YEREVAN STATE UNIVERSITY

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Integrable Systems with Kähler Phase Space

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Theoretical Physics

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Contents

1 Introduction

Integrble models play an important role in the modern theoretical and mathematical physics. Due to the fact that different physical phenomena can have similar mathematical description, exactly solvable models can be used in many different areas. One can see that using these models huge amount of (both macroscopic and microscopic) physical phenomena can be described. Moreover integrable models can have applications even in other disciplines, due to the fact that system of integrable differential equations arise in other subjects e.g. mathematics, computer science, biology etc. The thesis is devoted to superintegrable extensions of oscillator and Coulomb models with an inverse square potential. Integrable models with inverse square potential are studied for few decades. Due to this fact they are well studied and there are many important results about these systems. Namely the Calogero-model has unique properties and due to that nowadays this is an important system in mathematical physics. On the other hand projective spaces have also interesting properties . Due to the fact that they are maximally symmetric spaces it is important to consider physical systems on these spaces. Unfortunately these two branches of mathematical physics are disconnected now. Complex analogs of Calogero model are not studied well and attempts to construct complexification of Calogero-like models haven't succeeded yet.

The N-dimensional mechanical system, i.e. system with N degrees of freedom, will be called *integrable* if it has N mutually commuting and functionally independent constants of motion[10, 11]. In addition to these constants of motion the system may have more. In that case we will say that the system is *superintegrable*. In particular if N-dimensional mechanical system has $2N - 1$ functionally independent constants of motion it will be called *maximally superintegrable*. In case the system has N+1 conserved quantities it is called *minimally superintegrable*. While integrable models possess separation of variables in one coordinate system, superintegrability guarantees separation of variables in many coordinate systems. For example two-dimensional oscillator is superintegrable, which allows us to separate variables in Cartesian and polar coordinates. In classical mechanics maximal superintegrability guarantees the closeness of trajectories. Quantum mechanically energy spectrum of integrable models depend on N quantum numbers. If the system has K additional conserved quantities (superintegrable) energy spectrum depends on $N - K$ quantum numbers. For maximal superintegrability we have that the energy spectrum contains only one quantum number. So we can conclude that superintegrability leads to degeneracy of energy spectrum in quantum level. Well known examples of maximally superintegrable models are N-dimensional Coulomb system and N-dimensional harmonic oscillator. Another important but recently discovered model is the Calogero model.

In the *Section 2* we discuss the general notion of Kähler manifolds, namely we speak about Kähler potential, the metric and the Killing potentials. We give us a simple example of Kähler manifolds (pseudo)Euclidean complex spaces as well as compact and non-compact complex projective spaces.

In the *Section 3* we inviting The Klein model of Lobachevsky space and mapping a conformal mechanics to it in one dimensional case first. Than we extend Klein model for the N-dimensional case and writing down the $SU(1.N)$ algebra of Killing potentials. We got an interesting results that isometry generators of the phase space, namely Killing potentials are appear to be the constants of motion of the system.

In the *Section 4* we gave as an example the harmonic oscillator, and show that this family of integrable systems with Lobachevsky space being their phase space, contains it.

2 Kähler structure

Kähler manifolds play an important role in modern theoretical physics and mathematics [15, 18]. In algebraic geometry a class of algebraic varieties are Kähler manifolds. In supersymmetry the target space can be sometimes viewed as a Kähler manifold. Moreover, in string theory some compactification schemes are based on Kähler manifolds , e.g Calabi-Yau manifolds is a compact Kähler manifold with vanishing first Chern class, that is also Ricci flat. We will mainly focus on the role of Kähler spaces in Hamiltonian mechanics. Kähler manifolds have three mutually compatible structures, namely complex structure, Riemannian structure and symplectic structure. Kähler manifold is a special case of general Hermitian manifold ($g_{a\bar{b}}du^a d\bar{u}^b$). For the general Hermitian metric one can define a 2-form

$$\omega = \imath g_{a\bar{b}} du^a \wedge d\bar{u}^b \tag{1}$$

This two form is called a fundamental form of Hermitian manifold. Hermitian manifold is called Kähler manifold if the two form maintained above is symplectic, that is closed and non-degenerate. This requirement allow us to write Kähler metric as a second derivative of some function $K = K(u, \bar{u})$ which is called Kähler potential.

$$g_{a\bar{b}} = \frac{\partial^2 K(u, \bar{u})}{\partial u^a \partial \bar{u}^b} \tag{2}$$

This function is not defined uniquely, a holomorphic or antiholomorphic functions can be added to it.

$$K(z, \bar{z}) \to K(z, \bar{z}) + U(z) + \bar{U}(\bar{z}) \tag{3}$$

Due to natural symplectic structure of Kähler manifolds they can be equipped with Poisson brackets

$$\{f(u, \bar{u}), g(u, \bar{u})\} = \imath g^{a\bar{b}} \left(\frac{\partial f}{\partial u^a} \frac{\partial g}{\partial \bar{u}^b} - \frac{\partial f}{\partial \bar{u}^b} \frac{\partial g}{\partial u^a} \right), \qquad g^{\bar{a}b} g_{b\bar{c}} = \delta^a_c, \quad g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K. \tag{4}$$

Let us consider the Lagrangian

$$\mathcal{L} = \frac{\imath}{2}(\dot{z}^a \partial_a K - \dot{\bar{z}}^a \partial_{\bar{a}} K) - \mathcal{H}(z, \bar{z}), \tag{5}$$

where $K(z, \bar{z})$ is Kähler potential defining the metrics $g_{a\bar{b}} dz^a d\bar{z}^b$, $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$. It describes Hamiltonian mechanics with Hamiltonian $\mathcal{H}$ and symplectic structure $\frac{\imath}{2} g_{a\bar{b}} dz^a \wedge d\bar{z}^b$.

Since the symplectic structure relates functions (Hamiltonian) and vector fields (Hamiltonian vector fields), we can introduce functions that generate a Killing vector fields. The isometries of Kähler structure are *Hamiltonian holomorphic vector fields*,

$$\mathbf{V}_{(\mu)} = V^a_{(\mu)}(u)\frac{\partial}{\partial u^a} + \bar{V}^a_{(\mu)}(\bar{u})\frac{\partial}{\partial \bar{u}^a}, \quad V^a(u) = \imath g^{\bar{a}b}\partial_{\bar{a}} h_{(\mu)}(u,\bar{u}) \ : \{h_\mu, h_\nu\} = C^\lambda_{\mu\nu} h_\lambda \tag{6}$$

$$V_{(\mu)} = \{h_\mu, \quad \}. \tag{7}$$

and the Hamiltonian functions (generators) h_μ are known as the "Killing potentials". Using Killing Equations one can derive restrictions on Killing potentials. They should be real and they have to satisfy the following equation

$$\frac{\partial^2 h_\mu}{\partial z^a \partial z^b} - \Gamma^c_{ab}\frac{\partial h_\mu}{\partial z^c} = 0. \tag{8}$$

2.1 (Pseudo)Euclidean space as a Kähler manifold

The well known metric of N+1-dimensional complex Euclidean space is (from now on Einstein sum notation is assumed)

$$ds^2 = u_i\bar{u}_i \equiv u\bar{u}, \quad g_{i\bar{j}} = \delta_{i\bar{j}} \quad i = 0, ..., N. \tag{9}$$

for the further analogy with pseudo-Euclidean space let us keep in mind the following way of writing the metric above,

$$ds^2 = u_0\bar{u}_0 + u_a\bar{u}_a \equiv u\bar{u}, \quad a = 1, ..., N. \tag{10}$$

It is easy to see that the following Kähler potential will bring to this metric:

$$K(u,\bar{u}) = u\bar{u}, \quad \omega = -\imath du\wedge d\bar{u}, \quad \{u_i, \bar{u}_j\} = \imath\delta_{i\bar{j}}, \quad i,j = 0, ..., N. \tag{11}$$

The metric of N+1-dimensional pseudo-Euclidean space $\mathbf{C}^{1.N}$ is

$$ds^2 = u_0\bar{u}_0 - u_a\bar{u}_a, \quad g_{i\bar{j}} = \gamma_{i\bar{j}}, \quad i,j = 0, a \quad a = 1, ..., N. \tag{12}$$

$$\gamma = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots \end{pmatrix}. \tag{13}$$

The Kähler potential that brings to this metric and Kähler structure are as follows

$$K(u,\bar{u}) = u_0\bar{u}_0 - u_a\bar{u}_a, \quad \omega = -\imath\gamma_{i\bar{j}} du_i\wedge d\bar{u}_j, \quad \{u_i, \bar{u}_j\} = i\gamma_{i\bar{j}}, \quad i,j = 0, ..., N. \tag{14}$$

2.2 Compact and non-compact complex projective spaces

The N-dimensional complex projective space $\mathbf{CP}^N$, compact and non-compact, can be obtained by reduction from Euclidean complex space $\mathbf{C}^{N+1}$ and pseudo-Euclidean complex space $\mathbf{C}^{1.N}$ respectively. We can get to the compact $\mathbf{CP}^N$ from the Euclidean complex space by imposing the constraint $u_i\bar{u}_j = u_0\bar{u}_0 + u_a\bar{u}_a = 1$. In the same way, for the non-compact $\mathbf{CP}^N$ we need to have a constraint $u_0\bar{u}_0 - u_a\bar{u}_a = 1$. The coordinates of $\mathbf{CP}^N$ for both cases (compact and non-compact) are defined as follows

$$z_a = \frac{u_a}{u_0}, \quad \left(\bar{z}_a = \frac{\bar{u}_a}{\bar{u}_0}\right) \tag{15}$$

For the compact and non-compact complex projective spaces $\mathbf{CP}^N$, K̈hler structure and respective metrics are defined by expressions (the upper sign presents the compact case, the lower one non-compact)

$$K = \pm\log(1 \pm z\bar{z}), \quad g_{a\bar{b}}dz^a d\bar{z}^b = \frac{dzd\bar{z}}{1 \pm z\bar{z}} \mp \frac{(\bar{z}dz)(zd\bar{z})}{(1 \pm z\bar{z})^2}, \quad g^{\bar{a}b} = (1 \pm z\bar{z})(\delta^{\bar{a}b} \pm \bar{z}^a z^b). \tag{16}$$

The isometry algebra of $\mathbf{CP}^N$ is $su(N+1)/su(N.1)$. It is defined by the Killing potentials

$$h_{a\bar{b}} = \frac{z_a\bar{z}_b \mp \delta_{a\bar{b}}}{1 \pm z\bar{z}}, \qquad h_a = \frac{2z_a}{1 \pm z\bar{z}}, \qquad h_{\bar{a}} = \frac{2\bar{z}_a}{1 \pm z\bar{z}} \tag{17}$$

The generators $h_{a\bar{b}}$ form $u(N)$ symmetry algebra, and all together- $su(N+1)$ algebra for the upper sign, and $su(N.1)$ for the lower one, with

$$\{h_a, h_b\} = 0, \quad \{h_a, h_{\bar{b}}\} = -4\imath h_{a\bar{b}}, \quad \{h_a, h_{b\bar{c}}\} = \pm i\left(h_a\delta_{b\bar{c}} + h_b\delta_{a\bar{c}}\right), \tag{18}$$

$$\{h_{a\bar{b}}, h_{c\bar{d}}\} = \mp\imath(h_{a\bar{d}}\delta_{c\bar{b}} - h_{c\bar{b}}\delta_{a\bar{d}}). \tag{19}$$

Remark. This system can be interpreted as a "large mass limit" of the particle on Kähler manifold moving in the constant magnetic field $B_{ab} = \imath B g_{ab}$. Indeed, consider first order Lagrangian

$$\mathcal{L} = \pi_a\dot{z}^a + \bar{\pi}_a\dot{\bar{z}} - \frac{\imath B}{2}(\dot{z}^a\partial_a K - \dot{\bar{z}}^a\partial_{\bar{a}}K) - \frac{1}{\mu}g^{\bar{a}b}\bar{\pi}_a\pi_b - V(z, \bar{z}), \tag{20}$$

It describes particle with mass μ moving on Kähler space with metric $g_{a\bar{b}}dz^a d\bar{z}^b$ in the presence of potential field $\mathcal{H}$ and magnetic field with "vector" potential (1-form) $\mathcal{A} = \frac{\imath B}{2}(dz^a\partial_a K - d\bar{z}^a\partial_{\bar{a}}K)$. Its strength is equal to $d\mathcal{A} = \imath B g_{a\bar{b}}dz^a \wedge d\bar{z}^b$. Respectively, the magnitude of this magnetic field is equal to B, i.e. the magnetic field is constant. Hence in the "large mass limit" $\mu \to \infty$ the Lagrangian results in.

Most known example of integrable system with Kähler phase space is (compactified) Ruijsenaars-Schneider system, or the so-called "relativistic Calogero model" was suggested in [19]. It has trigonometric, elliptic variants and hyperbolic variants. Hyperbolic variant is dual to rational Calogero model. Trigonometric Ruijsenaars-Schneider model is periodic both on coordinates and momenta, and therefore, has a compact phase space. Pompactifying momenta in the trigonometric Ruijsenaars model one get that phase space to complex projective space $\mathbf{CP}^N$. Tthe explicit mapping of that phase space to complex projective space $\mathbf{CP}^N$, as well as formulation of the system in terms of action-angle variables was done by Ruijeersnaars in [20]. Then Van Dejen and Vinet quantised these system by the use of geometric quantization method [21], while Gorbe and Feher extended this result to elliptic Ruijsenaars-Schneider systems [22].

3 Non-compact complex projective space as a Klein model of Lobachevsky space

3.1 One dimensional case

In the one dimensional case the reduction is performed from the 2-dimensional pseudo-Euclidean space $\mathbf{C}^{1.1}$

$$ds^2 = u_0\bar{u}_0 - u_1\bar{u}_1, \quad K = u_0\bar{u}_0 - u_1\bar{u}_1, \quad \{u_i, \bar{u}_j\} = \imath\gamma_{i\bar{j}} \tag{21}$$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

The coordinates in $\mathbf{CP}^1$ are

$$z = \frac{u_1}{u_0}, \quad \left(\bar{z} = \frac{\bar{u}_1}{\bar{u}_0}\right) \tag{23}$$

The Kähler potential that defines Poincare model is

$$K = -\log(1 - z\bar{z}). \tag{24}$$

With the Poisson brackets for the one dimensional Poincare model

$$\{z, \bar{z}\} = -\imath(1 - z\bar{z})^2 \tag{25}$$

The mapping of conformal mechanics to Lobachevsky space(non-compact complex projective space) is convenient to perform by the use of Klein model. Let us perform following coordinate transformations in $\mathbf{C}^{1.1}$

$$u_0 = \frac{v_1 + \imath v_0}{\sqrt{2}}, \quad u_1 = \frac{v_1 - \imath v_0}{\sqrt{2}}. \tag{26}$$

it will bring us to to

$$ds^2 = \imath(v_0\bar{v}_1 - v_1\bar{v}_0), \quad K = \imath(v_0\bar{v}_1 - v_1\bar{v}_0), \quad \{v_i, \bar{v}_j\} = \imath\Gamma_{i\bar{j}} \tag{27}$$

$$\Gamma = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \tag{28}$$

Now we can make a reduction of $\mathbf{C}^{1.1}$, and parameterize $\mathbf{C}^{1.1}$ in the following way

$$w = \frac{v_1}{v_0}, \quad \left(\bar{w} = \frac{\bar{v}_1}{\bar{v}_0}\right). \tag{29}$$

This will bring to the Klein model of Lobachevsky space.It defined by Kähler potential

$$K = -\log \imath(\bar{w} - w). \tag{30}$$

The Poisson brackets in Klein model in one dimensional case are defined by

$$\{w, \bar{w}\} = i(\bar{w} - w)^2. \tag{31}$$

It can be obtained from Poincare model by transformation

$$z = \frac{w - \imath}{w + \imath}. \tag{32}$$

In this notation Killing potentials read

$$h_{1\bar{1}} = \frac{w\bar{w} + 1}{\imath(\bar{w} - w)}, \quad h_1 = \frac{(w - \imath)(\bar{w} - \imath)}{\imath(\bar{w} - w)}, \quad h_{\bar{1}} = \frac{(w + \imath)(\bar{w} + \imath)}{\imath(\bar{w} - w)} \tag{33}$$

Then we introduce the canonical phase space variables by the expression

$$w = \frac{p}{x} + \imath\frac{g}{x^2}. \tag{34}$$

In these terms the standard conformal mechanics reads

$$H = 2g\frac{w\bar{w}}{\imath(\bar{w} - w)} = p^2 + \frac{g^2}{x^2}, \quad K = 2g\frac{1}{\imath(\bar{w} - w)} = x^2, \quad D = 2g\frac{\bar{w} + w}{\imath(\bar{w} - w)} = px. \tag{35}$$

$$\{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K \tag{36}$$

3.2 N-dimensional case

Now, let us try to find the analog of this structure in the N-dimensional Klein model. In this case, as we have said above, for $\mathbf{C}^{1.N}$ space the metric is $ds^2 = u_0\bar{u}_0 - u_a\bar{u}_a$, the Kähler potential $K = u_0\bar{u}_0 - u_a\bar{u}_a$ and the Poisson brackets

$$\{u_i, \bar{u}_j\} = \imath\gamma_{i,\bar{j}}, \quad i, j = 0, N, \alpha \quad \alpha = 1, ..., N-1. \tag{37}$$

Here we have changed some labeling of the rows and columns of the matrix γ, namely we replace 1st row and 1st column by Nth ones and the next indexes are just shifted by 1 $(2, ..., N \to 1, ..., N-1)$.

$$\gamma = \left(\begin{array}{cc|ccc} 1 & 0 & & & \\ 0 & -1 & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right) \tag{38}$$

The coordinates in Poincare model of $\mathbf{CP}^N$ where defined as follows

$$z_a = \frac{u_a}{u_0}. \quad (39)$$

Performing an analogous transformations of coordinates as in one dimensional case

$$u_0 = \frac{v_N + \imath v_0}{\sqrt{2}}, \quad u_N = \frac{v_N - \imath v_0}{\sqrt{2}}, \quad u_\alpha = v_\alpha, \quad \alpha = 1, ..., N-1. \quad (40)$$

we will arrive to the metric $ds^2 = \imath(v_0\bar{v}_N - v_N\bar{v}_0 - v_\alpha\bar{v}_\alpha)$, Kähler potential $K = \imath(v_0\bar{v}_N - v_N\bar{v}_0 - v_\alpha\bar{v}_\alpha)$ and Poisson brackets

$$\{v_i, \bar{v}_j\} = \imath\Gamma_{i\bar{j}}, \quad i, j = 0, N, \alpha \quad \alpha = 1, ..., N-1. \quad (41)$$

$$\Gamma = \left(\begin{array}{cc|ccc} 0 & -i & & & \\ i & 0 & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right) \quad (42)$$

Then we define the coordinates in N-dimensional Klein model of Lobachevsky space by the following

$$w = \frac{v_N}{v_0}, \quad \tilde{z}_\alpha = \frac{v_\alpha}{v_0}, \quad \alpha = 1, ..., N-1. \quad (43)$$

N-dimensional Klein model is defined by the Kähler potential

$$K = -\log\left[\imath(\bar{w} - w) - \tilde{z}_\alpha\bar{\tilde{z}}_\alpha\right], \qquad \alpha = 1, \ldots N-1. \quad (44)$$

It can be obtained from the N-dimensional Poincare one by transformation

$$z_N = \frac{w - \imath}{w + \imath}, \quad z_\alpha = \sqrt{2}\frac{\tilde{z}_\alpha}{w + \imath}. \quad (45)$$

Poisson brackets in Klein model are defined by the relations

$$\{w, \bar{w}\} = \imath[(\bar{w} - w) - z_\gamma\bar{z}_\gamma](\bar{w} - w), \quad \{w, \bar{z}_\alpha\} = [\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma]\bar{z}_\alpha, \quad (46)$$

$$\{z_\alpha, \bar{z}_\beta\} = -\imath[\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma]\delta^{\bar{\alpha}\beta}. \quad (47)$$

The Killing potentials of respective Kähler structure are defined by the expressions (instead of $\tilde{z}_\alpha$ we use the old notation z_α)

$$h_{N\bar{N}} = \frac{w\bar{w} + 1}{\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma}, \quad h_{\alpha\bar{N}} = \frac{1}{\sqrt{2}}\frac{z_\alpha(\bar{w} + \imath)}{\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma}, \quad h_{\alpha\bar{\beta}} = \frac{z_\alpha\bar{z}_\beta}{\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma} \quad (48)$$

$$h_N = \frac{(w-\imath)(\bar w-\imath)}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \quad h_\alpha = \sqrt{2}\frac{z_\alpha(\bar w-\imath)}{\imath(\bar w - w) - z_\gamma \bar z_\gamma} \quad . \tag{49}$$

These potentials form $su(1.N)$ algebra, which in given notation reads the same as (6) with $a = N, \alpha$.

For our purposes it is more convenient to use the linear combination of above generators, vis

$$H_0 = \frac{w\bar w}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \quad K_0 = \frac{1}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \quad D_0 = \frac{w+\bar w}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \tag{50}$$

$$H_{\alpha\bar N} = \frac{z_\alpha \bar w}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \quad H_\alpha = \frac{z_\alpha}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \quad H_{\alpha\bar\beta} = \frac{z_\alpha \bar z_\beta}{\imath(\bar w - w) - z_\gamma \bar z_\gamma}, \tag{51}$$

In these terms the $SU(1.N)$ algebra reads

$$\{H_\alpha, H_\beta\} = \{H_{\alpha\bar N}, H_{\beta\bar N}\} = 0, \tag{52}$$

$$\{H_0, K_0\} = -D_0, \quad \{H_0, D_0\} = -2H_0, \quad \{K_0, D_0\} = -2K_0, \tag{53}$$

$$\{H_0, H_\alpha\} = H_{\alpha\bar N}, \quad \{H_0, H_{\alpha\bar N}\} = 0, \quad \{H_0, H_{\alpha\bar\beta}\} = 0 \tag{54}$$

$$\{K_0, H_\alpha\} = 0, \quad \{K_0, H_{\alpha\bar N}\} = -H_\alpha, \quad \{K_0, H_{\alpha\bar\beta}\} = 0, \tag{55}$$

$$\{D_0, H_\alpha\} = H_\alpha, \quad \{D_0, H_{\alpha\bar N}\} = -H_{\alpha\bar N}, \quad \{D_0, H_{\alpha\bar\beta}\} = 0, \tag{56}$$

$$\{H_\alpha, H_{\bar\beta}\} = -\imath K_0 \delta_{\alpha\bar\beta}, \quad \{H_\alpha, H_{N\bar\beta}\} = H_{\alpha\bar\beta} - \frac{1}{2}(H_0 + K_0 + \imath D_0)\delta_{\alpha\bar\beta}, \tag{57}$$

$$\{H_{\alpha\bar N}, H_{N\bar\beta}\} = -\imath H_0 \delta_{\alpha\bar\beta}, \tag{58}$$

$$\{H_\alpha, H_{\beta\bar\gamma}\} = -\imath H_\beta \delta_{\alpha\bar\gamma}, \quad \{H_{\alpha\bar N}, H_{\beta\bar\gamma}\} = -\imath H_{\beta\bar N}\delta_{\alpha\bar\gamma}, \tag{59}$$

$$\{H_{\alpha\bar\beta}, H_{\gamma\bar\delta}\} = \imath(H_{\alpha\bar\delta}\delta_{\gamma\bar\beta} - H_{\gamma\bar\beta}\delta_{\alpha\bar\delta}). \tag{60}$$

Notice that we have the following generators $H_{\alpha\bar N}, H_{N\bar\alpha}, H_{\alpha\bar\beta}$ that commute with Hamiltonian.

With these expressions at hands we can construct the variety of conformal mechanical systems defined by Killing potentials.

To transit to canonical coordinates let us write down the symplectic one-form

$$\mathcal{A} = \frac{1}{2}\frac{dw + d\bar w + \imath(z_\alpha d\bar z_\alpha - \bar z_\alpha dz_\alpha)}{\imath(\bar w - w) - z_\beta \bar z_\beta}. \tag{61}$$

Then we transit from the complex coordinates to the real ones $w = x + \imath y$, $z_\alpha = q_\alpha e^{\imath\varphi_\alpha}$ and require $\mathcal{A} = p_x dx + \pi_\alpha d\varphi_\alpha$. This yields the following canonical coordinates and momenta:

$$p_x = \frac{1}{2y - q^2}, \quad \pi_\alpha = \frac{q_\alpha^2}{2y - q^2} \quad \Leftrightarrow \quad q_\alpha = \sqrt{\frac{\pi_\alpha}{p_x}}, \quad y = \frac{\pi + 1}{2p_x}, \quad \text{with} \quad \pi = \sum \pi_\alpha \tag{62}$$

Thus, the complex coordinates express via canonical ones as follows

$$w = x + \imath\frac{\pi+1}{2p_x}, \qquad z_\alpha = \sqrt{\frac{\pi_\alpha}{p_x}}\mathrm{e}^{\imath\varphi_\alpha}. \tag{63}$$

This prompt us to perform the trivial canonical transformation $(p_x, x) \to (x, -p_x)$ and rewrite above expression in a more convenient form

$$w = -p_x + \imath\frac{\pi+1}{2x}, \qquad z_\alpha = \sqrt{\frac{\pi_\alpha}{x}}\mathrm{e}^{\imath\varphi_\alpha}. \tag{64}$$

Notice that

$$\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma = \frac{1}{x}. \tag{65}$$

For complete analogy with one-dimensional case we perform further canonical transformation $(p_x, x) \to (p_r, r)$ with

$$(p_x, x) \to (p_r, r) \;: \qquad p_x = \frac{p_r}{r}, \quad x = r^2. \tag{66}$$

Then we get

$$w = \frac{p_r}{r} + \imath\frac{\pi+1}{2r^2}, \qquad z_\alpha = \frac{\sqrt{\pi_\alpha}}{r}\mathrm{e}^{\imath\varphi_\alpha} \tag{67}$$

and

$$\imath(\bar{w} - w) - z_\gamma\bar{z}_\gamma = \frac{1}{r^2}. \tag{68}$$

Thus, the Killing potentials reads

$$H_0 = p_r^2 + \frac{(\pi+1)^2}{4r^2}, \quad K_0 = r^2, \quad D_0 = 2p_r r \tag{69}$$

$$H_{\alpha\bar{N}} = \left(p_r - \imath\frac{\pi+1}{2r}\right)\sqrt{\pi_\alpha}\mathrm{e}^{\imath\varphi_\alpha}, \quad H_\alpha = r\sqrt{\pi_\alpha}\mathrm{e}^{\imath\varphi_\alpha}, \quad H_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{\imath\left(\varphi_\alpha-\varphi_\beta\right)} \tag{70}$$

So, it describes the conformal mechanics with separated "radial" and "angular" sectors. Assuming that $(\pi_\alpha, \phi_\alpha)$ are action-angle variables we get that this set of systems includes rational Calogero models, as well as genic maximally superintegrable deformations of oscillator and Coulomb systems [7]. Let us mention the recent paper [3] where these system were in terms of phase space $\widetilde{\mathbf{CP}}^1 \times \mathcal{M}$, where $\mathcal{M}$ is the phase space of "angular" sector. Present description is completely geometrical, and allows to use geometric quantization etc. Moreover, in these terms we can construct the $\mathcal{N}$-extended superconformal extension of these systems, as it was done in [6] for one-dimensional case.

4 Holomorphic factorization

In [3] were discussed some aspects of "holomorphic factorization" and some hidden symmetries of deformed oscillator and Coulomb systems were found by guessing the view of the constants of motion. Having the algebra (52-60) in our hands we can do it easier due to explicit views of conserved quantities. Here we give an oscillator example by writing its hidden symmetries, i.e. "Fradkin Tensor" in terms of our generators. Let us write the algebra $so(1.2)$ of conformal generator in the following way

$$H_0 = \frac{p_r^2}{2} + \frac{(\pi+1)^2}{8r^2}, \quad K_0 = \frac{r^2}{2}, \quad D_0 = p_r r \tag{71}$$

and define the Casimir element

$$\mathcal{I} = 2H_0K_0 - \frac{1}{2}D_0^2 : \quad \{\mathcal{I}, H_0\} = \{\mathcal{I}, D_0\} = \{\mathcal{I}, K_0\} = 0 \tag{72}$$

It is obviously a constant of motion independent on radial coordinate and momentum, and thus could be expressed via appropriate angular coordinates ϕ_α and canonically conjugate momenta π_α which are independent on radial ones: $\mathcal{I} = \mathcal{I}(\phi_\alpha, \pi_\alpha)$.In these terms the generators of conformal algebra read

$$H_0 = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}, \quad K_0 = \frac{r^2}{2}, \quad D_0 = p_r r \tag{73}$$

Hence, such a separation of angular and radial parts could be defined for any system with dynamical conformal symmetry, and for those with additional potentials be function of conformal boost K. In particular, such a generalized oscillator and Coulomb systems assume adding of potential

$$V_{osc} = \frac{\omega r^2}{2}, \quad V_{Coul} = -\frac{\gamma}{r} \tag{74}$$

so that their Hamiltonian takes the form

$$H_{osc/Coul} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + V_{osc/Coul} \tag{75}$$

Analyzing these deformations in terms of actionangle variables, it was found that they are superintagrable iff the spherical part has the form

$$\mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1} k_\alpha I_\alpha + c\right)^2 \tag{76}$$

with c be arbitrary constant and k_a be rational numbers.

Define the action-angle variables

$$\mathcal{I} = \mathcal{I}(I_\alpha), \quad \omega = I_\alpha \wedge \phi_\alpha, \quad \phi_\alpha \in [0, 2\pi) \tag{77}$$

Comparing this with the results we got in previous section (68) we see that

$$\mathcal{I} = \frac{(\pi+1)^2}{8} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1} \frac{1}{2}\pi_\alpha + \frac{1}{2}\right)^2 \tag{78}$$

so it coincides with the formula (73) when $k_\alpha = \frac{1}{2}$ and $c = \frac{1}{2}$. Note that the action variables I_α are the conserved quantities $H_{\alpha\bar{\alpha}}$ in (50) or (67). Obviously we have more conserved quantities which are $H_{\alpha\bar{N}}$.The algebra of these integrals of motion is given by (51)-(57).

Now let as consider as an example the Harmonic oscillator given by Hamiltonian

$$H_{osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \omega r^2 = H_0 + \omega K \tag{79}$$

In this case an additional conserved quantities are

$$M_\alpha = H^2_{\alpha\bar{N}} + \omega H^2_\alpha, \quad \{H_{osc}, M_\alpha\} = 0 \tag{80}$$

where H_α-s are given by (50) or (67).

5 Conclusion

We are inviting non-compact complex projective space as the Klein model of Lobachevsky space to be Kähler phase space first in on dimensional case. Then we map the conformal mechanics to it, with generators H_0, K_0, D_0 forming $so(1.2)$ algebra. In N dimensional case we choose pretty handy coordinates that brings as to conclusion that the symmetries of the phase space are the same as the symmetries of the system. Moreover, It seems that very large family of integrable systems may be described in this way, namely all the deformations of N-dimensional oscillator, Coulomb systems etc.. This description is purely geometrical so it can be quantized by the geometric quantization. Also, we are planning to build some super generalization, which will be some unified description of a large family of supersymmetric systems.

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