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Applications of Split Octonions for Some Problems of Field Theory

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Abstract

In this PhD dissertation split octonionic fields and their symmetries are studied. The property of $(4+4)$-space known as triality, which manifests as equivalence of the 8-dimensional chiral spinors and the vector, is explored in the context of physics. Split octonionic representations of $SO\,(4,4)$ and $Spin\,(4,4)$ groups are found and are compared to the Clifford algebraic matrix representations. By utilizing group invariant forms, Lagrangian that generalizes Dirac and Maxwell theories is constructed, extending these fundamental theories onto exceptional mathematical structures. In addition, the dissertation investigates automorphism group of split octonion algebra, the noncompact G_2, corresponding Lie algebra, its Casimir operator and geometrical application of these algebraic structures in physics. These results contribute to the development of a theoretical framework for split octonionic field theories and open up new avenues for exploration of non-associativity in physics.

Contents

Chapter 1

Introduction

1.1 Research topic and its importance

The standard model of elementary particles, which was formed in the 1970s, is the most successful theory in particle physics. It classifies all known particles and describes all interactions between them except gravity. The big unsolved problem related to it is the so-called hierarchy problem, a problem concerning weakness of gravity compared to other forces on the one hand, and lightness of Higgs boson, with the mass of $\mu \approx 125$ GeV, compared to the Planck mass $M_{\text{Pl}} = \sqrt{\hbar c/G} \approx 1.22 \times 10^{19}$ GeV on the other.

One possible solution to these problems are supersymmetric theories, according to which particles have superpartners. For fermions, these superpartners are bosonic particles, and for bosons they are fermionic. Although none of the supersymmetric theories have been experimentally confirmed to date, they remain a subject of active research.

This thesis discusses the structures needed to construct a theory similar to supersymmetric theories. In particular, the Dirac equation describing fermionic particles and the system of Maxwell's equations describing bosonic particles with the corresponding Lagrangians are derived. Both systems are obtained in a particular limits of the same trilinear split octonionic Lagrangian. Unlike supersymmetric theories, this theory may not need to introduce additional Grassmannian variables, since the split octonion algebra itself contains zero divisors. It may also not be necessary to postulate additional particles. The spectrum of particles will be dictated from the algebra itself.

Advancements in mathematics often preceded the development of theoretical physics. For example, hypercomplex numbers had significant impact on theoretical physics. In particular, we can mention Clifford algebras and finite-dimensional normed division algebras. According to Hurwitz's theorem, there are only four number systems: $\mathbb{R}$ reals, $\mathbb{C}$ complex numbers, $\mathbb{H}$ quaternions, and $\mathbb{O}$ octonions. Historically, relatively less attention has been paid to the split versions of the last three algebras: $\mathbb{C}'$, $\mathbb{H}'$, and $\mathbb{O}'$, even though they have properties characteristic of physical systems. The broadest of the algebras listed, the octonions and their split version, are the least established in physics. This is partially explained by the fact that they do not have the associativity property $(ab)\,c = a\,(bc)$ which makes working with them particularly difficult. However, with the development of computer algebra systems, calculations with such structures became easier. One of the goals of this thesis is to fill these gaps in knowledge. Supersymmetric theories are mathematically related to the above-mentioned Hurwitz algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ [Kugo & Townsend 1983, Baez & Huerta 2009, Baez & Huerta 2011]. In particular, the octonion spin [Gamba 1968] is the subject of research in relation to supersymmetries [Schray 1994]. This work proposes a mathematical structure, the split octonionic triality, as a possible basis for a supersymmetric theory.

For more than half a century, group theory, a branch of mathematics that studies symmetries, has become extremely important to physics. Based on the theory of groups, it was possible to predict the existence of some subatomic particles, which was later confirmed experimentally. In addition, it was possible to combine electromagnetic and weak interactions in the so-called electro-weak interaction. For these reasons, many physicists base their hopes for a grand unification on the use of group theory.

Out of all groups Lie groups are the most important in particle physics and relativity. The elements of these groups are often represented as parameter-dependent matrices. Lie groups correspond to Lie algebras, which represent the tangential space of the group manifold at the identity element. Algebra elements are called generators because they allow us to generate group elements [Georgi 1999].

For physical applications, it is important to find the corresponding Casimir operators of the Lie algebra. These operators are expressed through algebra elements and commute with all algebra elements. As an example for why it's important, the eigenvalues of the Poincaré group Casimir operators correspond to the mass and spin of a particle.

At the end of the 19th century, Wilhelm Killing and Élie Cartan classified simple

Lie algebras [Agricola 2008]. It turned out that they fall into four infinite families and five exceptional cases that do not belong to any family.

Out of exceptional Lie groups G_2 has the smallest rank. Its existence was first surmised by Friedrich Engel in 1886 [Agricola 2008]. Cartan discovered that G_2 is the automorphism group of octonions. It was later found that the noncompact G_2 group describes two rolling spheres of relative radius 1:3 [Bor & Montgomery 2009]. In physics, there is an opinion that the group G_2 can also be used to classify elementary particles [Silagadze 1994, Gogberashvili 2016b], or to describe the symmetries of time and space [Zhevlakov 1982, Gogberashvili & Sakhelashvili 2015].

In this work, a noncompact variant of G_2 is studied, which is an automorphism group of $\mathbb{O}'$ split octonions. The possibility of its geometrical use is also discussed. The second-order Casimir operator of this group is found and it's shown that in a special case it can be represented as a sum of Casimir operators of Lorentz and Poincaré groups.

1.2 Research goals and objectives

The goal of the research is to create a supersymmetric-like theory based on the split octonion algebra and related eight-dimensional rotations. The aim of the thesis is to construct Maxwell and Dirac Lagrangians based on rotation invariant trilinear form.

The triality symmetry with respect to the split version of octonions is discussed. In this case, the space is non-Euclidean, but has a $(4,4)$ metric. Unlike the algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, their split versions $\mathbb{C}'$, $\mathbb{H}'$ and $\mathbb{O}'$ are not division algebras because they contain zero divisors. However, this is what brings them closer to physics, because zero dividers have the structure of a light cone and are thus related to Minkowski space. Of these, only the split octonion algebra $\mathbb{O}'$contains Minkowski space as a subspace. In such a space, the symmetry group of vectors $SO\left(8\right)$ is replaced by the pseudo-orthogonal noncompact group $SO\left(4,4\right)$, and its double coverage $Spin\left(8\right)$, which is the symmetry group of spinors, is replaced by the group $Spin\left(4,4\right)$. It is important to note that $SO\left(4,4\right)$ contains the Lorentz group as a subgroup. Because of this property, the study of split octonions is interesting in a geometrical context [Gogberashvili 2002, Gogberashvili 2005, Gogberashvili & Sakhelashvili 2015, Gogberashvili 2016a, Gogberashvili & Gurchumelia 2019].

The long-term goal is to build a theory of supersymmetric type. The aim of this particular study is to obtain a possible Lagrangian for the theory, or its interaction term, whose candidate is the split octonionic trilinear form $\mathcal{F}:\mathbb{O}'\times\mathbb{O}'\times\mathbb{O}'\to\mathbb{R}$. Also to determine the split octonionic representation of pseudo orthogonal groups.

The triality symmetry was investigated with respect to the split version of octonions [Gurchumelia & Gogberashvili 2021]. Triality exists between spinors and vectors in eight-dimensional space. In the discussed case, eight-dimensional space has 4 time-like dimensions and 4 space-like dimensions, which is why it is called (4+4) space. The symmetry groups of spinorial and vectorial objects are $SO\left(4,4\right)$ and $Spin\left(4,4\right)$. The invariant forms of these transformations are bilinear and trilinear forms on the split octonionic vector and on the two split octonionic chiral components of the spinor. In the thesis, these forms are used to express the Lagrangian.

A gradient operator is also needed to construct the Lagrangian. This requires analysis on split octonion functions, for which there is no mathematical literature, and literature on analysis of ordinary octonion functions is sparse [Kauhanen & Orelma 2018, Sudbery 1979].

Dirac and Maxwell equations have been written in terms of split octonions in the following papers [Gogberashvili 2006a, Gogberashvili 2006b, Chanyal, Bisht & Negi 2011], but their corresponding Lagrangian was not written so far, and therefore it was not linked to the rotation invariant trilinear form.

Chapter 2

Literature review

2.1 General overview

Octonions are eight-dimensional algebras and are the largest of the normal division algebras. It is the least established in physics despite various attempts [Okubo 1995, Gürsey & Tze 1996, Lõhmus, Paal & Sorgsepp 1994, Lõhmus, Paal & Sorgsepp 1998]. The use of octonions has been proposed for the color symmetry of quarks [Günaydin, M., & Gürsey 1973, Morita 1981], in grand unified theories [Sudbery 1984, Dixon 1990, Castro 2007], in quantum mechanics [Günaydin, Piron & Ruegg 1978, Dzhunushaliev 2006], string theory and M-theory [Chung & Sudbery 1987, Lukierski & Toppan 2002, Kuznetsova & Toppan 2006, Boya 2003], signal analysis [Gao & Li 2021, Błaszczyk 2020] and etc.

The eight-dimensional Euclidean space in which octonions reside has a unique property. In particular, the dimensions of the vector and chiral spinors coincide in this space, and there is a trilinear invariant form on them in which they are indistinguishable from each other. This property is called triality symmetry [Gamba 1968, Dray & Manogue] and is often formulated using spin group automorphisms and D_4 Dynkin diagram symmetries [Lounesto 2001].

2.2 Casimir operator and Poincaré group

An algorithm for finding Casimir operators of semisimple compact Lie algebras is given in the article [Gruber & O'Raifeartaigh 1964]. In the thesis, this algorithm was successfully used to find a Casimir operator of the noncompact $\mathfrak{g}_2$ Lie algebra. According to the algorithm, for some the Lie algebra $\mathfrak{h}$ following matrix should be found

$$Q = \sum_{m,n} g_{mn}\, \hat{z}_m \otimes Z_n \tag{2.1}$$

where $\hat{z}_k$ is a linearly independent basis element of $\mathfrak{h}$, Z_k is its corresponding element in some matrix representation, and

$$g_{mn} = \mathrm{tr}\left(X_m X_n\right) \tag{2.2}$$

is called Killing metric. Using this matrix Q , the Casimir operator of order p can be found as

$$\mathcal{C}_p = \mathrm{tr}\left(Q^p\right), \tag{2.3}$$

which means that for every $\hat{z}_k$ element the following holds:

$$[Q^p, \hat{z}_k] = 0\,. \tag{2.4}$$

For example, the Casimir operator of the Lorentz group has a form [Liu, Tang & Xun 2011, Bekaert et al 2021]

$$\begin{aligned} \mathcal{C}_{\text{Lorentz}} &= \frac{1}{2} M_{\mu\nu} M^{\mu\nu} \\ &= \sum_{n=1}^{3} \left(L_n - K_n\right), \end{aligned} \tag{2.5}$$

where the angular momentum operator

$$L_k = \sum_{ij} i\epsilon_{ijk} x_i \frac{\partial}{\partial x_j}\,, \tag{2.6}$$

and the Lorentz boost operator

$$K_n = -i\left(x_n\frac{\partial}{\partial t} + t\frac{\partial}{x_n}\right) \,, \tag{2.7}$$

can be written in 4-vector notation using

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \,,$$

where

$$P_\mu = -i\frac{\partial}{\partial x} \,, \tag{2.8}$$

is a 4-momentum operator.

The relation between the above two notations is

$$\begin{aligned} M_{mn} &= \sum_k \epsilon_{mnk} L_k \,, \\ M_{0\ell} &= -M_{\ell 0} = K_\ell \,. \end{aligned} \tag{2.9}$$

The Poincaré group has two Casimir operators [Bekaert et al 2021], of which the second-order operator is $P_\mu P^\mu$ with Klein-Gordon equation as its eigenvalue equation.

2.3 Cayley-Dixon constructions in physics

Using the involution[1] algebra $\mathbb{A}_n$, a new involutional algebra can be constructed using the generalized Cayley-Dixon construction [Albert 1942]

$$\mathbb{A}_{n+1} = \mathbb{A}_n \oplus \mathbb{A}_n \tag{2.10}$$

which defines multiplication as

$$\begin{aligned} (a_1, a_2)\,(b_1, b_2) &= (a_1 b_1 - \gamma b_2^* a_2,\ b_2 a_1 + a_2 b_1^*) \\ (a_1, a_2)^* &= (a_1^*, -a_2) \end{aligned} \tag{2.11}$$

where $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are elements of $\mathbb{A}_{n+1}$ algebra, a_1, a_2, b_1 and b_2 are the elements of the $\mathbb{A}_n$, $\gamma = \pm 1$ and the $*$ symbol denotes involution. A sequence of involutional algebras is obtained using (2.10) formula iteratively. If $\mathbb{A}_0 = \mathbb{R}$ and $\gamma = 1$ then a sequence of algebras one obtains is: $\mathbb{R}$ real numbers, $\mathbb{C}$ complex numbers, $\mathbb{H}$ quaternions, $\mathbb{O}$ octonions $\mathbb{S}$ sedenions, and so on. At every doubling, properties of the field of real numbers are lost, for example starting from $\mathbb{C}$ we no longer have ordering, for $\mathbb{H}$ commutativity, for $\mathbb{O}$ associativity and for $\mathbb{S}$ alternativity. The first four in the sequence are normed division algebras. According to the Hurwitz theorem, these are the only four normed division algebras.

There is a deep connection between normed division algebras and supersymmetries. For example, for non-Abelian Young-Mills fields and Green-Schwartz superstrings, supersymmetry is only possible in 3, 4, 6, or 10 dimensions, which is 2 more than the dimensions of the division algebras on which they depend [Baez & Huerta 2009]. Poincaré-Lie superalgebras of the same dimension are obtained from normed division algebras by a certain systematic procedure [Baez & Huerta 2011].

[1]involution f is such a mapping, for which $f^{-1} = f$, for instance complex conjugation.

2.4 Clifford algebras and spinors in physics

Clifford algebras form important mathematical structures for physics. They are the most natural generalizations of complex and quaternion algebras for any dimensional spaces. Clifford algebra theory is directly related to orthogonal groups important to physics. Clifford algebra defined on the real number field is used in geometry [Hestenes & Sobczyk 2012], where it is known as geometric algebra.

Spinors are one of the most important classes of objects in physics. The language of Clifford algebras is most convenient for talking about them. With its help, it is possible to generalize the concept of spinor for any dimensional spaces with diagonal non-degenerate metric. In this subsection, spinor and vector representations of orthogonal groups are discussed, based on the book [Lounesto 2001].

Clifford algebras also combine the dot product and the cross product in the so-called geometric (or Clifford) product, thereby generalizing the standard Gibbs-Heaviside vector notation, as well as complex, quaternion and some matrix algebras and bringing them together in one system.

The defining algebraic relation of $\mathcal{C}\ell_{p,q}\left(\mathbb{R}\right)$ Clifford algebras on the real number field is given in the following form

$$e_i e_j + e_j e_i = 2g_{ij}\,, \qquad \text{where} \qquad i,j = 1,2,\ldots,d\,, \tag{2.12}$$

where $d = p + q$ and g_{ij} is a diagonal metric with the signature (p,q),

$$g = \text{diag}(\underbrace{1,1,\ldots,1}_{p},\underbrace{-1,-1,\ldots,-1}_{q})\,. \tag{2.13}$$

Notation $e_i e_j = e_{ij}$ is usually used.

Clifford algebras are isomorphic to certain matrix rings. The isomorphism between $\mathcal{C}\ell_{p,q}\left(\mathbb{R}\right)$ algebras of $d = p + q < 8$ dimensional spaces is given in (table 2.1). The notation ${}^{m}\mathbb{A}\left(N\right)$ stands for is a block-diagonal matrix with m number of $\text{Mat}_N\left(\mathbb{A}\right)$ blocks, while, $\mathbb{A}$ is any associative Hurwitz algebra: $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

$q\backslash p$	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
1	$\mathbb{C}$	$\mathbb{R}(2)$	$^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$^2\mathbb{H}(4)$	
2	$\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$		
3	$^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$^2\mathbb{R}(8)$			
4	$\mathbb{H}(2)$	$^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$				
5	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$^2\mathbb{H}(4)$					
6	$\mathbb{R}(8)$	$\mathbb{C}(8)$						
7	$^2\mathbb{R}(8)$							

Table 2.1: Isomorphism between matrices and $\mathcal{C}\ell_{p,q}(\mathbb{R})$ Clifford algebras.

The following formulas are used to obtain the matrix representation of algebras of higher dimensional $d > 7$ spaces

$$\mathcal{C}\ell_{p,q}(\mathbb{R}) \simeq \mathcal{C}\ell_{p-4,q+4}(\mathbb{R}) \,, \tag{2.14}$$

$$\mathcal{C}\ell_{p+8,q}(\mathbb{R}) \simeq \text{Mat}_{16}\left(\mathcal{C}\ell_{p,q}(\mathbb{R})\right) \,. \tag{2.15}$$

As an example, the matrix representation of generators of the algebra $\mathcal{C}\ell_{2,0}(\mathbb{R}) \simeq \text{Mat}_2(\mathbb{R})$ can taken as

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.16}$$

from which it follows that

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,. \tag{2.17}$$

In the language of Clifford algebras, a spinor is defined as a minimal left or right ideal. It is easy to find in the matrix representation. A left (right) ideal is a subspace that is closed under left (right) side action by the general element of the algebra. An example of an ideal is a matrix whose only first column is non-zero

$$\psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \frac{1}{2}(1 + e_1)\psi_1 + \frac{1}{2}(e_1 - e_{12})\psi_2 \,, \qquad (\psi_1, \psi_2 \in \mathbb{R}) \tag{2.18}$$

because acting on it from the left with any matrix $A \in \mathrm{Mat}_2(\mathbb{R})$ yields a matrix of the same type. This left ideal is minimal because it contains no smaller subideals.

In order to find the minimal ideal, one must find an idempotent, that is, an element f of the algebra for which $f^2 = f$. This is an example of an idempotent for an algebra

$$f = \frac{1}{2}(1 + e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.19}$$

To get the left ideal from an idempotent, it is simply necessary to multiply it from the left by a general element of the algebra. The result of this can be clearly seen from the matrix representation. It will result in a matrix with only the first column being non-zero.

A spinor is transforms with respect to the group $Spin(p,q)$ in the following manner

$$\psi \to \psi' = L_{ij}(\vartheta)\,\psi \tag{2.20}$$

where

$$L_{ij}(\vartheta) = \exp\left(-\frac{1}{2}e_{ij}\vartheta\right). \tag{2.21}$$

A vector is an object whose basis is given by the generating elements of an algebra

$$\mathbf{x} = x_1e_1 + x_2e_2 + \cdots + x_de_d\,. \tag{2.22}$$

It transforms under the $SO(p,q)$ group as follows:

$$\mathbf{x} \to \mathbf{x}' = L_{ij}(\vartheta)\,\mathbf{x}L_{ij}(-\vartheta)\,. \tag{2.23}$$

Chapter 3

Original results

3.1 Noncompact G_2 group

The automorphism group of the split octonionic $\mathbb{O}'$ algebra is an exceptional Lie group called noncompact G_2. The Lie group is 14 dimensional, which means that its element is parametrized by 14 real numbers and thus it has 14 generators. These generators were first provided by Élie Cartan in his thesis [Cartan 1894] in the differential operator form. They can be written as:

$$\begin{aligned} Y_{ij} &= y_i \frac{\partial}{\partial y_j} - z_j \frac{\partial}{\partial z_i} + \frac{1}{3}\delta_{ij} \sum_n \left(z_n \frac{\partial}{\partial z_n} - y_n \frac{\partial}{\partial y_n} \right), \\ Y_{k0} &= y_k \frac{\partial}{\partial t} - 2t \frac{\partial}{\partial z_k} + \frac{1}{2} \sum_{m,n} \epsilon_{mnk} \left(z_m \frac{\partial}{\partial y_n} - z_n \frac{\partial}{\partial y_m} \right), \\ Y_{0k} &= z_k \frac{\partial}{\partial t} - 2t \frac{\partial}{\partial y_k} + \frac{1}{2} \sum_{m,n} \epsilon_{mnk} \left(y_m \frac{\partial}{\partial z_n} - y_n \frac{\partial}{\partial z_m} \right), \end{aligned} \tag{3.1}$$

where indices take values $1, 2, 3$ while $y_n, t, z_n \in \mathbb{R}$. Even though expressions (3.1) show 15 generators, Y_{11}, Y_{22} and Y_{33} are not linearly independent because

$$Y_{11} + Y_{22} + Y_{33} = 0\,. \tag{3.2}$$

To convert them to a linearly independent basis, the generator Y_{nn} can be replaced by

$$\begin{aligned} H_1 &= Y_{11} - Y_{22}\,, \\ H_2 &= \sqrt{3} Y_{33}\,, \end{aligned} \tag{3.3}$$

which is called the Cartan-Weyl basis.

The smallest faithful real representations of the noncompact group G_2 is provided by 7-dimensional matrices acting on the vector $p \in \mathbb{R}^7$

$$p = \begin{pmatrix} y_1 & y_2 & y_3 & t & z_1 & z_2 & z_3 \end{pmatrix}^T . \tag{3.4}$$

Weighted sum of (3.1) by α group parameters can be determined to have the following matrix form:

$$\begin{aligned} Y &= \sum_k \left(\alpha_{0k} Y_{0k} + \alpha_{k0} Y_{k0}\right) + \sum_{m,n} \alpha_{mn} Y_{mn} \\ &= \begin{pmatrix} A\left(\alpha\right) & 2d & B\left(b\right) \\ -b^T & 0 & -d^T \\ B\left(d\right) & 2b & -A^T\left(\alpha\right) \end{pmatrix}, \end{aligned} \tag{3.5}$$

where

$$b = \begin{pmatrix} \alpha_{10} & \alpha_{20} & \alpha_{30} \end{pmatrix}^T , \tag{3.6}$$

$$d = \begin{pmatrix} \alpha_{01} & \alpha_{02} & \alpha_{03} \end{pmatrix}^T , \tag{3.7}$$

A is a weighted sum of $SU\left(3\right)$ generators

$$A\left(\alpha\right) = \frac{1}{3} \begin{pmatrix} -2\alpha_{11} + \alpha_{22} + \alpha_{33} & -3\alpha_{21} & -\alpha_{31} \\ -3\alpha_{12} & \alpha_{11} - 2\alpha_{22} + \alpha_{33} & -3\alpha_{32} \\ -3\alpha_{13} & -3\alpha_{23} & \alpha_{11} + \alpha_{22} - 2\alpha_{33} \end{pmatrix}, \tag{3.8}$$

and B is a weighted sum of $SO\left(3\right)$ generators

$$\begin{aligned} B\left(b\right) &= \begin{pmatrix} 0 & \alpha_{30} & -\alpha_{20} \\ -\alpha_{30} & 0 & \alpha_{10} \\ \alpha_{20} & -\alpha_{10} & 0 \end{pmatrix}, \\ B\left(d\right) &= \begin{pmatrix} 0 & \alpha_{03} & -\alpha_{02} \\ -\alpha_{03} & 0 & \alpha_{01} \\ \alpha_{02} & -\alpha_{01} & 0 \end{pmatrix}. \end{aligned} \tag{3.9}$$

The generators of a noncompact G_2 group form a simple noncompact $\mathfrak{g}_2$ Lie algebra with respect ring commutator $\left[\,\cdot\,,\,\cdot\,\right] : \left(a, b\right) \mapsto ab - ba$. The commutative relations of the algebra are

$$\begin{aligned}
[Y_{ij}, Y_{i'j'}] &= \delta_{ji'} Y_{ij'} - \delta_{ij'} Y_{i'j} \, , \\
[Y_{ij}, Y_{0k}] &= -\delta_{ik} Y_{0j} + \frac{1}{3} \delta_{ij} Y_{0k} \, , \\
[Y_{ij}, Y_{k0}] &= \delta_{jk} Y_{i0} - \frac{1}{3} \delta_{ij} Y_{k0} \, , \\
[Y_{0k}, Y_{k'0}] &= 3 Y_{k'k} \, , \\
[Y_{k0}, Y_{k'0}] &= -2 \sum_{\ell} \epsilon_{kk'\ell} Y_{0\ell} \, , \\
[Y_{0k}, X_{0k'}] &= -2 \sum_{\ell} \epsilon_{kk'\ell} Y_{\ell 0} \, .
\end{aligned} \tag{3.10}$$

In matrix form the general element of the group is obtained by exponentiation of the matrix (3.5).

The invariant quadratic form $\mathcal{Q}_\eta : \mathbb{R}^7 \to \mathbb{R}$ of the group has the form:

$$\mathcal{Q}_\eta (p) = -t^2 - \sum_n z_n y_n \, . \tag{3.11}$$

Since the geometric interpretation of the noncompact group G_2 is needed, such a basis of $\mathbb{R}^7$ space needs to be found where the quadratic form is diagonal. It is possible to change to this basis by transforming the coordinates as:

$$\begin{aligned}
y_k &= \lambda_k + x_k \, , \qquad & \frac{\partial}{\partial y_k} &= \frac{1}{2} \left(\frac{\partial}{\partial \lambda_k} + \frac{\partial}{\partial x_k} \right) , \\
z_k &= \lambda_k - x_k \, , \qquad & \frac{\partial}{\partial z_k} &= \frac{1}{2} \left(\frac{\partial}{\partial \lambda_k} - \frac{\partial}{\partial x_k} \right) .
\end{aligned} \tag{3.12}$$

As a result, the vector p (3.4) is converted into a vector

$$q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & t & x_1 & x_2 & x_3 \end{pmatrix}^T , \tag{3.13}$$

and the operators rewritten for these variables (3.1) will be denoted with X instead of Y. X generators obey the same (3.10) commutation relations.

In order to simplify the formula of the Casimir operator which is to be obtained, it is convenient to introduce a new basis for the noncompact $\mathfrak{g}_2$ Lie algebra:

$$
\begin{aligned}
\Theta_k &= -X_{k0} + X_{0k} \, , \\
\mathrm{B}_k &= -X_{k0} - X_{0k} \, , \\
\Phi_k &= X_{kk} \, . \\
\Gamma_k &= \sum_{m,n} |\epsilon_{mnk}| \, X_{mn} \, , \\
\mathrm{R}_k &= \sum_{m,n} \epsilon_{mnk} X_{mn} \, ,
\end{aligned}
\tag{3.14}
$$

In this basis of algebra, (3.1) the differential operators have the form:

$$
\begin{aligned}
\Theta_k &= -2\left(x_k \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_k}\right) - \sum_{m,n} \epsilon_{mnk} \left(\lambda_m \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial \lambda_m}\right), \\
\mathrm{B}_k &= -2\left(\lambda_k \frac{\partial}{\partial t} + t \frac{\partial}{\partial \lambda_k}\right) - \sum_{m,n} \epsilon_{mnk} \left(\lambda_m \frac{\partial}{\partial \lambda_n} - x_n \frac{\partial}{\partial x_m}\right), \\
\Gamma_k &= \sum_{m,n} |\epsilon_{mnk}| \left(x_m \frac{\partial}{\partial \lambda_n} + \lambda_n \frac{\partial}{\partial x_m}\right), \\
\mathrm{R}_k &= \sum_{m,n} \epsilon_{mnk} \left(\lambda_m \frac{\partial}{\partial \lambda_n} + x_n \frac{\partial}{\partial x_m}\right), \\
\Phi_k &= \left(x_k \frac{\partial}{\lambda_k} + \lambda_k \frac{\partial}{\partial x_k}\right) - \frac{1}{3} \sum_n \left(x_n \frac{\partial}{\partial \lambda_n} + \lambda_n \frac{\partial}{\partial x_n}\right).
\end{aligned}
\tag{3.15}
$$

The matrix representation in the new basis is:

$$
\sum_k \left(\vartheta_k \Theta_k + \beta_k \mathrm{B}_k + \gamma_k \Gamma_k + \rho_k \mathrm{R}_k + \varphi_k \Phi_k\right) =
\begin{pmatrix}
B(\rho) - B(\beta) & -2\beta & M(\gamma, \varphi) - 3B(\vartheta) \\
-2\beta^T & 0 & 2\vartheta^T \\
M(\gamma, \varphi) - 3B(\vartheta) & 2\vartheta & B(\rho) + B(\beta)
\end{pmatrix}
\tag{3.16}
$$

where $\vartheta_k, \beta_k, \gamma_k, \rho_k, \varphi_k \in \mathbb{R}$,the matrix B is given by the formula (3.9),

$$
\beta = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \end{pmatrix}^T , \tag{3.17}
$$

$$
\vartheta = \begin{pmatrix} \vartheta_1 & \vartheta_2 & \vartheta_3 \end{pmatrix}^T , \tag{3.18}
$$

and matrix M is

$$M\left(\gamma,\varphi\right)=\frac{1}{3}\begin{pmatrix}-2\varphi_1+\varphi_2+\varphi_3 & -3\gamma_3 & -3\gamma_2\\ -3\gamma_3 & \varphi_1-2\varphi_2+\varphi_3 & -3\gamma_1\\ -2\gamma_2 & -3\gamma_1 & \varphi_1+\varphi_2-2\varphi_3\end{pmatrix}. \tag{3.19}$$

Second-order Casimir operator

In order to apply the algorithm for finding Casimir operators discussed in subsection 2.2 to the algebra (3.15), it is additionally necessary to change to the Cartan-Weyl basis (3.3) so that the generators are linearly independent, because otherwise Killing metric (2.2) will be degenerate. A matrix representation (3.15) of generators (3.16) is also needed.

The compact $\mathfrak{g}_2$ Lie algebra has non-zero Casimir operators of order $p = 2$ and $p = 6$, whose form for a specific basis of the algebra is found in the article [Bincer & Riesselmann 1993]. For a noncompact Lie algebra $\mathfrak{g}_2$, the Casimir operator of order $p=2$ in bases (3.1) and (3.15) have the form:

$$\begin{aligned}\mathcal{C}_2&=2\sum_{i,j}X_{ij}^2-\frac{2}{3}\sum_k\left(X_{k0}X_{0k}+X_{0k}X_{k0}\right)\\&=\sum_k\left(\frac{1}{3}\Theta_k^2-\frac{1}{3}\mathrm{B}_k^2+\Gamma_k^2-\mathrm{R}_k^2+2\Phi_k^2\right),\end{aligned} \tag{3.20}$$

By substituting the differential operators it becomes:

$$\begin{aligned}\mathcal{C}_2=6&\left(t\frac{\partial}{\partial t}+\sum_k\left(\lambda_k\frac{\partial}{\partial\lambda_k}+x_k\frac{\partial}{\partial x_k}\right)\right)\\&+2t\sum_k\left(x_k\frac{\partial^2}{\partial t\partial x_k}+\lambda_k\frac{\partial^2}{\partial t\partial\lambda_k}\right)-\frac{2}{3}\sum_{i,j}\lambda_i x_j\frac{\partial^2}{\partial\lambda_i\partial x_j}\\&+\sum_{i,j,k}\left|\epsilon_{ijk}\right|\left(x_ix_j\frac{\partial^2}{\partial x_i\partial x_j}-x_i^2\frac{\partial^2}{\partial x_j^2}+\lambda_i\lambda_j\frac{\partial}{\partial\lambda_i\partial\lambda_j}-\lambda_i^2\frac{\partial^2}{\partial\lambda_j^2}\right)\\+x^2&\left(\frac{\partial^2}{\partial t^2}+\sum_k\frac{\partial^2}{\partial\lambda_k^2}\right)-t^2\sum_k\left(\frac{\partial^2}{\partial\lambda_k^2}-\frac{\partial^2}{\partial x_k^2}\right)-\lambda^2\left(\frac{\partial^2}{\partial t^2}-\sum_k\frac{\partial^2}{\partial x_k^2}\right),\end{aligned} \tag{3.21}$$

where $x^2=x_1^2+x_2^2+x_3^2$ and $\lambda^2=\lambda_1^2+\lambda_2^2+\lambda_3^2$.

Eigenvalue equation written for the Casimir (3.21) operator

$$\mathcal{C}_2\psi = -m^2\psi\,, \tag{3.22}$$

is invariant to noncompact G_2 transformations, similarly to the Klein-Gordon equation, which is invariant to transformations of the Poincaré group and is an eigenvalue equation of the Casimir operator of the Poincaré algebra.

The first term in the operator (3.21)

$$c_1 = t\frac{\partial}{\partial t} + \sum_k \left(\lambda_k \frac{\partial}{\partial \lambda_k} + x_k \frac{\partial}{\partial x_k}\right) \tag{3.23}$$

itself also commutes with (3.15) generators, which is why the coefficient in front of it can be chosen arbitrarily, for example 3 instead of 6. In the limit where the λ variables are constants, all derivatives with these variables vanish from the operator (3.15) and what remains is

$$\mathcal{C}_2|_{\lambda=\text{const}} = \mathcal{C}_{\text{Lorentz}} - \lambda^2 P_\mu P^\mu\,, \tag{3.24}$$

where

$$P_\mu P^\mu = \frac{\partial^2}{\partial t^2} - \sum_k \frac{\partial^2}{\partial x_k} \tag{3.25}$$

is the Casimir operator of the Poincaré algebra, and

$$\begin{aligned}\mathcal{C}_{\text{Lorentz}} = 3\left(t\frac{\partial}{\partial t} + \sum_k x_k \frac{\partial}{\partial x_k}\right) + x^2\frac{\partial^2}{\partial t^2} + t^2\sum_k \frac{\partial^2}{\partial x_k^2} \\ + 2t\sum_k x_k \frac{\partial^2}{\partial t \partial x_k} + \sum_{i,j,k} |\epsilon_{ijk}| \left(x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - x_i^2 \frac{\partial^2}{\partial x_j^2}\right)\end{aligned} \tag{3.26}$$

is a $\mathfrak{so}\,(1,3)$ Lorentz algebra Casimir operator (2.5) in an explicit form.

The operator $\mathcal{C}_2|_{\lambda=\text{const}}$ (3.24) can be considered in two additional limits: when λ is small, it reduces to the Casimir operator of the $\mathfrak{so}\,(1,3)$ Lorentz algebra (2.5, 3.26), and when λ is large it reduces to the Casimir operator of the Poincaré algebra (3.25).

Noncompact G_2 transformations

By setting to zero all but one variable in the (3.16) matrix and then exponentiating this matrix, the group element is obtained, which is parameterized by the chosen variable. By acting on the vector (3.13) by the obtained matrix, the transformation of the noncompact G_2 Lie group is performed. Six matrices parametrized by the variables ρ_k and β_k contain trigonometric functions and describes transformations similar to rotations, while the nine matrices parametrized by the variables ϑ_k, φ_k and γ_k contain hyperbolic trigonometric functions and describe transformations similar to Lorentz boosts.

- **Rotations.** The three R_k generators corresponding to the ρ_k parameters are the Euler angles that simultaneously rotate (x_1, x_2, x_3) and $(\lambda_1, \lambda_2, \lambda_3)$ subspaces. For example $q' = \exp(\rho_1 \mathrm{R}_1)\, q$ is the transformation:

$$
\begin{aligned}
\lambda_1' &= \lambda_1 \,, \\
\lambda_2' &= \lambda_2 \cos\rho_1 + \lambda_3 \sin\rho_1 \,, \\
\lambda_3' &= \lambda_3 \cos\rho_1 - \lambda_2 \sin\rho_1 \,, \\
t_1' &= t_1 \,, \\
x_1' &= x_1 \,, \\
x_2' &= x_2 \cos\rho_1 + x_3 \sin\rho_1 \,, \\
x_3' &= x_3 \cos\rho_1 - x_2 \sin\rho_1 \,.
\end{aligned}
\tag{3.27}
$$

- **Boosts.** The three parameters corresponding to the generators Θ_k are hyperbolic angles $2\vartheta_k$. However, these transformations are not pure Lorentz boosts, because with them also hyperbolic rotation occurs in pairs of coordinates $(i, j \neq k)$ between

(x_1, x_2, x_3) and $(\lambda_1, \lambda_2, \lambda_3)$ subspaces. For example $q' = \exp(\vartheta_1 \Theta_1)\, q$ is:

$$
\begin{aligned}
\lambda_1' &= \lambda_1\,, \\
\lambda_2' &= \lambda_2 \cosh\vartheta_1 + x_3 \sinh\vartheta_1 \\
\lambda_3' &= \lambda_3 \cosh\vartheta_1 - x_2 \sinh\vartheta_1\,, \\
t_1' &= t \cosh(2\vartheta_1) + x_1 \sinh(2\vartheta_1)\,, \\
x_1' &= x_1 \cosh(2\vartheta_1) + t \sinh(2\vartheta_1)\,, \\
x_2' &= x_2 \cosh\vartheta_1 - \lambda_3 \sinh\vartheta_1\,, \\
x_3' &= x_3 \cosh\vartheta_1 + \lambda_2 \sinh\vartheta_1\,.
\end{aligned}
\tag{3.28}
$$

3.2 Clifford algebra $\mathcal{C}\ell_{4,4}(\mathbb{R})$

Since Clifford algebras are easier to work with than non-associative algebraic structures, spinors, vectors, pseudo-orthogonal groups, and rotations are first explored through them.

It is convenient to use $\mathcal{C}\ell_{4,4}(\mathbb{R})$ Clifford algebra to describe geometric objects in pseudo-Euclidean $(4+4)$ space. Metric of this space

$$g = \text{diag}\,(1,1,1,1,-1,-1,-1,-1) \tag{3.29}$$

will be denoted as $g_{\mu\nu}$, where the Greek alphabet indices take on the values $0, 1, \ldots, 7$. The $\mathcal{C}\ell_{4,4}(\mathbb{R})$ algebra is associative like other Clifford algebras. It can be defined by the anti-commutative relation:

$$e_\mu e_\nu + e_\nu e_\mu = 2g_{\mu\nu}\,, \tag{3.30}$$

where e_μ are the pseudo-orthonormal basis units of the $(4+4)$ space. The $e_\mu \mapsto \Gamma_\mu \in \text{Mat}_{16\times16}(\mathbb{R})$ representation of the basis elements will be used, which can be determined using the generating matrices of the $\mathcal{C}\ell_{8,0}(\mathbb{R})$ algebra [Gamba 1968], given that $\mathcal{C}\ell_{4,4}(\mathbb{R})$ algebra has a real representation [Lounesto 2001].

In the $(4+4)$-space vectors are represented using

$$\chi = \sum_{\beta=0}^{7} \chi_\beta \mathit{\Gamma}_\beta\,, \tag{3.31}$$

where $\chi_\beta \in \mathbb{R}$. By multiplying this matrix by itself, we get the diagonal $\mathcal{Q} : \mathbb{R}^8 \to \mathbb{R}$ quadratic form

$$\begin{aligned}\mathcal{Q}(\chi) &= \chi^2 \\ &= \mathbf{1}_{16\times16}\left(\chi_0^2+\chi_1^2+\chi_2^2+\chi_3^2-\chi_4^2-\chi_5^2-\chi_6^2-\chi_7^2\right)\,,\end{aligned} \tag{3.32}$$

where the right side of the equation is multiplied by the identity matrix. Continuous transformations, for which the form (3.31) is invariant, constitute $SO(4,4)$ group and are written as:

$$\chi' = L_{\mu\nu}(\vartheta)\,\chi L_{\mu\nu}^{-1}(\vartheta)\,, \tag{3.33}$$

where $\vartheta \in \mathbb{R}$ and

$$L_{\mu\nu}(\vartheta) = \exp\left(-\frac{1}{2}\vartheta\Gamma_\mu\Gamma_\nu\right) . \tag{3.34}$$

For spinors and rotations, the matrix $B = \Gamma_4\Gamma_5\Gamma_6\Gamma_7$ will play an important role due to the following property:

$$\chi^T = B\chi B . \tag{3.35}$$

The η spinor can be obtained in the way discussed above. Since η represents the first column of a 16×16 matrix, it will be treated as a 16×1 matrix for simplicity. The chiral parts of the $\eta = \phi + \psi$ spinor are

$$\begin{aligned} \phi &= \begin{pmatrix} 0 & 0 & \cdots & 0 & \phi_0 & \phi_1 & \cdots & \phi_7 \end{pmatrix}^T , \\ \psi &= \begin{pmatrix} \psi_0 & \psi_1 & \cdots & \psi_7 & 0 & 0 & \cdots & 0 \end{pmatrix}^T . \end{aligned} \tag{3.36}$$

The rule for transforming spinors is given as:

$$\eta' = L_{\mu\nu}(\vartheta)\,\eta . \tag{3.37}$$

This constitute the $Spin\,(4,4)$ group which is a double cover of $SO\,(4,4)$. The quadratic form

$$\eta^T B\eta = \phi^T B\phi + \psi^T B\psi , \tag{3.38}$$

is invariant with respect to $Spin\,(4,4)$ transformations. Under the transformations chiral spinors do not mix with each other and their quadratic forms $\phi^T B\phi$ and $\psi^T B\psi$ are preserved independently.

From (3.36) it can be seen that the dimension of the chiral spinors, like for χ vector, is 8. However, there is a greater symmetry behind this, which can be demonstrated by changing the basis of the spinors. In the new basis, the quadratic forms of the chiral spinors become diagonal and coincide with the quadratic form of the vector (3.31)

$$\begin{aligned} \phi^T B\phi &= \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2 - \phi_5^2 - \phi_6^2 - \phi_7^2 , \\ \psi^T B\psi &= \psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2 - \psi_5^2 - \psi_6^2 - \psi_7^2 . \end{aligned} \tag{3.39}$$

There exists trilinear form $\mathcal{F} : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}$ which is invariant under the joint $SO(4,4)$ and $Spin(4,4)$ transformations

$$\mathcal{F}(\phi,\chi,\psi) = \phi^T B\chi\psi\,, \tag{3.40}$$

To see a more important and deeper symmetry, one can for example observe the transformation $L_{01}(\vartheta)$

$$\begin{cases} \phi'_0 = \phi_0 + \frac{1}{2}\vartheta\phi_1 \\ \phi'_1 = \phi_1 - \frac{1}{2}\vartheta\phi_0 \\ \phi'_2 = \phi_2 - \frac{1}{2}\vartheta\phi_3 \\ \phi'_3 = \phi_3 + \frac{1}{2}\vartheta\phi_2 \\ \phi'_4 = \phi_4 - \frac{1}{2}\vartheta\phi_5 \\ \phi'_5 = \phi_5 + \frac{1}{2}\vartheta\phi_4 \\ \phi'_6 = \phi_6 + \frac{1}{2}\vartheta\phi_7 \\ \phi'_7 = \phi_7 - \frac{1}{2}\vartheta\phi_6 \end{cases}, \qquad \begin{cases} \chi'_0 = \chi_0 - \vartheta\chi_1 \\ \chi'_1 = \chi_1 + \vartheta\chi_0 \\ \chi'_2 = \chi_2 \\ \chi'_3 = \chi_3 \\ \chi'_4 = \chi_4 \\ \chi'_5 = \chi_5 \\ \chi'_6 = \chi_6 \\ \chi'_7 = \chi_7 \end{cases} \qquad \begin{cases} \psi'_0 = \psi_0 + \frac{1}{2}\vartheta\psi_1 \\ \psi'_1 = \psi_1 - \frac{1}{2}\vartheta\psi_0 \\ \psi'_2 = \psi_2 + \frac{1}{2}\vartheta\psi_3 \\ \psi'_3 = \psi_3 - \frac{1}{2}\vartheta\psi_2 \\ \psi'_4 = \psi_4 + \frac{1}{2}\vartheta\psi_5 \\ \psi'_5 = \psi_5 - \frac{1}{2}\vartheta\psi_4 \\ \psi'_6 = \psi_6 - \frac{1}{2}\vartheta\psi_7 \\ \psi'_7 = \psi_7 + \frac{1}{2}\vartheta\psi_6 \end{cases}. \tag{3.41}$$

As expected, one complete turn for the vector χ is only half a turn for the spinors ϕ and ψ. Here, in all planes except the one in which the vector rotates, ϕ and ψ rotate in opposite directions to each other, which is the manifestation of their different chiralities. However, since the dimensions of the chiral spinors and the vector coincide and the $L_{\mu\nu}$ matrices form a group with respect to multiplication, it is possible to construct such a transformation for the χ vector that exactly repeats, for example, the transformation of the ϕ chiral spinor (3.41)

$$\begin{aligned} &L_{10}\left(\frac{\vartheta}{2}\right) L_{23}\left(\frac{\vartheta}{2}\right) L_{54}\left(\frac{\vartheta}{2}\right) L_{67}\left(\frac{\vartheta}{2}\right) \\ &\simeq 1 - \frac{1}{4}\vartheta\left(\mathit{\Gamma}_1\mathit{\Gamma}_0 + \mathit{\Gamma}_2\mathit{\Gamma}_3 + \mathit{\Gamma}_5\mathit{\Gamma}_4 + \mathit{\Gamma}_6\mathit{\Gamma}_7\right). \end{aligned} \tag{3.42}$$

It represents the following interesting transformation:

$$\begin{cases} \phi_0' = \phi_0 + \frac{1}{2}\vartheta\phi_1 \\ \phi_1' = \phi_1 - \frac{1}{2}\vartheta\phi_0 \\ \phi_2' = \phi_2 + \frac{1}{2}\vartheta\phi_3 \\ \phi_3' = \phi_3 - \frac{1}{2}\vartheta\phi_2 \\ \phi_4' = \phi_4 + \frac{1}{2}\vartheta\phi_5 \\ \phi_5' = \phi_5 - \frac{1}{2}\vartheta\phi_4 \\ \phi_6' = \phi_6 - \frac{1}{2}\vartheta\phi_7 \\ \phi_7' = \phi_7 + \frac{1}{2}\vartheta\phi_6 \end{cases}, \quad \begin{cases} \chi_0' = \chi_0 + \frac{1}{2}\vartheta\chi_1 \\ \chi_1' = \chi_1 - \frac{1}{2}\vartheta\chi_0 \\ \chi_2' = \chi_2 - \frac{1}{2}\vartheta\chi_3 \\ \chi_3' = \chi_3 + \frac{1}{2}\vartheta\chi_2 \\ \chi_4' = \chi_4 - \frac{1}{2}\vartheta\chi_5 \\ \chi_5' = \chi_5 + \frac{1}{2}\vartheta\chi_4 \\ \chi_6' = \chi_6 + \frac{1}{2}\vartheta\chi_7 \\ \chi_7' = \chi_7 - \frac{1}{2}\vartheta\chi_6 \end{cases}, \quad \begin{cases} \psi_0' = \psi_0 - \vartheta\psi_1 \\ \psi_1' = \psi_1 + \vartheta\psi_0 \\ \psi_2' = \psi_2 \\ \psi_3' = \psi_3 \\ \psi_4' = \psi_4 \\ \psi_5' = \psi_5 \\ \psi_6' = \psi_6 \\ \psi_7' = \psi_7 \end{cases}. \tag{3.43}$$

Here it is peculiar that the χ vector and the ϕ and ψ spinors have cyclically exchanged roles, as shown in the diagram (fig. 3.1). This is a property of 8-dimensional spaces and is called triality, similar to duality for vector spaces.

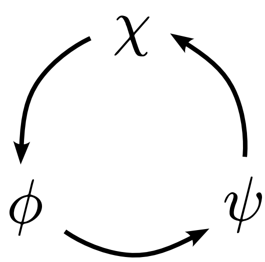

Figure 3.1: The sequence of exchange of chiral spinors and vector roles when going from the $L_{10}(\vartheta)$ transformation to the (3.42) transformation.

3.3 Split octonionic numbers $\mathbb{O}'$

Split octonions $\mathbb{O}'$ represent a non-associative algebra that can be constructed as a Cayley-Dixon construction (2.10) and (2.11). They can also be defined through algebraic relations:

$$\begin{aligned}
&I^2 = 1\,, && (\mathbb{C}' \text{ subalgebra})\\
&j_m j_n = -\delta_{mn} + \sum_{\ell=1}^{3} \epsilon_{\ell mn} j_\ell\,, && (\mathbb{H} \text{ subalgebra})\\
&J_m J_n = \delta_{mn} + \sum_{\ell} \epsilon_{\ell mn} j_\ell\,, && ,\\
&J_m j_n = \delta_{mn} I - \sum_{\ell} \epsilon_{\ell mn} J_\ell, && j_n I = J_n\,,
\end{aligned} \tag{3.44}$$

together with right and left alternativity properties $x\,(xy) = (xx)\,y$ and $(xy)\,y = x\,(yy)$ where $x, y \in \mathbb{O}'$.

A general split octonionic number $x \in \mathbb{O}'$ and its split octonionic conjugate $\overline{x} \in \mathbb{O}'$ are

$$\begin{aligned}
x &= x_0 + I x_4 + \sum_n \left(j_n x_n + J_n x_{4+n}\right),\\
\overline{x} &= x_0 - I x_4 - \sum_n \left(j_n x_n + J_n x_{4+n}\right),
\end{aligned} \tag{3.45}$$

where $x_0, x_1, \ldots, x_7 \in \mathbb{R}$.

Since this algebra is non-associative and 8-dimensional, algebraic manipulation of large expressions is difficult, which is why a computer algebra system for working with split octonions [Gurchumelia 2023] was created. Most of the following calculations are performed using this library.

It is possible to conjugate the imaginary part u separately, which will be denoted by $\mathrm{conj}_u : \mathbb{O}' \to \mathbb{O}'$ and calculated as:

$$\mathrm{conj}_u x = u^2 \left(u \overline{x} u\right)\,. \tag{3.46}$$

The product of the split octonion x by its conjugate defines the quadratic form $\mathcal{Q} : \mathbb{O}' \to \mathbb{R}$

$$\mathcal{Q}\,(x) = \overline{x} x = \sum_{n=0}^{3} \left(x_n^2 - x_{4+n}^2\right)\,. \tag{3.47}$$

Using the quadratic form, a symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle : \mathbb{O}'\times\mathbb{O}' \to \mathbb{R}$ can be constructed as:

$$\begin{aligned}\langle x,y\rangle &= \frac{1}{2}\mathcal{Q}(x+y) - \frac{1}{2}\mathcal{Q}(x) - \frac{1}{2}\mathcal{Q}(y)\\ &= \frac{1}{2}\left(\overline{x}y+\overline{y}x\right) = \sum_{n=0}^{3}\left(x_n y_n - x_{4+n}y_{4+n}\right) .\end{aligned} \tag{3.48}$$

Split octonionic gradient

For functions of the type $f:\mathbb{O}'\to\mathbb{O}'$ there are gradient operators

$$\partial = \frac{\partial}{\partial x} \qquad \text{and} \qquad \overline{\partial} = \frac{\partial}{\partial \overline{x}} . \tag{3.49}$$

When acting on the linear functions $f(x) = x$ and $f(x) = \overline{x}$ the split octonionic gradients have the properties

$$\partial x = \overline{\partial}\overline{x} = 1 \qquad \text{and} \qquad \overline{\partial} x = \partial \overline{x} = 0\,, \tag{3.50}$$

if defined as:

$$\partial = \frac{1}{2}\left(\partial_0 + I\partial_4\right) + \frac{1}{2}\sum_{n=1}^{3}\left(j_n\partial_n + J_n\partial_{4+n}\right) , \tag{3.51}$$

$$\overline{\partial} = \frac{1}{2}\left(\partial_0 - I\partial_4\right) - \frac{1}{2}\sum_{n=1}^{3}\left(j_n\partial_n + J_n\partial_{4+n}\right) . \tag{3.52}$$

These operators do not satisfy the properties of derivative, for instance the product rule.

Since split octonions are non-commutative, it will be necessary to specify the direction of the action of the operator when writing the gradient. Because in general $\overrightarrow{\partial} f \neq f\overleftarrow{\partial}$.

3.4 Split octonionic field theories

The linear forms (3.47) and (3.48) are required to construct the Lagrangian for split octonionic fields. Another such form that will be important for constructing the Lagrangian is the split octonion representation of the trilinear form (3.40) $\mathcal{F} : \mathbb{O}' \times \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$, which can be defined through the bilinear form (3.48) as:

$$\mathcal{F}(\phi, \chi, \psi) = \left\langle \overline{\phi}, \chi\psi \right\rangle . \tag{3.53}$$

Explicitly it is

$$\mathcal{F}(\phi, \chi, \psi) = \frac{1}{2}\phi(\chi\psi) + \frac{1}{2}\left(\overline{\psi}\overline{\chi}\right)\overline{\phi} . \tag{3.54}$$

Representation of symmetry groups of (4+4)-space

The $\mathcal{Q}$ and $\mathcal{F}$ quadratic and trilinear forms (3.47) and (3.53), as mentioned in section 3.2, are $SO(4,4)$ and $Spin(4,4)$ pseudo-orthogonal group invariants. Similar to the $\mathbb{O}$ octonionic representation of $SO(8)$ and $Spin(8)$ groups [Dray & Manogue], the split octonionic representation of pseudo-orthogonal groups is achieved by composition of the following transformations

$$\begin{aligned} \chi' &= T_{uv}(\vartheta)(u\chi u)\, T_{uv}(\vartheta) \\ \phi' &= u^2(\phi u)\, T_{uv}(s_u\vartheta) \\ \psi' &= u^2 T_{uv}(s_u\vartheta)(u\psi) \end{aligned} \tag{3.55}$$

where

$$T_{uv}(\vartheta) = \begin{cases} u\cos\frac{\vartheta}{2} + v\sin\frac{\vartheta}{2}, & u\overline{u} = v\overline{v}\,, \\ u\cosh\frac{\vartheta}{2} + v\sinh\frac{\vartheta}{2}, & u\overline{u} = -v\overline{v}\,. \end{cases} \tag{3.56}$$

and $s_u \in \{-1, 1\}$ is calculated as:

$$s_u = \left| u\overline{u} - u^2 \right| - 1\,. \tag{3.57}$$

Lagrangian

Using the trilinear form (3.53), a Lagrangian can be constructed by replacing the χ split octonion a right-hand (3.51) gradient operator

$$\mathcal{L} = \left\langle \overline{\phi}, \overrightarrow{\partial} \psi \right\rangle . \tag{3.58}$$

By stationaryizing the corresponding action, the conditions of right and left analyticity on the ϕ and ψ fields are obtained

$$\begin{cases} \phi \overleftarrow{\partial} = 0 \, , \\ \overrightarrow{\partial} \psi = 0 \, . \end{cases} \tag{3.59}$$

These equations are generalizations of Cauchy-Riemann and Cauchy-Riemann-Futter [Sudbery 1979] equations to split octonions. Taking the split octonionic conjugate of the first equation of (3.59) gives

$$\begin{cases} \overline{\overrightarrow{\partial}} \, \overline{\phi} = 0 \, , \\ \overrightarrow{\partial} \psi = 0 \, . \end{cases} \tag{3.60}$$

Adding quadratic terms to the Lagrangian

$$\mathcal{L} = \left\langle \overline{\phi}, \overrightarrow{\partial} \psi \right\rangle + \frac{1}{2} \lambda_1 \left\langle \phi, \phi \right\rangle + \frac{1}{2} \lambda_2 \left\langle \psi, \psi \right\rangle \tag{3.61}$$

results in equations becoming interdependent

$$\begin{cases} \overline{\overrightarrow{\partial}} \, \overline{\phi} = \lambda_2 \psi \, , \\ \overrightarrow{\partial} \psi = -\lambda_1 \overline{\phi} \, . \end{cases} \tag{3.62}$$

If the coefficient is set to $\lambda_2 = 0$, then using the property of alternativity (3.62) the equation reduces to eight independent wave equations (massless Klein-Gordon equations) in (4+4) space

$$\left\langle \overrightarrow{\partial}, \overrightarrow{\partial} \right\rangle \psi = 0 \, . \tag{3.63}$$

3.5 Dirac and Maxwell equations

To obtain the Dirac and Maxwell equations, it is necessary to define a new gradient operator D

$$D = I \partial I \; . \tag{3.64}$$

The imaginary parts of this operator j_n and J_n have a negative sign

$$D = \frac{1}{2}\left(\partial_0 + I\partial_4\right) - \frac{1}{2}\sum_n \left(j_n \partial_n + J_n \partial_{4+n}\right) \; . \tag{3.65}$$

Fields that will also be considered are

$$A = \mathcal{C}_0 + j_1\mathcal{A}_1 + j_2\mathcal{A}_2 + j_3\mathcal{A}_3 + I\mathcal{A}_0 + J_1\mathcal{C}_1 + J_2\mathcal{C}_2 + J_3\mathcal{C}_3 \; , \tag{3.66}$$

$$F = \overrightarrow{D} A \; . \tag{3.67}$$

By fixing the following parameters in the quadratic Lagrangian defined above (3.61)

$$\lambda_2 = 0 \qquad \text{and} \qquad \lambda_1 = -1 \, , \tag{3.68}$$

by setting the fields

$$\phi = \overline{F} \tag{3.69}$$

and

$$\psi = A \tag{3.70}$$

and using the D operator instead of the ∂ operator the following Lagrangian is obtained

$$\mathcal{L} = \left\langle F, \overrightarrow{D} A \right\rangle - \frac{1}{2}\left\langle \overline{F}, \overline{F} \right\rangle . \tag{3.71}$$

Using definition of the field F field (3.67) and the fact that

$$\left\langle \overline{F}, \overline{F} \right\rangle = \left\langle F, F \right\rangle , \tag{3.72}$$

the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{4}\left\langle F, F \right\rangle \; . \tag{3.73}$$

The equation of its motion is

$$\left\langle \overrightarrow{D}, \overrightarrow{D} \right\rangle A = 0 \, . \tag{3.74}$$

In the limit when

$$D \to \mathscr{D} = \frac{1}{2} \left(-j_1 \partial_x - j_2 \partial_y - j_3 \partial_z + I \partial_t \right) \tag{3.75}$$

the equation (3.74) reduces to the free Dyonic Maxwell equation in Minkowski space

$$\left\langle \overrightarrow{\mathscr{D}}, \overrightarrow{\mathscr{D}} \right\rangle A = 0 \, , \tag{3.76}$$

where $\mathcal{A}_n$ and $\mathcal{C}_n$ are electromagnetic and dyonic 4-potentials, whose split octonionic form was first introduced in [Chanyal, Bisht & Negi 2011, Chanyal 2014].

By stationarization of a Lagrangian with mass parameter m

$$\mathcal{L} = \left\langle \overline{\phi}, \overrightarrow{D} \psi \right\rangle - \frac{1}{2} m \left\langle \overline{\phi}, J_3 \psi \right\rangle = \left\langle \overline{\phi}, \left(\overrightarrow{D} - \frac{1}{2} m J_3 \right) \psi \right\rangle \, , \tag{3.77}$$

the following two independent equations of motion are obtained

$$\begin{cases} \left(\overrightarrow{\overrightarrow{D}} - \frac{1}{2} J_3 m \right) \overline{\phi} = 0 \, , \\ \left(\overrightarrow{D} - \frac{1}{2} J_3 m \right) \psi = 0 \, , \end{cases} \tag{3.78}$$

of which the second reduces to the Dirac equation in the limit $D \to \mathscr{D}$.

It is also possible to write the Lagrangian for a single field by setting $\phi = \overline{\psi} J_3$

$$\mathcal{L} = \left\langle -J_3 \psi, \left(\overrightarrow{D} - \frac{1}{2} J_3 m \right) \psi \right\rangle \, , \tag{3.79}$$

which in the limit $D \to \mathscr{D}$ walks on the split octonion form of the Dirac Lagrangian

$$\mathcal{L} = \left\langle -J_3 \psi, \left(\overrightarrow{\mathscr{D}} - \frac{1}{2} J_3 m \right) \psi \right\rangle \, . \tag{3.80}$$

This Lagrangian is equivalent to Dirac's Lagrangian and therefore its equation of motion

$$\overrightarrow{\mathscr{D}} \psi = \frac{1}{2} J_3 m \psi \, , \tag{3.81}$$

is equivalent to the Dirac equation.

It is noteworthy that equation obtained by stationarization of the (3.79) Lagrangian does not result in Dirac equation in the $D \to \mathscr{D}$ limit. In order to finally obtain the Dirac equation the limit needs to be taken at the Lagrangian level, before the action is stationarized. On the other hand (3.77) Lagrangian is not equivalent to Dirac Lagrangian, because it contains an additional ϕ field, but in this case it does not matter if the limit $D \to \mathscr{D}$ is taken at the Lagrangian level or at the level of the equation after the stationarizing the corresponding action, because in both cases an equation equivalent to the Dirac equation is obtained with respect to split octonion valued ψ function.

Dirac equation inside an external potential field $A = j_1\mathcal{A}_1 + j_2\mathcal{A}_2 + j_3\mathcal{A}_3 + I\mathcal{A}_0$ can be written as

$$\left(\mathscr{D} - \frac{1}{2}J_3 m\right)\psi = \frac{1}{2}J_3\left(\mathrm{conj}_{Ij}\left(A\psi\right)I\right). \tag{3.82}$$

3.6 Signal analysis

As a result of the development of computers, the use of division algebras has become convenient in data analysis, aerospace, and other practical tasks, since because of divisibility property the occurrence of zeros in the denominator are excluded

Computer algebra system created for working with split octonions [Gurchumelia 2023] can be used for data analysis.

Analysis was performed for Mrk 501 active galactic nucleus X-ray spectrum [Kapanadze, Gurchumelia & Aller 2023], which was primarily based on the Swift data in the time range of 2021 February to 2022 December. The data indicated a significant boost in X-ray emissions, marked by a sustained rise in the baseline flux level within the 0.3–10 keV range. This increase is further punctuated by occasional short bursts, occurring over periods of a few weeks to around two months. At certain points in time, Mrk 501 stood out as the most luminous blazar in the X-ray spectrum, and it also displayed rapid fluctuations that were occasionally observed within exposures lasting just a few hundred seconds.

The source displayed unusual spectral characteristics, including a prominent presence of spectral curvature, frequent instances of hard photon indices in both the 0.3–10 keV and 0.3–300 GeV energy ranges, and a peak in its synchrotron spectral energy distribution within the hard X-ray region.

These characteristics illustrate the significance of several factors, including relativistic magnetic reconnection, the first-order Fermi mechanism operating in magnetic fields with varying confinement efficiencies, stochastic acceleration, and hadronic processes. The distribution of X-ray and γ-ray fluxes follows a lognormal pattern, suggesting that accretion disc instabilities might influence the blazar jet. Surprisingly, the variations in optical-UV and γ-ray emissions do not show a strong correlation with the X-ray flares, challenging simple leptonic models and indicating the need for more complex explanations involving particle acceleration, emission mechanisms, and variability patterns.

Chapter 4

Conclusion

The geometric interpretation of the automorphism group of $\mathbb{O}'$ split octonionic algebra, the noncompact $G_2$and its similarity to the Poincaré group were discussed. It was shown that despite the absence of translations, it is possible to simulate them using additional dimensions. The second-order Casimir operator of the corresponding noncompact $\mathfrak{g}_2$ Lie algebra was found. In the limiting case where additional three λ dimensions are held constants, the differential operator form of the Casimir operator was shown to reduce to the sum of the Casimir operators of the Lorentz and Poincaré algebras [Gogberashvili & Gurchumelia 2019].

Split octonionic $\mathbb{O}'$ representation was found for pseudo-orthogonal $SO\left(4,4\right)$ and $Spin\left(4,4\right)$ Lie groups of exotic $\left(4+4\right)$-space and one-to-one correspondence with $\mathcal{C}\ell_{4,4}\left(\mathbb{R}\right)$ Clifford algebraic matrix representation of the same groups was shown. An invariant trilinear form of these groups defined on chiral spinors and vectors was also found [Gurchumelia & Gogberashvili 2021, Gogberashvili & Gurchumelia 2023]. Using the invariant bilinear and trilinear forms of the pseudo-orthogonal group, the Lagrangian with the symmetries of these groups was constructed and its corresponding equations of motion were calculated, which in some special cases reduces to Dirac or Maxwell equations [Gogberashvili & Gurchumelia 2023].

To work with split octonions, the SplitOct library was created, which makes it possible to computerize complex calculations. [Gurchumelia 2023].

List of dissertant's publications

- **Gogberashvili, M., & Gurchumelia, A. (2019). Geometry of the non-compact G(2). *Journal of Geometry and Physics*, 144, 308-313.**

- Gogberashvili, M., & Gurchumelia, A. (2023). Dirac and Maxwell Systems in Split Octonions. *Journal of Applied Mathematics and Physics*, 11(7), 1977-1995.

- Gurchumelia, A., & Gogberashvili, M. (2021). Split Octonions and Triality in (4+4)-Space. *Recent Advances in Mathematical Physics, Proceeding of Science Regio* 394, 008 doi: 10.22323/1.394.0008;

- **Kapanadze, B., Gurchumelia, A., & Aller, M. (2023). Long-term X-ray outbursts in the TeV- detected blazar Mrk 501 in 2021-2022: further clues for the emission and unstable processes. *Astrophys. J. Suppl. Ser.*, 268(1), 20.**

Data access and material availability

Based on SymPy computer algebra system [Meuer et al 2017] python library SplitOct was created for working with split octonions, which is are available at the following link along with computation examples in Jupyter environment [Granger & Grout 2016]:

- Gurchumelia, A. (2023). SplitOct.
  `https://github.com/EQUINOX24/SplitOct`

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