Yerevan State University

Master Thesis

Faculty of Physics

The Universal Eigenvalue of the Second Casimir Operator on the k-th Cartan Power of the X_2 Representation

Author:
Maneh Avetisyan

Supervisor:
Professor Ruben Mkrtchyan

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1 Introduction

The universal formulae for simple Lie algebras were first derived by P. Vogel in his Universal Lie Algebra [1, 2]. The main aim was to derive the most general weight system for Vassiliev's finite knot invariants. This program met difficulties, however, as a byproduct there appeared the uniform parameterization of simple Lie algebras by the values of Casimir operators on three representations, appearing in decomposition of the symmetric square of the adjoint representations:

$$S^2\mathfrak{g} = 1 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma) \tag{1}$$

One denotes the value of the second Casimir operator on the adjoint representation $\mathfrak{g}$ as $2t$, and parameterizes the values of the same operator on representations in (1) as $4t-2\alpha, 4t-2\beta, 4t-2\gamma$ correspondingly (hence notation of representations in (1)). It appears that $\alpha+\beta+\gamma = t$. The values of the parameters for all simple Lie algebras are given in the table 1, and in the table 2 in another form. According to the definitions, the entire theory is invariant with respect to rescaling of the parameters (which corresponds to rescaling of invariant scalar product in algebra), and with respect to the permutation of the universal (=Vogel's) parameters α, β, γ. So, effectively they belong to a projective plane, which is factorized w.r.t. its homogeneous coordinates, and is called Vogel's plane.

Table 1: Vogel's parameters for simple Lie algebras

Root system	Lie algebra	α	β	γ	$t = h^\vee$
A_n	$\mathfrak{sl}_{n+1}$	-2	2	$(n+1)$	$n+1$
B_n	$\mathfrak{so}_{2n+1}$	-2	4	$2n-3$	$2n-1$
C_n	$\mathfrak{sp}_{2n}$	-2	1	$n+2$	$n+1$
D_n	$\mathfrak{so}_{2n}$	-2	4	$2n-4$	$2n-2$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$	4
F_4	$\mathfrak{f}_4$	-2	5	6	9
E_6	$\mathfrak{e}_6$	-2	6	8	12
E_7	$\mathfrak{e}_7$	-2	8	12	18
E_8	$\mathfrak{e}_8$	-2	12	20	30

Table 2: Vogel's parameters for simple Lie algebras: lines

Algebra/Parameters	α	β	γ	t	Line
$\mathfrak{sl}_N$	-2	2	N	N	$\alpha+\beta=0$
$\mathfrak{so}_N$	-2	4	$N-4$	$N-2$	$2\alpha+\beta=0$
$\mathfrak{sp}_N$	-2	1	$N/2+2$	$N/2+1$	$\alpha+2\beta=0$
$Exc(n)$	-2	$2n+4$	$n+4$	$3n+6$	$\gamma=2(\alpha+\beta)$

For the exceptional line $n = -2/3, 0, 1, 2, 4, 8$ for $\mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, respectively.

As an example of application of this parametrization universal formulae [1, 3], for dimensions of representations from (1) are presented below:

$$
\begin{aligned}
\dim \mathfrak{g} &= \frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma} && (2) \\
\dim Y_2(\alpha) &= \frac{(2t-3\alpha)\ (\beta-2t)\ (\gamma-2t)\ t\ (\beta+t)\ (\gamma+t)}{\alpha^2\ (\alpha-\beta)\ \beta\ (\alpha-\gamma)\ \gamma} && (3)
\end{aligned}
$$

and other two (3) representations which are obtained by permutations of the parameters. These are typical universal formulae for dimensions: ratios of products of linear homogeneous functions of universal parameters.

There are a number of universal formulae for different objects in the theory and applications of simple Lie algebras. E.g. Vogel [1] found complete decomposition of third power of the adjoint representation in terms of Universal Lie Algebra, defined by himself, and universal dimension formulae for all representations involved. Landsberg and Manivel [3] present a method which allows derivation of certain universal dimension formulae for simple Lie algebras and derive those for Cartan powers of the adjoint, $Y_2(.)$, and their Cartan products.Sergeev, Veselov and Mkrtchyan derived [4] a universal formula for generating function for the eigenvalues of higher Casimir operators on the adjoint representation.

In subsequent works applications to physics were developed, particularly the universality of the partition function of Chern-Simons theory on a sphere [13, 6, 7], and its connection with q-dimension of $k\Lambda_0$ representation of affine Kac-Moody algebras [8] were shown, the universal knot polynomials for 2- and 3-strand torus knots [9, 10, 11, 12] were calculated.

The antisymmetric square of the adjoint representation of semisimple Lie algebras is known to be decomposed in the following universal form:

$$\Lambda^2 g = g \oplus X_2$$

First of all, let's suppose that for each algebra the square of the long root is equal to 2. This corresponds to the set of Vogel's parameters with $\alpha = -2$. Having this normalization in mind, we calculate the eigenvalues of the Casimir operator on X_2 and it's Cartan powers. The eigenvalue of the Casimir operator on an irrep with λ highest weight is equal to $(\lambda + 2\rho, \lambda)$. The corresponding Dynkin diagrams and highest weights are given in the Figure 1. and Table 3 respectively.

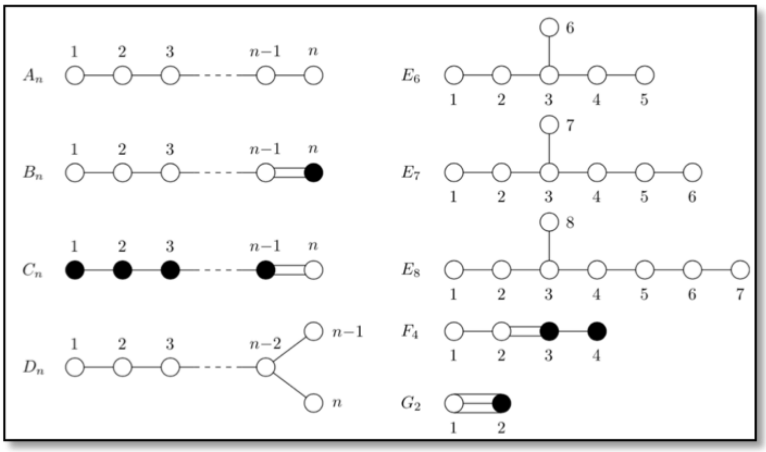


Figure 1: Dynkin diagrams

Table 3: $X_2(\alpha, \beta, \gamma)$

$A_n, n \geq 3$	$(2\omega_1 + \omega_{n-1}) \oplus (\omega_2 + 2\omega_N)$
$B_n, n \geq 4$	$\omega_1 + \omega_3$
$C_n, n \geq 3$	$2\omega_1 + \omega_2$
$D_n, n \geq 5$	$\omega_1 + \omega_3$
G_2	$3\omega_1$
F_4	ω_2
E_6	ω_3
E_7	ω_2
E_8	ω_6

2 Calculation of Casimir's Eigenvalues for X_2 Representations

Taking into account the above mentioned data, we carry out the direct calculations for X_2 first.

A_N

$$\alpha = -2, \beta = 2, \gamma = N+1, \lambda_1 = 2\omega_1 + \omega_{N-1}, \lambda_2 = \omega_2 + 2\omega_N,$$

For $\lambda_1 = 2\omega_1 + \omega_{N-1}$ case

$$C = (2\omega_1 + \omega_{N-1}, 2\omega_1 + \omega_{N-1}) + 2(\omega_1 + \cdots + \omega_N, 2\omega_1 + \omega_{N-1}) =$$
$$\frac{6N+6}{N+1} + \frac{2}{N+1}\left(N(N+1) + (N-1)(N+1)\right) = 6 + 2N + 2N - 2 = 4N + 4$$

For $\lambda_2 = \omega_2 + 2\omega_N$ irrep the Casimir eigenvalue coincides with the one calculated above, so the eigenvalue on the direct sum of these two irreps will be C.

B_N

$$\alpha = -2, \beta = 4, \gamma = 2N-3, \lambda = \omega_1 + \omega_3$$

$$C = (\omega_1 + \omega_3, \omega_1 + \omega_3) + 2(\omega_1 + \cdots + \omega_N, \omega_1 + \omega_3)$$
$$= F_{11} + F_{31} + F_{13} + F_{33} + 2\left((F_{11} + F_{12} + \cdots + F_{1N}) + F_{31} + \cdots + F_{3N}\right) =$$
$$6 + 2N - 1 + 9 + 6(N-3) = 2N + 14 + 6N - 18 = 8N - 4$$

Where $F_{i,k} = (\omega_i, \omega_k)$ (MetricTensor[] in LieART).

C_N

$$\alpha = -2, \beta = 1, \gamma = N+2, \lambda = 2\omega_1 + \omega_2$$

$$C = (2\omega_1 + \omega_2, 2\omega_1 + \omega_2) + 2(\omega_1 + \cdots + \omega_N, 2\omega_1 + \omega_2)$$
$$= 4F_{11} + 4F_{12} + F_{22} + F_{33} + 2\left(2(F_{11} + F_{12} + \cdots + F_{1N}) + F_{21} + \cdots + F_{2N}\right)$$
$$= 2 + 2 + 1 + 2(N + 1/2 + N - 1) = 5 + 2N + 1 + 2N - 2 = 4N + 4$$

D_N

$$\alpha = -2, \beta = 4, \gamma = 2N-4, \lambda = \omega_1 + \omega_3$$

$C=(\omega_1+\omega_3,\omega_1+\omega_3)+2(\omega_1+\cdots+\omega_N,\omega_1+\omega_3)=6+2(N-1+6+3N-12)=8N-8$

G_2

$$\alpha=-2, \beta=10/3, \gamma=8/3, \lambda=3\omega_1$$

$C=(3\omega_1,3\omega_1)+2(\omega_1+\omega_2,3\omega_1)=9F_{11}+2\left(3F_{11}+3F_{12}\right)=6+4+6=16$

F_4

$$\alpha=-2, \beta=5, \gamma=6, \lambda=\omega_2$$

$C=(\omega_2,\omega_2)+2(\omega_1+\omega_2+\omega_3+\omega_4,\omega_2)=6+30=36$

E_6

$$\alpha=-2, \beta=6, \gamma=8, \lambda=\omega_3$$

$C=F_{33}+2\sum_{k=1}^{6}F_{3k}=6+2(6+15)=48$

E_7

$$\alpha=-2, \beta=8, \gamma=12, \lambda=\omega_2$$

$C=F_{22}+2\sum_{k=1}^{6}F_{2k}=6+2(9+14+10)=72$

E_8

$$\alpha=-2, \beta=12, \gamma=20, \lambda=\omega_6$$

$C=F_{66}+2\sum_{k=1}^{6}F_{6k}=6+2\cdot 57=120$

It can be easily noticed, that for each of the algebra the obtained value can be expressed as $4t=4(\alpha+\beta+\gamma)$.

In the work of M.Cohen and R. de Man ([14]) the Casimir eigenvalues on each of the irrep appearing in the decomposition of up to 4th square of the adjoint representation for the exceptional Lie algebras have been computed. So we can check the correspondence between our formula and that values for the exceptional algebras. If we scale the Casimir eigenvalue to be equal to 1 on the adjoint representation, as it is done in [14], our formula will be $C_c=C/2t=4t/2t=2$, which coincides with the value, presented in that work.

3 Calculation of Casimir's Eigenvalues for Cartan Powers of X_2 Representations

Now we turn to the Cartan power case. The highest weight is now $k\lambda$. Substituting these new highest weights in the expressions for the Casimir eigenvalue, one obtains the expressions shown in the following table:

Table 4: Casimir Eigenvalues

$A_n, n \geq 3$	$6k^2 + k(4N-2)$
$B_n, n \geq 4$	$6k^2 + k(8N-10)$
$C_n, n \geq 3$	$5k^2 + k(4N-1)$
$D_n, n \geq 5$	$6k^2 + k(8N-14)$
G_2	$6k^2 + 10k$
F_4	$6k^2 + 30k$
E_6	$6k^2 + 42k$
E_7	$6k^2 + 66k$
E_8	$6k^2 + 114k$

One can easily check, that for each of the cases (except for the C_n) the eigenvalue can be expressed as

$$C = -3\alpha k^2 + (4t + 3\alpha)k = 3\alpha(k - k^2) + 4tk.$$

Notice, that for $k = 1$ case $C = 4t$, as expected.
In the recent work of M.Avetisyan and R.Mkrtchyan ([15]) a universal formula for dimensions of the k-th Cartan power of the X_2 representation has been obtained. A notable quality of the $X_2(k, \alpha, \beta, \gamma)$ formula is that for the parameters, corresponding to the C_n algebra it gives 0 for any $k \geq 2$. So, the situation is such that for the C_n algebra the $X_2(k, \alpha, \beta, \gamma)$ gives 0, while the universal Casimir eigenvalue on the corresponding representation does not.
A similar case regarding A_2 algebra is worth recalling. The universal decomposition of the symmetric square of the adjoint representation writes as follows:

$$S^2 g = \mathbb{1} + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)$$

The $Y_2(\beta)$ for A_2 is 0, whilst the Casimir eigenvalue on the same representation is $4t - 2\beta$. In either case discussed above the product of the universal Casimir and universal dimension is 0. These examples suggest an idea, that in the universal description (or, maybe, in general) of semisimple Lie algebras the second Casimir operators appear in product with the universal dimensions. In support of this idea we bring a formula, presented by Deligne in [16]:

$$Tr(C_2, [R]V) = \frac{1}{n!} \sum_{\sigma} \chi(\sigma) m(\sigma) (dimV)^{n(\sigma)-1} Tr(C_2, V)$$

where V is a representation of the algebra, R is a representation of the S_n group, $[R](V) := Hom_{S_n}(R, \otimes^n V)$, σ is an element of S_n, $\chi(\sigma)$ is the character on that element, $m(\sigma)$ is the sum of the squares of the lengths of cycles of σ, $n(\sigma)$ is the number of cycles of σ.
For the symmetric square of the adjoint we rewrite this formula explicitly:

$$1 \cdot C_2(\mathbb{1}) + Y_2(\alpha)C_2(Y_2(\alpha)) + Y_2(\beta)C_2(Y_2(\beta)) + Y_2(\gamma)C_2(Y_2(\gamma)) = (2+g)\cdot gC_2(g)$$

Substituting the corresponding universal formulae, one can check, that for A_2 algebra this formula is true.
Now we turn to the comparison of our universal expression with the values presented in [14]. For the $k=2$ case we rewrite our formula in the corresponding scaling.

$$C_c = \frac{3\alpha(k-k^2)+4tk}{2t} = \frac{-3\alpha+4t}{t} = \frac{6+4t}{t}.$$

In the following table the Casimir eigenvalues on X_2^2 (denoted as H in [14]) for the exceptional algebras and the values obtained by our formula are shown.

Table 5: Casimir on X_2^2

	a	$\gamma(H) = 4+6a$	t	C_c
A_1	1/2	7	2	7
A_2	1/3	6	3	6
G_2	1/4	11/2	4	11/2
D_4	1/6	5	6	5
F_4	1/9	14/3	9	14/3
E_6	1/12	9/2	12	9/2
E_7	1/18	13/3	18	13/3
E_8	1/30	21/5	30	21/5

Thus, we see that the Casimir eigenvalues coincide.

4 On $sp(-2n) = so(2n)$ duality

In ([13]) R.Mkrtchyan and A.Veselov have discussed the duality of Casimirs for $So(2n)$ and $Sp(2n)$ groups. Using the Perelomov and Popov ([17]) formula for the generating function for the Casimir spectra and parametrizing the Young diagrams in a different way ([13]), they have explicitly shown the $C_{Sp(2n)}(\lambda, z) = -C_{SO(-2n)}(\lambda', -z)$ duality for rectangular Young diagrams. Here, using the same ([17]) formula, we write the expressions for the corresponding eigenvalues of the second Casimir operator (C_2) for $so(2n)$ and $sp(2n)$ algebras, in the A, B parametrization, used in [13].

$so(2n)$

For $so(2n)$ the Casimir spectra writes as follows

$$C_{so(2n)}(z, A, B) = \sum_{p=0}^{\infty} C_{p_{so(2n)}} z^p = \frac{(1-zn)(2-z(4n-3))}{z(1-z(n-1))(2-z(4n-2))} \times$$
$$\prod_{i=0}^{k} \frac{1-z(-A_{k-i}+B_i+2n-1)}{1-z(A_{k-i}-B_i)} \times \prod_{i=1}^{k} \frac{1-z(A_{-i+k+1}-B_i)}{1-z(-A_{-i+k+1}+B_i+2n-1)}$$

After a proper expansion of $C_{so(2n)}(z, A, B)$ into series in the vicinity of the $z_0 = 0$ point, one can check, that the coefficient of z^2, i.e. $C_2 so(2n)$, can be expressed as follows:

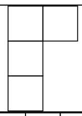
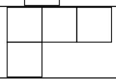
$$\begin{aligned} C_{2_{so(2n)}}(A,B) = \sum_{i=1}^{k} \big(&4nA_i(B_{-i+k+1} - B_{k-i}) + 2A_i^2(B_{k-i} - B_{-i+k+1}) + \\ &+ 2A_i(B_{k-i} - B_{-i+k+1}) + 2B_i^2(A_{-i+k+1} - A_{k-i})\big) - 4nA_0B_k + \\ &+ A_0^2(2B_k + 4B_0) + 2A_0(B_k - B_0) - B_0^2(2A_k + 4A_0) - \\ &- n(A_0 - B_0) + 2n\left(A_0^2 + B_0^2\right) + 2\left(B_0^3 - A_0^3\right) + 1/2(A_0 - B_0). \end{aligned}$$

$sp(2n)$

The Casimir spectra for this case is

$$C_{sp(2n)}(z,A, B) = \sum_{p=0}^{\infty} C_{p_{sp(2n)}} z^p = \frac{(1-zn)(2-z(4n+3))}{z(1-z(n+1))(2-z(4n+2))} \times$$
$$\prod_{i=0}^{k} \frac{1-z(B_{k-i}-A_i+2n+1)}{1-z(-B_{k-i}+A_i)} \times \prod_{i=1}^{k} \frac{1-z(-B_{-i+k+1}+A_i)}{1-z(B_{-i+k+1}-A_i+2n+1)}$$

Table 6: Comparison

Algebra	*Diagram*	A, B	$C_2(A,B)$	C_2
$so(2n)$		$A_1 = B_1 = 1, A_2 = 3, B_2 = 2$	$16n - 16$	$8n - 8$
$sp(2n)$		$A_1 = B_1 = 1, A_2 = 2, B_2 = 3$	$16n + 16$	$4n + 4$

And for $C_{2sp(2n)}$ one has

$$C_{2_{sp(2n)}}(A,B) = -\sum_{i=1}^{k}\big(- 4nB_i(A_{-i+k+1} - A_{k-i}) + 2A_i^2(B_{-i+k+1} - B_{k-i}) +$$
$$2B_i^2(A_{k-i} - A_{-i+k+1}) + +2B_i(A_{k-i} - A_{-i+k+1})\big) - 4nB_0A_k +$$
$$+ A_0^2(2B_k + 4B_0) - 2B_0(A_k - A_0) - B_0^2(2A_k + 4A_0) -$$
$$- n(B_0 - A_0) + 2n\left(A_0^2 + B_0^2\right) + 1/2(A_0 - B_0) - 2\left(A_0^3 - B_0^3\right).$$

Therefore, we have obtained formulae for second Casimir eigenvalues on irreps of $sp(2n)$ and $so(2n)$ algebras, corresponding to any Young diagram (any (A, B) set).
It can be checked, that

$$C_{2_{so(2n)}}(A, B) = -C_{2_{sp(-2n)}}(B, A)$$

i.e. the Casimir duality for the second Casimir holds for any Young diagram (for any A, B set). In particular, for X_2 one has the values, shown in the Table 4. It can be observed, that $C_{2_{so(2n)}} = 2C_{2_{sp(2n)}} = 1/2C_{2_{so(2n)}}(A, B)$, which indicates the difference of the definition of the Killing form in [13] [1].

In [15] it has been shown, that when permuting the Vogel parameters corresponding to the $so(2n)$ algebra in this way: $(\alpha, \beta, \gamma) \to (\beta, \alpha, \gamma)$, the $X_2(k)$ formula gives dimensions for some representations of the $sp(2n)$ algebra. More precisely, that permutation specifies a correspondence between $\lambda^{so(2n)} = k(\omega_1 + \omega_3)$ and $\lambda^{sp(2n)} = 2\omega_k + \omega_{2k}$ representations. One can notice, that the Young diagrams, associated with these representations are conjugate with each other. Indeed, in A, B parametrization the associated sets are

$$\lambda^{so(2n)} \leftrightarrow A_0 = B_0 = 0, A_1 = 1, B_1 = k, A_2 = 3, B_2 = 2k,$$

$$\lambda^{sp(2n)} \leftrightarrow A_0 = B_0 = 0, A_1 = k, B_1 = 1, A_2 = 2k, B_2 = 3.$$

Therefore, it is reasonable to check the Casimir duality for these representations. Substituting

[1] in [13] the Killing form is defined as $Tr(\hat{X}^a, \hat{X}^b)$ in the fundamental representation, while our normalization (so called Cartan-Killing normalization) corresponds to the Killing form, defined as $Tr(ad\hat{X}^a, ad\hat{X}^b)$, i.e. in the adjoint representation.

the corresponding (A, B) sets into the expressions for $C_2(A, B)$ written above, one gets

$$C_{2_{so(2n)}}(A, B) = 12k^2 + k(16n - 28),$$

$$C_{2_{sp(2n)}}(B, A) = -12k^2 + k(16n + 28) = -(12k^2 + k(16(-n) - 28) = -C_{2_{so(2n)}}(A, B).$$

So, the Casimir duality holds for representations, associated with the
$X_2(k, -2, 4, 2n-4) \leftrightarrow X_2(k, 4, -2, 2n-4)$ transformation of the $X_2(k, \alpha, \beta, \gamma)$ universal formula [15].

For the same representations in the Cartan-Killing normalization we have

$$C_{2_{so(2n)}} = 6k^2 + k(8n - 14),$$

$$C_{2_{sp(2n)}} = -3k^2 + k(4n + 7),$$

i.e.

$$C_{2_{so(2n)}}(\lambda) = -2C_{2_{sp(-2n)}}(\lambda'),$$

as expected.

5 Conclusion

In the present work new universal formulae for the second Casimir operator on X_2 representations and its Cartan powers are presented. Then, the correspondence of these formulae with previously known particular cases is checked.The correlation of obtained formulae with $Sp(2n) \leftrightarrow So(-2n)$ duality is also discussed.
In general, these formulae do assure the importance of the universality in mathematical physics. The further analysis and particularly the investigation of applications of these formulae in Chern-Simons theory is planned.

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