\phi$ twice for the kinetic term.

We simply had no other scalars or vectors to work with!

With fermions, described by 4-spinors ψ, one can build the kinetic term of the Lagrangian (which needs to be a scalar!) using $\bar{\psi}\gamma_\mu\psi$ to "neutralise" the vector index μ of the derivative.

I.e., we can build a kinetic term like this: $\bar{\psi}\gamma_\mu\partial^\mu\psi$

Clearly, $\bar{\psi}\psi$ is a good candidate for the mass term of the Lagrangian.

Free fermionic field

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From these two terms, one can build the Lagrangian of a free fermionic field:

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$$

Constants i and m are arbitrary to some extent, but this choice makes perfect sense.

E.g.: treating ψ and $\bar{\psi}$ as separate classical fields, one can use E-L equation to obtain Dirac's equation:

$$i\gamma_\mu\partial^\mu\psi - m\psi \ = 0$$

Squaring this equation, one should be able to recover Klein-Gordon, as fermions too must satisfy the general relativistic relation $p^2 = m^2$ which it represents.

When squared, i^2 takes care of the sign, and the mass term is correct, but the rest will only converge into the expected $\partial^\mu\partial_\mu$ if

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$$

which is the major defining property of the Dirac's matrices γ_μ.

Coincidentally, one cannot find matrices with this property among 2×2 or 3×3, so these have to be at least 4×4. Hence, ψ has to have at least 4 components — which it has...

Symmetry of the Lagrangian

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Assume that there is some kind of transformation of ψ, and a matching transformation of $\bar{\psi}$, which leave the Lagrangian $\mathcal{L}(\psi, \bar{\psi}, \partial_\mu\psi, \partial_\mu\bar{\psi})$ invariant.

An example would be an infinitesimally small transformation

$$\psi \to (1 + i\epsilon)\psi, \qquad \bar{\psi} \to (1 - i\epsilon)\bar{\psi},$$

If, as we said, $\mathcal{L}$ is invariant under this transformation, then $\delta\mathcal{L} = 0$, and we have:

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) + \ldots$$

where "..." contain similar terms with $\psi \to \bar{\psi}$. Now, $\delta(\partial_\mu\psi) = \partial_\mu(\delta\psi)$, and the first term $\frac{\partial\mathcal{L}}{\partial\psi}$ can be replaced by its equivalent from the equations of motion:

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}$$

Then the two terms form a derivative of the product:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\psi\right)$$

Noether's theorem

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Very similar things happen to the terms containing derivatives w.r.t. $\bar{\psi}$, and we have:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \, \delta\psi + \delta\bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = 0$$

So, we have a (Lorentz vector) object, in brackets, which forms a conserved "current", associated with this particular symmetry of the Lagrangian. In particular, if we integrate that current over the 3D space, we get "something" conserved in time.

We just "proved" **Noether's theorem** from classical field theory, stating that **for any continuous symmetry of a Lagrangian, there is an associated integral of motion** (i.e. a conserved current). And it even **provides a recipe to calculate that current**!

For the fermionic Lagrangian, this current can be shown to be (up to a constant)

$$\epsilon \bar{\psi} \gamma_\mu \psi$$

which is usually (again, up to a constant) associated to the electromagnetic current of a charged fermionic field.

Gauge transformation

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The symmetry we just looked at is **not** related to space-time transformations. It was an infinitesimal version of the full gauge transformation, given by

$$\psi \to \psi' = \exp(ie\chi)\psi$$

where e is a constant introduced for "future convenience", while χ is the parameter of the gauge transformation. For the charge-conjugate wave function $\bar{\psi}$, one has accordingly:

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\exp(-ie\chi)$$

The "operator" $\exp(ie\chi)$ represents the group $U(1)$, so this transformation has the gauge group $U(1)$.

In a **global** gauge transformation, the parameter χ is the same for all space-time points, i.e. remains constant, and hence its derivatives vanish.

In a **local** gauge transformation, the parameter χ may depend on space-time coordinates, hence its derivatives may be non-zero.

Note: the parameter χ may depend on x_μ, but this does **not** make this a space-time transformation, as the space-time coordinates x_μ are not being transformed.

Global gauge invariance

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Lagrangian of a free fermionic field

$$L = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$$

stays invariant under **global gauge transformations**:

$$\gamma_\mu\partial^\mu\psi \to \gamma_\mu\partial^\mu\psi' = \exp[ie\chi]\gamma_\mu\partial^\mu\psi$$

So the exponent cancels with respective exponent from $\bar{\psi}$.

For a local transformation

$$\gamma_\mu\partial^\mu\psi \to \gamma_\mu\partial^\mu\psi' = \exp[ie\chi]\gamma_\mu(\partial^\mu\psi + ie\psi\partial^\mu\chi)$$

$$\bar{\psi}'\gamma_\mu\partial^\mu\psi' = \bar{\psi}\gamma_\mu\partial^\mu\psi + (ie\partial^\mu\chi)\bar{\psi}\gamma_\mu\psi$$

Where the existence of the extra term means that **this Lagrangian is not locally gauge-invariant.**

Covariant derivative

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Enter the **gauge covariant derivative**

$$D_\mu \equiv \partial_\mu + ieA_\mu$$

where A_μ is a vector "gauge" field. Let's see how this field can be used to restore the gauge invariance in the case of local transformations.

With normal derivative ∂_μ replaced by the covariant derivative, the local gauge transformation for the above part of the lagragian now looks like

$$\begin{aligned} \bar{\psi}'\gamma^\mu D'_\mu \psi' &= \bar{\psi}\gamma^\mu(\partial_\mu + ieA'_\mu + ie\partial_\mu\chi)\psi \\ &= \bar{\psi}\gamma^\mu D_\mu \psi, \end{aligned}$$

provided

$$A'_\mu + \partial_\mu \chi = A_\mu$$

which gives the transformation properties of field A_μ under local gauge transformations.

The field A_μ is the potential of the electromagnetic field.

A new term has emerged in the Lagrangian: $-e\bar{\psi}\gamma^\mu A_\mu \psi$

It corresponds to the interaction between the fermionic field ψ and the electromagnetic field A_μ.

From gauge invariance to QED

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The previous slide is one of the most important in this course.

It explains how the requirement of **local** gauge invariance shows the way to introduce an interaction into the Lagrangian.

The introduction of the gauge field with the required transformation properties under local gauge transformations leads to a new Lagrangian which is locally gauge invariant.

The covariant derivative was not used to actually replace the normal derivative, it just provided a recipe how the gauge field can/should be introduced.

This method will be used to track down the effects of local gauge invariance on other terms containing derivatives in the Lagrangians we will deal with in the future.

The gauge field A_μ will need its own kinetic term, which has to be added to the Lagrangian.

After this procedure — the introduction of covariant derivative and the kinetic term for the gauge field — the free fermion Lagrangian becomes the Lagrangian of Quantum Electrodynamics.

QED Lagrangian

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The Lagrangian of Quantum Electrodynamics (QED), which describes a world consisting of charged fermions of one type (say, electrons, operator ψ) and bosons carrying the interaction (photons, operator A) looks like this:

$$L = L_{\text{fermion}} + L_{\text{boson}} + L_{\text{interaction}}$$

$$\begin{aligned} L_{\text{fermion}} &= i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \\ L_{\text{boson}} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \qquad F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \\ L_{\text{interaction}} &= -e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \end{aligned}$$

where m and e are electron mass and charge, γ_{μ} are Dirac's matrices, and the summation over repeated indices $\mu, \nu = 0, 1, 2, 3$ is understood.

Starting from this Lagrangian, by applying the Euler-Lagrange formalism, one can **derive** the equations which describe a big part of the dynamics of this world: Dirac's equation and Maxwell's equations.

We have done some of this already, and you know by now that this is not as hard as it may seem. Finding solutions to any of those equations is a lot harder...

Maxwell's equations: quick reference

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A_μ is a 4-vector with components $(V, \mathbf{A})$, i.e. unites the scalar and vector potentials into a single quantity. It's not surprising then, that the tensor $F_{\mu\nu}$ contains the familiar fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

In the relativistic covariant form, Maxwell's equations look like this:

$$\begin{aligned} \partial^\mu F_{\mu\nu} &= j_\nu \\ \varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= 0 \end{aligned}$$

where j_ν is the electromagnetic current, and ε is the 4D analog of ϵ_{ijk}.

The first equation replaces the "sourced" pair

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mathbf{j}$$

while the second one replaces the homogeneous pair:

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{E} = \frac{\partial\mathbf{B}}{\partial t}$$

QED: brief summary so far

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Let's have another look at the QED Lagrangian:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu}, \qquad F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

This stays invariant under the following local gauge transformation:

$$\begin{aligned}
\psi(x) \rightarrow \psi'(x) &= e^{ie\chi(x)}\psi(x) \\
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) &= \bar{\psi}(x)e^{-ie\chi(x)} \\
A_{\mu}(x) \rightarrow A'_{\mu}(x) &= A_{\mu}(x) - \partial_{\mu}\chi(x)
\end{aligned}$$

The symmetry group, corresponding to this transformation, is $U(1)$: the simplest unitary group, which is abelian. Its basic representation can be multiplicative (as with $\psi, e^{i\chi}$), or additive (as with A_{μ}). The trick is that the function $\chi(x)$ is the same, otehrwise the Lagrangian will not be invariant!

Side note: invariance of currents

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We have seen, that the invariance of the Lagrangian is linked, through Noether's theorem, to conservation of the electromagnetic current of the fermionic field, given by

$$j_\mu = e\bar{\psi}\gamma_\mu\psi$$

In addition, this current itself is gauge-invariant: as γ matrices are constants, the two conjugate exponentials in the gauge transformation simply cancel:

$$j'_\mu = e\bar{\psi}'\gamma_\mu\psi' = e\bar{\psi}\gamma_\mu\psi = j_\mu$$

Building QED: overview

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Historically, the development of QED was very different from the view below, but here is how it could have been derived:

- A minimal free fermion Lagrangian built from the two possible invariant quadratic forms, $\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$ and $\bar{\psi}\psi$.
- Postulating gauge invariance of the Lagrangian, and hence introducing a massless vector field (photons) to keep the Lagrangian locally gauge-invariant.
- This introduces non-linearity into equations of motion, which means interaction between fields.
- At this point we already have a correct classical theory of electromagnetism, Maxwell's equations and all the rest. Dirac's equation is recovered along the way.
- Quantisation of the free fermionic filed by Fourier-decomposition, and assigning creation- and annihilation- operator powers to the coefficients of this decomposition (i.e. postulating their commutation relations).
- Similarly, quantisation applied to the photon field. Things are rather complicated here, mostly due to the fact that a massless vector filed does not need 4 degrees of freedom provided by a 4-vector, but only needs 2. These difficulties have been sorted out (with gauge invariance playing an important role), so we do not care about them here.
- Develop **perturbation theory** to be able to do meaningful calculations.

This last step was quite painful, and we have not touched it yet. Let's have a brief look.

Heisenberg and Schrödinger pictures

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Without interactions, all fields in QFT satisfy some sort of a wave equation, with solutions described by a combination of plane waves.

Time evolution of the states can be described in either of the well-established "pictures":

- Heisenberg picture: operators are time-dependent, states are not. This is the picture used so far, in particular, for scalar fields:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega}}[\hat{a}(k)e^{-ikx} + \hat{a}^\dagger(k)e^{ikx}]$$

  Very useful if the solution is known — which is (only?) the case for free fields.

- Schrödinger picture: all time dependence carried by the wave function of the state $|s\rangle$:

$$i\frac{d}{dt}|s\rangle = \hat{H}|s\rangle = (\hat{H}_{\rm free} + \hat{H}_{\rm int})|s\rangle$$

  Solution is obtained relative to time moment $t = 0$, at which the above Fourier expansion is evaluated (at that moment, the two pictures match).

Interaction picture

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With interactions, which are (or can be made) limited to certain range in space and/or time, one can still view the initial and final multi-particle states as plane waves, but in-between certain changes ("scattering") take place due to the interaction of those waves with each other. This is best done in the **interaction picture**:

- **operators** have time dependence generated by the **free** Hamiltonian
- the **interaction term** in the Hamiltonian (or Lagrangian) is responsible for time dependence in the state **wave functions**.

This way, the interaction is effectively considered as a **perturbation** to the free field case.

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This whole procedure is in general described by the elements of the scattering matrix, so called S-matrix, which transforms an initial state $|i\rangle$ into a final state $|f\rangle$:

$$|f\rangle = \hat{S}|i\rangle \qquad \Rightarrow \qquad S_{\rm fi} = \langle f|\hat{S}|i\rangle$$

S martix is unitary, $\hat{S}^\dagger \hat{S} = 1$.

At 0-th order there is no interaction at all, $\hat{S} = 1$ and hence $|f\rangle = |i\rangle$.

At the first order $\hat{S} \sim 1 + i\mathcal{L}_{\rm int}$:

$$|f\rangle = |i\rangle + \int dx \int dt_1 \; i\mathcal{L}_{\rm int}(t_1)|i\rangle$$

And so on, one gets a **perturbative expansion** for the S operator:

$$\begin{aligned} \hat{S} &= 1 + \int d^4x_1 \; i\mathcal{L}_{\rm int}(x_1) + \int d^4x_1 \int_{t_1>t_2} d^4x_2 \; (i\mathcal{L}_{\rm int}(x_1))(\; i\mathcal{L}_{\rm int}(x_2)) + \ldots \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \ldots \int d^4x_1 \ldots d^4x_n \; T\left\{\mathcal{L}_{\rm int}(x_1) \ldots \mathcal{L}_{\rm int}(x_n)\right\} \end{aligned}$$

where T stands for **time-ordering** of the operators involved in $\mathcal{L}_{\rm int}$.

Theory and practice

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So, all we have to do is calculate as many terms as we can, right?

Following this recipe, even the very simplest cases appeared to be far too complicated to work with, until Feynman came up with his diagrammatic technique.

But even armed with Feynman's toolbox, things are not all good:

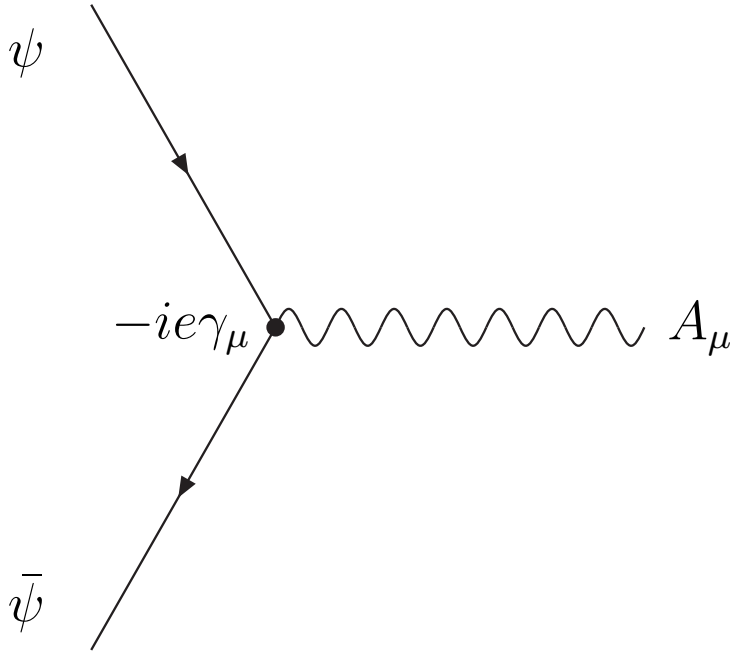
- There is no guarantee, that the series converges; on the contrary, it's likely to be divergent.
- May still be useful, if the next few terms are smaller than the ones being calculated.
- So far, even in QED, hardly beyond 4th order...

The Basic Vertex in QED

Here is the interaction term in Quantum Electro Dynamics again:

$$L_{\rm int} \quad = \quad -e\bar{\psi}\gamma_\mu\psi A_\mu$$

According to **Feynman Rules**, this interaction term is represented graphically by the basic vertex of QED:



The arrow on the solid line represents the flow of the fermionic quantum numbers, **not** the directions of momenta.

Every vertex must include the δ-function guaranteeing 4-momentum conservation, e.g. $\delta^4(p_1 + p_2 + k)$ where p_1, p_2, k are 4-momenta flowing in the electron, positron and photon lines **into** the vertex.

This vertex is the basic block in building matrix elememts of various processes in QED.

One Diagram — Many Processes

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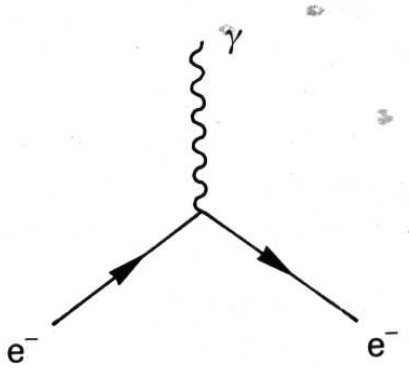
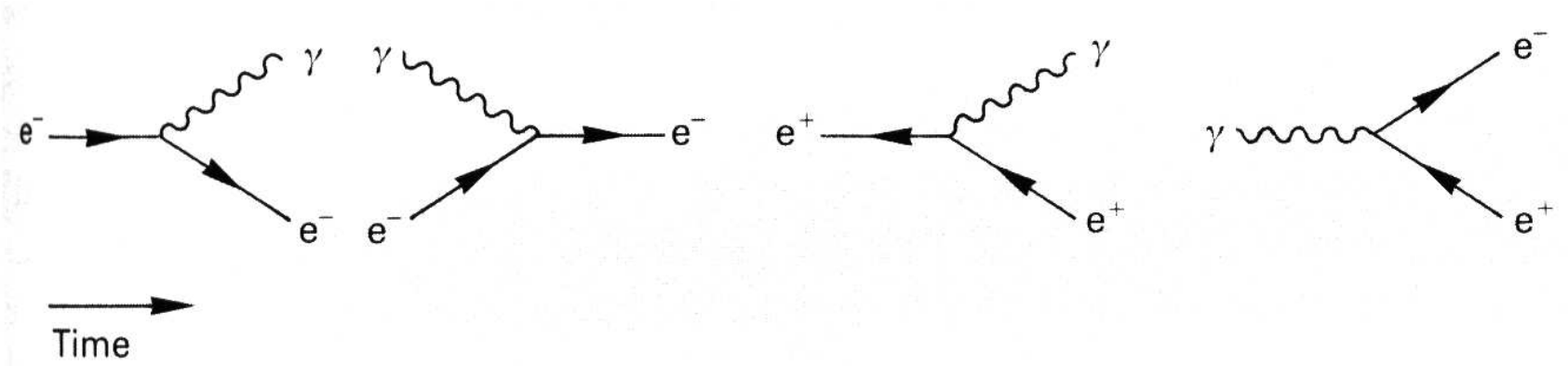
So, the basic vertex of QED looks like this ⇒



Depending on the orientation of the lines, it can represent a number of different processes, e.g.:



Neither of these can actually take place because of energy-momentum conservation, but they can be valid parts of more complicated diagrams.

Feynman rules!

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1. Time flows from left to right: lines coming in from the left describe incoming particles, lines leaving the picture at right describe outgoing particles.
2. Whenever the direction of the fermionic line coincides with the direction of momentum, the line describes the fermion (e^-, μ^-).
3. Whenever the direction of the fermionic line is opposite to the direction of momentum, the line describes the antifermion (e^+, μ^+).
4. The fermionic line should be continuous, i.e. no colliding arrows are allowed when joining vertices together.
5. The coupling constant — electric charge e in QED — is present at each vertex.
6. Both energy and momentum (i.e. 4-momentum) must be conserved at each vertex.
7. External, initial-state and final-state lines represent "real" particles, for which $p_i^2 = m_i^2$. They are described by the wave functions with appropriate spin (scalar for spin 0, spinor for spin $1/2$, vector for spin 1 etc.).
8. Internal lines connecting two vertices represent "virtual" particles, for which (4-momentum)$^2 \neq m^2$. They are said to be "off-mass-shell". They are represented by special functions describing their propagation from one vertex to the other - the **propagators**.

Propagators

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Propagation of a particle from one vertex to another is, according to Feynman rules, described by its respective **propagator**:

Propagator of a photon with 4-momentum k:

$$i \int d^4k \, \frac{-g^{\mu\nu}}{k^2}$$

Propagator of an electron (positron) with 4-momentum p $(-p)$

$$i \int d^4p \, \frac{p_\mu \gamma^\mu + m}{p^2 - m^2}$$

Every propagator also includes the integration over all possible values of the momentum.

Line ends: wave functions

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An initial state electron with 4-momentum p and spin state s is described by a spinor $u(p,s)$

A final state electron with 4-momentum p and spin state s is described by a conjugate spinor $\bar{u}(p,s)$

An initial state positron with 4-momentum p and spin state s is described by a conjugate spinor $\bar{v}(p,s)$

A final state positron with 4-momentum p and spin state s is described by a spinor $v(p,s)$

An initial state photon with 4-momentum k and polarisation λ is described by a polarisation vector $\epsilon_\mu(k,\lambda)$

A final state photon with 4-momentum k and polarisation λ is described by a polarisation vector $\epsilon^*_\mu(k,\lambda)$

When applying Feynman rules to a diagram, you should always start from the **end** of the fermionic line, and move against the arrow.

If there is no end of fermionic line in the diagram, then start from any vertex and still move against the arrow.

Invariant amplitude

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Rather than calculate the actual element of the S-matrix, it's more convenient to obtain a closely related quantity, the invariant amplitude $\mathcal{M}$, which is more directly linked to measurable processes:

$$S_{fi} \sim \delta_{fi} + \mathcal{M}_{fi}$$

Fermi's "golden rules" show the formal way. For the two simplest types of processes,

$$\begin{aligned} \Gamma(1 \to 2) &= \frac{p_f}{8\pi M^2} \left|\mathcal{M}_{1\to 2}\right|^2 \\ \frac{d\sigma(2 \to 2)}{dt} &= \frac{1}{64\pi s} \frac{1}{p_i^2} \left|\mathcal{M}_{2\to 2}(s,t)\right|^2 \end{aligned}$$

Quantities p_i, p_f are the moduli of the 3-momenta of initial and final particles, respectively, in the c.m.s.:

$$\begin{aligned} p_i &= \frac{1}{2\sqrt{s}} \sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]} \\ p_f &= \frac{1}{2\sqrt{s}} \sqrt{[s-(m_3+m_4)^2][s-(m_3-m_4)^2]} \end{aligned}$$

For decays, s should be replaced by M^2. Obviously, at high energies one neglects masses and $p_i = p_f = \sqrt{s}/2$.

Feynman Rules: Building Amplitudes

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Once you figure out what reaction you want to study, the next step is to draw the corresponding Feynman diagram(s). If the process is electromagnetic, the only type of vertex you have is the photon-fermion-antifermion vertex shown above.

If the process is allowed, you should be able to draw a valid Feynman diagram. If you cannot, this means the process is firbidden (usually because it violates one or more conservation laws).

The next step is to apply Feynman Rules to write down the amplitude(s) for the process in question. Every external line, every internal line, and every vertex corresponds to a term (function) which should be writted in the right sequence. For one, each vertex in QED contributes a constant factor e into the amplitude.

Then the modulus squared of (the sum of) the amplitude(s) should be calculated.

(the most tedious part of the procedure, especially when fermions are involved).

Electron-Muon scattering: the Matrix Element

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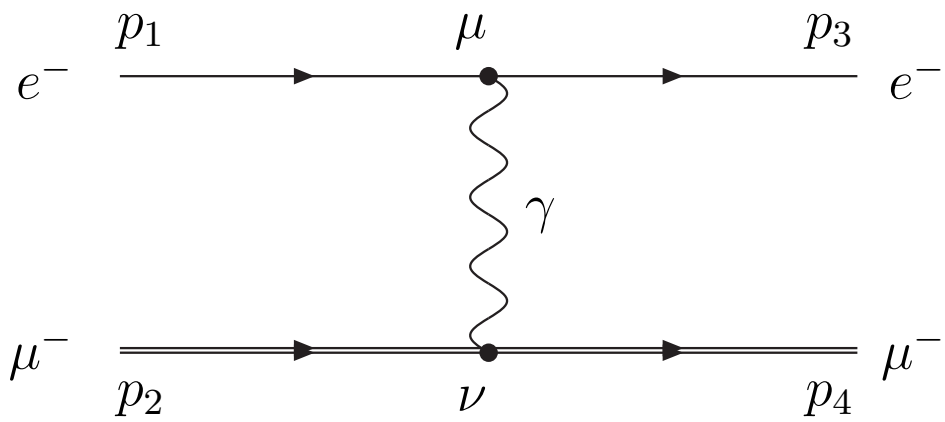
Surprisingly enough, the simplest real process in QED involves two types of fermions, say electrons and muons.

The Feynman graph for Electron-Muon scattering looks like this:



. . . and the corresponding matrix element will look something like this:

$$\mathcal{M}_{e^-\mu^-\to e^-\mu^-} = \bar{u}(p_3)\,(-ie\gamma_\mu)\,u(p_1)\,\frac{-ig^{\mu\nu}}{k^2}\,\overline{U}(p_4)\,(-ie\gamma_\nu)\,U(p_2)$$

where the photon 4-momentum $k = p_1 - p_3 = p_4 - p_2$ if directed from top to bottom, spinors $u, \bar{u}$ describe electrons, while $U, \overline{U}$ describe muons.

Squaring the Matrix Element

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The next step involves writing out the conjugate matrix element $\mathcal{M}^{\dagger}$:

$$\mathcal{M}^{\dagger}_{e^-\mu^-\to e^-\mu^-} = \bar{u}(p_1)\,(ie\gamma_{\mu'})\,u(p_3)\,\frac{ig^{\mu'\nu'}}{k^2}\,\overline{U}(p_2)\,(ie\gamma_{\nu'})\,U(p_4)$$

and calculating $|\mathcal{M}|^2 \equiv \mathcal{M}^{\dagger}\mathcal{M}$, while summing over various spin states of electrons and muons with

$$\sum_{\text{spin}} u(p_1)\bar{u}(p_1) = p_1^{\mu}\gamma_{\mu} + m \qquad etc.$$

This stage ends up with a few traces of products of γ-matrices to be calculated, which is sometimes quite tedious. The final expression will include invariant dot-products of 4-momenta like (p_1p_2) etc, which can be related to Mandelstam variables

$$s = (p_1+p_2)^2 = (p_3+p_4)^2, \quad t = (p_1-p_3)^2 = (p_4-p_2)^2, \quad u = (p_1-p_4)^2 = (p_3-p_2)^2$$

which obey the wonderful relation $s+t+u = 2m_e^2 + 2m_{\mu}^2$.

Electron-Muon scattering: the cross section

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Here is the end result of these calculations:

$$\left|\mathcal{M}_{e^-\mu^-\to e^-\mu^-}(s,t)\right|^2 = 32\pi^2\,\alpha^2\,\frac{s^2+u^2}{t^2}, \qquad \alpha \equiv \frac{e^2}{4\pi}$$

Now, using the golden rule, we have the differential cross section:

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{s^2}\,\frac{s^2+u^2}{t^2}$$

The result looks remarkably simple. We could have guessed some gross features of it:

- α^2 — the diagram is second-order, hence it must contain the coupling squared;
- $1/t^2$— the photon exchange happens in t-channel (i.e. $k^2 = t$), hence the photon propagator in $\mathcal{M}$ is $\sim 1/t$, which should be squared in the cross section.

However, there is no way of obtaining the full result without completing the calculation outlined above.

Cross-symmetry

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Now we can immediately find the matrix element of a different, but related process: $e^+ + e^- \to \mu^+ + \mu^-$
This is accomplished using cross-symmetry. Here is the recipe:

- Take the reaction $e^- + \mu^- \to e^- + \mu^-$.
- Replace initial μ^- with its antiparticle μ^+ and change the sign of its 4-momentum p_2. Now the μ^- in the in itial state is transformed into a μ^+ in the final state.
- Replace final e^- with its antiparticle e^+ and change the sign of its 4-momentum p_3. Now the e^- in the final state is transformed into a e^+ in the initial state.
- Now we have the reaction $e^+ + e^- \to \mu^+ + \mu^-$.

The interesting thing is that this new process is still described by the same matrix element, with the momenta changed accordingly:

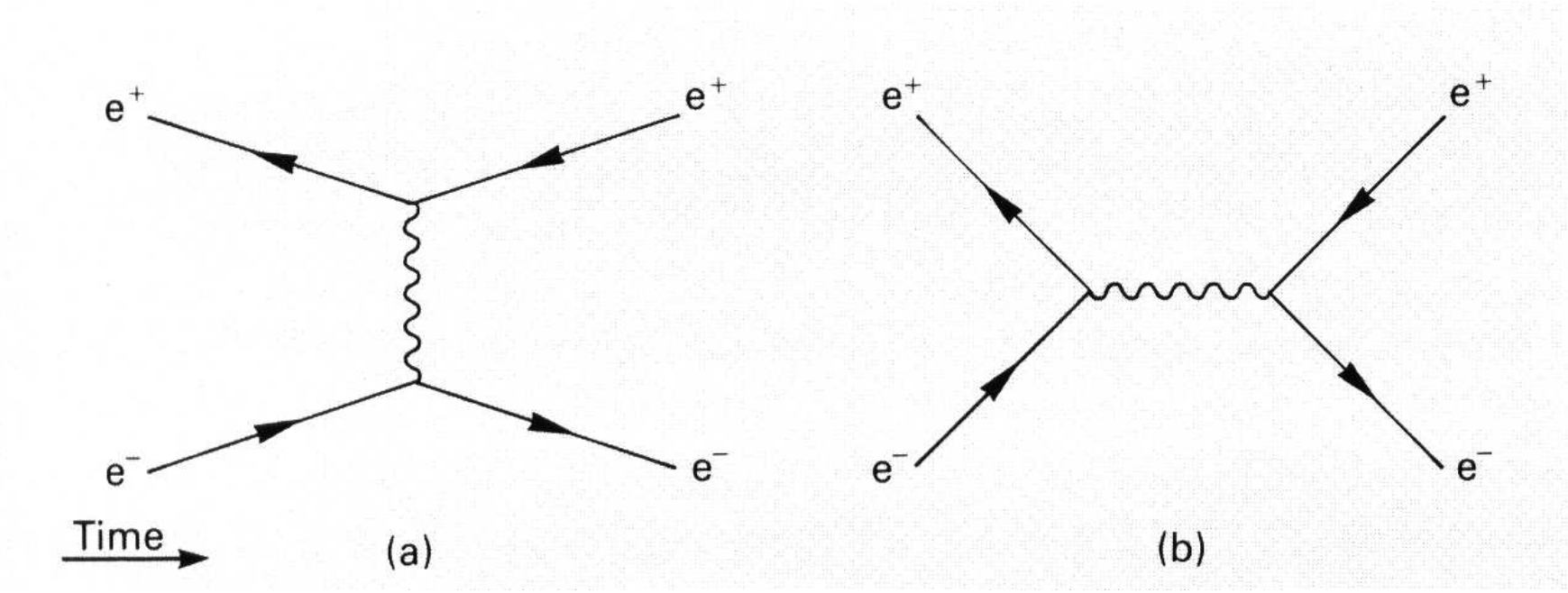
$$\mathcal{M}_{e^-\mu^-\to e^-\mu^-}(p_1, p_2, p_3, p_4) = \mathcal{M}_{e^+e^-\to\mu^+\mu^-}(p_1, -p_3, -p_2, p_4)$$

But this change of momenta is equivalent to swapping s and t. Hence

$$\begin{aligned} \left|\mathcal{M}_{e^+e^-\to\mu^+\mu^-}(s,t)\right|^2 &= 32\pi^2\,\alpha^2\,\frac{t^2+u^2}{s^2} \\ \frac{d\sigma}{dt}(e^+e^-\to\mu^+\mu^-) &= \frac{2\pi\alpha^2}{s^2}\,\frac{t^2+u^2}{s^2} \end{aligned}$$

One Process — Two Diagrams

Here is another example: the process of elastic electron-positron scattering (Bhabha scattering):



'the t-channel exchange' 'the s-channel annihilation'.

For a calculation to be valid, one should calculate the square of the modulus of the sum of these two matrix elements.

. . . and the cross section

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Here is the square of the matrix element

$$|\mathcal{M}_{e^+e^-\to e^+e^-}|^2 = 32\pi^2\alpha^2\left[\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2} + \frac{2u^2}{st}\right]$$

- The first term is exactly the same as in $e\mu \to e\mu$ elastic scattering.
- The second term is exactly the same as in $e^+e^- \to \mu^+\mu^-$ annihilation.
- The last term is new. It is due to **interference** between the two amplitudes represented by the two Feynman diagrams (it's quantum mechanics!).

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Feynman Diagram Construction Kit

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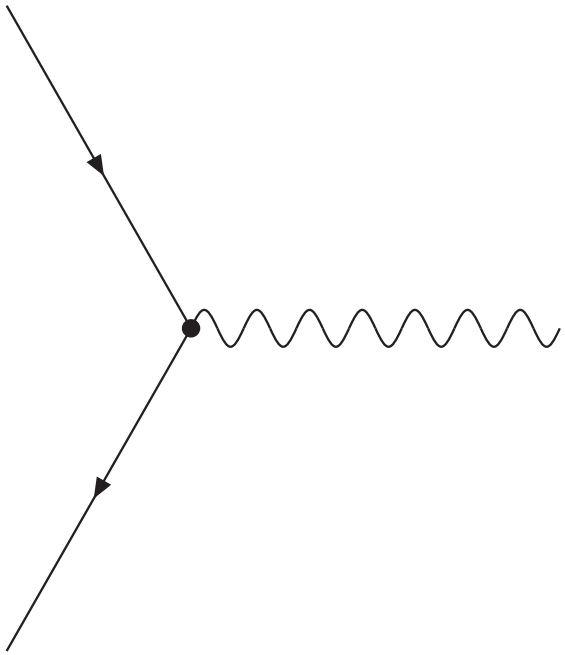

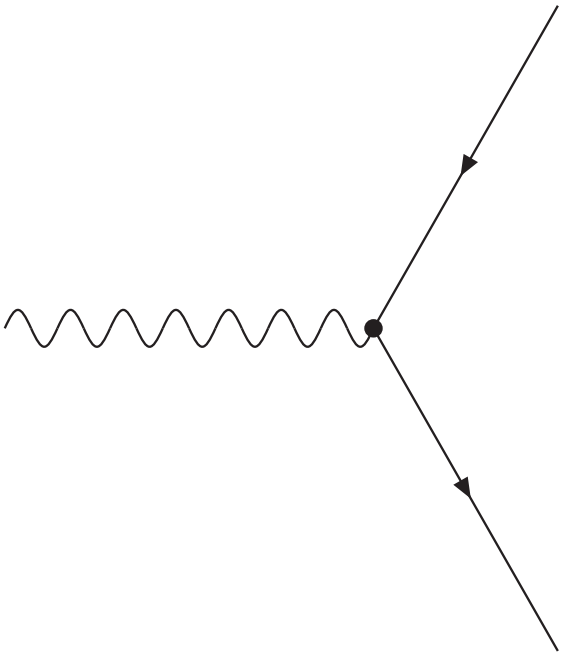

Higher Orders

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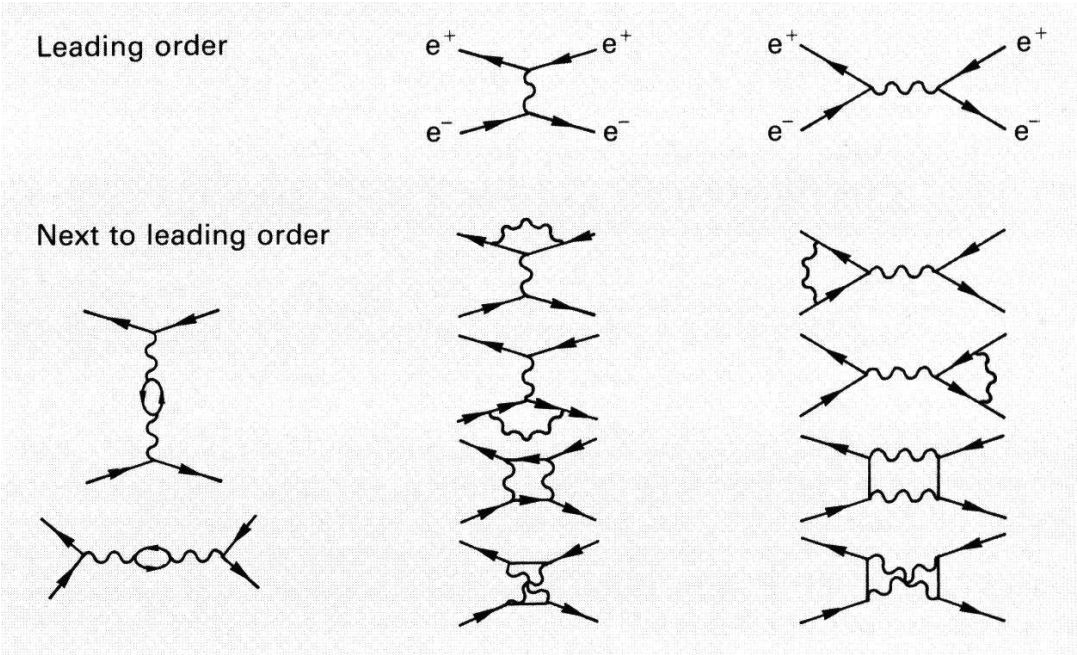
The lowest order non-zero (and non-trivial) diagram is **the leading order** diagram, LO. Then come next-to-leading order diagrams NLO, next-to-next-to-leading order NNLO etc.



At higher orders, number of diagrams increases substantially, and soon the calculations become unmanageable. But we are happy as long as higher order corrections are "small".

Note: at higher orders diagrams contain loops, and some integrals become divergent. A procedure called **'renormalisation'** is then in order, to fight those divergences.

Convergence of the Perturbation Theory

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So, every vertex used to draw the diagram introduces the factor e (charge of the fermion) which then transforms into $\alpha = e^2/4\pi$ in the square of the matrix element. But $\alpha = 1/137 = 0.0073$ is a small number. So the more vertices we use, the smaller the corresponding contribution.

The number of vertices used is called **the order** of the Feynman diagram.

If the same process can be described by two diagrams of different orders, the one with higher number of vertices usually introduces a small (sometimes negligible) correction.

One might argue that the "true exact" amplitude should contain all orders, but that is beyond our reach at the moment (and the series probably starts diverging after about $\sim 1/\alpha = 137$ terms anyway). I believe the fourth order is as far as we've got so far...

In the leading order one should use as few vertices as possible. This is the basis of the perturbation theory in QED.

Loops and divergences

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Some diagrams, especially at higher orders, will contain **loops**.

Now, remember Feynman rules?

- Every **propagator** contains an integration over the 4-momentum of that line, d^4k
- Every **vertex** contains a δ-function of energy-momentum conservation at that vertex, e. g. $\delta^4(p_1 - p_3 - k)$.
- In a **tree** diagram, the number of δ-functions balances out the number of integrations (one δ-function will be left to take care of the overall energy-momentum conservation).
- But if a diagram contains **loops**, the number of δ-functions is not enough, and one (or more) integration is left "alive" and needs to be performed.
- These "loop integrals" are very dangerous: more often than not, they are divergent (i.e. give rise to infinities).

Infrared and ultraviolet divergences

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In the process of calculating Feynman amplitudes, two types of divergences can take place.

- Some of the divergent integrals have the singularities at the low-end of the energy-momentum range, i.e. in the area around zero momenta.

- These are called **'infrared'** divergences. They often occur with photons (or other massless particles) being present. If careful, these usually cancel out.

- In fact, one can show that IR divergences cancel once the amplitudes with **real** and **virtual** photons are (carefully) added together.

- **The divergences happening at high momentum limits of integration are called 'ultraviolet'. These are very persistent, and require a special procedure to deal with them. This procedure is called renormalisation.**

Counting rules for ultraviolet divergences

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The convergence of a generic integral over some 4-momentum k

$$I(a) = \int^{\Lambda} \frac{d^4k\ k^m}{(a+k)^n}$$

depends on the balance of powers m and n, i.e. on index $\omega = 4 + m - n$

The integral will converge at high k, i.e. will stay finite when $\Lambda \to \infty$, if $\omega < 0$.

For $\omega = 0$ it diverges *logarithmically*, e.g. $I \sim \log(\Lambda/a)$.

For $\omega = 1$ it diverges linearly ($I \sim \Lambda$), for $\omega = 2$ it diverges quadratically ($I \sim \Lambda^2$) etc.

For a given process in QED, described by a Feynman diagram with n_e external fermionic lines and n_γ external photonic lines, once can, knowing the corresponding Feynman rules, calculate ω in advance:

$$\omega = 4 - \tfrac{3}{2}n_e - n_\gamma$$

and hence identify potentially dangerous diagrams.

Examples of divergent diagrams

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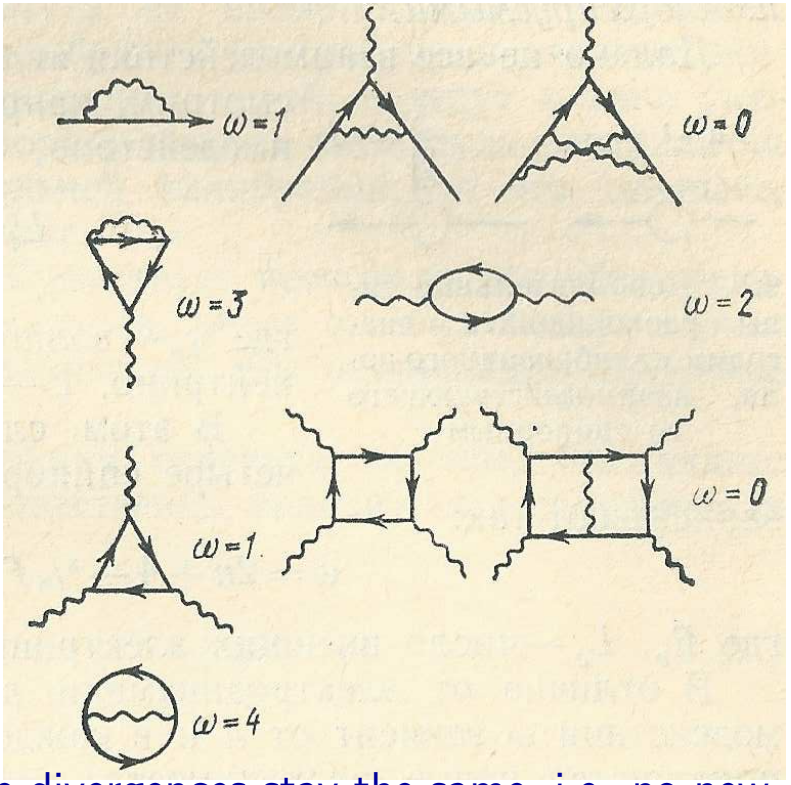
Some of these vanish, or are irrelevant, or
happen to be finite after the renormalisation procedure.



The important ones in QED are the corrections to

- the electron propagator,
- the photon propagator,
- the electron-positron-photon vertex

The important thing is that at higher and higher orders in α, the divergences stay the same, i.e. no new types of divergences appear.

This means that one can eliminate all divergences in all orders of perturbation theory in one go. Theories like this are said to be **renormalisable**.

Renormalisation

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In order for any quantum field theory (QFT) to make sense, i.e. be viable as a fundamental theory, it needs to be **renormalisable**.

This term means that there is a well-defined mathematical procedure to get rid of those ultraviolet divergences.

This is usually done in two steps:

- At regularisation stage, the integrals are somehow made finite, e.g. by limiting the range of integration to some (large) value, or, more recently, by pretending that the space has non-integer number of dimensions $d = 4 - \epsilon$ (the integrals in such space happen to be finite: the infinities show up as singularities like $\Gamma(d-4)$ etc.)
  This is called **dimensional regularisation**.

- Then, the "pure divergent part", i.e. the singularity at $\epsilon = 0$, is subtracted, and the remaining finite part is calculated (back in $d = 4$ dimensions).

Renormalisation —II

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This looks rather arbitrary, but here is the difference between renormalisable and non-renormalisable theories: in a good (renormalisable) theory, one only needs to regularise several (finite number) types of integrals, to keep things divergence-free in all orders of perturbation theory. A bad (non-renormalisable) theory needs one or more new "counter-terms" (i.e. subtractions) at each order.

In any case, even in good theories, after the regularisation, there is still some freedom left in the choice of **regularisation scale**. This should not affect the results once "all orders" are included, but it sure can change the result of a finite-order calcluation!

QED has a natural regularisation scale, corresponding to zero momenta (i.e. large distances), where the classical limit should work, and the Coulomb's law should be reproduced.

QED has been proven to be renormalisable. The facts that the photon is massless, and that the coupling α is dimensionless, play an important role here — but the real big player is gauge invariance.

Renormalisation and gauge invariance

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QED has played a hugely important role as a test-bed, because it's a well established theory with extremely precise predictions, checked in great detail in various experimental measurements, and proven to be successful. It is also shown to be a gauge theory.

Gauge-invariant theories are guaranteed to be renormalisable.

Gauge invariance is the guardian of renormalisability.

This can be shown through the use of "Ward identities", which are based on the concept of the **gauge-invariant derivative** we met before.

They essentially claim that multiplying the matrix element by a momentum k^μ, which essentially means taking a (spatial) derivative, is equivalent to adding a vertex with a photon of momentum k^μ to the diagram. So more divergent terms can be expressed through less divergent terms, with more vertices.

This paved the way for other successful gauge theories, including the quatum chromodynamics (QCD) and the Electroweak theory.

However, if the bosonic fields are massive, adding an extra bosonic propagator to a loop would not in general be "divergence-neutral", but rather would make things more and more divergent at higher orders, thus making the theory non-renormalisable.

We will see how this problem can be avoided, if the boson masses are introduced in the gauge-invariant way, through spontaneous breaking of the gauge symmetry.

Regularisation scale dependence

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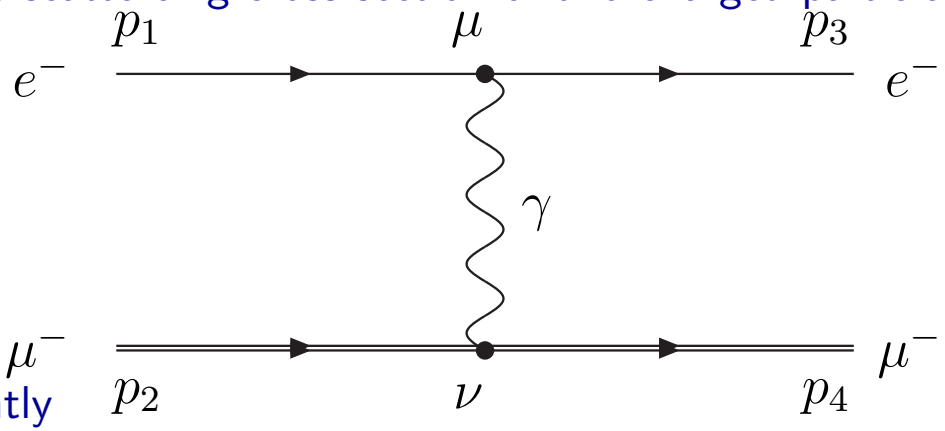
As mentioned above, a calculation to all orders should be independent of regularisation scale. However, at a finite order of perturbation theory there is still some dependence on the regularisation scale — which, in fact, can be exploited to achieve some useful results.

The three basic objects in QED, which are being renormalised/regularised, are electron mass, photon mass, and the coupling. The latter is the most interesting.

At leading order, coupling is defined as the α in the scattereing cross section of two charged particles at $t \to 0$:

$$\left[\frac{d\sigma}{dt}\right]_{t\to 0} = \frac{4\pi}{t^2}\,\alpha^2$$



At one-loop level this gets modified, most importantly by the fermion-antifermion loop emerging in the middle of the photon propagator.

The contribution of this diagram can be calculated, regularised, renormalised, and fed back to the expression defining the strength of the coupling. The interesting thing here is that the effective value of α will depend on the **regularisation** scale μ^2, i.e. the typical value of momenta of interacting particles at which the regularisation (i.e. subtraction of infinities) took place. In this case $\mu^2 \simeq |t|$.

Beta-function in QED

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Evolution of the coupling α is given by the **renormalisation group equation**

$$\mu^2 \frac{\partial \alpha}{\partial \mu^2} = \beta(\alpha)$$

where μ is the regularisation scale, and $\alpha = \alpha(\mu^2)$ is the QED coupling "constant".

It now depends on μ^2: the "real" fine structure constant we know and love is $\alpha(0)$.

β is the **beta-function**, which determines the behaviour of α at various scales.

In QED, calculations show that to the leading order, $\beta(\alpha) = \frac{2}{3\pi}\alpha^2$. Thus we get a simple differential equation to solve:

$$\begin{aligned} \mu^2 \frac{\partial \alpha}{\partial \mu^2} &= \frac{2}{3\pi}\alpha^2 \\ \frac{3\pi}{2}\frac{d\alpha}{\alpha^2} &= d(\log \mu^2) \\ -\frac{3\pi}{2}\left(\frac{1}{\alpha(q^2)} - \frac{1}{\alpha(\mu^2)}\right) &= \log q^2 - \log \mu^2 \end{aligned}$$

Regularisation scale dependence in QED

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So, given the coupling constant α at some μ^2, we can now estimate its value at some (say) higher value q^2:

$$\alpha(q^2) \quad = \quad \frac{\alpha(\mu^2)}{1 - \frac{2}{3\pi}\alpha(\mu^2)\log\frac{q^2}{\mu^2}}$$

- Because of that negative sign in the denominator, with q^2 increasing, $\alpha(q^2)$ also increases, maybe even infinitely (Landau singularity).
- The $\log$ increases very slowly, and it's likely that higher-order corrections will kick in before the singularity is reached.
- In any case, the trend is clear: moving to higher q^2, i.e. probing deeper into an electron, one sees a larger charge.
- A useful interpretation is that the point-like electron charge, buried very deeply at very large q^2, is quite strong, but virtual e^+e^- dipoles are shielding it, effectively reducing it when viewed from larger distances (smaller q^2).
- What we see at "classical" distances is the fine structure constant, $\alpha = 1/137$.
- I must add here that the q^2-dependence of α has been **measured**: at $q^2 \simeq m_W^2$ one has $\alpha \simeq 1/129$.

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Colour of quarks

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Properties of hadrons imply that quarks and antiquarks have a special property (i.e. "quantum number"), called colour, which leptons do not have.

- This means that the quark wave function ψ_n, apart from being a 4-spinor (like an electron), also has an additional index n (and hence extra degrees of freedom).
- This index n can have one of three possible values, or "colours", $n = R, G, B$.
- Antiqiarks, resepctively, carry anticolours, $\bar{R}, \bar{G}, \bar{B}$.
- No quark or antiquark is colourless!

So, a free Lagrangian for one particular quark flavour (e.g. u, or d, etc.) will look like this:

$$\mathcal{L} = i\bar{\psi}_n \gamma^\mu \partial_\mu \psi_n - m\bar{\psi}_n \psi_n$$

where summation over the new indices is implied.

E.g., the mass term, when expanded, looks like this:

$$\begin{aligned} m\,\bar{\psi}_n\psi_n &\equiv \sum_{n=R,G,B} m\,\bar{\psi}_n\psi_n \\ &= m(\bar{\psi}_R\psi_R + \bar{\psi}_G\psi_G + \bar{\psi}_B\psi_B) \end{aligned}$$

Electromagnetic interactions of quarks

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Lagrangian for a free coloured quark field is invariant under 3 separate $U(1)$ gauge transformations, $\psi_n \to \psi'_n = \exp(ie\chi_n)\psi_n$, where each colour n has its own phase χ_n.

If this transformation is made local, one can introduce three separate vector gauge fields (one for each colour) to make the Lagrangian locally gauge invariant. This would mean that each quark colour has its own photon-like field which **only** interacts with quarks of this particular colour. Note that such "photon" would still be colourless, as it is not authorised to change the quark's colour. Such theory would contain three separate kinetic terms for the three types of "photons", and three separate interaction terms. This kind of interaction does not correspond to any known phenomenology.

Another possibility is to have a **common** phase for all three colours, and hence one common gauge field for all three colours, $\psi_n \to \psi'_n = \exp(ie\chi)\psi_n$. This is **the right way** to introduce the electromagnetic interactions of quarks, with interaction vertex similar to QED with an added δ_{mn} (which forbids colour change at the interaction vertex)

$$-ie\bar{\psi}_n(\delta_{nm}\gamma^\mu)\psi_m A_\mu = -ie\bar{\psi}_n\gamma^\mu\psi_n A_\mu$$

and one common kinetic term for the photon.

However, the Lagrangian of a free 3-colour quark field also has a much bigger symmetry.

Useful definitions

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In a gereric $SU(N)$ special unitary group, two of the lowest order irreducible representations play special roles.

One is the "founding father" representation $N \times N$. This is called the **fundamental** representation. These act on basic objects of dimension N. In $SU(2)$ these are spinors with 2 (complex) components, which are "rotated" by 2×2 unitary matrices. In $SU(3)$ these are columns with 3 components ("3-spinors?"), "rotated" by 3×3 matrices. In $SU(N)$, the basic objects are columns with N components.

The other important representation is the **adjoint** representation, which is represented by matrices of size $(N^2-1) \times (N^2-1)$, i.e. 3×3 for $SU(2)$ and i.e. 8×8 for $SU(3)$.

In fact, the dimensions of adjoint representation are due to the number of parameters — and hence number of generators — in the group:

There are $2^2 - 1 = 3$ generators in $SU(2)$ and $3^2 - 1 = 8$ generators in $SU(3)$.

General element of $SU(N)$

Lancaster University

A general element of the $SU(N)$ group **in any representation** can be parameterised in the familiar form of an exponentiated matrix:

$$U(\eta) = e^{\frac{i}{2}\eta_a \Lambda_a}$$

where η denotes the collection of group parameters η_a ("angles" ?), whose number is $N^2 - 1$ (I.e. $a = 1, \ldots, N^2 - 1$), matched by the number of generators Λ_a.

There is no dependence on a in the l.h.s., because summation is implied over index a.
(There are 3 terms in the sum in $SU(2)$, 8 terms for $SU(3)$, $(N^2 - 1)$ terms for $SU(N)$).

The dimension of matrix U is the same as the dimension of the generators Λ_a.

Both dimensions depend on which representation we are in:

- $N \times N$ in the fundamental representation
- $(N^2 - 1) \times (N^2 - 1)$ in the adjoint representation
- as appropriate in other representations

But: **In any representation of the same group**, the number of terms in the sum over a **remains the same**. It's the **size** of generator matrices Λ_a which changes from representation to representation, **not their number**, which is always $(N^2 - 1)$ for any representation of $SU(N)$.

And, of course, **the commutation relations of the generators Λ_a stay the same** for any representation of the same group!

Fundamentals of $SU(3)$

Here are the 8 generators of $SU(3)$, in fundamental 3×3 representation:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- All these are traceless.
- λ_3 and λ_8 are diagonal, representing two useful quantities: respectively 'isospin' and 'hypercharge', which commute withe ach other.
- $\lambda_1, \lambda_4, \lambda_6$ represent exchanges of initial quarks: $1 \leftrightarrow 2, 1 \leftrightarrow 3, 2 \leftrightarrow 3$ resepectively.
- $\lambda_1, \lambda_2, \lambda_3$ are Pauli matrices extended by zeroes, hence $SU(3)$ has at least one $SU(2)$ as a subgroup (in fact, three of them!)

Fundamentals of $SU(3)$ — II

Lancaster University

The **algebra** of $SU(2)$ generators — commutation relations between its three generators — looked like this:

$$\left[\frac{\sigma_k}{2}, \frac{\sigma_l}{2}\right] = i\epsilon_{klm}\frac{\sigma_m}{2}$$

with $k, l, m = 1, 2, 3$.

The **algebra** of $SU(3)$ generators looks similar, but is much more complicated:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = if_{abc}\frac{\lambda_c}{2}$$

where $a, b, c = 1, 2, \ldots, 8$.

The **structure constants** f_{abc} are also fully antisymmetric, but some of them now have different non-zero values:

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$

Representations of $SU(3)$

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The fundamental representation of $SU(3)$ was already shown. Basic "3-spinors" — triplets of quarks — transform according to these.

Antiquarks are described by conjugate anti-3-spinors.

Quark-antiquark states may belong to either singlet or octet:

$$3 \otimes \bar{3} = 1 \oplus 8$$

Three-quark states things are obviously more involved:

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

Note those "1" in the r.h.s. of both triplet-antitriplet and three-triplet states

(There is also a similar one with three antitriplets).

These – and their combinations – are **the only possibilities** to assemble an $SU(3)$-singlet (invariant) if your basic components are all triplets and/or antitriplets.

$SU(3)$ as the colour symmetry: QCD

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Historically, $SU(3)$ was introduced into particle physics by Gell-Mann and Neeman as a flavour symmetry of three types of quarks – u, d, s. It was exceptionally successful to explain the mass spectrum of hadrons (more later).

But $SU(3)$ has a second role in particle physics, as the **gauge group** for Quantum Chromodynamics (QCD).

The important thing for us now is that the wave function ψ describing each quark flavour belongs to the **triplet representation of of the colour group** $SU(3)$, operating in some "colour space".

This means that under the $SU(3)$-"rotations" in that space, each quark field ψ and the antiquark field $\bar{\psi}$ transform as 3-component $SU(3)$-objects:

$$\psi_n \to \exp(i\eta \cdot \lambda_{nk}/2)\,\psi_k \qquad \bar{\psi}_k \to \bar{\psi}_n \exp(-i\eta \cdot \lambda_{nk}/2) \qquad n,k = R,G,B$$

Here λ^a are the $SU(3)$ generators in the fundamental triplet representation of $SU(3)$ — the λ-matrices, and the "dot product" $(\eta \cdot \lambda_{nk})$ is a shorthand for

$$(\eta \cdot \lambda_{nk}) = \sum_{a=1\ldots8} \eta^a \lambda^a_{nk}$$

where η^a are the eight "rotation angles" in $SU(3)$ colour space.

Colour of quarks and hadrons

Lancaster University

The key point here is that coloured (i.e. non-singlet in colour $SU(3)$) objects are not allowed to travel macroscopic distances — **"colour confinement"**.

Leptons are fine — they are colour-singlet objects as they are.

But in the quark world, we are only ever able to directly observe composite objects which are colour-singlets, as all quarks and antiquarks carry colour.

And **there are only three possible ways to construct colour-singlet objects from colour-triplet quarks and antiquarks:**

$$\frac{1}{\sqrt{3}}|R\bar{R}+G\bar{G}+B\bar{B}\rangle$$

$$\frac{1}{\sqrt{6}}|RGB-GRB+BRG-RBG+GBR-BGR\rangle$$

$$\frac{1}{\sqrt{6}}|\bar{R}\bar{G}\bar{B}-\bar{G}\bar{R}\bar{B}+\bar{B}\bar{R}\bar{G}-\bar{R}\bar{B}\bar{G}+\bar{G}\bar{B}\bar{R}-\bar{B}\bar{G}\bar{R}\rangle$$

- The first one clearly needs a quark and an antiquark - this will be a **meson**.
- The second can only be formed from three quarks; this is a **baryon**.
- The last one must contain three antiquarks, and will be an **antibaryon**.
- These are the three types of **hadrons** — strongly interacting particles.

There is a compact way of writing the colour part of the wave functions describing these particles:

$$\frac{1}{\sqrt{3}}\delta_{nk}q_n\bar{q}_k, \qquad \frac{1}{\sqrt{6}}\epsilon_{nkl}q_nq_kq_l, \qquad \frac{1}{\sqrt{6}}\epsilon_{nkl}\bar{q}_n\bar{q}_k\bar{q}_l, \qquad n,k,l=R,G,B.$$

Gauge group $SU(3)$

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So the combination of colours $(\bar{R}R + \bar{G}G + \bar{B}B)$ is **invariant** under general rotations in colour space, defined by

$$\psi_n \to \exp[\frac{ig}{2}(\eta \cdot \lambda_{nk})]\psi_k \qquad \bar{\psi}_k \to \bar{\psi}_n \exp[-\frac{ig}{2}(\eta \cdot \lambda_{nk})] \qquad n,k = R,G,B$$

where λ^a are $SU(3)$ λ-matrices, the 8 generators of $SU(3)$ in the fundamental 3-component representation, while the "dot product" $(\eta \cdot \lambda_{nk})$ is a shorthand for

$$(\eta \cdot \lambda_{nk}) = \sum_{a=1\ldots8} \eta^a \lambda^a_{nk}$$

where η^a is an octet of "rotation angles" in $SU(3)$ colour space.

The way things are defined so far, both terms in the original free-quark Lagrangian will be invariant under such **global gauge transformations**, as the "positive" rotations of ψ will be cancelled out by "negative" rotations of $\bar{\psi}$, very similarly to the $U(1)$ case.

Local $SU(3)$ transformation

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Clearly, if the angles η^a are changing from point-to-point in space, i.e. for the **local** gauge transformations, the differentiation operator in the kinetic term will cause non-invariance (again!).

Suppressing the quark colour indices, we have

$$\begin{aligned} \psi' &= \exp(\frac{ig}{2}\,\eta(x)\cdot\lambda)\psi \\ \bar{\psi}' &= \bar{\psi}\exp(-\frac{ig}{2}\,\eta(x)\cdot\lambda) \end{aligned}$$

and it is clear that $\partial_\mu\psi'$ will contain an unwanted term $\partial_\mu\eta^a(x)$, which violates local gauge invariance.

But we already know the way to combat this: introduce the covariant derivative

$$D_\mu = \partial_\mu + \frac{ig}{2}(G_\mu\cdot\lambda)$$

where G^a_μ are the new gauge fields. There are 8 of them, because you need one for each angle η^a, as the colour dot-product implies. They are called **gluons**.

Local $SU(3)$ invariance

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Now, just as in the case of $U(1)$ in QED, the condition that the fermionic part of the Lagrangian, now with the modified derivative, stays invariant under local gauge transformations,

$$\bar{\psi}' D'_\mu \gamma^\mu \psi' = \bar{\psi} D_\mu \gamma^\mu \psi$$

dictates the transformation properties of the gauge fields G:

$$G^a_\mu \quad \rightarrow \quad G'^a_\mu = G^a_\mu - \partial_\mu \eta^a - g f^{abc} \eta^b G^c_\mu$$

This again looks similar to the $U(1)$ case, except for the last term, which shows up because $SU(3)$ is a non-ableian group. It emerged from the invariance condition, because otherwise the terms with λ would not cancel.

Lagrangian for gluons

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The Lagrangian of the gauge fields proper still looks quite innocent

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^{a\mu\nu} F^{a}_{\mu\nu}$$

But the price to pay is an extra term in the definition of F:

$$F^{a}_{\mu\nu} \equiv \partial_\mu G^{a}_{\nu} - \partial_\nu G^{a}_{\mu} - g f^{abc} G^{b}_{\mu} G^{c}_{\nu}$$

The effect of this term is quite dramatic:

In addition to quadratic terms, there will now be terms containing three G fields, and even terms containing four G fileds.

This means that gluons must carry colour charges, and can interact with each other directly.

And this is all due to the fact that $SU(3)$ is non-ableian: all those new interaction terms vanish if $f^{abc} = 0$.

Quantum Chromodynamics

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So, the interaction term between quarks ψ_n and gluons G^μ_a looks like this:

$$-ig\bar{\psi}_n \left(\frac{\lambda^a}{2}\right)_{nk} \gamma^\mu \psi_k G^a_\mu$$

Once again, due to non-Abelian nature of $SU(3)$, the interaction resulting from the requirement of local $SU(3)$ gauge inveriance exists not only between quarks and gluons, but also between gluons themselves, through 3-gluon and 4-gluon couplings.

Only "coloured" particles — quarks, antiquarks ang gluons — participate in this interaction, hence all leptons, photons, W and Z are colour-neutral (i.e. are $SU(3)$ scalars) and hence do not participate in strong interactions.

So far, there is no indication that gluons have mass. Like photons, they are considered to be massless.

According to the **colour confinement** hypothesis, only colour-neutral ("white") particles can exist as asymptotic (observable) states, hence quarks and gluons remain locked inside hadrons: mesons, baryons and antibaryons.

Colour of gluons

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For completeness, let's list possible colours of gluons.

There are eight different gluons. Here is the usual choice:

$$|R\bar{G}\rangle,\ |R\bar{B}\rangle,\ |G\bar{R}\rangle,\ |G\bar{B}\rangle,\ |B\bar{R}\rangle,\ |B\bar{G}\rangle,\ \frac{1}{\sqrt{2}}|R\bar{R}-G\bar{G}\rangle,\ \frac{1}{\sqrt{6}}|R\bar{R}+G\bar{G}-2B\bar{B}\rangle$$

There is no such thing as a white, colourless gluon (or quark, for that matter).

On the other hand, leptons, photons, W and Z are all colourless.

Gluons are massless. They do not carry electric charge, and have no coupling with photons, $W^{\pm}$ or Z^0. They only participate in strong (QCD) interactions.

QCD Vertices

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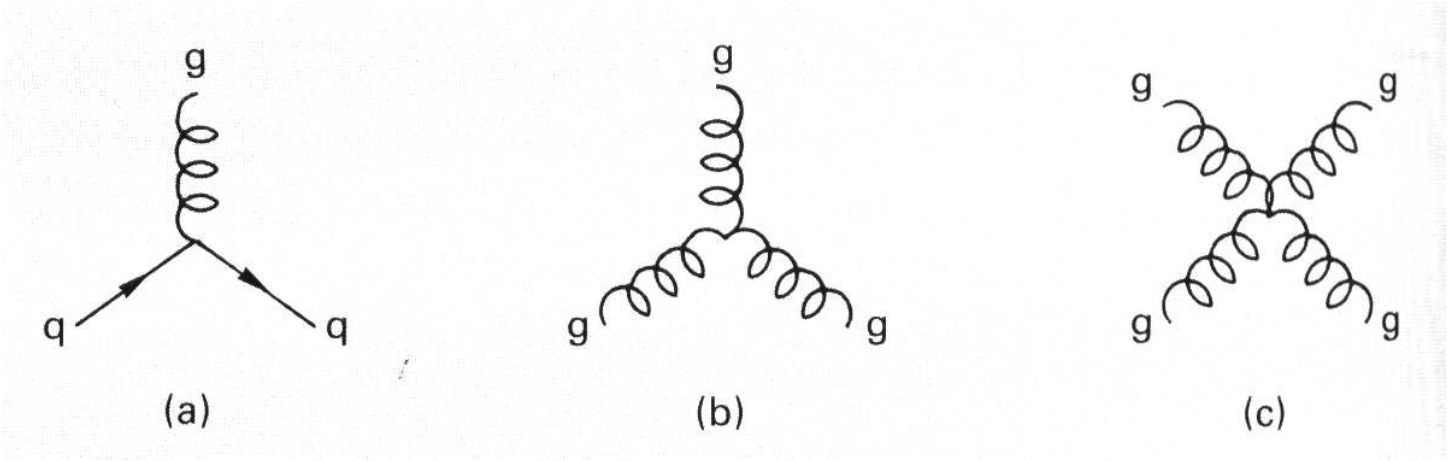
So, gauge invariance leads us to the following interaction vertices:



a) A quark-(anti)quark-gluon vertex, somewhat similar to that in QED with coupling g.

b) A three-gluon vertex: the trademark QCD interaction, with coupling g.

c) A four-gluon vertex, with higher-order coupling g^2.

There are Feynman rules for these vertices, as well as for quark and gluon propagators. The structure of the matrix elements is quite similar to that in QED, but in addition to spatial part with traces of γ matrices and momemtum integration, there are also colour factors, made up from traces of λ matrices and colour structure functions f^{abc}.

You should be able to use the QCD vertices to draw all kinds of Feynman diagrams for the various QCD processes.

Strong coupling α_s

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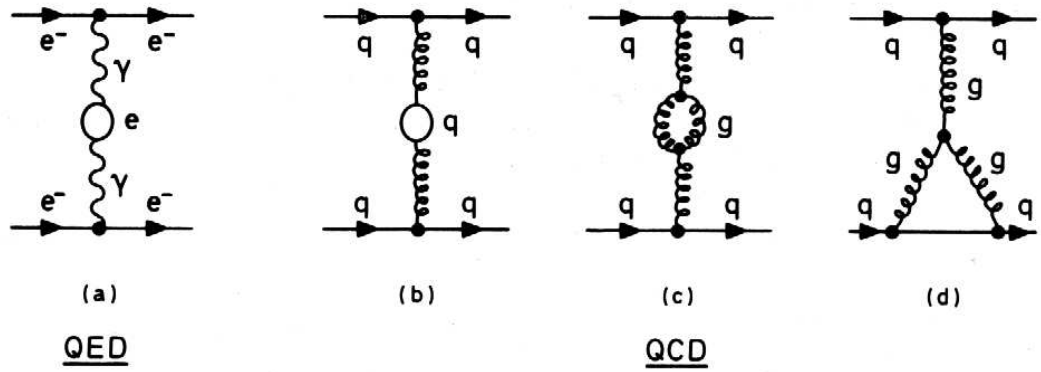
Strong coupling α_s plays in QCD the role of α in QED.

A simple one-gluon exchange in t-channel, between two scattering quarks, could have been used as the definition at $t \to 0$, if such limit existed — but because of quark confinement, no such limit exists. Hence, a different reference momentum scale is needed.

$$\frac{e^2}{4\pi} \equiv \alpha \qquad \Rightarrow \qquad \frac{g^2}{4\pi} \equiv \alpha_s$$



Loop diagrams, like these, result in q^2 dependence of the strong coupling.

QED the electron-positron loop as in (a) leads to Landau singularity, and increase of the coupling at large q^2 (small distances).

In QCD, there are not one, but many one-loop corrections to α_s, as in (b,c,d). Some of them involve quarks, and some involve gluon loops.

Regularisation scale dependence in QCD

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Once again, the evolution of the coupling α_s with regularisation scale μ^2 is given by the equation

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s)$$

where β is the beta-function for QCD, which now should include both quark and gluon loop contributions.

In QCD, $\beta(\alpha_s) = -b\alpha_s^2$ where $b = \frac{33-2N}{12\pi}$.

This 33 is due to gluon loops, while each quark flavour contributes -2 (so all N quark flavours contribute $-2N$).

As in QED, using the equation above, one can find the expression for $\alpha_s(q^2)$ in terms of $\alpha_s(\mu^2)$ by solving this differential equation:

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = -b\alpha_s^2$$

The solution process is identical to that in QED case, as only b is different.

Regularisation scale dependence in QCD (cont.)

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Following that derivation, we get

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + b\alpha_s(\mu^2)\log\frac{q^2}{\mu^2}}, \qquad b = \frac{33 - 2N}{12\pi}$$

Now, if a quark is really-really heavy, like top quark, it's not likely to significantly contribute. At each q^2, only those quarks contribute, whose typical masses m_q are smaller than q. Usually, at momentum scale of order of tens of GeV, this means no more than 5 "active" quark flavours (u, d, s, c, b). This seems to be the case in QCD so far.

So, as $N < 16$, coefficient b is **positive**, and our effective constant becomes weaker and weaker at high q^2. This property of QCD is called **asymptotic freedom**: at asymptotically large values of the trasferred momenta, the "strong" coupling "constant" becomes small, and the quarks are essentially free...

On the other extreme, of very low $q^2 \ll \mu^2$, i.e. large distances, the whole $\log$ term changes sign, and α_s becomes stronger and stronger.

This may be interpreted as the onset of colour confinement, but as α_s increases, higher order diagrams become more and more important and the above approximation becomes invalid.

Confinement and Asymptotic Freedom

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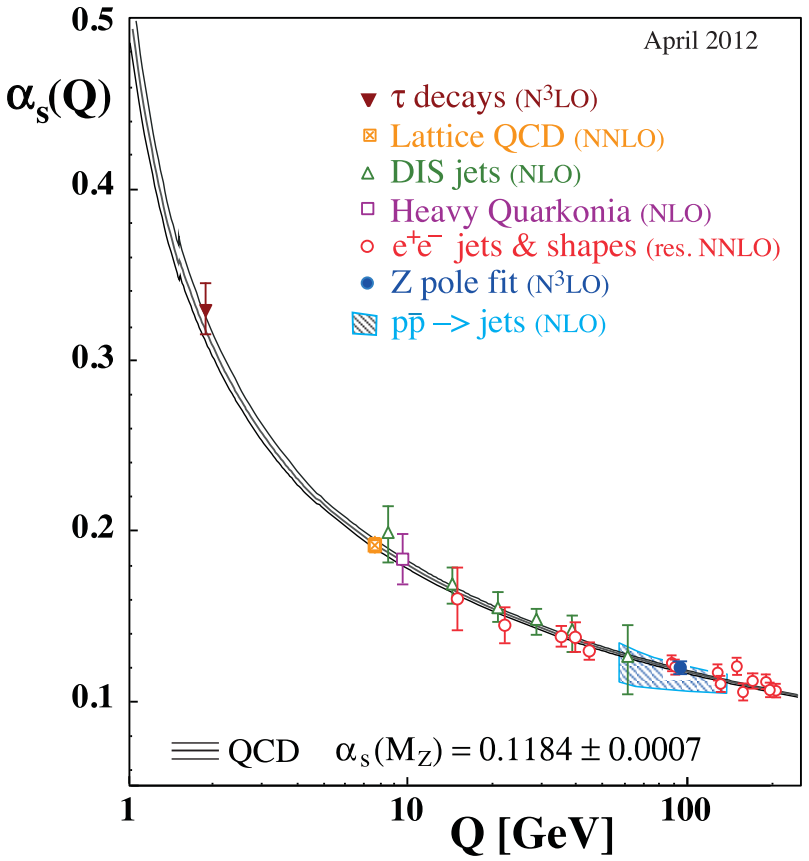
The formula describing q^2-dependence of the strong coupling α_s can be re-written in a different form:

$$\alpha_s(q^2) = \frac{4\pi}{\beta_0 \log(q^2/\Lambda^2)} \qquad \beta_0 = 11 - \frac{2}{3}N$$



Here Λ is the QCD scale parameter, which replaces the reference point μ. Its measured value is about 100 MeV, which corresponds to $\alpha_s(q^2 = M_Z^2) \simeq 0.12$.

There are various ways of measuring α_s, such as the rate of gluon radiation off quarks, heavy quarkonium decays, deep inelastic lepton-hadron scattering, and others.

The plot here shows results of some experimantal measurements, compared to the theoretical expectations.

Electromagnetic Interactions of Quarks

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EM interactions of all refmions — quarks and leptons — are very similar: coupling with the photon cannot change the type of the fermion, just its 4-momentum, and the coupling strength is proportional to the electric charge of the fermion.

All Feynman diagrams — and hence matrix element calculatiuons — describing EM interactions of leptons will also be valid for quarks - just remember to use the correct charge in each vertex

Additional difference may come from colour: some matrix elements acquire an additional (perfectly calculable) factor.

Example: The differential cross section of the process $e^+ + e^- \to u + \bar{u}$ is

$$\frac{d\sigma}{dt}(e^+e^- \to u\bar{u}) = \frac{2\pi\alpha^2}{s^2} \times \left(\frac{2}{3}\right)^2 \times 3 \times \frac{t^2+u^2}{s^2}$$

where the $(2/3)^2$ are due to the u-quark charge, and 3 is due to colour.

(Essentially, there are three possible $u\bar{u}$ pairs one can create: red-antired, blue-antiblue and green-antigreen, which the photon cannot distinguish).

Masses of Fundamental Fermions

Lancaster University

The three generations (families) of fundamental fermions have quite different masses (all in MeV):

$$\begin{pmatrix} e & 0.51 \\ \nu_e & \lesssim 1 \cdot 10^{-6} \end{pmatrix} \qquad \begin{pmatrix} \mu & 105.7 \\ \nu_\mu & \lesssim 1 \cdot 10^{-6} \end{pmatrix} \qquad \begin{pmatrix} \tau & 1777.0 \\ \nu_\tau & \lesssim 1 \cdot 10^{-6} \end{pmatrix}$$

Leptons become heavier with each new generation!

All masses on this slide are in MeV

Quark masses, although not very well defined, are still meaningful and important quantities (see later). And here the same pattern is even more pronounced:

$$\begin{pmatrix} u & \sim 5 \\ d & 7 \end{pmatrix} \qquad \begin{pmatrix} c & 1200 \\ s & 120 \end{pmatrix} \qquad \begin{pmatrix} t & 175000 \\ b & 4300 \end{pmatrix}$$

In order to produce a quark-antiquark pair $q\bar{q}$, the energy $\sqrt{s}$ should be higher than the respective threshold, $E_{\rm th} \simeq 2m_q$. So producing a $t\bar{t}$ pair is obviously a bit of a problem...

e^+e^- Annihilation into Quarks

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We've seen that away from both $\mu\mu$ and $q\bar{q}$ thresholds,

$$\frac{d\sigma}{dt}(e^+e^- \to u\bar{u}) = \frac{4}{9} \cdot 3 \cdot \frac{d\sigma}{dt}(e^+e^- \to \mu^+\mu^-)$$

The same should be true for the total cross section, and in fact all $q\bar{q}$ cross sections:

$$\sigma(u\bar{u}) = \sigma(c\bar{c}) = \sigma(t\bar{t}) = \frac{4}{9} \cdot 3 \cdot \sigma(e^+e^- \to \mu^+\mu^-)$$

$$\sigma(d\bar{d}) = \sigma(s\bar{s}) = \sigma(b\bar{b}) = \frac{1}{9} \cdot 3 \cdot \sigma(e^+e^- \to \mu^+\mu^-)$$

Then the total cross section of producing **any** quark-antiquark pair in e^+e^- annihilation (away from threshold regions) should be proportional to the cross section of producing a $\mu^+\mu^-$ pair:

$$R \equiv \frac{\sum \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum 3e_q^2$$

where the summation goes over 'kinematically allowed' quark flavours (those above threshold at given energy, so ignore $t\bar{t}$ for the time being).

So, R should increase with energy in steps, being roughly constant between thresholds.

Experimental Plot of $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$

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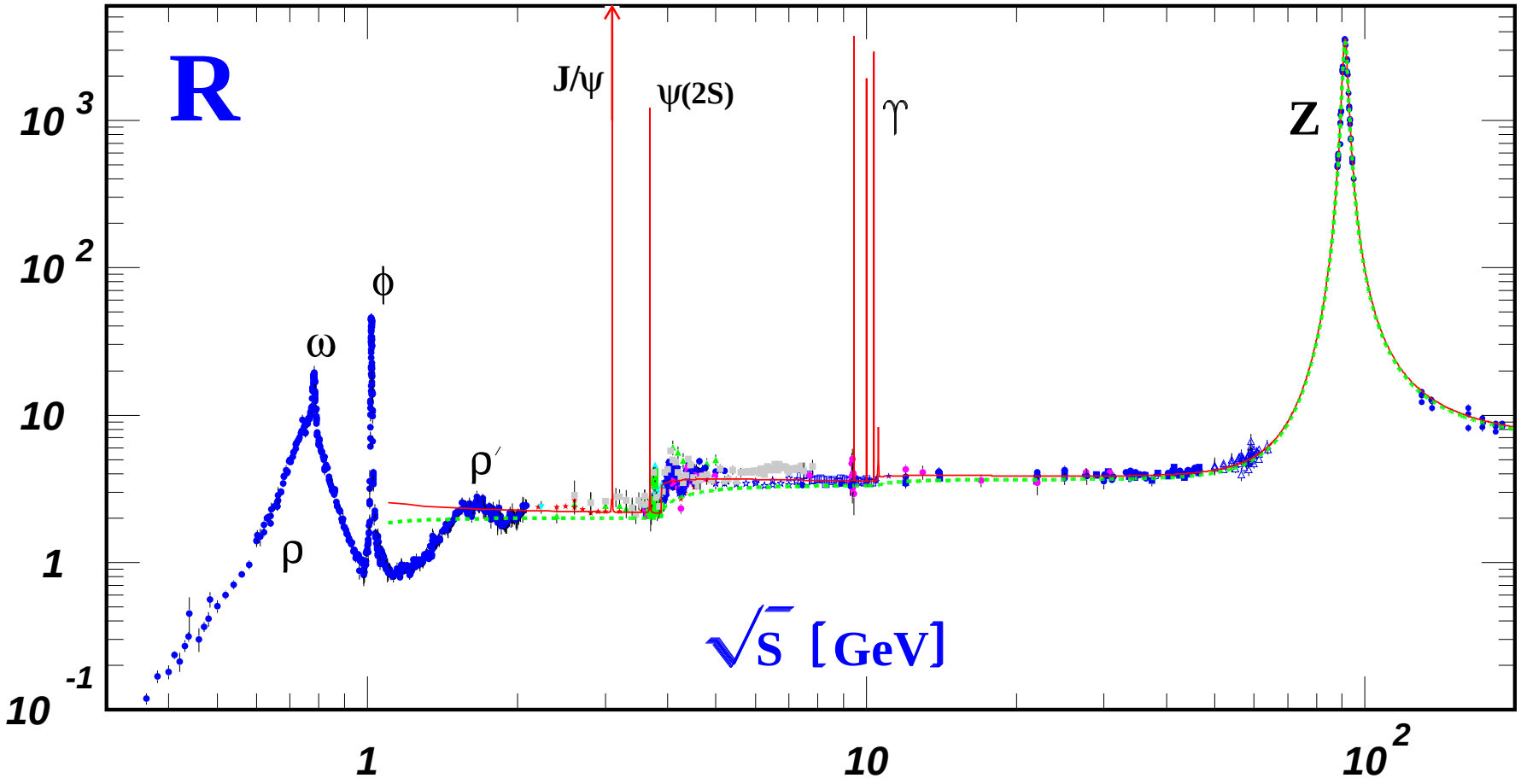

Q & A: Is it true that $q\bar{q} \equiv$ hadrons ? What's the difference between the red and the green lines? What are those blue peaks? Red peaks? The huge green Z peak?

Q & A Session

Lancaster University

- Is it true that $q\bar{q} \equiv$ hadrons ?

  The short answer is 'almost', as at high s QCD corrections must be small, since α_s is small

- What's the difference between the red and the green lines?

  The green line describes our prediction, which assumes that q and $\bar{q}$, unce produced, do not interact with each other any more. The red line takes into account this interaction as described by QCD. Including leading order corrections gives a factor $1 + \alpha_s(s)/\pi$.

- What are those blue peaks? Red peaks?

  There are hadronic resonances being produced (and then decayed) at/near each flavour threshold: ρ/ω for u, d, ϕ for s, J/ψ for c, Υ for b. All these are bound states of the respective quark and antiquark and are called **quarkonium**.

- What's the huge green Z peak?

  That's a completely different story. That peak corresponds to the production and subsequent decay of the Z^0-boson, one of the carriers of weak interactions.

  And that will be our next topic!

Propagator of an Unstable Particle

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The propagator of a stable particle with 4-momentum k and mass m is:

$$\sim \frac{1}{k^2 - m^2}$$

The propagator has a singularity at $p^2 = m^2$. Theorists may even say that a particle's mass (squared) is defined as the value of k^2 where the propagator has that singularity.

Unstable particles: mass acquires an imaginary part proportional to the (total) decay rate:

$$m \quad \rightarrow \quad m - i\Gamma/2$$

Propagator of an unstable particle:

$$\frac{1}{k^2 - (m - i\Gamma/2)^2} = \frac{1}{k^2 - (m^2 - \Gamma^2/4) + i\Gamma m}$$

The position of the singularity is shifted slightly along the real axis (but usually $\Gamma^2/m^2 \ll 1$, so this does not matter much).

More importantly, the singularity is shifted away from the real axis, because of the imaginary part. As long as particle momenta remain real, the denominator is never zero and the propagator of an unstable particle will remain finite.

The Breit-Wigner Distribution

Lancaster University

Usually $\Gamma^2/4 \ll m^2$, so let's neglect that term for simplicity.

Calculate the modulus squared of the propagator:

$$\left| \frac{1}{(k^2 - m^2) + i\Gamma m} \right|^2 = \frac{1}{(k^2 - m^2)^2 + m^2\Gamma^2}$$

This function clearly has a peak — but not a singularity — at $k^2 = m^2$ (check this!).

Let's move to the decay frame of the particle, where, by definition, $\mathbf{k} = 0$. But the particle is **not** real, and $E \neq m$ in general. Then

$$k^2 - m^2 = E^2 - m^2 = (E + m)(E - m)$$

In the vicinity of the peak $E + m \approx 2m$ and we finally have:

$$|\text{Propagator}|^2 \sim \frac{1}{4m^2(E - m)^2 + m^2\Gamma^2} \sim \frac{\Gamma^2}{(E - m)^2 + \Gamma^2/4}$$

This formula, as a function of E, describes the Breit-Wigner distribution. It has a peak at $E = m$, whose width at half-height is Γ (check this!).

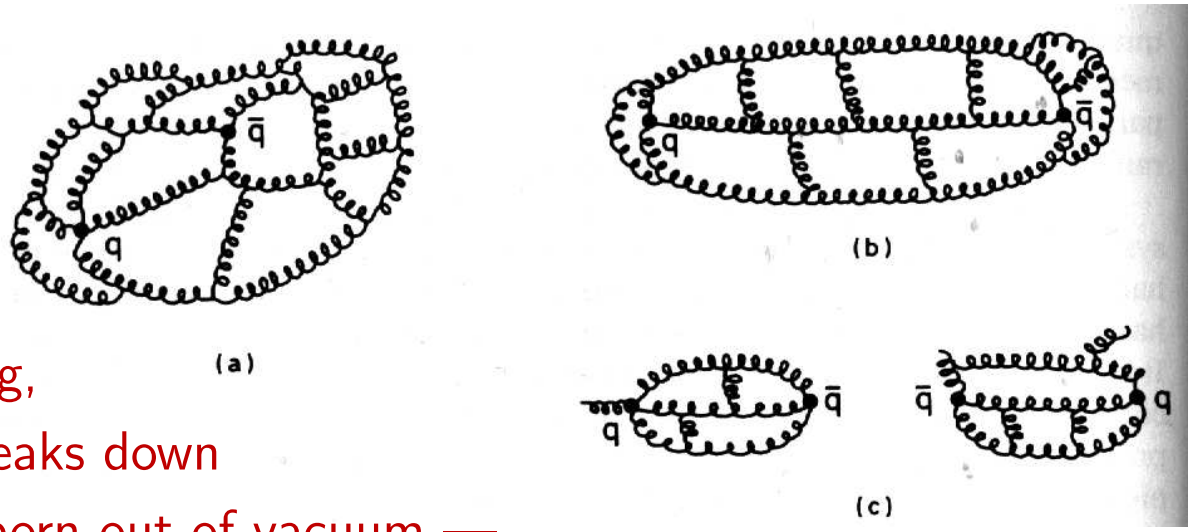
All resonances in physics, including particle physics, are described by a Breit-Wigner distribution.

Hadronisation

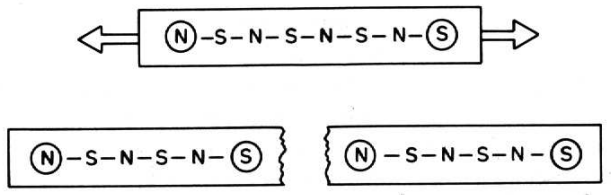
Lancaster University

Process $e^+e^- \to q\,\bar{q}$, via an s-channel photon and/or Z^0, at high energy $\sqrt{s} \gtrsim 10$ GeV.
The $q\bar{q}$ system is colourless (as was the γ/Z^0), but there are very strong colour forces acting between q and $\bar{q}$, (fig. (a)), especially when their invariant mass $\sqrt{s}$ is high and hence they have to move away from each other with big relative momentum.

These forces are such that some kind of "colour string" is formed between the two quarks (fig. (b)). The energy of this string increases (linearly) with the length of the string, up to the point when the vacuum breaks down and another quark-antiquark pair is born out of vacuum — or, in other words, a virtual pair becomes real, thanks to the energy of the string.



It's a bit like a long straight magnet: if you pull the ends, it will break into two (or more) smaller straight magnets. . . .



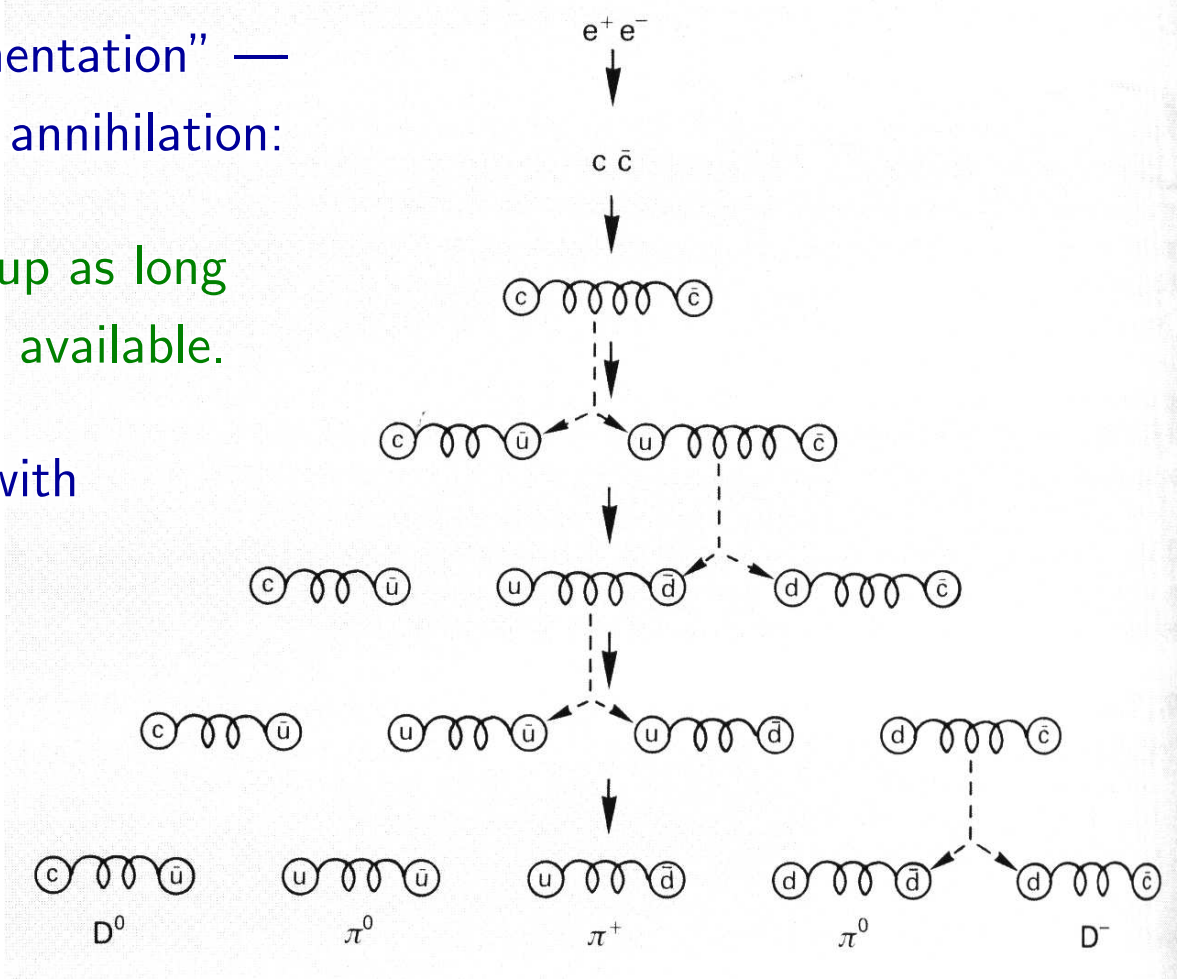
Hadronisation — II

Let's observe the evolution — "fragmentation" — of say, a $\to c\bar{c}$ pair produced in e^+e^- annihilation:

The colour strings continue breaking up as long as there is enough energy (inv. mass) available.

Then, the produced quarks combine with "nearest" antiquarks (diquarks) to produce final mesons (baryons).

Those secondary quark-antiquark pairs are usually light, i.e. $u\bar{u}$ or $d\bar{d}$, sometimes $s\bar{s}$.



That's why the leading hadrons — those at the ends of the chain — usually contain the initial quarks (whichever flavour they were), while the rest are nearly always pions.

Feynman vs Quark Flow

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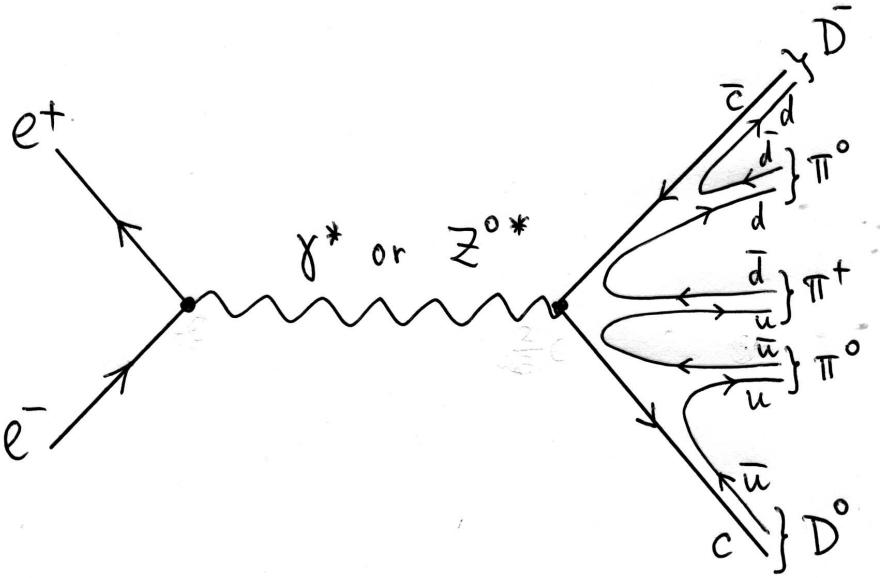
The Feynman diagram (or the colsest thing to F.d. we can draw in QCD), equivalent to the process on the previous slide, looks like this:



The space between the quark-antiquark lines is understood to be filled with virtual gluon "fishnet". But counting vertices does not make sense: the coupling is too strong!

In these circumstances, including all those gluonic interactions in the Feynman diagrams is pointless, and those soft gluons are just ignored. Feynman diagrams thus become quark-flow diagrams, showing only constituent quarks, i.e. those which determine the quatum numbers of the hadrons, not the "sea" of virtual $q\bar{q}$ pairs and gluons which is always present inside hadrons. But electroweak vertices still have to be shown explicitly.

On the contrary, it is thought that the hadronisation proceeds without any suppression, i.e. the probability of ending up in **some** final hadronic state is equal to unity. The problem is, we cannot predict probabilities of **specific** final configurations very well...

Hadrons at High Energy

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So, at rest all mesons consist of a "constituent" quark and a "constituent" antiquark, while all (anti)baryons consist of three "constituent" (anti)quarks, of various flavours.

These "constituent" quarks and antiquarks are, however, in constant interaction with each other, emitting and swallowing gluons all the time. In their turn, these gluons can split into more gluons, or create virtual (same flavour) quark-antiquark pairs.

So, at any given moment, a hadron will contain its constituent (or valence) quarks, a few gluons and a few quark-antiquark pairs. The valence quarks carry all the quantum numbers (isospin, strangeness etc), but the gluons and the quark-antiquark "sea" can carry quantities like momentum, energy and/or spin.

In order to see the "sea", one needs a camera with a very fast shutter, to take a snapshot. At high energy, this is relatively easy to accomplish: a proton which has high energy in the lab frame will have its proper time slowed down a lot. If it is probed with the help of a high-energy electron, we indeed get a snapshot with very fast shutter speeds.

Electron-Proton Scattering

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At low energies, electrons can only scatter elastically off the protons, i.e. the reaction is

$$e^- + p \to e^- + p$$

Here the electron does not see the structure of the proton, it only sees it as a whole, and the cross section is similar to $e\mu$ scattering: a single-photon exchange in the t-channel gives rise to the formulae we've seen before.

At higher energies, the resolving power of the virtual photon improves, and now the electron scatters **not off the proton as a whole**, but off individual charged constituents, i.e. quarks (and antiquarks) inside the proton.

Now, the (anti)quark which took the blow will gain a lot of energy/momentum, which will most probably break up the proton. This would give rise to those strong colour forces stretching those colour strings, and we get lots of final $q\bar{q}$ pairs etc. recombining into numerous final hadrons.

This is called not simply inelastic, but deep-inelastic electron-proton scattering:

$$e^- + p \to e^- + X$$

where X stands for a multi-hadronic final state, which, incidentally, must have net baryon number of $+1$, zero strangeness etc., but number and type of final particles and their total invariant mass may vary in a wide range.

Deep Inelastic Kinematics

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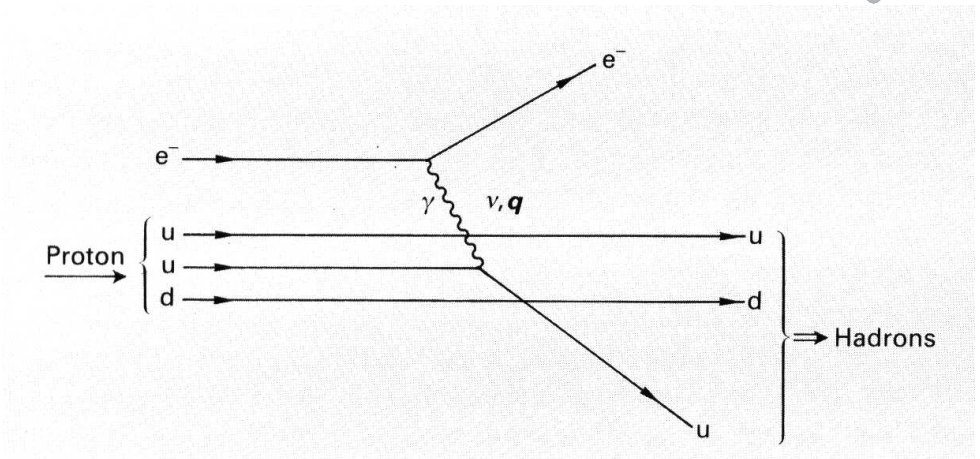
$$\begin{aligned} e^-(p_1) &+ p(p_2) \to e^-(p_3) + X(p_4) \\ q &= (\nu, \mathbf{q}) \equiv p_1 - p_3 \\ -q^2 &= \mathbf{q}^2 - \nu^2 \equiv Q^2 > 0 \end{aligned}$$



In deep inelastic scattering, the photon is deeply virtual, with Q^2 very large, $\gtrsim 10$ GeV^2.

The quantity $1/\sqrt{Q^2}$ is the measure of the spatial resolution and/or of the shutter speed achieved, and is hence very small. So, at large Q^2 we get information about the deep structure of the proton. But, in the best traditions of the uncertainty principle, this info comes at the price of completely destroying the proton...

If the quark hit by the photon was initially carrying the fraction x of the proton's 4-momentum, $k_1 = xp_2$, then its final 4-momentum is $k_2 = k_1 + q = xp_2 + q$. But in QCD, we expect the quark to be (almost) free, hence the electron-quark scattering is elastic and the quark's mass should remain the same, $k_1^2 = k_2^2$:

$$k_2^2 = (k_1 + q)^2 = k_1^2 + q^2 + 2x(p_2 q) \quad \Rightarrow \quad x = -\frac{q^2}{2(p_2 q)} = \frac{Q^2}{2M_p \nu}$$

$x = Q^2/M_p\nu$, is called Bjorken's scaling variable.

Structure Functions

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The cross section of the deep-inelastic ep scattering looks like this:

$$\frac{d^2\sigma}{d\cos\theta dx} = \frac{\pi\alpha^2 s}{q^4}\left(1+\cos^4\frac{\theta}{2}\right)F_2(x,Q^2)$$

which is identical to the scattering of electrons of momentum p_1 on a bunch of point-like particles, with momenta xp_2, times the probability of finding that pointlike particle.

$F_2(x)$ is called the structure function of the proton, and is closely related to the distributions of various quarks and antiquarks inside the proton:

$$F_2(x) = \frac{4}{9}x[u(x)+\bar{u}(x)] + \frac{1}{9}x[d(x)+\bar{d}(x)] + \frac{1}{9}x[s(x)+\bar{s}(x)]$$

Here $q(x)$ describe the probabilities of finding a quark of type q inside a proton, carrying the fraction x of proton's momentum. c, b, t quarks and antiquarks are usually neglected at this stage as they are too heavy.

Quark Disrtibutions in a Proton

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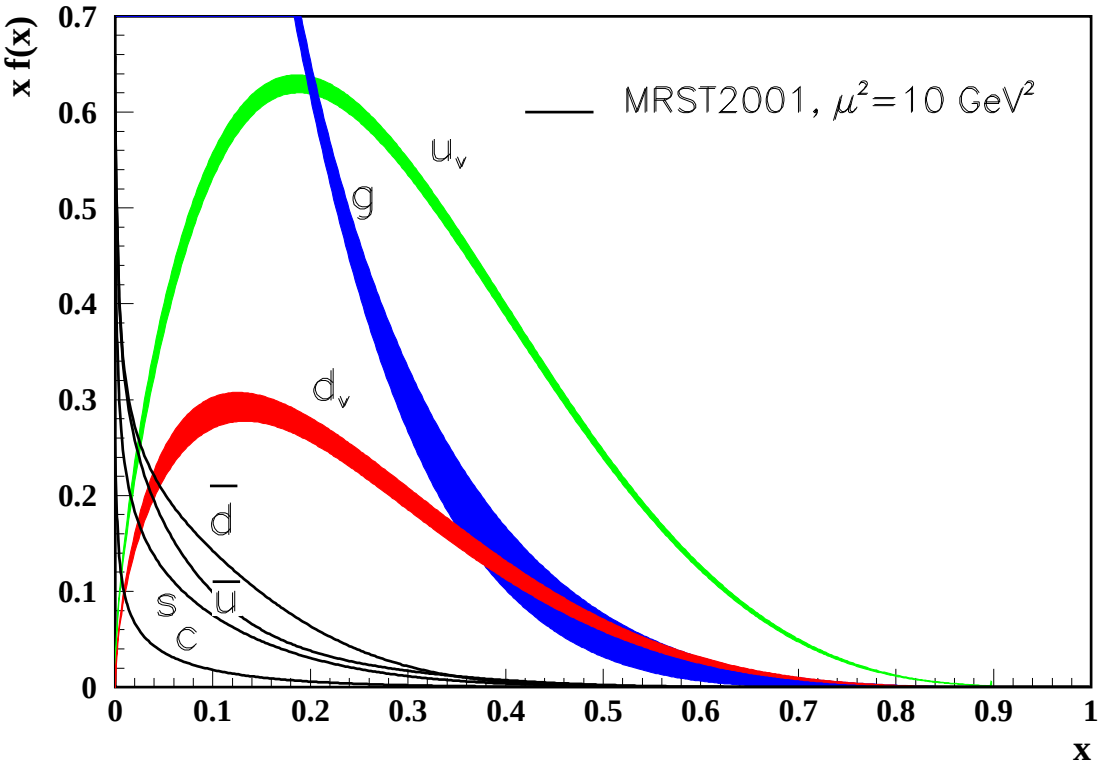
Due to isotopic symmetry, u-quarks in a proton have the same distribution as d-quarks in a neutron, and deep inelastic en scattering can be observed if deuterium target is used.

Similar deep inelastic processes are possible with initial neutrinos, via charged weak interactions. Here, ν ($\bar{\nu}$) will mainly see d and $\bar{u}$ (u and $\bar{d}$) quarks inside protons and neutrons, thus giving enough information to determine all quark and antiquark distribution functions independently.

Figure shows valence u, d and sea u, d, s, c distributions, as well as gluon distribution g. In the sea, $u = \bar{u}$ etc.

Roughly half of momentum is taken away by gluons, only about 30% by valence quarks.



Scaling Violations

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So far, the distribution functions — and hence the structure functions — depend only on x and do not depend on Q^2. This is called "scale invariance", or simply "scaling".

This is only the case if the quarks, antiquarks and gluons inside a proton are free, i.e. do not interact with each other.

When Q^2 is high, QCD coupling α_s is small due to asymptotic freedom. So, at high Q^2 quarks and gluons are almost free, but not quite.

Hence, the structure function F_2, apart from x, will have **some** dependence on Q^2 too, which would mean scaling violation.

In particular, if a photon of a certain Q^2 sees a quark carrying momentum fraction x, another photon of higher Q^2 may see the quark after it has radiated a gluon, and hence its momentum fraction is smaller. In general, at larger Q^2, QCD interaction would tend to make the distribution functions softer.

Q^2-dependence of the Structure Function

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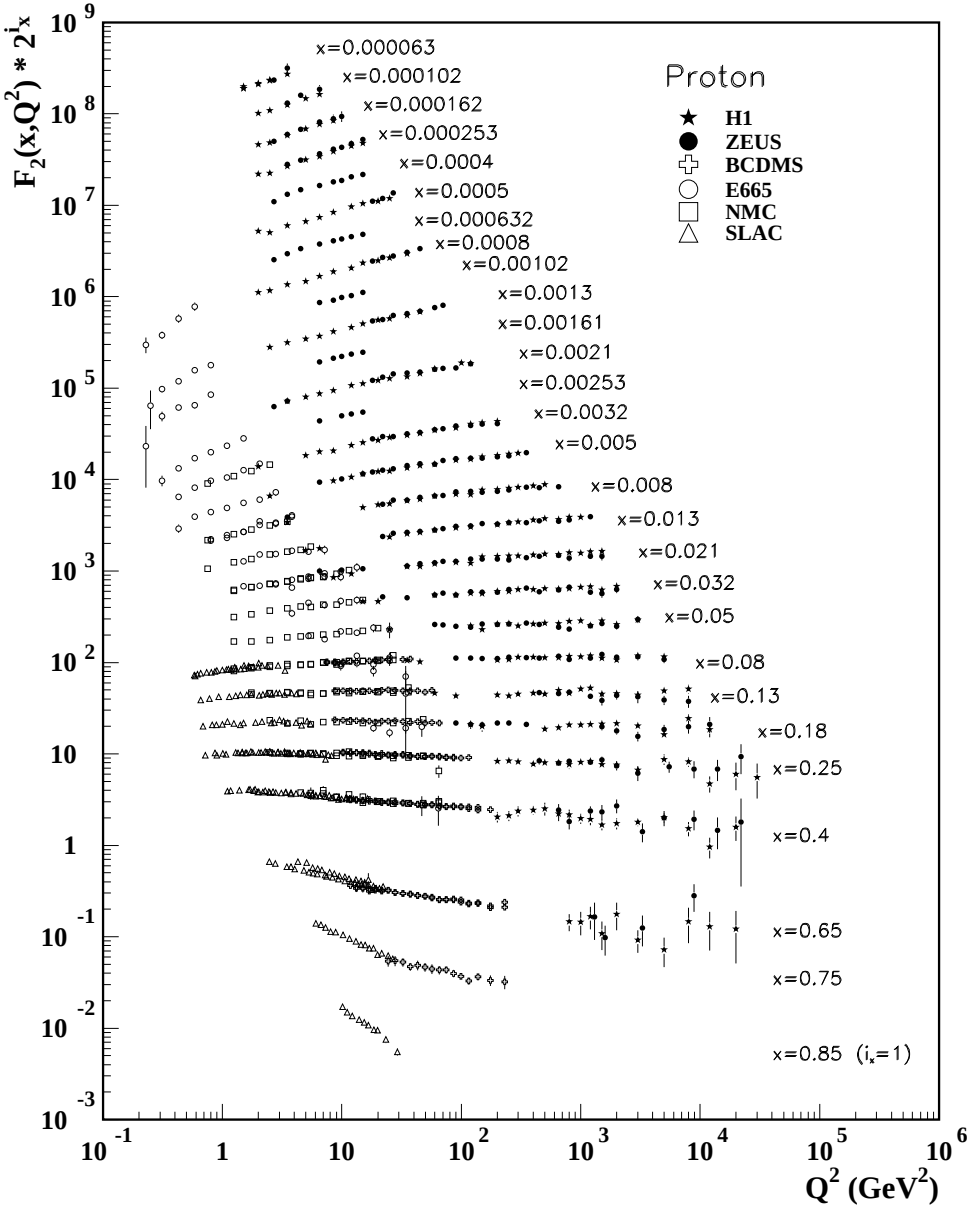
Figure shows $F_2(x, Q^2)$ at various x values, versus Q^2.

If scaling is exact, all the curves on the plot will be horizontal straight lines.
In fact, we see that at small x values, F_2 increases with increasing Q^2.
At large x F_2 decreases with Q^2, while at intermediate values around $x \simeq 0.1$ it is almost constant.

Perturbative QCD cannot predict the exact shape of F_2, but given $F_2(x, Q_0^2)$ at some $Q^2 = Q_0^2$, it can predict the evolution of F_2 with increasing Q^2.



Quantum Numbers of Quarks

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Property	d	u	s	c	b	t
$\mathcal{B}$ — baryon number	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$
Q — electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I — isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z — isospin z-component	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
S — strangeness	0	0	-1	0	0	0
C — charm	0	0	0	$+1$	0	0
B — bottomness (beauty)	0	0	0	0	-1	0
T — topness	0	0	0	0	0	$+1$

Gell-Mann-Nishijima formula: $Q = I_z + \frac{\mathcal{B}+S+C+B+T}{2}$

For antiquarks, all numbers change signs (except isospin).

Charmonium Decays

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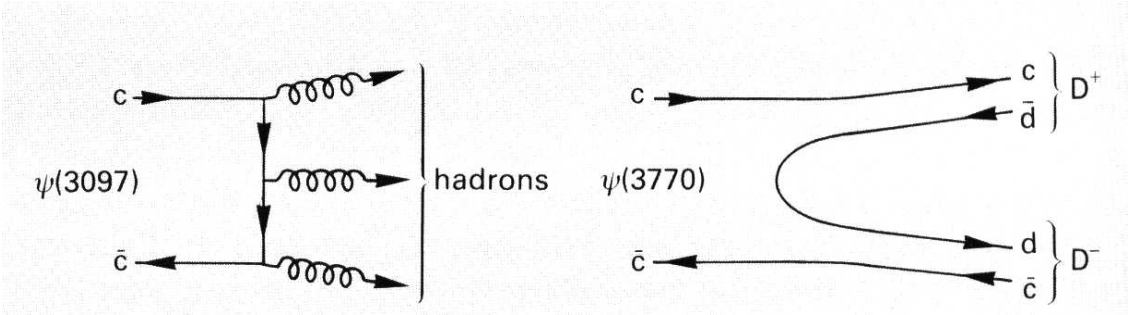
$J/\psi(3097)$ and $\psi(3770)$ are examples of **charmonium**, $c\bar{c}$ bound states, mesons with "hidden charm".



There is a rich spectrum of charmonium states, somewhat similar to positronium. $\psi(3770)$ is one of excited states, just heavy enough to decay into two "open charm" mesons D^+ and D^-, as shown in quark flow diagram on the right. Its total width is a few MeV, since it's very close to the threshold.

$J/\psi(3097)$ is well below this threshold, so it can only decay via $c-\bar{c}$ annihilation. Various selection rules forbid annihilation into one and two gluons, so the left diagram is the leading one in QCD. It's total width is about 60 keV, **much** smaller than a typical hadronic decay (Okubo-Zweig-Iizuka rule).

We draw a proper QCD diagram here because J/ψ is quite heavy, and some k^2 in this process are as large as ~ 10 GeV^2, so we would hope to be able to use QCD perturbation theory, with $\alpha_s \simeq 0.2$.

Further "hadronisation" of the three gluons happens "softly", i.e. they produce many $q\bar{q}$ pairs with small invariant masses, which somehow combine into a few final-state pions etc. This should not require any more "hard" (i.e. small!) QCD vertices.

Mesons in 4-quark World

Here are the 16-plets of pseudoscalar (a) and vector (b) mesons, built out of 4 quark flavours.

Each level ("floor") contains mesons with a particular value of charm.

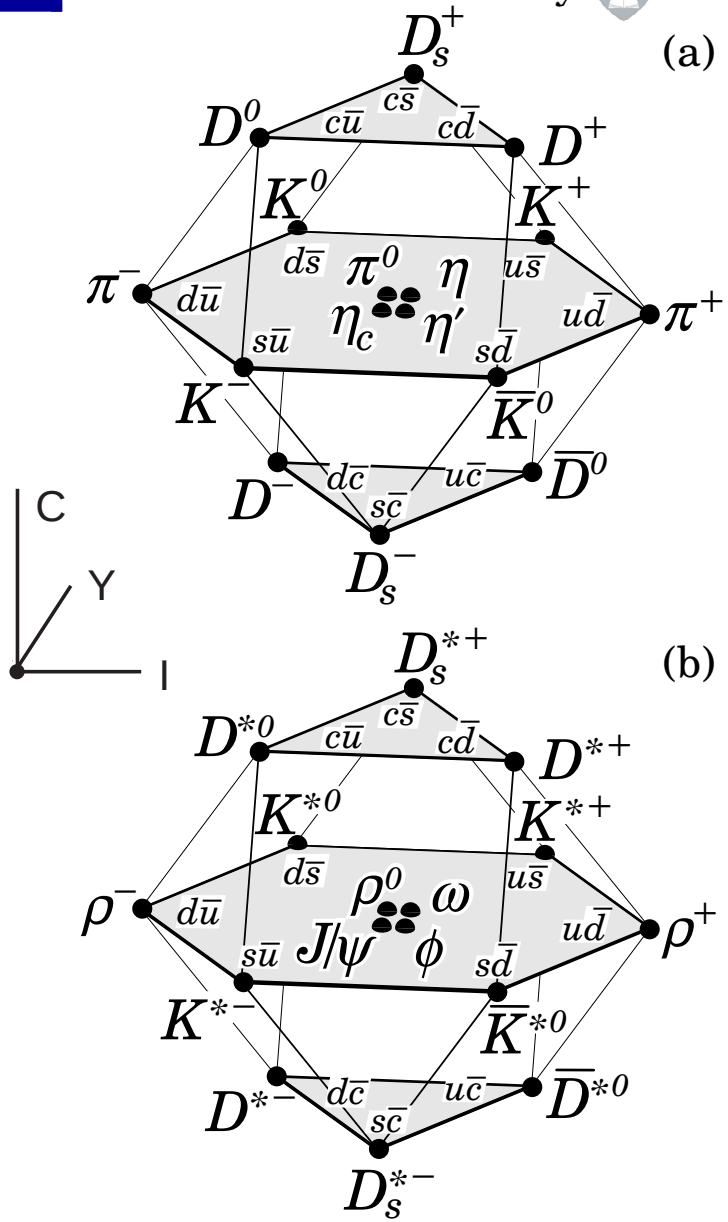
Adding b-quarks would make the plot 4-dimensional...

The two groups of four mesons in the centres are various mixtures of $q\bar{q}$ states: $u\bar{u}, d\bar{d}, s\bar{s}, c\bar{c}$.

In pseudoscalar sector, η_c is almost pure $c\bar{c}$, but η and η' are mixtures of $\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$ and $s\bar{s}$.

In vector mesons, the mixing is close to "ideal":

$$\omega = \frac{u\bar{u}+d\bar{d}}{\sqrt{2}}, \quad \phi = s\bar{s}, \quad J/\psi = c\bar{c}.$$



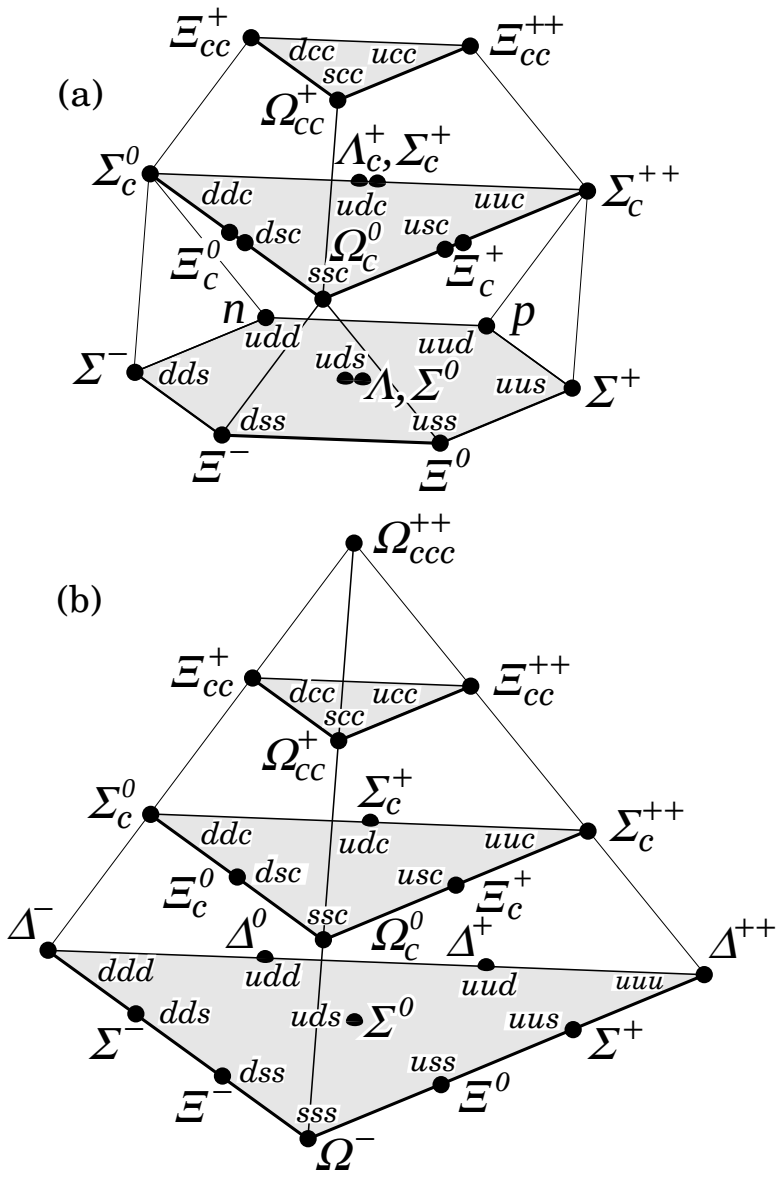
Baryons in 4-quark World

Here are the spin-1/2 baryons (a)
and spin-3/2 baryons (b).
Again, each floor contains baryons with
one value of charm.

The octet of charmless spin-1/2 baryons
lies on the ground floor in (a).
The decuplet of charmless spin-3/2 baryons
lies on the ground floor of (b).

In each multiplet, same letter denotes
similar quark content, but different
isospin projection.

Some baryons have identical quark contents,
but different spins. The one with spin-3/2
then acquires a star, e.g. Σ_c^{*++}.



Hadron-hadron scattering

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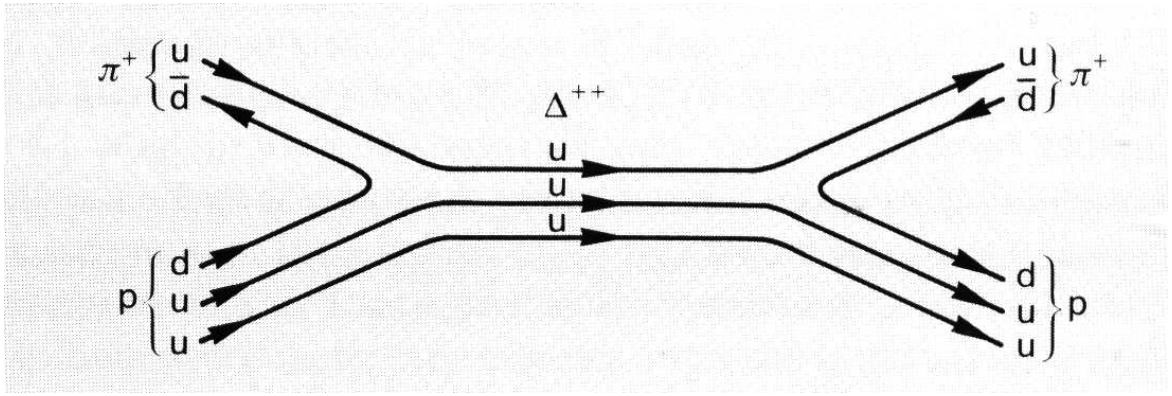
Let's try and draw a diagram describing $\pi^+ + p$ elastic scattering.



All the space between the quark and antiquark lines is understood to be filled with QCD interactions — a "fishnet" of gluon-quark-aniquark lines.

We cannot hope to use perturbation theory as α_s is rather large in "soft" processes, and we don't even attempt to count QCD vertices, but we know they are there.

Note that all rules used in drawing Feynman diagrams are still valid, simply gluons are not plotted and are just ignored, unless at least one line in the vertex is "hard", i.e. has a large k^2.

In this particular diagram, we expect to see a peak in the cross section at $s = M_\Delta^2$, but there are other possible diagrams too: e.g. t-channel exchange of a ρ meson, or simply two gluons exchanged in the t-channel (why not just one gluon?).

Incidentally, the right half of the above diagram represents quark flow in the main decay of Δ baryon: $\Delta \to \pi + p$.

πp Cross Section

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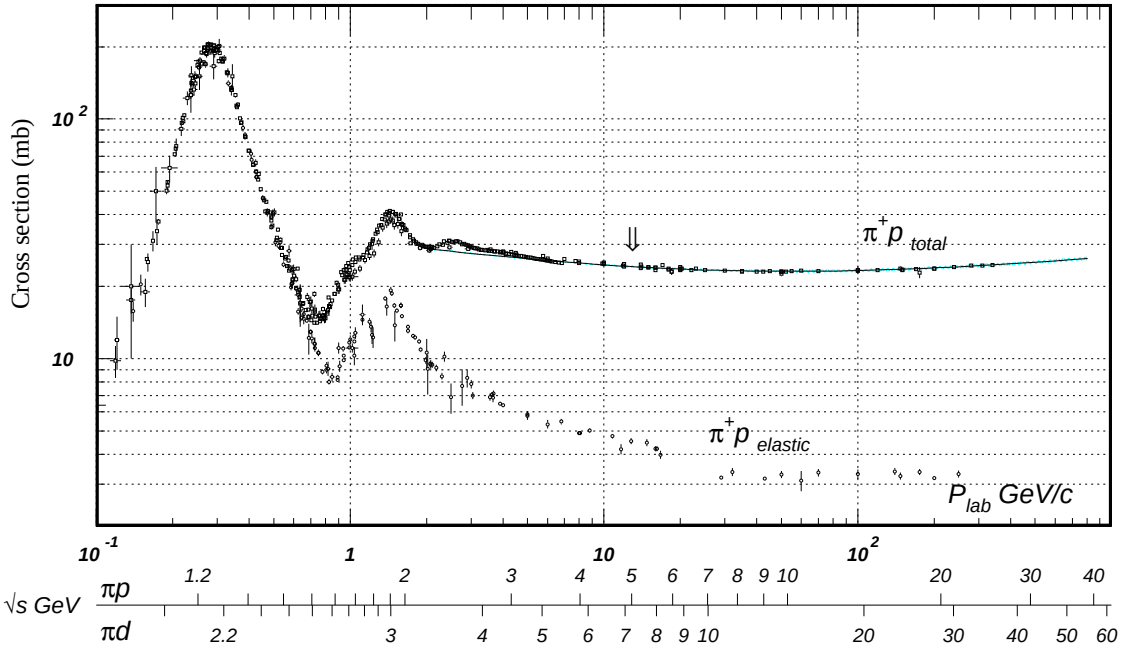
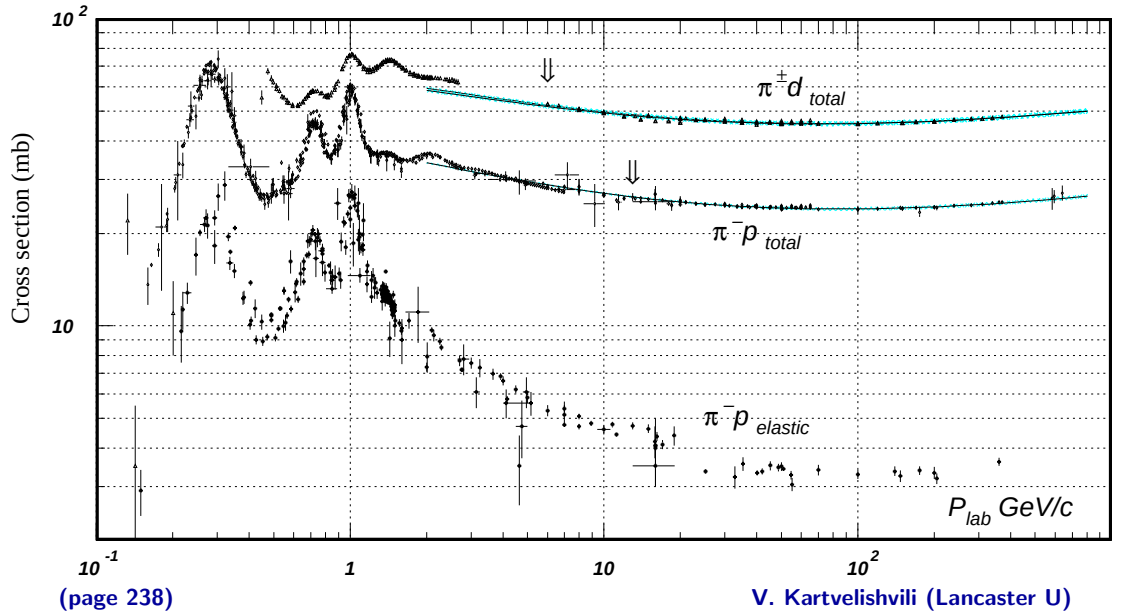
You can see the Δ^{++} resonance in $\pi^+ p$ cross sections, and also the Δ^0 resonance in $\pi^- p$ cross section, as well as a number of higher-mass baryonic resonances.

Why do you think σ_{elastic} at peak is much less than σ_{total} for $\pi^- p$, but **not** for $\pi^+ p$?

Note that the energies do not go as far as in pp, $\bar{p}p$ or even e^+e^- collisions, because all πp are fixed-target experiments: no colliding pion beams so far...





Quark-parton model and QCD

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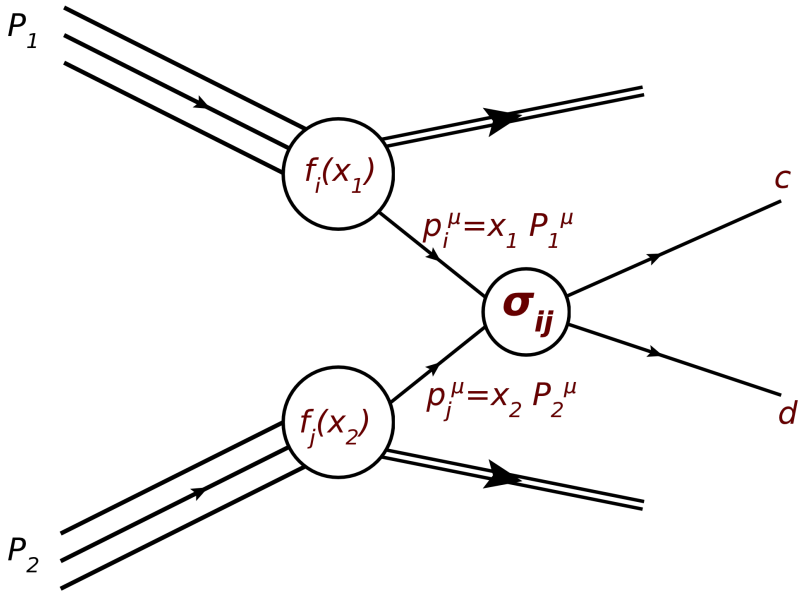
So, QCD perturbation theory (PT) can only work in processes with momentum transfer q^2 larger than a few GeV^2 (i.e. happening at small distances), where α_s is small. Hence, QCD PT cannot describe things like hadron masses, or strong decays or hadronic resonances, or hadronic scattering at low-energy.

It has been shown however, that in high energy hadronic collision, the cross section can be **factorised** into a large-distance, low q^2 part, described using phenomenological probability densities $f_i(x)$, and a small-distance, high q^2 part described by QCD PT.

This $f_i(x)$ is the probability of finding, inside a proton, a "parton" i (i.e. a quark, and antiquark or a gluon) carrying a fraction x of the proton's momentum.



The $\hat{\sigma}_{ij}$ is the cross section of some **subprocess** $i + j \to c + d$
Could be $gg \to gg$, or $q\bar{q} \to gg$, or $gg \to H \to W^+W^-$ etc.
The c.m.s. energy^2 of the subprocess is hence $\hat{s} = x_1 x_2 s$

The cross section of the whole process is then calculated with

$$\sigma(p + p \to c + d + X) = \sum_{i,j} \int dx_1 dx_2 f_i^{(1)}(x_1) f_j^{(2)}(x_2) \sigma(i + j \to c + d)$$

where "X" is whatever other "stuff" is produced in addition to c and d, and the sum is taken over all subprocesses contributing to the final state of interest, i.e. production of c and d.

Path Integrals and classical limit

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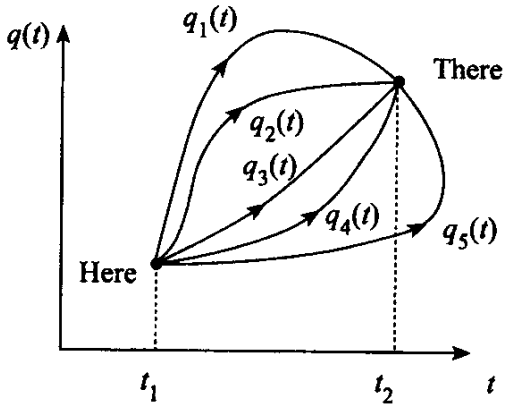
Remember the Least Action Principle? In classical physics, the system chooses **one** trajectory (path) corresponding to the **minimum** of S.

Important question: how is this generalised to the quantum case? "Particles" are now waves, covering the whole space...



Feynman gave a very useful generalisation of that principle, allowing to consider classical and quantum-mechanical systems on the same footing, with a smooth transition between the two:

In quantum mechanics, **all** paths are possible, and the evolution of a system is determined by integrating over all paths. But here is the interesting part: each path needs to be **weighted** with a factor $\exp(iS/\hbar)$, where S is the action along this path.

One can build a theory of interacting quantum-mechanical particles using this approach, without invoking all the complexities of the Quantum Field theory (but there are other difficulties, not least computational). Lattice theory is one example.

In classical limit, action S along a trajectory tends to be large, $S \gg \hbar$. This leads to very rapidly oscillating weights which tend to cancel out along all the paths, except the one corresponding to the smallest value of S — and that's the classical one!

From strong to weak

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We have used QED — the most successful quantum field theory so far — as a platform to move to QCD, the theory describing coloured quarks and their interactions. Along this way, we've postulated the existence of the colour quantum number in the quark wave functions, and the invariance of the Lagrangian under $SU(3)$ gauge transformations.

Once these gauge transformations are allowed to be local, the gauge invariance only holds if gluon fields, with very specific interaction properties, are introduced.

Now, because $SU(3)$ group is non-abeliean, gluon fields are required to carry colour charges and hence interact with each other.

Next step would be to try and build the theory of weak interactions. But here there are a few new hurdles to overcome:

- interactions changing quarks' flavour (and charge)
- parity violation
- massive gauge bosons

The main challenges of the weak theory

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- We know weak interactions require charged currents, i.e. the Lagrangian needs to include interaction terms where the fermion type is changing at the vertex. These were not allowed in QED and QCD.

- In QED and QCD, all therms in the Lagrangians were true scalars, and hence parity was a conserved quantum number. We know this is not the case in weak interactions, so the respective term in the Lagrangian may contain both scalar and pseudo-scalar terms, which means that ψ_R and ψ_L may not / should not be treated equally.

- Still, the biggest difference is that the weak interactions are (very) short-range, hence the bosons involved need to be massive. This is by far the biggest challenge, as theories with massive vector bosons are usually not renormalisable.

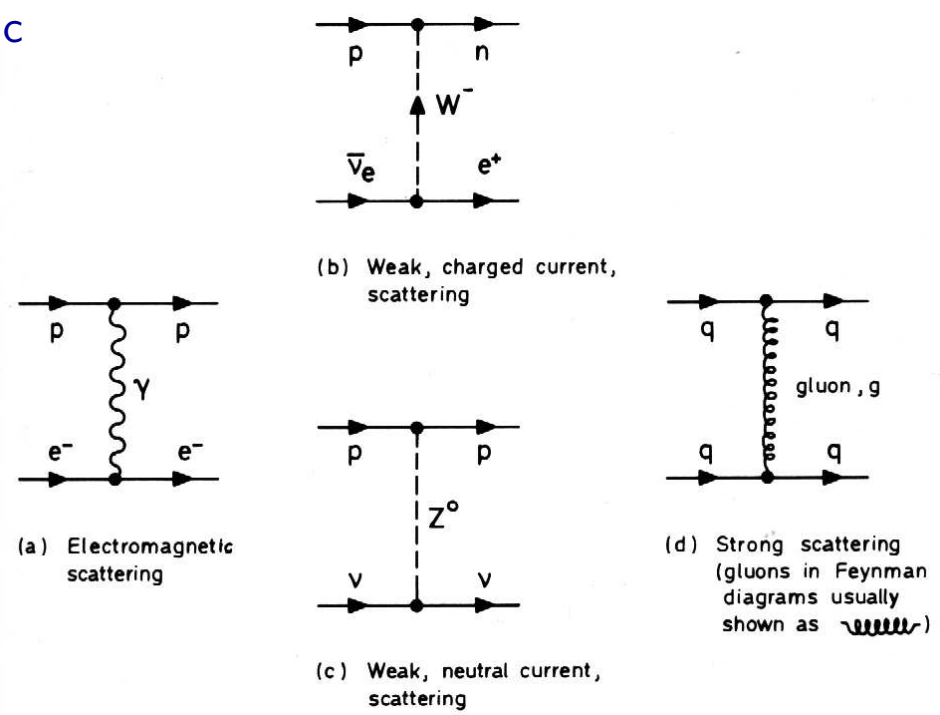
- On top of all these complciations, there are also weak neutral currents. These cannot change fermion flavours (just like in QED and QCD), but they do violate parity, and need massive bosons.

In the next few lectures, we will study the phenomenology of weak intereactions, and see how the gauge theory of electroweak interactions overcomes these hurdles.

Gauge Boson Exchange Potential

Here are four examples of fermion-fermion (quasi-)elastic scattering due to the four gauge boson exchange:
One can easily assess these amplitudes using simplified Feynman rules:
coupling $\times$ propagator $\times$ coupling.
A 3D Fourier transform of these amplitudes gives an equivalent potential $V(r)$.
In QED we thus have the Coulomb potential:

$$\mathcal{M}_{QED} \sim \frac{e^2}{q^2} \quad \Rightarrow \quad V_{QED}(r) = \frac{e^2}{4\pi r}$$



Gluons are much like photons as they are massless, but the three-gluon vertex — and colour confinement — result in important differences, as we have seen.

Weak amplitudes, with massive particles being exchanged, lead to $\mathcal{M}_{W,Z} \sim \frac{e^2}{q^2 - M^2_{W,Z}}$, which correspond to short range Yukawa-type potentials $V_{W,Z}(r) = \frac{e^2}{4\pi r} \exp(-M_{W,Z} r)$.

Diagonality of photon and gluon interactions

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So far, we've been considering flavours (types) of leptons and quarks one at-a-time.

This was OK, as neither QED not QCD are capable of changing the flavours.

EM interaction is fully diagonal:

the photon only couples to the fermion-antifermion pair of the same type, with strength proportional to its charge: 0 for neutrinos; -1 for e, μ, τ; $+2/3$ for u, c, t; $-1/3$ for d, s, b.

The photon is **unable** to change a quark's colour or flavour, or a lepton's type.

Similarly, **gluons are unable to change quark types, so strong interactions are flavour-diagonal too.**

Gluons **do** change quark colour though...

Left-right symmetry

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In terms of left and right components of the fermionic filed, the electromagnetic interaction term contains both

$$ej^{\mu}_{EM}\,A_{\mu} = e\bar{\psi}\gamma^{\mu}\psi\,A_{\mu} = e\left(\bar{\psi}_L\gamma^{\mu}\psi_L + \bar{\psi}_R\gamma^{\mu}\psi_R\right)\,A_{\mu}$$

The interaction term in QED (and also in QCD for that matter) is a true scalar, and hence parity is conserved in these interactions.

In order to describe **weak charged interactions**, three big differences are needed:

- There is maximum possible parity violation: only the first term in brackets is present in weak interactions. Right-haded particles do **not** participate in charged weak interactions.
- The interaction **has to** change the type of fermion, i.e. $\bar{\psi}$ on the left and ψ on the right **have** to describe different fermions. Neutrinos are transformed into electrons (and vice-versa), u-quarks into d-quarks, etc.
- Hence, the exchange boson has to be charged, i.e. there will be two hermitian-conjugate interaction terms in the Lagrangian. The bosons responsible for weak charged interactions are, as you should know by now, the $W^{\pm}$ bosons.

W Bosons

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We know that there are three families of leptons and quarks, organised into doublets:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} \quad \begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}$$

Neutrinos and u, c, t quarks are considered "up"-type, while charged leptons and d, s, b quarks are "down"-type. Then, the piece of electroweak Lagrangian describing charged weak interactions with coupling g should look like this:

$$\mathcal{L}_{\text{weak charged}} \sim g \left(\bar{\psi}_L^{\text{up}} \gamma_\mu \psi_L^{\text{down}}\right) W_\mu^+ + g \left(\bar{\psi}_L^{\text{down}} \gamma_\mu \psi_L^{\text{up}}\right) W_\mu^-$$

- There are two terms, one describing the interaction of down-type fermion and up-type antifermion with W^+, and the other, charge-conjugate term describing the interaction of up-type fermion and down-type antifermion with W^-.
- In both terms, **only left-handed fermions** are present. Right-handed fermions remain untouched by the charged weak interactions.
- The operator projecting out left handed components is $\frac{1}{2}(1 - \gamma_5)$, i.e. the Lorentz structure of the charged weak current is "vector minus axial-vector", i.e. $V - A$.

Feynman vertices for charged current interactions

Here are the Feynman vertices matching the various interaction terms from the part of the Lagrangian that describes charged current interactions of fundamental fermions with $W^{\pm}$ bosons.

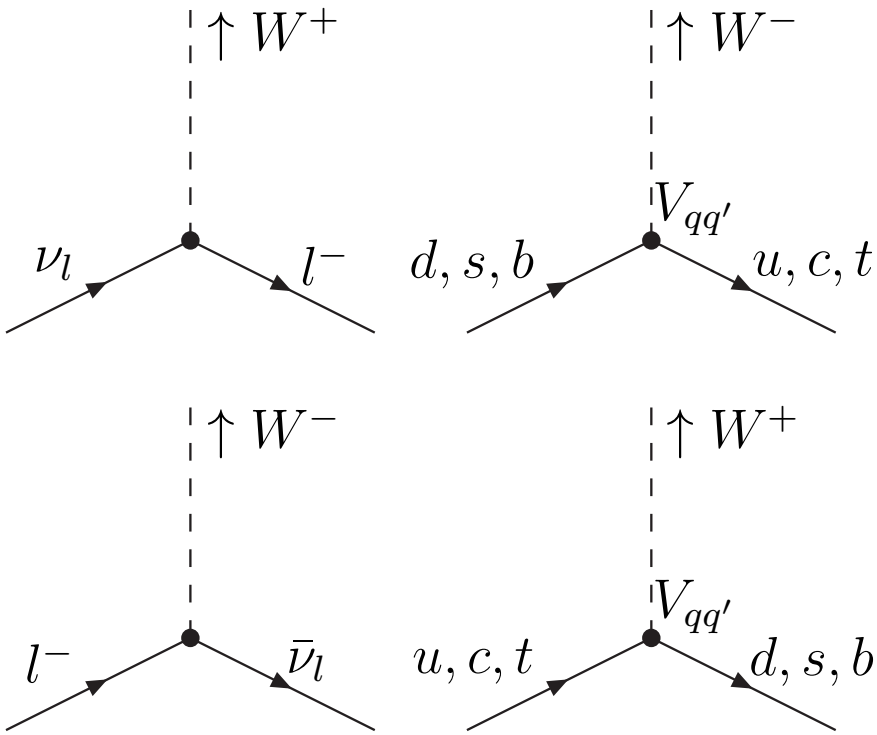
- At each vertex, there is a structure

$$-ig\, V_{ff'}\, \gamma_\mu(1-\gamma_5)$$

  and the usual energy-momentum δ function.
- The charge of the W depends on the direction of its momentum. Arrows show the correct flow, but the charge will be opposite if momentum direction is inverted.
- The propagator of a (massive) W boson with momentum q is

$$i\frac{-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_W^2}}{q^2-m_W^2}$$

  accompanied by the inevitable integration over d^4q.



Weak charged interactions

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So here is again the piece of electroweak Lagrangian describing the interaction of charged fermionic currents with the W bosons:

$$\mathcal{L}_{\rm W} \sim -ig\left\{V_{ff'}\bar{\psi}^{\rm down}\gamma_\mu(1-\gamma_5)\psi^{\rm up}W^-_\mu + V^\dagger_{ff'}\bar{\psi}^{\rm up}\gamma_\mu(1-\gamma_5)\psi^{\rm down}W^+_\mu\right\}$$

There are as many terms like this as there are up-type/down-type combinations in the Standard Model.

This time I also show the constants $V_{ff'}$, that determine relative strengths of various terms in the Lagrangian. These constants in fact form unitary matrices, one for quarks, one for leptons.

For quarks, this is the Cabibbo-Kobayashi-Maskawa matrix

$$V_{ff'} = V_{CKM} = \begin{array}{c} \\ u \\ c \\ t \end{array}\begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \end{array} \approx \begin{array}{c} \\ u \\ c \\ t \end{array}\begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} 1-\lambda^2/2 & \lambda & \sim A\lambda^3 \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ \sim A\lambda^3 & -A\lambda^2 & 1 \end{pmatrix} \end{array}$$

where $\lambda \simeq \sin\theta_{\rm Cabibbo} \simeq 0.22$, and elements are expanded up to $\mathcal{O}(\lambda^4)$. Hidden inside the corner terms A there is a complex phase which, if not zero, will allow CP violation.

There is a similar Pontecorvo-Maki-Nakagawa-Sakata matrix describing neutrino mixing, but it is (just) outside the scope of this course.

Basic Charged Current Processes: Leptons

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Here are the basic first-order leptonic processes corresponding to the basic vertices:

- A neutrino can radiate a W^+ (or "swallow" a W^-) and become the negatively charged lepton from the same family.
- An antineutrino can radiate a W^- (or "swallow" a W^+) and become the positively charged lepton from the same family.
- A negatively-charged lepton can radiate a W^- (or "swallow" a W^+) and become the neutrino the same family.
- A positively-charged lepton can radiate a W^+ (or "swallow" a W^-) and become the anti-neutrino the same family.
- In any case, charge should be conserved at each vertex.
- For massless neutrinos, transitions between families are not allowed, and lepton numbers of each type are conserved separately.

Basic Charged Current Processes: Quarks

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Here are the basic first-order quark processes corresponding to the basic vertices:

- A down-type quark can radiate a W^- (or "swallow" a W^+) and become an up-type quark, possibly from a different family.
- A down-type anti-quark can radiate a W^+ (or "swallow" a W^-) and become an up-type anti-quark, possibly from a different family.
- An up-type quark can radiate a W^+ (or "swallow" a W^-) and become a down-type quark, possibly from a different family.
- An up-type anti-quark can radiate a W^- (or "swallow" a W^+) and become a down-type anti-quark, possibly from a different family.
- In any case, charge should be conserved at each vertex.
- Any quark-flavour-specific quantum numbers (although conserved in other interactions) will **not** be conserved in weak charged current interactions.
- Whenever kinematically allowed, transitions within the same quark family are still more likely, because diagonal elements of the CKM matrix still dominate, as λ is a small number.

W Decays

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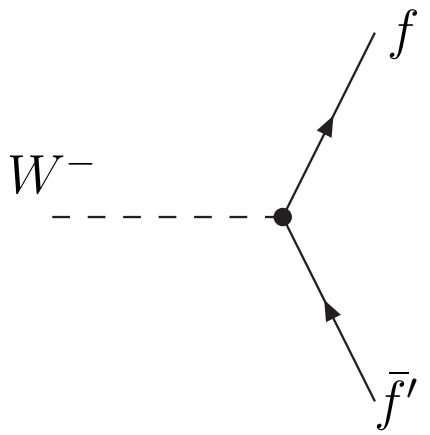
W bosons decay into a fermion-antifermion pair, but the fermions must be different, such that the sum of two fermion charges is equal to the charge of the decaying W.

Hence, possible decay modes of W^- are: $e^-\bar{\nu}_e, \mu^-\bar{\nu}_\mu, \tau^-\bar{\nu}_\tau, d\bar{u}, s\bar{u}, b\bar{u}, d\bar{c}, s\bar{c}, b\bar{c}$.

(why not $d\bar{t}$ etc?):



So there are many decay modes here, with effective branching fractions proportional to this combination of couplings:

$$\Gamma(W^- \to f\bar{f}') = N_{\text{colour}} |V_{ff'}|^2 \, \Gamma_1$$

where N_{colour} is 1 for leptonic, and 3 for $q\bar{q}'$ decay modes, and Γ_1 is a constant common for all decays.

This is the case of a first-order Feynman diagram actually corresponding to a real process — but again, remeber that the W^- in the initial state is usually virtual, at least slightly away from its mass shell.

It is not that difficult to calculate that

$$\Gamma_1 = \frac{g^2}{48\pi} M_W = 226 \text{ MeV}$$

Knowing the total width $\Gamma_{\text{tot}}(W) = 2.1$ GeV, can you calculate the separate branching fractions and partial decay widths?

Some Interesting Processes

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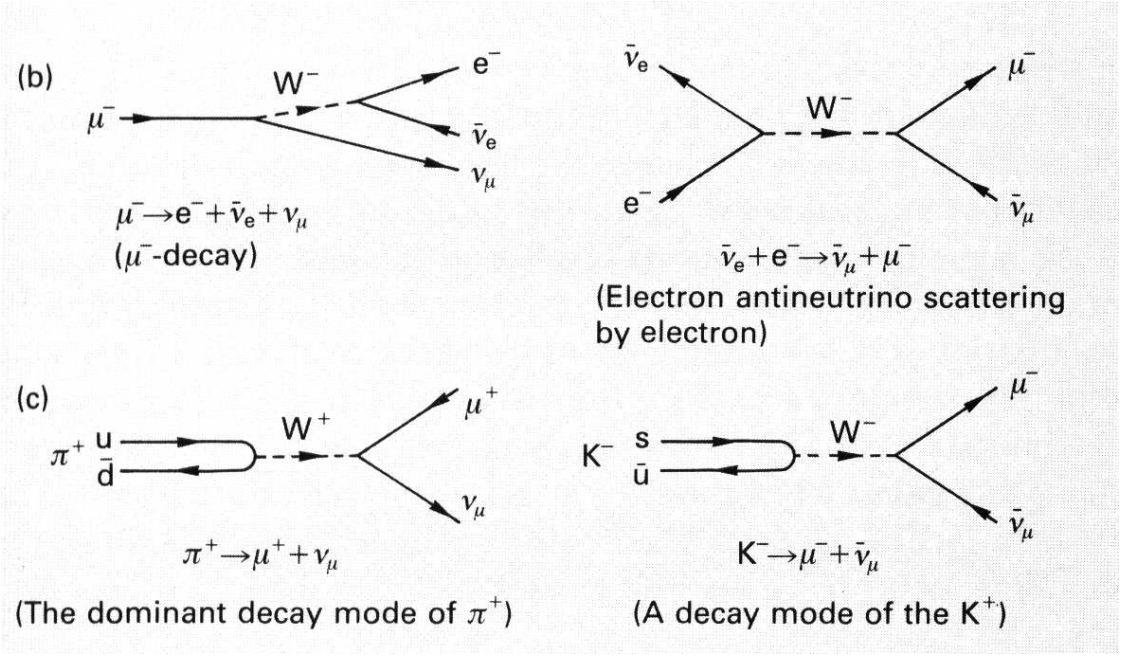
Muon decay rate:

$$\Gamma_\mu = \tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

where $\tau_\mu = 2.2 \times 10^{-6}$ s.

Our simplified Feynman rules would "predict" that the amplitude describing this decay is

$$\mathcal{M} \sim \frac{g^2}{M_W^2}$$



The width is proportional to $|\mathcal{M}|^2$, and one would expect m_μ^5 to keep dimensions right.
Indeed, G_F is the Fermi constant, which, in modern terms, is

$$G_F = \frac{g^2}{4\sqrt{2}M_W^2} = 1.166 \cdot 10^{-5}\text{GeV}^{-2}$$

So, our rules are doing very well!

τ Decay

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Decay of the τ lepton proceeds in a very similar way to the decay of the muon.

The Feynman diagram is identical, with $\mu - \nu_\mu$ line replaced by $\tau - \nu_\tau$ line.

However, in case of τ, the virtual W has enough invariant mass to decay not only into $e\nu_e$, but also into $\mu\nu_\mu$,ud and us — with appropriate couplings.

Leptonic decay mode, $\tau \to \nu_\tau e\nu_e$ has its partial decay rate calculated same way as the muon decay rate, only the mass of the decaying particle has to be changed.

Rates for other decay channels can be calculated similarly, but colour and CKM couplings should be taken into account for hadronic decays.

Full decay rate, or total width, is calculated as usual, as the sum of partial decay rates.

Quark Decay Chain

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Top quark is the heaviest fundamental particle (so far). Its main decay amplitude contains V_{tb}:

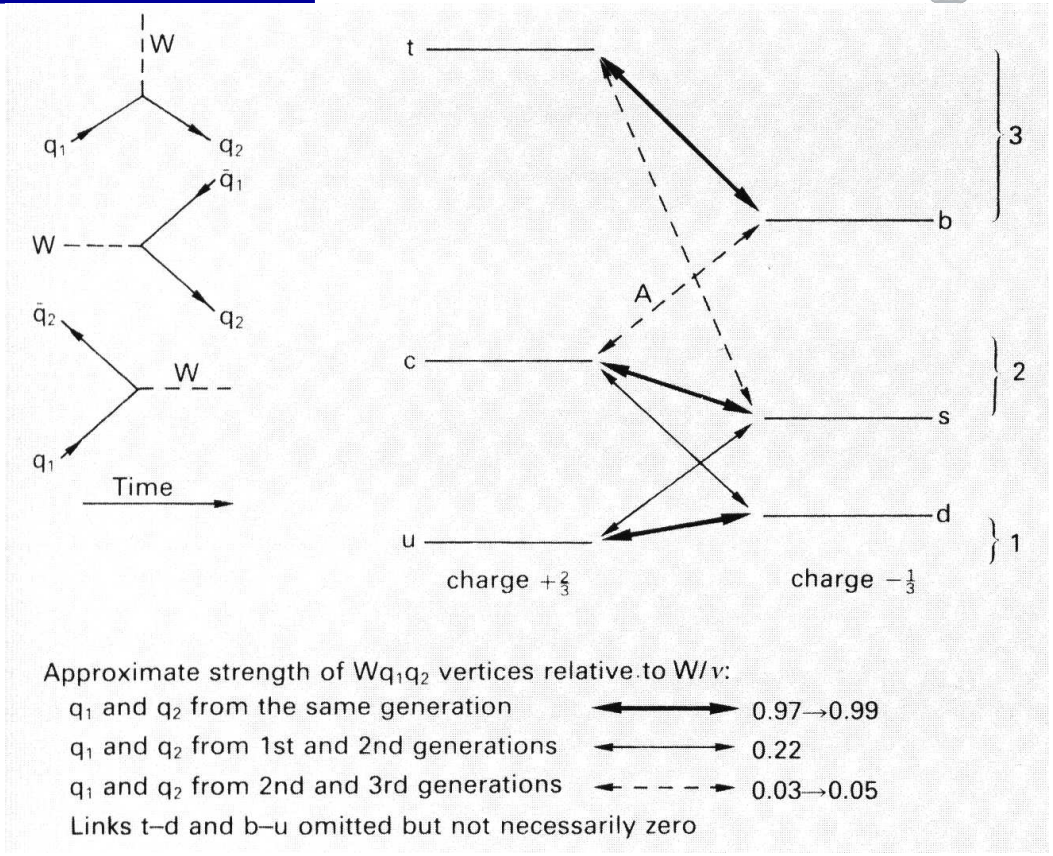
$$t \to b + W^+$$

Top quark is heavy enough for W to be "real". But W is unstable, and will decay "as usual". Main amplitude of b decay contains V_{bc}:

$$b \to c + W^{-*}$$

This time, W^- is virtual. It can only decay into states which are lighter than its actual invariant mass ($e\nu, \mu\nu, \tau\nu, d\bar{u}, s\bar{c} \ldots$).
Now, c-quark's main decay amplitude contains V_{cs}:

$$c \to s + W^{+*}$$



Of course, the lighter the quark, the fewer decay modes are available for W^*.
The decay amplitude of the s-quark contains V_{su}:

$$s \to u + W^{-*}$$

β decay

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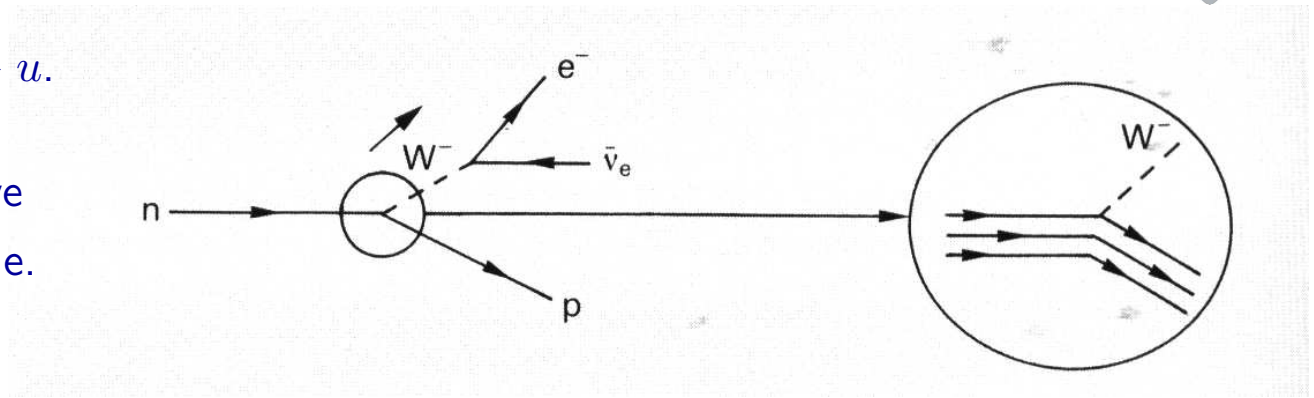
Final step in quark decay chain: $d \to u$.



In heavier quark doublets, the positive quark is heavier than the negative one.

The lightest quark family is an exception: u is lighter than d, and hence u is the stable one. d quark decays, with amplitude containing $V_{ud} \approx 0.975$:

$$d \to u + W^{-*} \to u + e^- + \bar{\nu}_e$$

Given the small $d - u$ mass difference, this is the only decay channel left available for the virtual W in this case.

From outside, d quark decay is usually seen as the β decay of the neutron.

Can we predict neutron lifetime, starting from muon decay rate?

What would the world look like, if u were slightly heavier than d?

Is the gauge group $SU(2)$?

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Introducing a doublet of left-handed fermionic fields $\psi_L \equiv \begin{pmatrix} \psi_L^{\rm up} \\ \psi_L^{\rm down} \end{pmatrix}$ one can rewrite the charged current interaction terms as

$$\mathcal{L}_{\text{weak charged}} \sim g\,\bar{\psi}_L \gamma^\mu \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} \psi_L \;\sim\; g\,\bar{\psi}_L \gamma^\mu \left(\sigma^+ W_\mu^+ + \sigma^- W_\mu^-\right) \psi_L$$

where $\sigma^\pm = (\sigma_1 \pm i\sigma_2)/2$ are combinations of σ matrices (again!).

This looks very much like $SU(2)$ invariance, but with one term missing:

$$\sigma^k W_\mu^k = \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}$$

with the off-diagonal combinations identified as $W_\mu^\pm \equiv (W_\mu^1 \mp iW_\mu^2)/\sqrt{2}$.

Now, if this is $SU(2)$, then we can naturally associate ψ_L with a **weak isospin doublet**, so that any up-type fermion has weak isospin projection $t_3 = +\frac{1}{2}$ while any down-type fermion has $t_3 = -\frac{1}{2}$.

In that formalism, W bosons form a 3-component **isovector** in weak isospin, $t = 1$.

The right-handed spinors then behave as invariants, i.e. **isoscalars** under weak isospin, $t = 0$.

Weak currents

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So, if $SU(2)$ is the gauge group of weak interactions, with left-handed up-type and down-type fermions forming the basic fundamental spinor representation, then, following our usual formalism:

global gauge invariance $\Rightarrow$ local gauge invariance $\Rightarrow$ covariant derivative $\Rightarrow$ interaction terms

we will end up with a theory with three W fileds, W^+, W^-, W^3. The two charged ones carry the weak charged currents, while the neutral one, W^3, carries a left-handed neutral current.

The right-handed fermions, either up-type or down-type, do **not** participate in these interactions.

The interaction Lagrangian looks like this:

$$\mathcal{L}_W = -gJ_L^{+\mu}W_\mu^+ - gJ_L^{-\mu}W_\mu^- - gJ_L^{3\mu}W_\mu^3, \qquad J_L^{\pm\mu} \equiv \bar{\psi}\,\frac{\sigma^\pm}{\sqrt{2}}\,\gamma^\mu\,\frac{1-\gamma_5}{2}\,\psi, \qquad J_L^{3\mu} \equiv \bar{\psi}\,\frac{\sigma^3}{2}\,\gamma^\mu\,\frac{1-\gamma_5}{2}\,\psi$$

Here L-components have been projected out, as required, by $(1-\gamma_5)$, so the first two terms represent the correct phenomenology of the weak charged interactions.

However, the third term does **not** show up in nature in this form.

Electroweak Gauge group $SU(2) \times U(1)$

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In order to describe the correct phenomenology of weak and electromagnetic interactions, including the correct properties of weak neutral currents, one needs to extend the electroweak gauge group by adding a $U(1)$ gauge symmetry, which includes both L and R components.

Through now familiar route, this would introduce one new neutral vector field B_μ:

Interaction	Gauge group	Fermion representation	Covariant derivative
QCD	$SU(3)$	ψ_n : $SU(3)$ triplet	$\partial_\mu + i\frac{g_s}{2}\lambda^a G^a_\mu$
Electroweak	$SU(2) \times U(1)$	ψ_L : $SU(2)$ doublet, $t = \frac{1}{2}$	$\partial_\mu + i\frac{g}{2}\sigma^i W^i_\mu + i\frac{g'}{2} y B_\mu$
		ψ_R : $SU(2)$ singlet, $t = 0$	$\partial_\mu + i\frac{g'}{2} y B_\mu$

The operator governing the $SU(2)$ part is the weak isospin t with coupling g, so $\frac{g}{2}\sigma^i$ represents this operator for weak isoboublets, while the matching term for weak isosinglets is zero.

The $U(1)$ part is governed by the operator of "weak hypercharge" y with coupling g'.

Weak isospin and weak hypercharge

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The SM fermions have a variety of properties which distinguish them from each other. In each generation, various electroweak couplings depend on whether the fermion is left-handed or right-handed, up-type or down-type (i.e. weak isospin projection t_3) and what its electric charge Q is.

Here is a table summarising the assignments of these quantum numbers to various fileds:

	t	t_3	Q	y
$\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$	$1/2$	$+1/2$	0	-1
$\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$	0	0	0	0
e_L, μ_L, τ_L	$1/2$	$-1/2$	-1	-1
e_R, μ_R, τ_R	0	0	-1	-2
u_L, c_L, t_L	$1/2$	$1/2$	$2/3$	$1/3$
u_R, c_R, t_R	0	0	$2/3$	$4/3$
d_L, s_L, b_L	$1/2$	$-1/2$	$-1/3$	$1/3$
d_R, s_R, b_R	0	0	$-1/3$	$-2/3$
ϕ^+	$1/2$	$+1/2$	1	1
ϕ^0	$1/2$	$-1/2$	0	1

The table also includes weak hypercharge $y = 2(Q - t_3)$ for various fields, and the quantum numbers for the scalar fields we will shortly introduce.

Bosonic part of the Lagrangian

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The new gauge fields need their own separate kinetic terms in the electroweak Lagrangian:

$$\mathcal{L}_{\rm boson} = -\frac{1}{4}F^{k\,\mu\nu}F^{k}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad\qquad k = 1,2,3$$

where

$$\begin{aligned} F^{k}_{\mu\nu} &= \partial_\mu W^k_\nu - \partial_\nu W^k_\mu - g\epsilon^{klm}W^l_\mu W^m_\nu \qquad\qquad k,l,m = 1,2,3 \\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned}$$

$F^k_{\mu\nu}$ corresponds to the triplet of W fields.

The term with ϵ^{klm} is, of course, due to non-abelianity of $SU(2)$ and eventually leads to self-interactions of the gauge fields.

$F_{\mu\nu}$ correspinds to the B field. Its kinetic term does not contain self-interaction, as this field corresponds to an abelian gauge symmetry.

Mixing gauge bosons

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There are four new gauge fields, $W_\mu^{1,2,3}$ and B_μ, but neither of these are physical fileds.

Their linear combinations are.

We have already introduced the two off-diagonal W boson combinations, which are each other's hermitian conjugates and are responsible for charged weak current interactions:

$$W_\mu^\pm \equiv \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}}$$

Both remaining two fields are neutral, but neither of them has the right properties to represent the good old, parity conserving, flavour-conserving photon.

Let's introduce two orthogonal linear combinations of W^3 and B, mixed with a certain angle θ_W — Weinberg angle, or simply "weak mixing angle":

$$\begin{aligned} A_\mu &= B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W \\ Z_\mu &= -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W \end{aligned}$$

These can be easily resolved for W^3 and B:

$$\begin{aligned} B_\mu &= A_\mu \cos\theta_W - Z_\mu \sin\theta_W \\ W_\mu^3 &= A_\mu \sin\theta_W + Z_\mu \cos\theta_W \end{aligned}$$

Mixing neutral currents

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Thus, the interaction terms between the electroweak gauge bosons and the fermions now look like this:

$$\mathcal{L}_{EW} = -gJ_L^{+\mu}W_\mu^+ - gJ_L^{-\mu}W_\mu^- - gJ_L^{3\mu}W_\mu^3 - g'J^{y\mu}B_\mu$$

Let's concentrate on the neutral-current terms, which include two neutral currents:

$$J_L^{3\mu} = \bar{\psi}\frac{\sigma^3}{2}\gamma^\mu\frac{1-\gamma_5}{2}\psi \qquad J^{y\mu} = \bar{\psi}\frac{y_L}{2}\gamma^\mu\frac{1-\gamma_5}{2}\psi + \bar{\psi}\frac{y_R}{2}\gamma^\mu\frac{1+\gamma_5}{2}\psi$$

where each fermionic field comes with appropriate values of t_3 and y.

In particular, for a neutrino $t_3 = +1/2, y_L = -1, y_R = 0$, the term $-\frac{1}{2}\gamma^\mu\frac{1-\gamma_5}{2}$ will have a coefficent

$$gW^3 - g'B = g(s_W A + c_W Z) - g'(c_W A - s_W Z) = (gs_W - g'c_W)A + (gc_W + g's_W)Z$$

But if A is the photon, its coupling to neutrinos should be zero. Hence

$$c_W \equiv \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \qquad s_W \equiv \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$$

Now, for electrons: $t_3 = -1/2, y_L = -1, y_R = -2$. It's not so hard to see that the couplings with the A field are the same for L and R terms. Thus the sum of these terms looks like $\bar{\psi}\gamma^\mu\psi$ again, parity is indeed conserved in electromagnetic interactions, and we can link g and g' to the electromagnetic coupling:

$$-\frac{2gg'}{\sqrt{g^2+g'^2}} = -2e \qquad \Rightarrow \qquad e = \frac{gg'}{\sqrt{g^2+g'^2}} = g\sin\theta_W = g'\cos\theta_W$$

Neutral currents (cont.)

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In general, the two terms describing neutral currents can be rewritten as

$$\begin{aligned}
\mathcal{L}_{\rm NC} &= -gJ_L^{3\mu}W_\mu^3 - g'J^{y\mu}B_\mu \\
&= -\left(gs_W J_L^{3\mu} + g'c_W J^{y\mu}\right) A_\mu - \left(gc_W J_L^{3\mu} - g's_W J^{y\mu}\right) Z_\mu \\
&= -ej_{\rm em}^\mu A_\mu - \frac{e}{2s_W c_W}\bar{\psi}_i\gamma^\mu(g_V - g_A\gamma_5)\psi_i Z_\mu
\end{aligned}$$

So, the t, y assignments for each fermion **were designed** in such a way that the combination of the neutral currents coupled to the photon field A_μ has the correct coupling and parity of the electromagnetic current $j_{\rm em}^\mu$.

The price to pay is that every fermion type i now has a non-trivial set of couplings g_V and g_A to the vector and axial-vector parts of the weak neutral current, coupled to the Z boson:

$$g_V = t_3 - 2Q\sin^2\theta_W, \qquad g_A = t_3$$

where $t_3 = +1/2$ for all neutrinos and all u-type quarks, $t_3 = -1/2$ for all charged leptons and all d-type quarks, and Q is the electric charge of the fermion (e.g. -1 for electrons, $+2/3$ for u quarks).

Properties of weak neutral current

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Let's look at weak neutral currents in some more detail.

For an individual fermionic field ψ, the interaction term in the Lagrangian is

$$\mathcal{L}_Z = -\frac{e}{2\sin\theta_W\cos\theta_W}\bar{\psi}\gamma^\mu(g_V - g_A\gamma_5)\psi Z_\mu$$

where $e^2/4\pi = \alpha,\ \sin^2\theta_W = 0.23120 \pm 0.00015$.

Weak neutral interactions are diagonal in fermion flavour (and colour!)

g_V and g_A are weak-neutral-vector-current and weak-neutral-axial-vector-current couplings, respectively, for each type of the fermion-antifermion pair, with the Z^0 boson. They depend on whether the fermion is a lepton or a quark, up-type or down-type:

$$g_V = t_3 - 2Q\sin^2\theta_W, \qquad g_A = t_3$$

where $t_3 = +1/2$ for all neutrinos and all u-type quarks, $t_3 = -1/2$ for all charged leptons and all d-type quarks, and Q is the electric charge of the fermion (e.g. -1 for electrons, $+2/3$ for u quarks).

In general, if either g_V or g_A were zero, parity would be conserved.

However, this is not the case, so in weak neutral interactions parity is not conserved.

In practice, this means, for example, that the angular distributions for the reaction $e^+e^- \to Z \to \mu^+\mu^-$ are not left-right symmetric, as opposed to $e^+e^- \to \gamma^* \to \mu^+\mu^-$.

Couplings

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Vector and axial-vector couplings to γ and Z^0 for various fermions

Carrier	Current type	Common factor	neutral ν_e, ν_μ, ν_τ	charged e, μ, τ	quarks u, c, t	quarks d, s, b
γ	Vector	$-e$	0	-1	$+\frac{2}{3}$	$-\frac{1}{3}$
γ	Axial	$-e$	0	0	0	0
Z^0	Vector, g_V	$\frac{-e}{2s_W c_W}$	$\frac{1}{2}$	$-(\frac{1}{2}-2s_W^2)$	$(\frac{1}{2}-2\frac{2}{3}s_W^2)$	$-(\frac{1}{2}-2\frac{1}{3}s_W^2)$
Z^0	Axial, g_A	$\frac{-e}{2s_W c_W}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

A partial decay width of Z into any of 11 allowed fermion-antifermion pairs (neglecting fermion masses) is

$$\Gamma(Z \to \bar{f} f) = \frac{\alpha M_Z}{12 s_W^2 c_W^2}(g_V^2 + g_A^2)\, N_c = (330\ \text{MeV})(g_V^2 + g_A^2)\, N_c$$

where the number of colours N_c is 1 or 3 for leptons or quarks, respectively.

Comparing γ and Z^0

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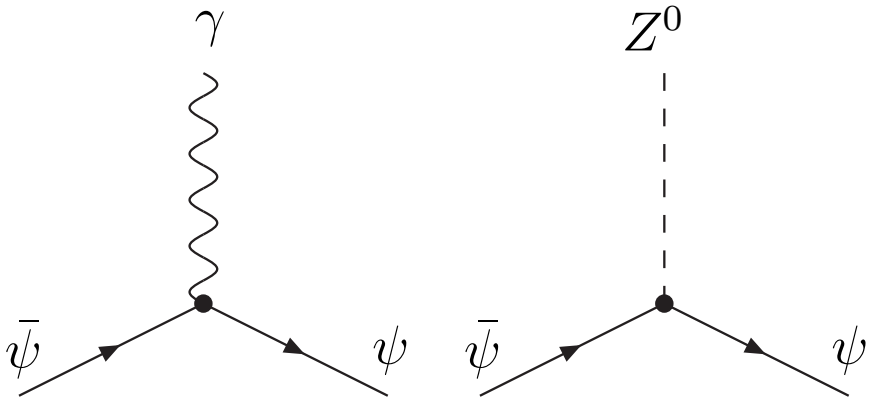

The vertices for QED and W.N.C are quite similar:

- Neither of the two can change fermion type.
- Couplings have similar strengths.

However, there are significant differences:

- The fermionic current paricipating in the QED interactions is a pure vector, and Parity is conserved.
- The fermionic current participating in the Weak Neutral-Current interation is a mixture of a vector and an axial vector, hence it may (and will!) violate Parity conservation.
- Only charged fundamental fermions participate in E.M. interactions.
- All fundamental fermions participate in W.N.C. interactions, including neutrinos.
- The photon is massless and stable.
- The Z^0 boson is very heavy, $M_Z = 91.2$ GeV, and decays quickly into a fermion-antifermion pair, with total width $\Gamma_Z = 2.5$ GeV.

Old Friends Again

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So, both γ and Z are couipled to e^+e^- and also to $\mu^+\mu^-$. Hence, we must add the two amplitudes (star * means that the particle is virtual):

$$e^+e^- \to \gamma^* \to \mu^+\mu^- \qquad \text{and} \qquad e^+e^- \to Z^{0*} \to \mu^+\mu^-$$

Draw Feynman diagrams for both processes, and use our simplified Feynman rules:

$$|\mathcal{M}_\gamma|^2 \sim \frac{\alpha^2}{s^2}$$

$$|\mathcal{M}_Z|^2 \sim \frac{\alpha^2}{16 s_W^4 c_W^4}(g_{Ve}^2 + g_{Ae}^2)(g_{V\mu}^2 + g_{A\mu}^2)\left|\frac{1}{s - M_Z^2 + i\Gamma M_Z}\right|^2$$

- Which one dominates at small s?
- At $s = M_Z^2$?
- At extremely large $s \gg M_Z^2$?

Note that at the Z peak, $s = M_Z^2$, the $\gamma - Z$ interference term vanishes (check this!).

Also, note the absence of terms like $g_{Ve}g_{Ae}$: vector and axial-vector currents do not mix!

$W - Z - \gamma$ Relations

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We have not looked at all interaction terms in the electroweak Lagrangian yet...

Consider once again the bosonic part:

$$\mathcal{L}_{\rm boson} = -\frac{1}{4}F^{k\,\mu\nu}F^{k}_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad k,l,m = 1,2,3$$

$$\begin{aligned} F^{k}_{\mu\nu} &= \partial_\mu W^k_\nu - \partial_\nu W^k_\mu - g\epsilon^{klm}W^l_\mu W^m_\nu \\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned}$$

there is only one way of producing a tri-linear interaction term here, through the ϵ^{klm} which is due to non-abelianity of $SU(2)$.

So, any such term needs to be proportional to

$$\begin{aligned} \mathcal{L}_{\rm tri-boson} &\sim g\epsilon^{123}W^1W^2W^3 \sim gW^+W^-W^3 \\ &\sim gW^+W^-(c_W Z + s_W A) \sim e(W^+W^-A + \cot\theta_W W^+W^-Z) \end{aligned}$$

Hence, there are only two trilinear interaction terms: $W^+W^-\gamma$ and W^+W^-Z

$W - Z - \gamma$ Relations – II

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- $W^{\pm}$ are charged, so the existence of the $W^+W^-\gamma$ vertex with coupling $\sim e$ is not surprising
- Moreover, theory predicts a vertex $W^+W^-Z^0$, with coupling $\sim e\cot\theta_W$.
- This means that we can draw three distinct Feynman diagrams for producing a W^+W^- pair in e^+e^- collisions: through a t-channel neutrino exchange, through an s-channel annihilation into a virtual γ, and through an s-channel annihilation into a virtual Z (do it!).
- The product of two ϵ^{klm} terms will give rise to terms quadri-linear in gauge fields, but, again, not all possible combinations will be allowed.
- There are only four types of quadrilinear vertices:

$$W^+W^-\gamma\gamma \quad W^+W^-\gamma Z^0 \quad W^+W^-Z^0Z^0 \quad W^+W^-W^+W^-$$

These are of second order in e, i.e. have couplings $\sim e^2$. Together with other second-order diagrams, these will be important while studying vector-boson scattering.

Almost there

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So far, we have succeeded in introducing the correct phenomenology of the charged weak interactions, kept the electromagnetic interactions and predicted the properties of weak neutral interactions, but:

All our fields are still massless – both fermions and bosons.

Because the gauge transformation properties for L and R components are different, even the usual mass terms like $m\bar{\psi}\psi$ will **not** be gauge invariant, to say nothing of the gauge boson mass terms like $M^2 B^\mu B_\mu$.

Even worse, mass terms like that will render the theory non-renormalisable!

So, what's the proper way to introduce masses into the theory?

It's spontaneous breaking of the gauge invariance and generation of masses through the Higgs mechanism.

Training exercise: scalar fields again

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Remember scalar fields, with a nice simple non-interacting Lagrangian (p. 36)

$$\mathcal{L}_\phi = (\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2$$

First of all, let's make the scalar field charged. This would mean making ϕ complex:

$$\mathcal{L}_\phi = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

Also, let us introduce a non-linear interaction term $\sim (\phi^* \phi)^2$, which can be easily switched on and off by choosing its coupling λ. Let's also include the Lagrangian of a separate free vector field A_μ:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. At this point, the scalar field ϕ is self-interacting (if $\lambda \neq 0$), but the scalar and the vector fields know nothing about each other.

Now, let's use our vast experience in building gauge-invariant theories to replace the normal derivative ∂_μ with the $U(1)$-gauge-covariant derivative:

$$\partial_\mu \quad \rightarrow \quad D_\mu \equiv \partial_\mu + ieA_\mu,$$

Is this photon mass?

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Our Lagrangian gets modified accordingly (note that the $F_{\mu\nu}F^{\mu\nu}$ part stays the same, as the new terms cancel, thanks to abelian nature of $U(1)$):

$$\begin{aligned}\mathcal{L} &= [(\partial_\mu - ieA_\mu)\phi^*][(\partial^\mu + ieA^\mu)\phi] - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*)A^\mu \\ &\quad - \left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e^2\phi^*\phi A_\mu A^\mu\right)\end{aligned}$$

Here we can identify the interaction term between the photon field and the electromagnetic current of the scalar field, as expected.

We also see a term $\sim A_\mu A^\mu$, which would indicate that the photons may have acquired something like a mass. Indeed, following Euler-lagrange formalism, the new Klein-Gordon equation for A_μ would look like

$$\partial_\nu\partial^\nu A_\mu + 2e^2(\phi^*\phi)A_\mu = 0$$

with a plane wave solution satisfying

$$k^2 = 2e^2(\phi^*\phi)$$

So, if, on average, $(\phi^*\phi) > 0$, our "photon" acquires mass as a result of interaction with the scalar field.

Vacuum expectation

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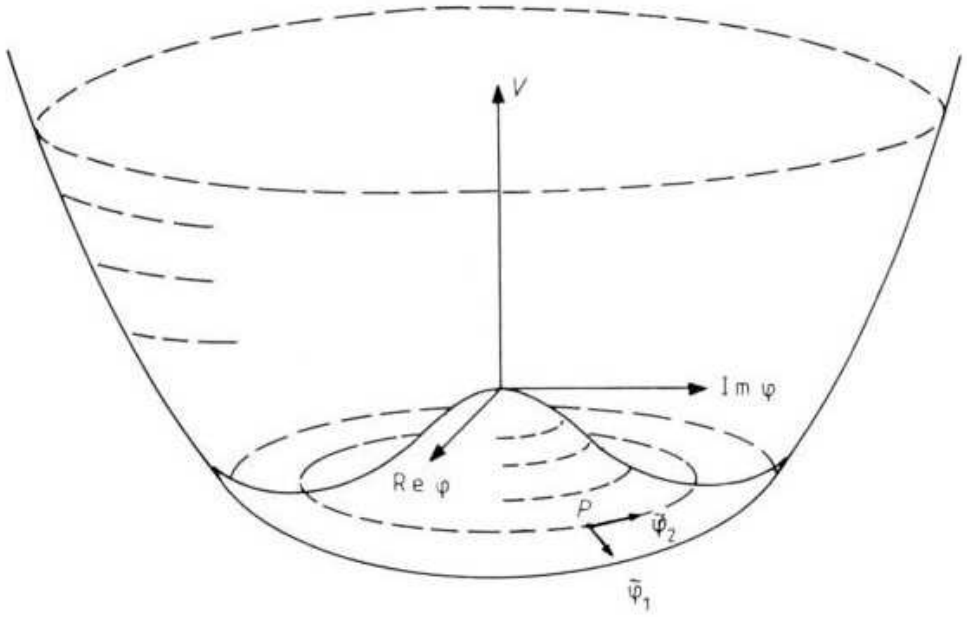
For a classical field ϕ, it looks natural that at its ground state (i.e. vacuum) the average value of $(\phi^*\phi) = 0$. Indeed, with the potential given by

$$V(\phi) = m^2\phi^*\phi + \lambda(\phi^*\phi)^2$$

the minimum is clearly achieved at $\phi = 0$.



But hang on, this is only true if both parameters m^2 and λ are positive!

If $\lambda > 0$ but $m^2 < 0$, then the point $\phi = 0$ is a local maximum, while the minimum is achieved at

$$\phi^*\phi = |\phi|^2 = \frac{-m^2}{\lambda} \equiv \frac{v^2}{2}$$

This is not a single point, but a circle in complex ϕ plane. Any point on this circle corresponds to a minimum of the potential. But whichever point the system chooses, the average value of $\phi^*\phi$, the **vacuum expectation value** will be the same:

$$\langle\phi^*\phi\rangle = v^2/2$$

Spontaneous symmetry breaking

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This is interesting: the Lagrangian only depends on combinations like $\phi^*\phi$, i.e. is clearly invariant under arbitrary rotations around the origin $\phi = 0$ on a complex ϕ plane, for whatever values of m^2 and λ.

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - \lambda(\phi^* \phi)^2 - ie(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) A^\mu - \left(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - e^2 \phi^* \phi A_\mu A^\mu\right)$$

However, for our new preferred choice, the lowest energy solution — the vacuum — in not unique. And **whichever direction is chosen**, it will violate the original symmetry.

Once again: any direction from the origin is equally probable, i.e. the symmetry is still there. But once we choose our vacuum, i.e. pick one point among the equally probable circle of points corresponding to the minimum of the potential, the physics stops being explicitly invariant under rotations in ϕ plane.

This is an example of **spontaneous symmetry breaking**, when the Lagrangian is symmetric under some transformations, but the vacuum is not. The description of the physical system built on top of that vacuum will have that symmetry explicitly broken, but the fundamental properties of the symmetric Lagrangian will remain intact.

Spontaneous symmetry breaking: examples

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Spontaneous symmetry breaking is all around us:

- when two cars are "palying chicken", to avoid collision they have to swerve — but which way? A priori, there is a left-right symmetry, which is spontaneously broken in different ways in the UK and Europe.

- Place a thin long knitting needle vertically, and push the tip down. It will bend, but which way? The direction is random, but whichever it chooses, the original axial symmetry is broken (despite the force being axially symmetric too).

- Find your own examples?

Moving to new vacuum

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What we need to do now is move our theory to a new vacuum: instead of the real and imaginary parts of the original scalar field ϕ, let's introduce two real new fields, h and θ, relative to the new minimum:

$$\phi = \sqrt{\tfrac{1}{2}}(v + h + i\theta) \qquad \phi^* = \sqrt{\tfrac{1}{2}}(v + h - i\theta)$$

Gauge invariance can be used to totally eliminate the phase θ, thus only leaving one physical scalar field h (the price to pay is that we have no more freedom in choosing the gauge, we have exhausted our gauge-fixing capacity).

So, re-expanding the Lagrangian in terms of h, we obtain:

$$\begin{aligned}\mathcal{L} \;=\; & \tfrac{1}{2}\left(\partial_\mu h\, \partial^\mu h - 2\lambda v^2 h^2\right) - \lambda v h^3 - \tfrac{1}{4}\lambda h^4 \\ & - \left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}e^2 v^2 A_\mu A^\mu\right) + \tfrac{1}{2}e^2 h^2 A_\mu A^\mu + v e^2 h A_\mu A^\mu\end{aligned}$$

The first line is the Lagrangian of a massive (with mass $= \sqrt{2\lambda v^2}$) self-interacting scalar field. In the second line, the term in brackets is the Lagrangian of a massive vector field (with mass $= ev$), and the remaining two terms correspond to the interaction of photons with the field h.

What's the gain?

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So, if we wanted to introduce a mass to the photons, all we had to do was to fill the whole space with a complex scalar field, postulate that the scalar field has its gauge symmetry spontaneously broken, and re-locate the vacuum to a new position.

As a result, we get a self-interacting real massive scalar field, but more importantly our photons gain masses, and also gain the ability to interact with the remaining scalar fields.

Was it worth the effort? Yes, because the Lagrangian is still the same, it never changed, it's gauge-invariant and hence renormalisable, despite the apparent "massiveness" of the photon.

But we know that the photon is massless!!!

We did all this to demonstrate the machinery at work, now we will leave the photons alone and outline how to do this where it really matters, i.e. in the weak interaction domain.

The real photon will remain massless, but the W and Z bosons — and the fermions too! — will gain masses.

Back to Electroweak Lagrangian

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The EW Lagrangian, with all participating fields still free, contains terms for each fermion ψ^f (with index f running through all colour and flavour degrees of freedom, for all six quarks and all six leptons, but with no mass terms) and terms for each vector field ($A, W^{\pm}, Z$ or equivalently, before mixing, W^1, W^2, W^3, B, with index b running through all necessary degrees of freedom, again with no mass terms):

$$\mathcal{L}_{\text{fermion}} = \bar{\psi}^f \gamma^\mu \partial_\mu \psi^f \qquad \mathcal{L}_{\text{gauge}} = -\tfrac{1}{4} F^b_{\mu\nu} F^{b\mu\nu}$$

$$\mathcal{L} = \sum_{\text{fermion}=q,l} \mathcal{L}_{\text{fermion}} + \sum_{\text{boson}=W,B} \mathcal{L}_{\text{gauge}}$$

The Lagrangian is invariant under global gauge transformations from group $SU(2) \times U(1)$, with W fields associated to $SU(2)$ and B field linked to $U(1)$.

Now we request that this Lagrangian is invariant under **local** gauge transformations from $SU(2) \times U(1)$, and this is achieved by replacing the derivative ∂_μ with the covariant derivative, which for this symmetry group will be

$$\partial_\mu \quad \rightarrow \quad D_\mu \equiv \partial_\mu + i\frac{g}{2}\sigma^k W^k_\mu + i\frac{g'}{2} y B_\mu$$

Adding scalar fields

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Now we want to add a new term to the Lagrangian, describing a set of complex scalar fields ϕ, a doublet in weak isospin with hypercharge $y = 1$:

$$\mathcal{L} = \sum_{\text{fermion}=q,l} \mathcal{L}_{\text{fermion}} + \sum_{\text{boson}=W,B} \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{scalar}}$$

where the scalar field Lagrangian has the familiar form:

$$\mathcal{L}_{\text{scalar}} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2$$

except the ϕ fields have multiple components:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_U \\ \phi_D \end{pmatrix} = \begin{pmatrix} \phi_{U1} + i\phi_{U2} \\ \phi_{D1} + i\phi_{D2} \end{pmatrix}$$

so the simple complex conjugate ϕ^* becomes the hermitian conjugate, row $\phi^\dagger$.

To achieve local gauge invariance, the covariant derivative should be used throughout the Lagrangian, **including the kinetic term in the scalar field Lagrangian**. For the fermion and gauge parts of the Lagrangian, we have seen this introduce fermion-antifermion-gauge and gauge-gauge interactions, and then the mixing between W^3 and B yileds familiar A and Z fields as before. But so far, everything remains massless.

Breaking spontaneously

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For the scalar part of the Lagrangian, the introduction of the covariant derivative also introduces various gauge-scalar couplings, similarly to the spontaneous-symmetry-breaking example exercise we did before. Following that example, we will consider the case $\mu^2 < 0$ and hence the non-zero vacuum expectation value for the ϕ fields:

$$\begin{aligned} \langle \phi^\dagger \phi \rangle &= \frac{1}{2} \langle \phi_{U1}^2 + \phi_{U2}^2 + \phi_{D1}^2 + \phi_{D2}^2 \rangle \\ &= -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \end{aligned}$$

Thus the minimum of the scalar field potential is situated in ϕ space somewhere along the circle with radius $v/\sqrt{2}$. Anywhere on this "sphere" will do, but:

We should use this ambiguity to make sure that while W (and hence Z) bosons gain masses due to spontaneous symmetry breaking, the symmetry tied to the photon stays unbroken and the photon remains massless.

Higgs field

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These aims can be achieved by choosing the following new vacuum:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix}$$

where H is the only remaining physical field (out of initial 4 components of ϕ).

This neutral scalar field is called the Higgs field.

Plugging this form of ϕ into the Lagrangian will reveal what the existence of this field does to the other fields: vector bosons and fermions.

Let's start from the vector bosons, where the non-zero mass terms emerge even without propagating Higgs fields, i.e. for $H = 0$.

When we decide that time has come to study not just the masses of the vector bosons but also their couplings to the Higgs field, we can substitute $v + H$ instead of v.

The terms with v, which are quadratic in the fields, describe the masses. The terms with H are the interaction terms.

Weak gauge boson masses

Lancaster University

So we have the scalar field Lagrangian with local gauge invariance

$$\mathcal{L}_{\text{scalar}} = (D_\mu \phi^\dagger)(D^\mu \phi) + \lambda \frac{v^2}{2} \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

where

$$D_\mu = \partial_\mu + i\frac{g}{2}\sigma^k W_\mu^k + i\frac{g'}{2}B_\mu, \qquad \phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix}$$

and we can put $H = 0$ for the time being. The two "potential" terms simply give the value of the potential at minimum (by construction), which is an irrelevant constant. The important terms are those which contain gauge and scalar fields:

$$\frac{1}{8} \left| \begin{pmatrix} gW_\mu^3 + g'B_\mu & gW_\mu^1 - igW_\mu^2 \\ gW_\mu^1 + igW_\mu^2 & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 = \frac{1}{8} \left| \begin{pmatrix} \sqrt{g^2 + g'^2} A_\mu & g\sqrt{2}W_\mu^+ \\ g\sqrt{2}W^- \mu & -\sqrt{g^2 + g'^2} Z_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \frac{(g^2 v^2)}{4} \left(W_\mu^+ W^{\mu -} + W_\mu^- W^{\mu +} \right) + \frac{1}{2} \frac{(g^2 + g'^2) v^2}{4} Z_\mu Z^\mu$$

So, the two charged Ws gain the mass $M_W = g(v/2)$ each, the Z boson gains the mass $M_Z = \sqrt{g^2 + g'^2}(v/2)$, while the photon remains massless.

The ratio of masses is firmly predicted: $M_W / M_Z = g/\sqrt{g^2 + g'^2} = \cos\theta_W$

...and more

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So, we have predictions for the masses of W and Z:

$$M_W = \frac{vg}{2}, \qquad M_Z = \frac{v\sqrt{g^2+g'^2}}{2} = \frac{M_W}{\cos\theta_W}$$

Also, we remember that $g\sin\theta_W = g'\cos\theta_W = e$.

From some basic processes like beta decay or muon decay, we know that

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}, \qquad G_F = 1.16639\times 10^{-5}\ \text{GeV}^{-2}, \qquad \alpha(M_W^2) \equiv \frac{e^2}{4\pi} = \frac{1}{128.9}$$

so we have more constraints than unknowns, and we can check whether the measured values for masses satisfy the predicitions.

$$M_W = 80.40\ \text{GeV}, \quad M_Z = 91.19\ \text{GeV}, \quad \sin^2\theta_W = 0.231, \quad v \simeq 250\ \text{GeV}$$

It appears that they do, up to loop corrections. These are usually assigned to α, which changes to the above value at $q^2 = M_W^2$ from its value at zero momenta, $\alpha(0) = 1/137.036$, as mentioned before.

Note that we have thus a way of measuring $g, g', \cos\theta_W, v$ and some other combinations of constants, but have no handle on λ (except $\lambda > 0$).

Higgs mass etc.

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How about the mass of the Higgs itself? It also gains mass through spontaneous symmetry breaking (just like in our abelian example):

$$m_H = \sqrt{2\lambda v^2}$$

which remains unknown, as we do not know λ. In fact, it's much more convenient to replace λ with $m_H^2/(2v^2)$, as m_H has a direct physical meaning.

How about vector-boson interactions with the Higgs?

We could have obtained those terms, if we had not put $H = 0$. Here is the end result:

- There is a Higgs$-W-W$ vertex, $gM_W H W_\mu^+ W_\mu^-$, with a very strong coupling gM_W (which is very strong! Remember: $g\sin\theta_W = g'\cos\theta_W = e$).
- There is also a Higgs$-Z-Z$ vertex, $\frac{gM_Z}{2\cos\theta_W} H Z_\mu Z_\mu$, with the coupling as shown (which is very strong too!).
- There is no fundamental coupling of Higgs with photons — no surprise here, that part of the symmetry is not broken, and photon has no mass.
- Naturally, there are also $HHWW$ and $HHZZ$ couplings, but these are higher-order ($\sim g^2$).

Fermion masses

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In order to allow fermions to have masses (such that do not break the gauge invariance), one needs to introduce interactions with the (same) scalar fields, and then allow spontaneous symmetry breaking to work its magic.

By construction, the quantum numbers of the scalar fields are such, that the new interaction term should couple the $SU(2)$-doublet of left-handed fermions with the $U(1)$-singlet of the right-handed fermions (and vice-versa):

$$\mathcal{L}_{\rm fermion-scalar} = -g_D \left(\bar{\psi}_L^U, \bar{\psi}_L^D\right) \begin{pmatrix} \phi_U \\ \phi_D \end{pmatrix} \psi_R^D + \ldots$$

where $\ldots$ replace three more similar terms. After spontaneous symmetry breaking, with $\phi_U = 0$ and $\phi_D = (v+H)/\sqrt{2}$, we get two kinds of terms:

$$\mathcal{L} = -\frac{g_D v}{\sqrt{2}} \left(\bar{\psi}_L^D \psi_R^D + \bar{\psi}_R^D \psi_L^D\right) - \frac{g_D H}{\sqrt{2}} \left(\bar{\psi}_L^D \psi_R^D + \bar{\psi}_R^D \psi_L^D\right) = -m_D \bar{\psi}^D \psi^D - \frac{m_D}{v} H \bar{\psi}^D \psi^D$$

and a similar expression for up-type "U"-fermions. Clearly, the first one has the structure of the mass term for D-type fermion, while the second one is the fermion-antifermion-higgs interaction term.

Hence, introducing the mass $m_D \equiv \dfrac{g_D v}{\sqrt{2}}$, the Higgs-fermion-antifermion coupling becomes $\dfrac{m_D}{v}$.

Each fermion couples to the Higgs field with strength proportional to the fermion's mass.

Higgs decays

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It's not too hard to calculate (at leading order) the partial widths of the Higgs decays into any of the final states:

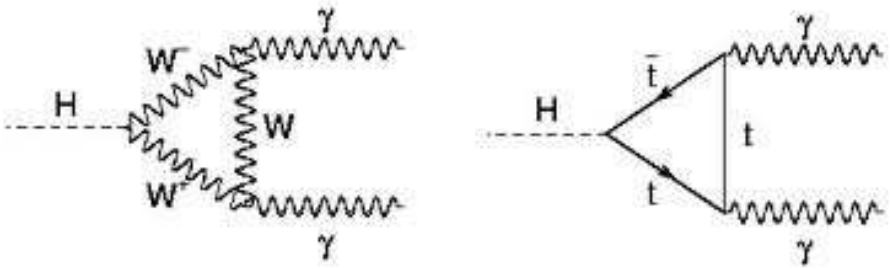
$$\Gamma(H \to WW) \simeq \frac{\alpha}{16 \sin^2 \theta_W} \frac{M_H^3}{M_W^2} \beta_W \qquad \Gamma(H \to \bar{l}l) \simeq \frac{\alpha}{8 \sin^2 \theta_W} \frac{m_f^2 M_H}{M_W^2} \beta_l^3$$

$$\Gamma(H \to ZZ) \simeq \frac{\alpha}{32 \sin^2 \theta_W} \frac{M_H^3}{M_W^2} \beta_Z \qquad \Gamma(H \to \bar{q}q) \simeq 3 \frac{\alpha}{8 \sin^2 \theta_W} \frac{m_f^2 M_H}{M_W^2} \beta_q^3$$

where I used $e = g \sin \theta_W$, $\alpha = e^2/4\pi$, and β_i are speeds of the final state particles in the Higgs decay frame.

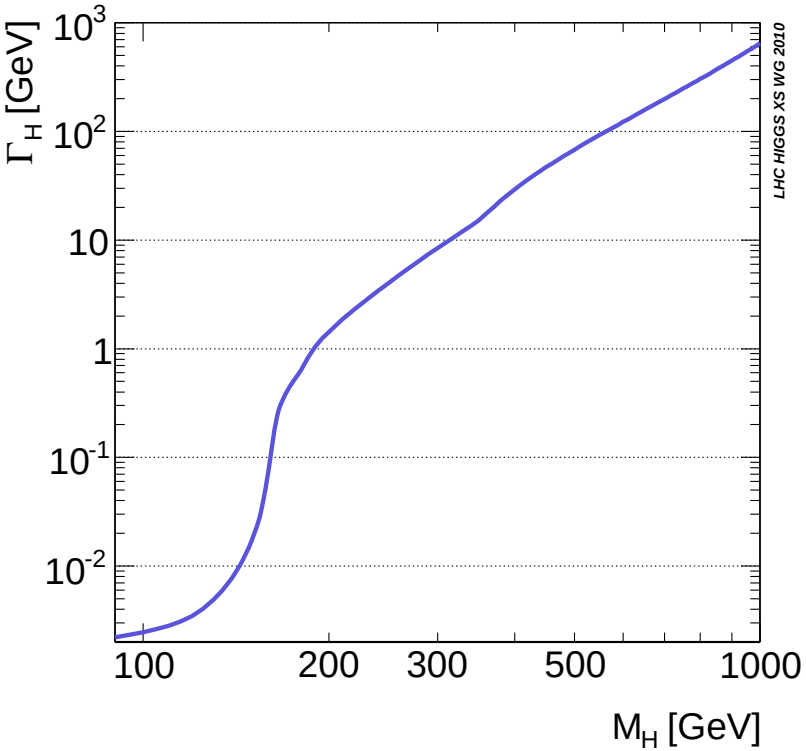
There are also sizeable decays into:

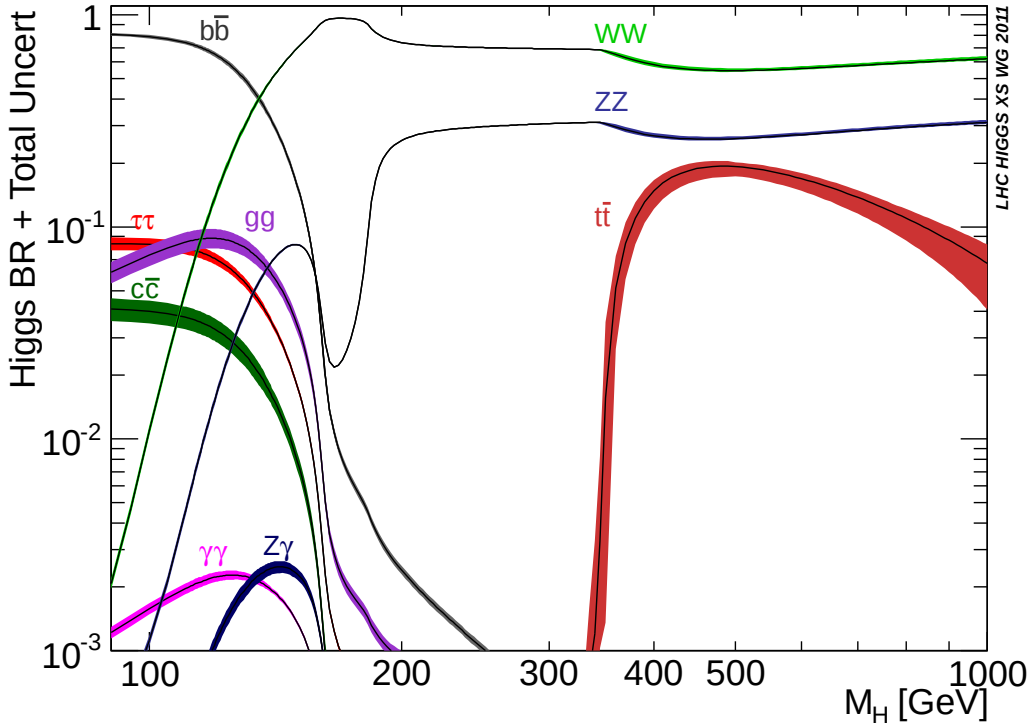
- $\gamma\gamma$ (both diagrams)
- two gluons (only the diagram on the right)



despite the fact that these come only at loop level

Higgs width and branching fractions

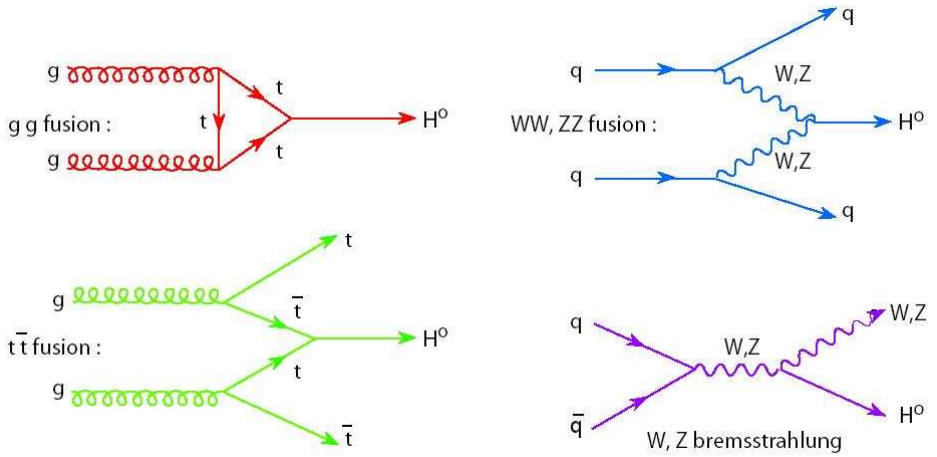


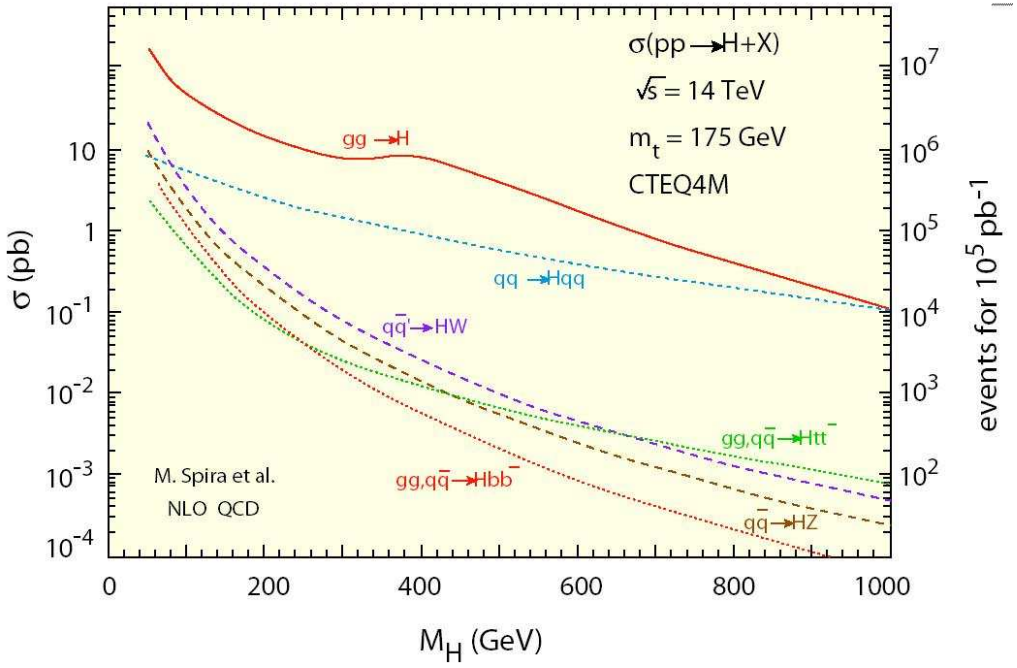


Full width of the Higgs is a very steep function of its mass – the whole concept loses sense at about 1 TeV...

Expected branching fractions vary hugely with mass too...

Higgs production at LHC





There are several subprocesses contributing to the Higgs production process in proton-proton collisions.

Their contributions depend on energy, and the Higgs mass...

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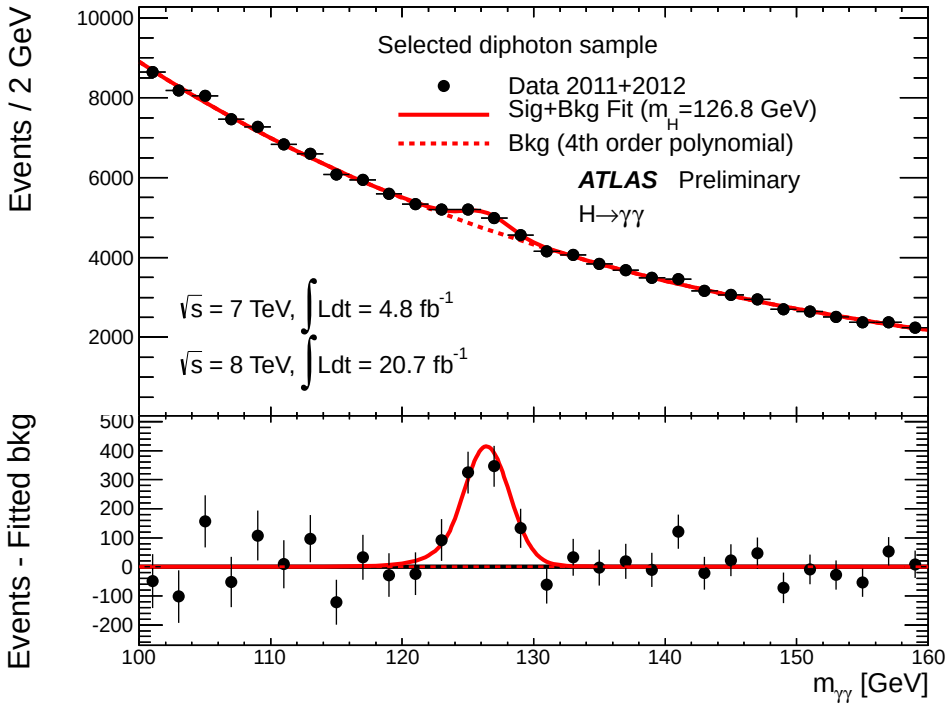
Fortunately, the Higgs – or at least a Higgs – has been observed, at the most interesting mass about 125 GeV, where many decay channels may be within reach!

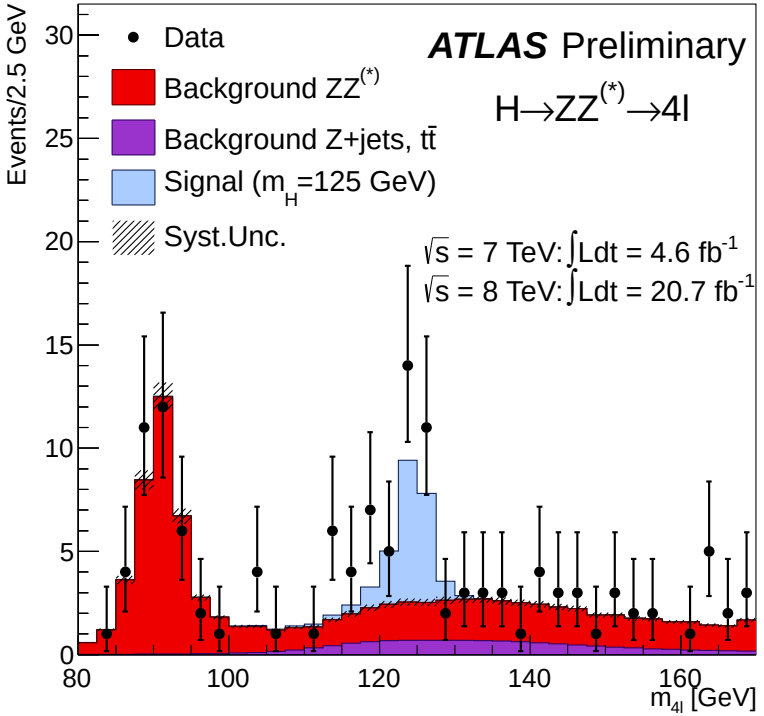
Make no mistake: it's a

HUGE achievement

for both theory and experiment!

Higgs discovery



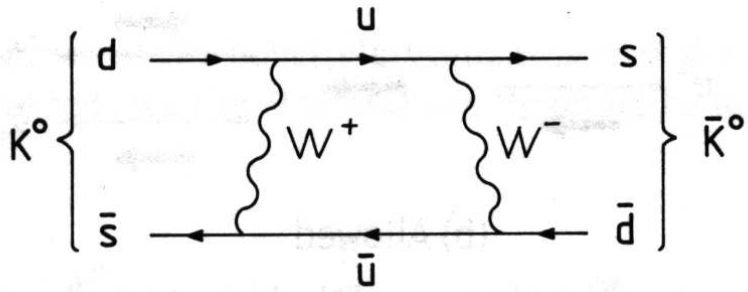


The object found at ~ 125 GeV looks very much like a Higgs, and is seen beyond reasonable doubt in both $\gamma\gamma$ and $ZZ^* \to 4$leptons decay modes.

Spin is also more compatible with zero (than with 2). Looking forward to further studies on measuring angular distributions and branching fractions into various channels!

Neutral Kaon System

Let's have a special look at the $K^0(498) = d\bar{s}$ and $\bar{K}^0(498) = s\bar{d}$. Strong interactions can tell them apart, because strangeness is conserved. But weak interactions may mix these two with each other!



Both neutral kaons are pseudoscalars, meaning that their internal parity is negative:

$$P|K^0\rangle = -|K^0\rangle, \qquad P|\bar{K}^0\rangle = -|\bar{K}^0\rangle.$$

Both kaons have strangeness and isospin, so a charge-conjugation operator transforms one into the other (minus occurs from swapping the quarks: parity is negative):

$$C|K^0\rangle = -|\bar{K}^0\rangle, \qquad C|\bar{K}^0\rangle = -|K^0\rangle,$$

Applying combined parity operator CP we have:

$$CP|K^0\rangle = |\bar{K}^0\rangle, \qquad CP|\bar{K}^0\rangle = |K^0\rangle,$$

neither of these two is a CP eigenstate, but certain mixtures of the two will be:

$$CP|K^0 + \bar{K}^0\rangle = |K^0 + \bar{K}^0\rangle, \qquad CP|K^0 - \bar{K}^0\rangle = |\bar{K}^0 - K^0\rangle = -|K^0 - \bar{K}^0\rangle,$$

K-long and K-short

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Provided combined CP-parity is conserved, the CP-even eigenstate should be able to decay into a pair of pions (angular momentum is conserved, hence $\Rightarrow L_{\pi\pi} = 0$, which means $C = (-1)^L = +1$). This happens rather quickly as a lot of phase scpace is available. The respective CP-even kaon combination has a short lifetime, and is hence called K-short:

$$K_S \equiv \frac{K^0 + \bar{K}^0}{\sqrt{2}} \rightarrow \pi^+\pi^- \quad \text{or} \quad \pi^0\pi^0, \qquad \tau_S = 0.9 \cdot 10^{-10}\ \text{s} \qquad c\tau_S \simeq 3\ \text{cm}$$

The CP-odd eigenstate, however, must decay into three pions (here all orbital momenta are zero again, so it's the intrinsic negative parity of the pions wihc makes the difference). The phase space is tight here, the decay goes far slower, and the CP-odd kaon is called K-long:

$$K_L \equiv \frac{K^0 - \bar{K}^0}{\sqrt{2}} \rightarrow \pi^+\pi^-\pi^0 \quad \text{or} \quad \pi^0\pi^0\pi^0, \qquad \tau_L = 0.5 \cdot 10^{-7}\ \text{s} \qquad c\tau_L \simeq 15\ \text{m}$$

Even masses of the two are slightly different: $M_{K_L} - M_{K_S} \simeq 3.5 \cdot 10^{-12}$ MeV.

The CPT theorem requires that masses of particles and antiparticles are equal, so $m(K^0) = m(\bar{K}^0)$. However K_S and K_L are **not** each other's antiparticles!

Kaon Regeneration

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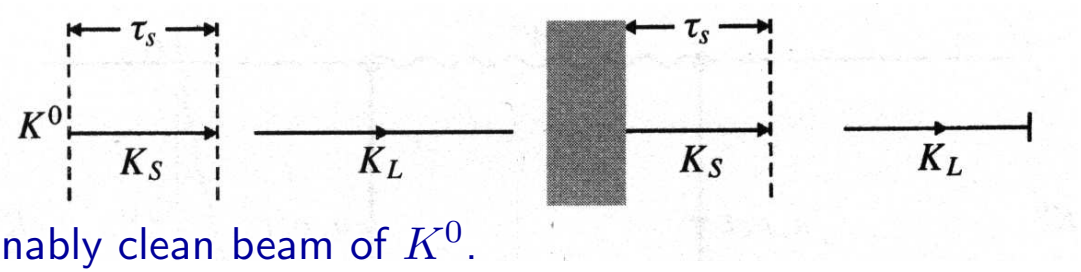
Consider a reaction of neutral kaon production, e.g. one of the following:

$$K^+ + p \to K^0 + \pi^+ + p, \qquad K^- + p \to \bar{K}^0 + \pi^- + p,$$

The neurtal kaons are produced in strong interactions as eigenstates of the strangeness operator, i.e. have definite strangeness, $+1$ or -1.

However, we know that $K^0 \equiv \frac{1}{\sqrt{2}}(K_S + K_L)$ and $\bar{K}^0 \equiv \frac{1}{\sqrt{2}}(K_S - K_L)$. So, after some time, all K_S will have decayed, while K_L will survive for significantly longer, and in either reaction we are left with a "beam" of K_L mesons, which is in fact a superposition of K^0 and $\bar{K}^0$:

If this "beam" hits a slab of material, the $\bar{K}^0$ component is absorbed much stronger (because it contains a light antiquark), and on exit we will have a reasonably clean beam of K^0.



Which, of course, is a superposition of K_S and K_L, and after some time most of K_S will have decayed and we get a clean beam of K_L (of course, with reduced intensity).

And then somebody asks why people called them strange particles...

Similar oscillations exist in $B^0 - \bar{B}^0$ and $B_s - \bar{B}_s$ systems.

CP-violation

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This is not the whole story, however...

In prehistoric days, everybody believed in conservation of parity, i.e left-right symmetry of fundamental laws of narure.

But particle physicists have a nasty habit of checking everything experimentally. And in 1957 it was shown that parity is violated in weak interactions, which means that if we view a weak decay in the mirror, the reflection will **not** look the same as the original.

Then it was discovered that C-parity is not conserved in weak interactions either, i.e. a particle's decay does not look the same as its antiparticle's decay (without any mirrors).

However, theorists expected that the combined, CP-parity was still conserved, i.e. a weak decay of a particle would look the same as the decay of its antiparticle viewed in the mirror. In which case, K_S only decays into two pions, while K_L only decays into three pions.

In 1964 another group of particle physicists discovered that very rarely — but most definitely — K_L does decay into two pions (BR $\simeq 10^{-3}$), thus violating CP-invariance!

Similar CP-violating decays have been observed in B-systems, too. In fact, two dedicated B-factory experiments (BaBar in California and Belle in Japan) have been built to study these decays.

Standard Model – looking back

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Starting from basic free-field Lagrangians for the various fermions, we applied gauge transformations and used gauge invariance to introduce all three types of interactions known in particle physics.

Through spontaneous breaking of the gauge symmetry and the Higgs mechanism, some of the gauge bosons became massive, but without jeopardising good properties of our model such as renormalisability.

The result is the Standard Model of particle physics.

This theory has explained a very wide variety of phenomena, and in many cases came up with successful predictions, thus making it by far the most successful theory of its kind.

After the discovery of the Higgs boson – a true triumph of both theory and experiment – the focus is now shifting to precision measurements and searches for any deviations from the Standard Model.

After all, the SM has far too many seemingly arbitrary parameters to be the true theory of everything...

If such a thing exists...

Summary of the Standard Model gauge group structure

- The gauge group of the Standard Model is $SU(2) \otimes U(1) \otimes SU(3)$.
- $SU(2)$ roughly corresponds to weak interactions. All fundamental fermions (quarks and gluons) belong to the doublet representation of this group. Bosons carrying weak interactions belong to the triplet ($SU(2)$-"vector", i.e. **adjoint**) representation.
- $U(1)$ roughly corresponds to electromagnetic interactions. All fundamental fermions (quarks and gluons) belong to the singlet representation of this group (well, it's an abelian group, everything is a singlet there...).
- More precisely, $SU(2) \otimes U(1)$ describes unified **electroweak** interactions, a mixture of weak and electromagnetic.
- $SU(3)$ describes strong interactions, Quantum Chromodynamics. Leptons do not feel these interactions, and hence are invariants (scalars, singlets) against this group; same is true about photons and W, Z bosons. Quarks belong to the fundamental (triplet) representation of $SU(3)$, while gluons, as carriers, belong to octet ($SU(3)$-"vector"), i.e. adjoint representation of $SU(3)$.

Summary of the Standard Model group structure — II

- Non-Abelian nature of $SU(2)$ means that weak interactions allow self-coupling of their gauge bosons, i.e W and Z interact with each other directly.

- Similarly, non-abelian nature of $SU(3)$ means that gluons carry colour charges, i.e. interact with each other directly.

How many different particles are there in the SM?

- 3 pairs of leptons and same number of antileptons;

- 3 pairs of quarks and 3 pairs of antiquarks, each in 3 colours;

- photon, 3 weak bosons, 8 gluons;

- The Higgs (recently discovered)

Let's try matching these numbers with dimensions of relevant representations of the subgroups of the overall gauge symmetry groups.

Standard Model: summary of group representations

As explained above, different fundamental particles belong to different representations of the subgroups of the Standard Model gauge group, $SU(2) \times U(1) \times SU(3)$.

This table attempts to give a schematic summary.

Particle	Electroweak $SU(2) \times U(1)$	Strong (QCD) $SU(3)$
leptons (each family)	2, 1	1
quarks (each family)	2, 1	3
W, Z, γ	3, 1	1
gluons	1	8
Higgs	1	1

Unification of Interactions?

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The progress of science goes in two directions:

1. We are able to describe mathematically and explain wider and wider range of phenomena by covering more and **more applications**.

2. More and more "laws" and "equations" are derived from **fewer** and more fundamental **basic principles**, thus offering unification of fundamental concepts.

Particle physics is at the forefront in both directions, but so far we (in this lecture course) were mainly busy trying to derive and explain various properties of particles from a variety of theories and models.

Let's now have a look in the other direction: unification of interactions.

From Electromagnetic to Electroweak Unification

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Without diving too deep into history, Maxwell's theory of electromagnetism is a very good exmple of unification.

Here, magnetic and electric phenomena were described within a single framework. Amazingly, only one 'external' new constant needed to be introduced: speed of light c.

Electroweak unification considers elctromagnetic and weak interactions on the same grounds, within a single gauge theory, with gauge symmetry described by the $SU(2) \times U(1)$ group of transformations.

Gauge theories are guaranteed to be renormalizable, which means that this theory may be more than just an "effective" low energy theory — it may be a part of a bigger theory, describing all interactions, with a bigger gauge symmetry group which becomes apparent at very high energies.

After all, even weak interactions are not exactly symmetric to electromagnetic ones, as γ, W and Z have rather different proeprties.

Grand Unification

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So, EW symmetry is spontaneously broken at low energies: photons are massless, W and Z rather massive, hence weak interactions seem week. But at $Q^2 \gg M_Z^2$ the symmetry is restored: W and Z propagators will not be suppressed by the masses any more.

The hope is that at even greater energies, the symmetry between strong and electroweak interactions will be restored:

- all three interactions will have equal strengths (the extrapolation of the Q^2-evolution of couplings predicts the meeting point somewhere at $\sqrt{Q^2} \simeq 10^{15}$ GeV);
- both leptons and quarks will belong to the same representations of a big symmetry group, which would enclose both electroweak $SU(2) \times U(1)$ and strong $SU(3)$;
- many more gauge bosons should exist (some with charges like $-\frac{4}{3}$ and "half-coloured" and imaginatively called X and Y, allowing — oh horror! — transitions between leptons and quarks.

Wonders of GUT Theories

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GUT theories offer natural explanations to a number of existing puzzles:

- Electric charge, presently a continuously-variable generator of the (Abelian) electromagnetic gauge group $U(1)$, is forced into commutation relations which allow only dicrete eigenvalues — so electric charge is quantised!
- Fractional charges of quarks occur because each quark comes in three colours, while the sum of all charges in a multiplet should add up to zero.
- The historic question why the hydrogen atom (and any other atom for that matter) is exactly neutral can now be answered.
- Similar patterns existing between quark and lepton doublets (like the difference of charges $\nu - e$ being equal to $u - d$ etc.) acquire natural explanations.

But the Grand Unified symmetry is badly broken, and those new X, Y bosons must have very high masses (of order of the scale of the symmetry breaking, just like in EW). That's why their existence has not been noticed so far.

The scale of their masses can be assessed by looking for proton decay. Many simpler models were excluded by the experimental fact that protons live longer than 10^{32} years.

Our Universe is "just" $\sim 10^{10}$ years old, but it contains many protons...

Hierarchy problem

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There is one problem in SM which cannot be addressed by GUT: the hierarchy problem.

If there is a GUT scale of, say, 10^{15} GeV, then quite a few fundamental particles will have the masses around that scale.

This makes it extremely improbable (will require a lot of fine-tuning of parameters) that our "good old" quarks, leptons and bosons have much-much smaller masses.

The problem shows up at its extreme for the Higgs itself: the Higgs mass depends quadratically on any such big scale! This is a serious theoretical problem, which needs a serious solution.

And the solution is . . . **Supersymmetry (SUSY).**

Maybe.

How to extend Poincaré group

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Remember how we built our space-time symmetry 'portfolio'?

We started with rotations, then we added translations, finally we added Lorentz boosts and got to the biggest group (so far): Poincaré group.

Is it possible to extend Poincaré group further, i.e. find a group for which Poincaré group is a subgroup? That would be a wonderful way to find a symmetry which is even more fundamental!

Not so easily! **Coleman-Mandula theorem** says, that the invariance group of any quantum field theory can only be extended beyond Poincaré group by a finite number of Lorentz-invariant generators.

This in fact means that whatever additional symmetry there may be, it does not mix with the space-time invariance.

Gauge symmetries are examples of such symmetries. Their generators belong to trivial (scalar) representation of the Poincaré group.

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Fortunately (or unfortunately?) **Haag, Lopuszanski and Sohnius** soon found a way out, thus opening the possibility of numerous new developments.

The algebra of a quantum filed theoretical model can be extended in a non-trivial way, without violating Coleman-Mandula theorem, by 'simply' (!) adding the possibility that the algebra may contain **anticommutators**. Then, some of the new generators Q must belong to the spinor representation. Their main properties are defined by

$$Q_\alpha \bar{Q}_\beta + \bar{Q}_\beta Q_\alpha = 2P_0 I_{\alpha\beta} - 2\sigma^i_{\alpha\beta} P_i$$

where P_0, P_i are Poincaré translation generators. The new Q's anticommute with themselves, and commute with P's:

$$Q_\alpha Q_\beta + Q_\beta Q_\alpha = 0 \qquad \bar{Q}_\alpha \bar{Q}_\beta + \bar{Q}_\beta \bar{Q}_\alpha = 0 \qquad P_i Q_\beta - Q_\beta P_i = 0 \qquad \text{etc.}$$

Inevitably, these spinor generators generate "rotations" which transform fermions into bosons, and vice-versa. This gives rise to various **supersymmetric** (SUSY) field theories, which happen to have some useful properties (from the field-theoretical point of view).

MSSM

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The simplest of these is MSSM - Minimal Supersymmetric Standard Model

Each of the known fundamental particles acquires here a **superpartner**, whose spin differs by $\frac{1}{2}$:

$$\begin{aligned}
\text{leptons } (\tfrac{1}{2}) &\leftrightarrow \text{sleptons } (0) \\
\text{quarks } (\tfrac{1}{2}) &\leftrightarrow \text{squarks } (0) \\
\text{photon } (1) &\leftrightarrow \text{photino } (\tfrac{1}{2}) \\
\text{gluons } (1) &\leftrightarrow \text{gluinos } (\tfrac{1}{2}) \\
W, Z\ (1) &\leftrightarrow \text{wino, zino } (\tfrac{1}{2}) \\
\text{Higgs } (0) &\leftrightarrow \text{higgsino } (\tfrac{1}{2})
\end{aligned}$$

In fact, one needs 4 physical Higgses in MSSM: two neutral, two charged. Hence, there are also 4 higgsinos.

Interestingly, the new particles must belong to the same representations of the gauge groups as their "old" partners, and hence must have the same interactions. E.g. gluinos, just like gluons, only participate in strong interactions.

Supersymmetry and the Hierarchy Problem

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In SM, fermions and bosons are playing very different roles and have very different properties. In SUSY theories, the symmetry between fermions and bosons is restored: for every existing fermion there will be a matching boson (**squark** or **slepton**, and for every existing boson — a matching fermion **photino, wino, zino, gluino, higgsino...**).

How does this help the hierarchy problem?

Mass corrections come from loop diagrams with all possible particles in the loops, and the contributions of bosons and fermions come with opposite signs. So, if all couplings are the same, the contributions from the superpartners will cancel out at high-energy end of the integration, even if the actual masses of the superpartners are not exactly equal.

Supersymmetry Searches

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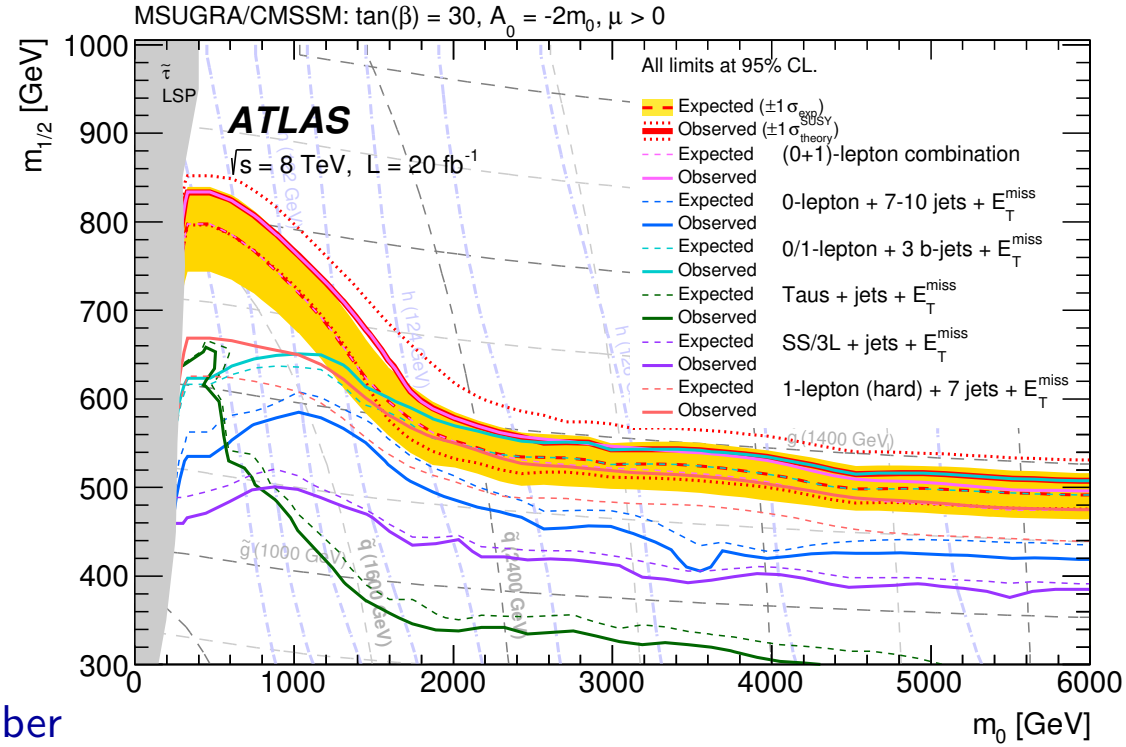
Supersymmetry is clearly broken: we have no scalar electrons with masses of 0.51 MeV, nor new charged fermion winos at 80 GeV. If they existed, we woluld have certainly seen them already! But the new superpartners must have masses below ~ 1 TeV — if higher, the cancellation mechanism will not work.

In any case, SUSY particles should be copiously produced in high energy colliders and will decay into ordinary ones (leptons, jets of hadrons), but with some specific observable signatures: missing energy, high number of energetic jets and/or leptons etc.



SUSY theories, however, contain a few free parameters, in terms of which the properties of SUSY particles are calculated. So the number of possible scenarios is enormous!

We are doing our best to cover as much as we can, and we believe we have a reasonably good chance of discovering SUSY at the LHC, if it exists. Nothing so far...

Looking for MSSM

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So far, no experimental evidence of supersymmetry exists, but there are numerous predictions of possible signatures that could/should be observable at LHC.

Theorists believe that typical masses of superpartners — at least some of them —
are below 1 TeV, i.e. such that they should be produced at LHC.

If LHC experiments fail to find SUSY (as is so far the case), much of the hard work by theorists over last few decades will be in vain.

We will need some new theories...

Maybe someone in this room will come up with one?