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Studies on Modified Gravity Theory

Thesis for acquiring the degree of candidate of physical-mathematical sciences

in division 01.04.02, Theoretical Physics

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Yerevan, 2020

Contents

Contents i

List of Figures iv

List of Tables v

Introduction 1

1 Newton Theorem and Dark Sector 11

1.1 Introduction . . . 11

1.2 On the common nature of dark matter and dark energy: galaxy groups . . . 13

1.2.1 Newton's theorem and General Relativity . . . 14

1.2.2 The Eq.(1.3), the cosmological constant and the dark matter . . . 15

1.2.3 Galaxy groups and Eq.(1.8) . . . 16

1.3 Gravity lens critical test for gravity constants and dark sector . . . 17

1.3.1 Newton's theorem and gravity lensing with Λ . . . 18

1.4 The invalidity of "negative mass" description of the dark sector . . . 21

1.5 Conclusions . . . 25

2 Cosmological Constant as a Fundamental Constant 28

2.1 Introduction . . . 28

2.2 Two fundamental constants of gravity unifying dark matter and dark energy 28

2.2.1 Newton's theorem and General Relativity 30

2.2.2 Group-theoretical analysis of Newton's theorem 32

2.2.3 Newton's theorem in d-dimensions 34

2.3 Cosmological constant as a fundamental constant 38

2.3.1 The 4 units and information 39

2.3.2 Information, time evolution and Weyl principle 41

2.3.3 CCC: rescaling of physical constants 43

2.4 Conclusions 45

3 Λ-gravity and Observations **48**

3.1 Introduction 48

3.2 The cosmological constant derived via galaxy groups and clusters 49

3.2.1 The Λ constant and galaxy systems 50

3.3 Dark matter extremal galaxies probing Lambda-gravity 57

3.3.1 Newton theorem and Λ-gravity 58

3.3.2 DM-extremal galaxies 60

3.3.3 DM deficient dwarf galaxies 61

3.3.4 DM-free dwarf spheroidals in the Local Group 62

3.3.5 Early type galaxies 64

3.4 H_0 tension: clue to common nature of dark sector? 64

3.4.1 Newton's theorem and Λ 66

3.4.2 Local and global Hubble flows with Λ 67

3.5 On the Lambda-evolution of galaxy clusters 73

3.5.1 Newton's theorem and Λ 74

3.5.2 Ricci curvature 75

3.5.3 Results 77

3.6 Conclusions 79

Conclusions **88**

Bibliography **90**

List of Figures

3.1 The Ricci curvature variation vs cosmological time for galaxy cluster parameters in Newtonian (green) and Λ-modified gravity (blue) regimes. 79

3.2 The difference of Ricci curvature values of curves in Fig.3.1, i.e. at Newtonian and Λ-modified gravity laws. 80

3.3 The same as in Fig.3.2 but for supercluster (Virgo) parameters. 81

3.4 The same as in Fig.3.3 but for $R = 18$ Mpc. 82

3.5 The same as in Fig.3.3 but for $R = 10$ Mpc. 83

List of Tables

1.1 Λ obtained using Eq.(1.8) for galaxy groups of the Hercules—Bootes region . 17

2.1 Background geometries for vacuum solutions 32

2.2 3D Background Geometries with O(3) as the stabilizer 33

2.3 Non-Relativistic Background Geometries 42

2.4 Spatial Geometries 42

3.1 Background geometries for vacuum solutions 51

3.2 SIS profile 52

3.3 NFW profile 52

3.4 Galaxy groups of Leo/Cancer region 53

3.5 Galaxy groups of Bootes strip region 54

3.6 CLASH Survey 55

3.7 Planck Clusters 55

3.8 LoCuSS Clusters 56

3.9 Error limit of Λ for CLASH Survey 57

3.10 Background geometries for GR solutions 59

3.11 Constraints on Λ for DM deficient dwarf galaxies 62

3.12 Constraints on Λ for 24 dwarf galaxies surrounding the Milky Way 63

3.13 Constraints on Λ for 20 dwarf spheroidals of LG 64

3.14 Constraints on Λ for early type galaxies 65

3.15 Critical distance for different objects 69

3.16 Distances of objects where the density is ρ_{loc} 71

Introduction

The modern era of cosmology started after Einstein's formulation of General Relativity (GR) in 1915 [1, 2, 3, 4]. The first attempts of describing the dynamics and evolution of Universe as a whole began by early works of Einstein himself. Einstein introduced a new constant called cosmological constant, Λ, in order to maintain the gravitational stability against the gravitational contraction of dust-like matter in the Universe [5]. Namely, he modified GR equations as follows

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1}$$

Although many years later Einstein himself concluded that introducing Λ was not needed, cosmological constant remained as the important and inevitable part of relativistic cosmology. Later, Russian mathematician Alexander Friedmann introduced some basic cosmological models based on GR [6, 7]. These models now known as Friedmann models, describe three different scenarios of our Universe i.e. contracting, expanding and accelerated expanding Universe, where the final state of our Universe depends on the amount of its matter content [8, 9, 10, 11, 12]. For all of these models it is assumed that Universe is spatially homogeneous and isotropic. Namely, the spacetime metric is written as

$$ds^2 = c^2 dt^2 - a^2(t)[\frac{1}{1-kr^2} dr^2 + r^2 \Omega^2] \tag{2}$$

where $a(t)$ is called the scale factor and k stands for spatial curvature and can be $\pm 1, 0$.

Consequently, the GR equations will be written as

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho}{3} \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{4}$$

In the above equations, $\dot{a} = \frac{da}{dt}$. Moreover, the so called Hubble constant is defined is defined as

$$H = \frac{\dot{a}}{a} \tag{5}$$

However, the models remained purely theoretical until Edwin Hubble and Georges Lemaître concluded that the galaxies outside our Local Group are moving away from us with velocities directly proportional to their distance [13, 8]. This conclusion changed the situation in favor of expanding models.

Later, the astronomical observations revealed a new feature of matter in the galaxies and clusters. Assuming that the galactic dynamics is governed by Newtonian theory, astrophysicists were able to find the mass of celestial objects based on virial theorem. However in 1933 Zwicky [85] noticed that the amount of dynamical mass in Coma cluster is roughly 10 times more than the luminous matter which can be detected via observations. In addition to this fact, later the flatness of rotation curves in spiral galaxies indicated that there must be some form of non-luminous matter [15]. These observational issues leaded scientists to a new concept in relativistic cosmology i.e. Dark Matter (DM) to be responsible for the missing matter in the Universe.

Nowadays, based on latest *Planck* satellite [16] data, it is concluded that the matter, including both baryonic and non-baryonic, contributes only 31.7 % to the content of our Universe. Among that, the 26.8 % is Cold Dark Matter (CDM). However, the nature of DM is still unknown. A number of models have been proposed for DM, among those primordial black holes, class of objects called massive compact halo objects (MACHOs). While for non-baryonic DM, they are hypothetical particles such as axions, sterile neutrinos, weakly interacting massive particles (WIMPs), gravitationally-interacting massive particles (GIMPs), supersymmetric particles etc. The search of DM particles is an active area of theoretical as well as observational astrophysics and experimental physics [18].

Considering DM in galaxies, it is commonly believed that DM halos envelope the whole galaxies. Since no direct observation has yet confirmed the exact shape of these halos, various DM halo density profiles have been introduced by astrophysicists. Here we list some of the commonly used profiles to model DM halos. The first introduced profile is called pseudo-isothermal profile which is written as [17]

$$\rho(r) = \rho_0 \left[1 + \left(\frac{r}{r_c}\right)^2\right]^{-1}, \tag{6}$$

where ρ_0 denotes the central density and r_c is the core radius. The other profile is Navarro-Frenk-White (NFW) [19] which has a wide range of applications for different galaxies. For this profile, we have

$$\rho(r) = \frac{\rho_{crit}\delta_c}{\left(\frac{r}{r_c}\right)\left(1 + \frac{r}{r_c}\right)^2}, \tag{7}$$

here ρ_{crit} is the critical density of the universe defined as

$$\rho_{crit} = \frac{3H^2}{8\pi G}, \tag{8}$$

in which H stands for Hubble constant. One of the important features of this profile is that, it depends directly on the cosmological parameter ρ_{crit}. The dimensionless parameter δ_c, which is called characteristic parameter, relates ρ_{crit} to ρ_0 for the halo under consideration i.e. $\rho_{crit} = \delta_c\rho_0$. Although NFW is widely used, other similar profiles are also considered for modeling the halos. Among them, Burkert [20],

$$\rho(r) = \frac{\rho_{crit}\delta_c}{\left(1+(\frac{r}{r_c})\right)\left(1+(\frac{r}{r_c})^2\right)} \tag{9}$$

and Moore profile [21]:

$$\rho(r) = \frac{\rho_{crit}\delta_c}{\left(\frac{r}{r_c}\right)^{\frac{3}{2}}\left(1+(\frac{r}{r_c})^{\frac{3}{2}}\right)} \tag{10}$$

are the most well-known profiles. All of the three profiles have been used to model DM halo of M 31 [22].

As mentioned above, the first indirect observation of DM was related to Virial Theorem

$$\sigma^2 = \frac{GM}{R}, \tag{11}$$

where σ, M and R stand for velocity dispersion, virial mass and virial radius respectively. Considering the above mentioned profiles for DM halos, the velocity dispersions read as (see [23] for more details)

$$\sigma^2_{NFW} = 4\pi G\rho_c\frac{r_c^3}{r}(ln\big(1+\frac{r}{r_c}\big) - \frac{\frac{r}{r_c}}{(1+\frac{r}{r_c})}), \tag{12}$$

$$\sigma^2_{Moore} = \frac{8}{3}\pi G\rho_c\frac{r_c^3}{r}(ln(1+(\frac{r}{r_c})^{\frac{3}{2}})), \tag{13}$$

$$\sigma^2_{Burkert} = 2\pi G\rho_c\frac{r_c^3}{r}((ln\big(1+\frac{r}{r_c}\big)\sqrt{1+\big(\frac{r}{r_c}\big)^2}) - \arctan(\frac{r}{r_c})). \tag{14}$$

Another widely used model is the Einasto profile which is written as [24]

$$\rho(r) = \rho_e e^{\left(-d_n\left((\frac{r}{r_e})^{\frac{1}{n}}-1\right)\right)} \tag{15}$$

Constructed based on Einasto model [25], d_n is a function of n such that ρ_e is the density at the radius r_e that defines a volume containing half of the total mass.

As mentioned above, the Planck satellite provides us the information on the content of our Universe, whence 68.3 % of the Universe is in the form of Dark Energy (DE) which causes the accelerated expansion of the Universe. The rest is mainly due to contribution of matter where the dominant part is in the form of DM.

Currently DE observationally favors the properties of the cosmological constant Λ. Then, by considering Λ in the GR equations, we will have

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3}, \tag{16}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{17}$$

According to ΛCDM cosmology, almost 96% of our Universe content is due to Dark Sector (DS) which is made up of DE and DM. Note, that besides ΛCDM cosmology there are also other alternative theories which try to explain the available observational data.

The underlying principle behind these theories is quite simple. In order to explain the observational results, physicists modify the original theories of Newtonian gravity and GR by introducing new parameters. For relativistic theories, such modifications date back to early works of Brans and Dicke where it is assumed that Newton's universal gravitational constant G is not actually a constant and a new scalar field ϕ is introduced instead of G. For such theories the action is written as

$$S = \frac{c^4}{16\pi}\int \left(\phi R - \frac{\omega}{\phi}\partial_a\phi\partial^a\phi\right)\sqrt{-g}d^4x, \tag{18}$$

where ω is a dimensionless constant known as Dicke coupling constant [26]. Besides that, it is also possible to modify GR by introducing some higher order terms in Einstein-Hilbert action

$$S = \frac{c^4}{16\pi G} \int R\sqrt{-g}\, d^4x, \tag{19}$$

where R is called Ricci scalar. This approach is generally called $f(R)$ theories of gravity [75, 58]. All these modified theories should reduce to ordinary GR at specific regimes.

On the other hand, Modified Newtonian Dynamics (MOND) [29, 30, 31] is one of the well known candidates for explaining the missing matter in galaxies to revisit the DM problem. In MOND theory, Newton's second law is modified as

$$F = ma\mu(\frac{a}{a_0}), \tag{20}$$

where $\mu(\frac{a}{a_0})$ is called the "extrapolating function" and depends on the new parameter $a_0 \approx 1.2 \times 10^{-12} ms^{-2}$. Thus, although for $a_0 << a$ Newton's second law remains valid, in the so-called "deep-MOND" regime, it is modified to

$$F = m\frac{a^2}{a_0}, \quad (a << a_0). \tag{21}$$

Consequetly, for the circular motion of an object with mass m around another object with mass M, one can write

$$\frac{GmM}{r^2} = m\frac{(\frac{v^2}{r})^2}{a_0}. \tag{22}$$

Considering the above relation, it turns out that we can explain the so-called "flat rotation curve" of galaxies without any need of dark matter. Meantime, it is possible to interpret Eq.(22) as the modification of Newtonian gravity, leaving Newton's law intact. In such case, the modified gravitational potential will be written as

$$\Phi = (GMa_0)^{\frac{1}{2}} lnr. \tag{23}$$

Thus, for DM paradigm, MOND proposes a modification of gravity according to Eq.(22) without any further need to DM. The question whether we have to accept MOND or DM should be solved with observations. To this aim, one of the analysis is studying the rotation curve of galaxies. One of the best cases for such study is galaxy M31. However, rotation curves based on MOND yields to totally different values seeming to be inconsistent with observations [32]. It should be noticed that, in addition to M31, there are several other galaxies i.e. NGC 2841, NGC 3198 [33], M33 [32, 34], UGC 4173 [35], UGC 6787 and UGC 11852 [36], for which rotation curves are poorly fitted with MOND.

In this thesis, we follow the same path of modification of gravity to describe the DS. Our fundamental principle to do such modification is the Newton theorem which is a mathematical theorem about the equivalency of gravitational field of sphere and that of a point-mass located in its center. Namely, in 1687, Newton published his book "Principia Mathematica" [37] where for the first time he introduced a universal law for gravitational effects based on some mathematical formulation. The formula is quite simple and familiar

$$F = -G\frac{m_1 m_2}{r^2}. \tag{24}$$

The interesting feature of this formula is the proportionality of gravitational force to inverse squared of the distance. However, it should be noticed here that the main reason that Newton started to think about the so called "Inverse Squared Law" was not some sort of intuitions but a profound foundational argument which he proved mathematically 20 years after starting to think about it. The argument is related to the fact that, in the real world we are always dealing with extended objects. However, Eq.(24), has been written for two point-like objects. So the above formulation won't be applicable to the physical reality, unless we prove that it is possible to consider extended spherical objects with mass M, as some point-like objects with all mass M concentrated at that point. This was the main challenging problem for Newton. However, he succeeded to prove that when the force is written as $\frac{C}{r^2}$, where C is a constant, it is possible to do such consideration. Although, Newton's gravitational theory fulfills the above mentioned principle, it is not the most general form. Namely, it has been proved in [38] that the most general form of the force is obtained by solving the following equation

$$\frac{r^2}{2}F''(r) + rF'(r) - F(r) = 0, \tag{25}$$

and has solutions in the form of

$$F(r) = \frac{C_1}{r^2} + C_2 r, \tag{26}$$

in which C_1 and C_2 are two constants. It is obvious that the first term in Eq.(26) is the ordinary Newtonian gravitational force, whereas the additional linear term can be considered as analogous to cosmological constant.

This thesis consists of three chapters.

In the first chapter we study the nature of both terms of Eq.(26). By considering the analogy between the second term C_2, and the cosmological constant, we show that it is possible to describe the DS via a unified theory of gravity [39]. Furthermore, we present a prospective test to detect the possible deviations of GR and the modified theory of gravity. Finally, based on the results we discuss the refutation of some the current alternative theories of gravity.

In the second chapter following the results of first chapter, we use the group-theoretical methods to present a mathematical proof for identification of C_2 as the cosmological constant. Then, we study the consequences of Λ considered as one of the fundamental constants of Nature.

In the third chapter we revisit the current main observational results according to our modified theory of gravity. Namely, we obtain the possible error limits over the cosmological constant, based on the data of astrophysical configurations. Then, we generalize these results to the recently discovered extreme galaxies regarding the nature of DM. After that, based on our modified theory of gravity we propose a possible solution to the so-called H_0 tension. Finally, we check the role of cosmological constant in the stability of N-body gravitating systems.

The main results of the thesis are formulated in Conclusions.

The results of this thesis are published in [40], [41], [42], [43], [44], [45], [46], [47], [48].

Acknowledgments

I would like to express my heartfelt gratitude to my supervisor, Professor V.G. Gurzadyan, who introduced me the field of modern Cosmology. His knowledge, overly enthusiasm and integral view on the research and his advice for providing ‘only high-quality work and never less’ have made a deep impression on me.

I am very grateful to my co-authors, especially to Dr A.A. Kocharyan.

My thanks are due to colleagues of the Alikhanian National Laboratory for their valuable advice and friendly help. The extensive discussions at various phases of my work have been very helpful.

Chapter 1

Newton Theorem and Dark Sector

1.1 Introduction

According to historians of science, Newton postponed the publication of his theory of gravity for 20 years. The reason was that, originally his theory had been written for point particles. However, in order to apply his theory of gravity to celestial dynamics, he had to prove that "The gravitational field of a sphere acting on external objects is equivalent to that of a point mass located at its center". The above statement which is called "Newton Theorem", ensured Newton to consider the spherical objects as points. Considering the "Newton Theorem" as a mathematical theorem to find the most general form of the function in which "Sphere=point" equivalency is correct, it has been shown in [38], that besides the ordinary Newtonian term, there is another additional linear term such that:

$$F(r) = -\frac{Gm_1m_2}{r^2} + Cr, \tag{1.1}$$

On the other hand, by considering the McCrea-Milne cosmological model, it can be shown that the behaviour of linear term will be similar to the one by cosmological constant. Namely, regarding Eq.(1.1), the equations of motion of a test particle on the surface of expanding sphere will be written as:

$$\ddot{a} = -\frac{4\pi a^3 \rho}{3a^2} + Ca \tag{1.2}$$

where a is the radius of the sphere and ρ is its density. This equation, will coincide the Friedmman equations in the GR written for FLRW metric in the of presence non-zero Λ.

However, besides the above analogy, the linear term in Eq.(1.1) has another important feature. Namely, it produces a non-force free field inside the sphere. But, this oscillator term does not change $O(4)$ symmetry of the Newtonian field. Moreover, it correponds to the only field possesing the Newotnian one: the closing of orbits at any negative value energy i.e. the coincidence of the period of variation of the value of radius vector with that of its revolution by 2π.

Although the contribution of the second term is so small that it excludes its checking by experimental methods, its contribution can be evaluated from the analysis of the structure of galaxy clusters, their halos etc. This is the topic which we will discuss with more details in next section. The results discussed in this chapter have been published in [41] and [45].

1.2 On the common nature of dark matter and dark energy: galaxy groups

The observational indications for the DM and DE have stipulated the active development of variety of models, including those of modifications of Newton's gravity and of GR. Various principles are taken as bases in those models, such as scalar-tensor, f(R) theories, MOND (see [49, 50, 51, 52] and refs therein) with different motivations and with the natural aim to satisfy the observational data or to suggest verification tests for future observations.

Below we reconsider a principle which is in the very roots of Newton's gravity, i.e. the theorem that a sphere is acting gravitationally as a point mass situated in its center. That theorem had enabled Newton to attribute the gravitational law r^{-2} to the motion of planets which are definitely extended spheres and are not point masses. Later the Newton's gravity became a key element GR, acting as its weak field limit and enabling to correspond the predicted effects with the observed ones. That concerns, however, the Einstein equations without the cosmological term and Einstein's motivation [5] for introducing the cosmological constant was the static Universe. In the approach below we show that the cosmological term appears in Einstein's equations from the above mentioned theorem proved in [54].

As shown in [38], the most general function satisfying that theorem, besides the usual r^{-2} term, contains also another term with a cosmological constant.

Taking that modified Newton's law with a cosmological constant as the weak field limit of GR one arrives to a modification of GR containing the cosmological term naturally! We then show that this approach both to Newton's gravity theory and to GR enables one to link two observational facts, i.e. the dark matter in the galaxies and the cosmological constant. The dark matter then appears as an observational contribution of repulsive gravity at large scales, i.e. in galactic halos and clusters of galaxies.

1.2.1 Newton's theorem and General Relativity

As shown in [38] the most general function for the force satisfying Newton's theorem i.e. the condition for the sphere to attract as a point of the sphere's mass and situated in its center has the form

$$f(r) = Ar^{-2} + \Lambda r, \tag{1.3}$$

as the solution of equation

$$\frac{r^2}{2} f''(r) + r f'(r) - f(r) = 0, \tag{1.4}$$

where A and Λ are constants.

The second term corresponds the cosmological constant term if one turns to the Newtonian form of the Friedmann cosmological equation [38]. Eq. (1.3), however, within the context of the shell theorem defines a force-free field only in the center of a shell, but preserves the O(3) symmetry.

Turning to the GR, instead of the usual Newtonian limit for its weak field approximation [65] now metric tensor components g_{00} and g_{rr} will be modified and one will have the metric for the point mass as

$$ds^2 = (1 - 2Ar^{-1} - \Lambda r^2/3)c^2 dt^2 + (1 - 2Ar^{-1} - \Lambda r^2/3)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.5}$$

The principal fact here is in the following. The Einstein equations without cosmological term are considered to have the usual Newtonian limit in weak-field approximation, while the Einstein equations with cosmological term will formally violate that limit, e.g. according to Weinberg "..Λ *must be very small so as not to interfere with the successes of Newton's theory of gravitation.*" (Chapter 7.1, [65]). We now see that via Eq.(1.3) that violation is removed. Eq.(1.5) is the covariant metric having its weak-field limit Eq.(1.3) [54].

Instead, we see that Eq.(1.3) ensures the weak-field limit for Einstein equations with cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.6}$$

The adoption of the Newton's "sphere = point mass" theorem and hence of the Eq. (1.3) will readily lead to renormalization in various predictions of GR. Although such modification of GR coincides with its original form of Einstein's equations, there is drastic difference in the motivations.

Here the starting point is the Newton's gravity and his sphere-point theorem and Λ is appearing in Einstein's equations readily and not as an extra term added by hand to fulfill a static universe concept. Namely, if Newton might have found the general function Eq. (1.3), then Einstein initially would have written GR equations with Λ i.e. with the link of O(3) and the Lorentz group.

Of course, the Λ-term in Eq.(1.5) has been considered previously (Schwarzschild – de Sitter metric), however, the Newton theorem's approach described here, as we will see below leads to insights on the common nature of the dark matter and dark energy (cosmological constant).

1.2.2 The Eq.(1.3), the cosmological constant and the dark matter

If the Einstein equations with the cosmological constant have the Newtonian limit Eq.(1.3), then one will have a link e.g. to the two currently adopted principal observational facts on the dark energy and the dark matter. Indeed, while the cosmological constant in the Einstein equations is considered to describe the acceleration of the Universe, the Newtonian potential and its modifications are attributed e.g. to the observational indications for the dark matter in the galaxies.

The value of the cosmological constant is deduced in several ways, the one indicated by the Planck data [16] is

$$\Lambda \simeq 1.1 \times 10^{-52} m^{-2}. \tag{1.7}$$

Regarding the dark matter in the galaxies, it is shown that the parameters of the halos determine the late disk and early spheroidal structures of the galaxies [55]. Then, for virialized (for "oscillator" term $\propto R^k$ with $k = 2$, see Eq. (10.7) in [56]) structures we have

$$\Lambda = \frac{3\sigma^2}{2c^2R^2} \simeq 3 \times 10^{-52} (\frac{\sigma}{50\, kms^{-1}})^2 (\frac{R}{300\, kpc})^{-2} m^{-2}, \tag{1.8}$$

normalized to the velocity dispersion at a given radius of halo.

Numerically Λs in Eq.(1.7) and Eq.(1.8) are close and this fact can be interpreted as follows. The positive cosmological constant corresponds to the accelerating Universe and hence to negative pressure and to repulsion as evidence of vacuum energy [85]. The crucial point regarding the dark matter is that Eq.(1.3) defines non-force-free field within the sphere except its center, increasing from center, thus mimicking increase of the central mass. Thus, within this interpretation the dark matter is a gravitating mass with force (1) and revealing its repulsion at large scales e.g. in galactic halos. The effective increase of the central attracting mass will support the effect of “flat rotation curves”, although, obviously, the description of given observational rotation curves will need extensive modeling and numerical simulations for the input parameters of the disk and halo. For the analysis of the dark matter problem within $f(R)$ theories see [58].

1.2.3 Galaxy groups and Eq.(1.8)

We now test this approach on Λ-nature of dark matter using the data by Karachentsev et al [59] for a sample of 17 galaxy groups of Hercules—Bootes region which (the data) include the *rms* galactic velocities σ and the harmonic average radii R_h of the groups. Then, using Eq.(1.8) we obtain the Λ as presented in the Table 1.1; the galaxy groups are denoted by

Table 1.1: Λ obtained using Eq.(1.8) for galaxy groups of the Hercules—Bootes region

Galaxy group	$\sigma(km/s^{-1})$	$R_h(kpc)$	$\Lambda(m^{-2})$
NGC4736	50	338	3.84E-52
NGC4866	58	168	2.09E-51
NGC5005	114	224	4.55E-51
NGC5117	27	424	7.12E-53
NGC5353	195	455	3.23E-51
NGC5375	47	66	8.91E-51
NGC5582	106	93	2.28E-50
NGC5600	81	275	1.52E-51
UGC9389	45	204	8.55E-52
PGC55227	14	17	1.19E-50
NGC5961	63	86	9.43E-51
NGC5962	97	60	4.59E-50
NGC5970	92	141	7.48E-51
UGC10043	67	65	1.87E-50
NGC6181	53	196	1.28E-51
UGC10445	23	230	1.76E-52
NGC6574	15	70	8.07E-52
Average			8.24E-51
St.deviation			1.15E-50

the name of the brightest galaxy (first column).

The correspondence of values of Λ in Table 1 to those in Eq.(1.7) and Eq.(1.8) is visible.

1.3 Gravity lens critical test for gravity constants and dark sector

High precision tests of the General Relativity (GR) and of its weak- and strong-field limits, along with of modified gravity models have gained new content with establishing of the dominating dark sector of the Universe. The models proposed to describe the dark energy and dark matter observations are being constrained by means of those tests. The weak-field General Relativity is continuously being tested in ever increasing accuracy [64, 65] including the frame-dragging measurements with laser ranging satellites [60].

The use of the strong gravity lens ESO 325-G004 [66] demonstrates the efficiency of the lens studies to constrain the weak-field GR in the intergalactic scales. Given the increasing accuracy and statistics of the gravity lensing observations (e.g. [67]), those studies can become laboratories to test GR modifications along with the structure and dynamics of the dark matter dominated configurations.

The idea of our approach to this problem is the following. In recent studies [40, 39] we have shown that the Newton theorem on the identity of the gravity of a sphere and of a point mass located in its center provides a natural way for the weak-field modification of GR. Consequently, the constant Λ appears both in the cosmological solutions describing the accelerated expansion and at galactic halo scales (cf. [62]) as weak-field GR thus linking dark energy and galactic scales.

Here we predict a critical value for the parameter of the parametrized post-Newtonian (PPN) formalism [65] $\gamma = 0.998$ (normalized to given lens mass and light impact distance) which if observed at gravity lenses with proper significance will, for the first time, reveal the weak-field modification of GR differing from conventional Newtonian limit. It is important that the critical γ does not depend on any open parameter but involves only the fundamental constants and well measured quantities.

1.3.1 Newton's theorem and gravity lensing with Λ

In [40] it is shown that from the Newton's theorem the weak-field limit of GR follows involving the cosmological constant Λ, so that the metric tensor components have the form

$$g_{00} = 1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}; \quad g_{rr} = (1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3})^{-1}. \tag{1.9}$$

While this metric (Schwarzschild – de Sitter) was obviously known before (e.g. [68, 54]), with the Λ introduced by Einstein to get a static cosmological model, the motivation based on the Newton's theorem essentially differs from that and ensures the Newton's potential with Λ term as the weak-field limit of GR.

Namely, as follows from the consideration of the general function for the force satisfying Newton's theorem on the identity of sphere's gravity and that of a point of the sphere's mass situated in its center. That function besides the r^{-2} term contains also a second term [38]

$$f(r) = Ar^{-2} + \Lambda r. \tag{1.10}$$

When the modified Newtonian law (for the potential) is taken as weak-field limit of GR, one has the constant Λ as a second gravity constant along with the classical Newtonian constant G [40]. Thus, the second constant acts as cosmological constant in the solutions of Einstein equations, at the same time enters the low-energy limit of GR.

Turning to the strong lensing and following [66], it is convenient to deal with the parameter representing the ratio

$$\gamma = \Psi/\Phi \tag{1.11}$$

of the two perturbing functions - of the Newtonian potential Φ and curvature potential Ψ - entering the weak field metric

$$ds^2 = (1 + 2\Phi)c^2dt^2 - (1 - 2\Psi)dr^2 - r^2d\Omega^2. \tag{1.12}$$

For GR $\gamma = 1$, obviously.

In the weak-field limit of GR following from the Newton's theorem Eq.(3.33) [39, 40], we obtain

$$\Phi = \Psi = -\frac{GM}{rc^2} - \frac{\Lambda r^2}{6}. \tag{1.13}$$

To find out γ the authors of [66] introduced the lense dynamical mass

$$M_{dyn} = \frac{1+\gamma}{2} M, \tag{1.14}$$

where consequently the light bending angle in the vicinity of a mass of surface density Σ is achieved is:

$$\alpha = \frac{2G(1+\gamma)}{c^2} \int d^2x' \Sigma(x') \frac{x - x'}{|x - x'|^2}. \tag{1.15}$$

Upon the analysis of the observational data of ESO 325-G004 they finally obtain the value $\gamma \simeq 0.97 \pm 0.09$.

Now, within our approach of the weak field metric Eq.(3.33) the bending angle will be [69]

$$\alpha = \frac{4GM}{c^2 r} - \frac{\Lambda c^2 r^3}{6GM}. \tag{1.16}$$

Here, important difference arises between Newtonian and Λ-modified case. Namely, the authors in [66] have obtained

$$\alpha = 2(1+\gamma)\frac{GM}{c^2 r}. \tag{1.17}$$

Comparing this with Eq.(8), we get for the γ-parameter

$$\gamma = 1 - \frac{\Lambda c^4 r^4}{12 G^2 M^2}. \tag{1.18}$$

Inserting the current value of the cosmological constant e.g. that of the Planck satellite [16] $\Lambda = 1.11 \times 10^{-52} m^{-2}$ we obtain

$$\gamma_{cr} = 1 - 0.002 = 0.998 \; (\frac{M}{1.5\; 10^{11}\; M_\odot})^{-2} (\frac{r}{2\, kpc})^4, \tag{1.19}$$

where the data of [66, 70] for ESO 325-G004, i.e. the Einstein radius and the estimated mass inside that radius, were used for the normalization. Obviously, the normalization and hence the precise numerical value of γ_{cr} will vary from one lens to another, and the principal point is the existence of a well defined parameter enabling to reveal the weak-field limit of GR with the lensing effect.

The other key point of using ESO 325-G004 is that, within other available observational means it is technically impossible to detect the contribution of Λ term in the gravitational lensing. For example, for the same effect within the Solar System the value of γ_{cr} will be approximately (1- 9.6×10^{-25}). Note, that a limitation on γ also will emerge due to the proper motion of the lens for it affects the measured value of γ as shown in [71].

1.4 The invalidity of "negative mass" description of the dark sector

The paper [75] aimed to describe the nature of dark sector in a unified model, where the author's essential new notion is the "negative mass". That approach is already criticized in [76]. Since, the "negative mass" concept was publicized in media as a new theory to explain the "missing" part of cosmos (e.g. https://phys.org/news/2018-12-universe-theory-percent-cosmos.html), we will outline the shortcomings of that approach.

First, the author claims that by considering the negative cosmological constant Λ in GR equations as the "negative mass" from one side, and in the modified Newtonian equation from the other side, it is possible to solve both the dark matter and dark energy problems, simultaneously. To do that, the author writes the equations of GR in the form

$$G_{\mu\nu} = \frac{8\pi G}{c^4}(T^{+}_{\mu\nu} + T^{-}_{\mu\nu} + C_{\mu\nu}), \tag{1.20}$$

where according to the author "*... the conventional $\Lambda g_{\mu\nu}$ term is now represented by a combination of $T^{-}_{\mu\nu}$ (an exotic matter term) and $C_{\mu\nu}$ (a modified gravity term)*". So, in order to explain the accelerated expansion of the universe, the author replaces the notion of positive cosmological constant Λ with a new term in the right hand side of the field equations above. This is severe misunderstanding of GR since the author's statement entirely rejects the very essence of Einstein equations as explained in any textbook! (See e.g. the strong critics by t'Hooft [77] of such attempts to interpret the Einstein equations). That very essence of Einstein equations has been triumphantly confirmed by now experimentally, it is enough to mention the discovery of gravitational waves [78] and the frame dragging measurements [60], testing the strong and weak-field GR, respectively.

Second, in the Newtonian regime, the author obtains the following relation for the gravitational force exerted on a particle of mass m

$$\frac{mv^2}{r} = \frac{GMm}{r^2} - \frac{\Lambda c^2 rm}{3}. \tag{1.21}$$

To ensure the "negative mass" as the "missing matter" in galaxies i.e. to ensure flat rotation curves, the author claims that the negative cosmological constant has to be

$$\Lambda \simeq -0.3 \times 10^{-52} m^{-2}.$$

This contradicts a variety of observational data including the SN surveys, CMB, lensing etc when interpreted via cosmological constant and GR. Namely, the observations indicate the positive Λ and fix its value $\Lambda \simeq +1.1 \times 10^{-52} m^{-2}$, as of the Planck data [16]. Furthermore, with the negative value of Λ there will be in contradiction with the gravitational potential

$$\Phi = -\frac{GM}{r} - \frac{\Lambda c^2 r^2}{6}, \tag{1.22}$$

which is equally important on introducing the notion of "missing matter". In fact, it should be noticed that there are two main analyses for the existence of dark matter i.e. of the "virial theorem" and the "flat rotation curves". Historically, the concept of "missing matter" was introduced by Zwicky [85] using the virial theorem. Decades later this concept was used to describe the "flatness" of galactic rotation curves [15]. In contrast to virial analysis where the gravitational potential is used, to study the rotation curves of galaxies one uses the gravitational force and there is an important difference between those analyses. Namely, if one considers the additional Λ-term in Eq.(1.22), then the Λ-term will correspond to "mass" term in Eq.(1.22) with density

$$\rho = \frac{\Lambda c^2}{8\pi G}. \tag{1.23}$$

While for Eq.(1.21), in view of $F_g = mE_g = -m\nabla\Phi$, the sign of Λ-term changes and it should be interpreted as "negative mass" of

$$\rho = -\frac{\Lambda c^2}{4\pi G}. \tag{1.24}$$

In [75] to solve the dark matter problem the author has focused only on the "rotation curve" analysis based on Eq.(1.21) and has claimed that considering the "negative mass" defined via the negative Λ it is possible to have flat rotation curves for galaxies. In fact, although it will act to flatten the rotation curves, the negative Λ term cannot explain the other main analysis i.e. of the "virial theorem" [79]. Indeed, in such case the Λ-term in the gravitational potential will become positive

$$\Phi = -\frac{GM}{r} + \frac{|\Lambda| c^2 r^2}{6}, \tag{1.25}$$

where $|\Lambda|$ is the absolute value of the cosmological constant. Consequently, for the virial theorem the "negative mass" not only will not help but even will decrease the amount of necessary mass to have a virialized configuration.

Thus, what the author calls a "unifying theory" for dark energy and dark matter via negative value of Λ actually is just a matter of reparametrization of Λ, c and G based on Eq. (1.21) (cf. [64]), which has its own theoretical and observational constraints, as it is clear comparing Eq.(1.23) with Eq.(1.24) and recalling that both the sign and the value of Λ are confirmed by various observations.

Third, the author claims that Anti de Sitter (AdS) universe undergoes a cycle of expansion and contraction with a timescale of

$$\sqrt{\frac{-3\pi^2}{\Lambda c^2}},$$

where the author states that "*...a universe with a negative cosmological constant would eventually recollapse due to this extra attractive force*". However, negative cosmological constant is neither necessary nor sufficient condition to have a Big Bang/Big Crunch scenario and AdS spacetime is totally unrelated to a collapsing universe and the negative Λ cannot ensure any so-called "cyclic cosmology", see e.g. [80].

Finally, let us mention that there is still a way to incorporate the cosmological constant within Newtonian gravity without rejecting basic principles and observations, namely via Newton theorem on the equivalency of attractions of a sphere and a point mass. The latter leads to the modification of the weak-field limit of GR as follows [40, 43]

$$g_{00} = 1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}; \quad g_{rr} = (1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3})^{-1}. \tag{1.26}$$

Then, the cosmological constant not only enters naturally in the gravity equations, both Newtonian and GR, but also enables one to describe via the observed value of the cosmological constant the common nature of the dark sector, without any need to change the sign of Λ.

1.5 Conclusions

In this chapter we demonstrated a natural way for the appearance of the cosmological constant in Einstein equations. This follows while adopting for the weak field approximation of General Relativity the Eq.(1.3) as the general function satisfying the Newton's theorem of 1687, that the gravitating sphere acts as a point mass located in its center. This drastically differs from Einstein's original motivation for the introduction of the cosmological constant.

This approach enables to draw the following conclusions:

(a) the nature of the dark matter in galaxies is in the gravity;

(b) the dark energy (cosmological constant) and the dark matter are of the common nature;

(c) the dark matter is the signature of repulsive gravity of the ordinary matter determined by Λ constant and dominating at large scales, i.e. larger than of galactic halos.

Thus, the key constituents of the dark universe are naturally linked here. The repulsive gravity nature of the dark matter is responsible for the observed increase of the mass-to-luminosity M/L ratio while moving from the scales of galaxies to those of galaxy clusters. The non-force-free shell (galactic halo) determines the internal structures of galaxies (disks), as indicate the observations. The observational data of a sample of galaxy groups [59], i.e. systems containing 3 or more galaxies, are shown to confirm Eq.(1.8) on the Λ-nature of the dark matter.

Obviously, far more consequences of this modified GR, i.e. with weak-field of Eq.(1.3), are of interest regarding the experimental and theoretical aspects. The current experimental tests for GR, including the recent 5% accuracy for the Lense-Thirring effect obtained via the LARES satellite [60], are obviously far from detecting the potential contribution of the Λ-term in GR. However, the observations of black holes and pulsars including the detections of gravitational waves via LIGO [61], on the one hand, and of galactic halos (e.g. [62]), dynamics of galaxy groups and clusters for weak-field gravity, on the other hand, can provide efficient tests. Then, the cosmological constant determines not only expansion of the Universe but also the weak-field gravity and even possibly is linked to the arrow of time [63]. The modified GR reveals also the obvious link to the AdS/CFT correspondence, since instead of Poincare group now one has a SO(2,3) group. Further consequences of this approach are in [40, 41].

Meantime, the accurate measurements of strong lensing of extragalactic objects provide important means for the study of profound cosmological problems. The recent study of the lensed object ESO 325-G004 [66] (cf. previous studies [67, 72, 73]) enables a remarkable testing of General Relativity in extragalactic scales. The observational surveys of lensing will definitely proceed further with ever increasing precision and statistics which will enable to improve the available accuracy of the value of the weak-field parameter γ.

In view of that, here we derive a critical γ_{cr} for the strong lensing of extragalactic objects which can be informative for gravity theories. It is remarkable that, γ_{cr} does not depend on any hypothetical parameter of modified gravity models (coupling constant, scalar field mass, etc) but is determined entirely by measured physical quantities. Since the needed accuracy for measuring of γ_{cr} seems not principally unreachable given the variability range of the lense mass and light impact scale, due to γ_{cr} the gravity lens measurements will get similar importance as the renown Solar eclipse of 1919 which enabled to distinguish GR from the classical Newtonian gravity. Thus, for the first time the detection of a discrepancy with the conventional General Relativity can become feasible, with further intriguing relation to dark sector. While the breakthrough study [66] reveals the possibility of obtaining of γ, the observations of more distant objects and hence the needed accuracy for γ certainly is a matter of future advances; however, let us recall the classical example, i.e. Einstein's scepticism as regards observing gravitational lenses [74].

Finally we discussed the possible shortcomings of "negative mass" approach. It should be noticed that before [75], there have been other attempts to use negative cosmological constant for flatness of rotation curve too (for example see [81]). However, for all such cases including “negative mass” approach, the results contradict basic physics and a bunch of observational data.

Chapter 2

Cosmological Constant as a Fundamental Constant

2.1 Introduction

In this chapter, we continue our investigation regarding the "Newton Theorem". Namely, based on the group theoretical analysis, we show that how the cosmological constant Λ introduced by Einstein according to principles of GR will be identified as the linear term of "Newton Theorem"'s most general function. In this sense, we conclude that Λ is the second fundamental constant of gravity. Then, we study the further consequences of Λ as one of the fundamental constant of Nature. The results of this chapter have been published in [40] and [43].

2.2 Two fundamental constants of gravity unifying dark matter and dark energy

The discovery of the dark sector as a dominant constituent of the Universe is one of outstanding recent astrophysical achievements and continues to be a key puzzle for physical theories.

Various modifications of the Newtonian gravity and of GR are being actively considered in that context.

Among the possible approaches to the modified gravity, including GR, is the one based on a theorem proved by Newton in "Principia" on the equivalence of the gravity of sphere and that of a point mass located in its center. The principal importance of that theorem was obvious, since the motion of the planets which were spheres and not point masses, could be considered explained by gravity law only upon the proof of that theorem. Now, it appears that this theorem provides a two-step path to modified gravity theories and directly to the dark sector problem [38, 39]:

1. The general function satisfying that theorem provides an additional term containing a constant and thus modifying the Newtonian gravity;

2. That modified Newtonian gravity leads to a modified GR with the former as its weak-field limit.

So, the modified GR has to initially include that additional constant, along with the gravitational constant. As shown in [38, 39] that additional constant entering both the modified Newtonian gravity and GR enables to describe by its sign and quantitative value both the dark matter and the dark energy. That constant appears to be the renown cosmological constant which was introduced by Einstein [5] in order to have static solutions to Einstein equations.

The fact that the two constants of the gravitational interaction are able to describe self-consistently, i.e. without postulation of additional scalar or other fields, the dark matter and dark energy, reveals their unified, gravitational nature. Namely, the dark matter appears as a result of pure gravitational interaction, but with a law containing the additional constant.

We analyze this approach and reveal the role of that additional universal physical constant in the classical and relativistic gravities. The consequences of this approach can have direct impact on the strategy of the observational studies of the dark energy and the search of the dark matter.

2.2.1 Newton's theorem and General Relativity

The general function to satisfy the Newton's theorem that a sphere acts as a point-mass located in its center has the following form for the force [38]

$$f(r) = C_1 r^{-2} + C_2 r, \tag{2.1}$$

where C_1 and C_2 are constants and $f(r)$ is the solution of equation

$$\frac{r^2}{2} f''(r) + r f'(r) - f(r) = 0. \tag{2.2}$$

Thus, one can conclude that according to Newton theorem the gravitational force should contain two terms i.e. an inversed square term and a linear one.

Considering the original formulation of gravity by Newton himself, it becomes clear that the constant C_1 is written as $C_1 = G\,m^2$, i.e. with the familiar gravitational constant G and mass m, the latter entering also the law of mechanics. In this sense, the Newtonian gravity can be regarded as a very special case i.e. $C_2 = 0$ of all possible forms of gravitational fields where one can consider spherical objects as points. Furthermore it should be noticed that, Newton himself did not consider the most general form of the force before formulation of his theory of gravity, although he proved that in the context of his theory it is possible to consider spheres as points.

In this context the presence of a linear term was forgotten for hundreds of years until the formulation of GR and introduction of cosmological constant Λ

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.3}$$

After that it became obvious that, by considering the Λ, as introduced above, the GR's weak field limit will contain an additional linear term. In this sense the metric tensor components for the sphere's gravity in the weak-field limit will be

$$g_{00} = 1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}; \;\; g_{rr} = 1 + \frac{2Gm}{rc^2} + \frac{\Lambda r^2}{3}. \tag{2.4}$$

So, one can conclude that although for the first time the existence of Λ was proposed by Einstein in the context of GR [13], it would be possible to find the full GR equations (Eq.(2.3)), if Newton had considered both terms and formulated his theory based on Eq.(2.1). In this sense, by considering the Newton's principle and the most general form of the force, the cosmological term appears in Einstein's equations not by principles of GR, but as the second linear term of Newtonian gravity.

It should be noticed that, although Eq.(2.4) has been considered previously in different contexts (e.g. [13] and [52] and references therein), the approach of [38],[39] from the roots of Newtonian gravity/GR provides an insight to the unified nature of the dark matter and the dark energy. Namely, the presence of Λ as an additional linear term in Newtonian regime i.e. Eq.(4) enables one to describe the dark matter in galaxies, as the cosmological constant in GR describes the dark energy, and as shown in [39] both values of Λ, i.e. those describing the dark matter and dark energy (cosmological constant) quantitatively agree with each other. In the case of the dark matter, it is of principal importance since Eq.(2.1) describes a non-force-free field inside a shell except its center, while for Newtonian law the force-free field is entirely inside the shell. This fact agrees with the observational evidence that the galactic halos determine the properties of galactic disks [55].

Table 2.1: Background geometries for vacuum solutions

Sign	Spacetime	Isometry Group	Curvature
$\Lambda > 0$	de Sitter (dS)	O(1,4)	+
$\Lambda = 0$	Minkowski (M)	IO(1,3)	0
$\Lambda < 0$	Anti de Sitter (AdS)	O(2,3)	-

2.2.2 Group-theoretical analysis of Newton's theorem

In previous section we have shown that, it is possible to justify the existence of second term in Eq.(2.1) as the weak-field limit of GR equations written with Λ. However, as mentioned above Λ was introduced not by Newton's theorem but according to conservation of Energy-Momentum tensor and the fact that $\partial^{\mu} g_{\mu\nu} = 0$. So it seems quite reasonable that, to make a more powerful justification, we try to infer the Newton's theorem based on above relativistic considerations. Thus we turn to the isometry groups.

In Eq.(2.3), depending on Λ's sign - positive, negative or zero - one has three different vacuum solutions (three different asymptotic limits) for the field equations as shown in Table 2.1.

The interesting feature of all these 4-dim maximally symmetric Lorentzian geometries is that, for all of them the stabilizer subgroup of isometry group is the Lorentz group O(1,3). This means that at each point of all these spacetimes, one has an exact Lorentz symmetry. Since O(1,3) is the group of orthogonal transformations, one can conclude that all above spacetimes possess spherical symmetry (in Lorentzian sense) at each point. Speaking in terms of geometry, for above three spacetimes we have

$$dS = \frac{O(1,4)}{O(1,3)}, \quad M = \frac{IO(1,3)}{O(1,3)}, \quad AdS = \frac{O(2,3)}{O(1,3)}. \tag{2.5}$$

It is clear that in non-relativistic limit the full Poincare group IO(1,3) is reduced to Galilei group Gal(4)=(O(3)$\times$R)$\ltimes$R^6, which is the action of O(3)$\times$R (as the direct product of spatial orthogonal transformations and of time translation) on group of boosts and spatial transla-

Table 2.2: 3D Background Geometries with O(3) as the stabilizer

Spacetime	Isometry Group	Curvature
Spherical	O(4)	+
Euclidean	E(3)	0
Hyperbolic	$O^+(1,3)$	-

tions R^6. In the same way one can find the non-relativistic limit of O(1,4) and O(2,3) groups

$$O(1,4) \to (O(3) \times O(1,1)) \ltimes R^6, \quad O(2,3) \to (O(3) \times O(2)) \ltimes R^6. \tag{2.6}$$

Furthermore, considering the fact that the Galilei spacetime is achieved via quotienting Gal(4) by $O(3)\ltimes R^3$ (the group generated by orthogonal transformations and boosts), one can continue the analogy and find the so-called Newton-Hooke $NH(4)^{\pm}$ spacetimes by same quotient group but now for groups of Eq.(2.6) (see [82][83][84]). In this sense, depending on the sign of Λ, we can not only find the general form of the Newtonian modified gravity (according to section 2), but also the non-relativistic background geometries of the Lorentzian spacetimes in Table 2.1 and their symmetries.

To complete the proof, one has to check whether it is possible to apply the Newton's theorem to these spacetimes or not. As stated above, to apply the gravity law to planets (spheres) Newton considered them as points. Speaking in terms of mathematics it means that at each point one should have O(3) symmetry. This statement is similar to what we showed for 4-dim geometries of Table 2.1 and the Lorentz group O(1,3). The possible 3-geometries with such property are listed in Table 2.2.

Recalling that for non-relativistic theories we have two absolute notions of space and time geometry (in contrast to relativistic theories where space and time are unified in spacetime geometry), the last step is to check whether the spatial geometry of two $NH(4)^{\pm}$ spacetimes, as well as the Galilei spacetime are equal to one of the geometries mentioned in Table 2.2 or not. There are several ways to check this statement, however the most straightforward one is to check the algebraic structure of spatial geometry. Recalling the fact that for both $NH(4)^{\pm}$ spacetimes and Galilei spacetime the spatial algebra is identical and equal to Euclidean algebra $E(3)=R^3 \rtimes O(3)$, we can conclude that for all above spacetimes we have an exact O(3) symmetry at each point of spatial geometry. In this sense we will arrive at Newton's theorem based on group theoretical analysis of GR equations.

2.2.3 Newton's theorem in d-dimensions

To throw more light on the constant Λ we consider the higher dimensional cases which simply means that the gravitational field defined on S^{d-1} should be equal to that defined for a single point at d-dimensional space. For the potential one has

$$\Delta_{S^{d-1}}\Phi = C_1, \tag{2.7}$$

where $\Delta_{S^{d-1}}$ denotes the Laplace operator defined on S^{d-1} and the constant C_1 defines the mathematical feature of geometrical point. Now due to spherical symmetry we can write

$$\frac{1}{r^{d-1}}(\frac{d}{dr}r^{d-1}\frac{d}{dr}\Phi) = C_1. \tag{2.8}$$

So the most general form of the gravitational potential Φ of sphere in d-dimensional case according to Newton's theorem is

$$\Phi(r) = C_1\frac{r^2}{2d} + \frac{C_2}{(d-2)r^{d-2}}. \tag{2.9}$$

In this equation C_1 is the constant of Eq.(2.7) and the constant C_2 arises during solving the equation. Note, that for $d = 2$ the second term becomes logarithmic but the first one remains unchanged.

The potential Φ in Eq.(2.9) at $d = 3$ is not only in full agreement with Eq.(2.1), but also leads to further insights. One can identify the C_1 in Eq.(2.9) with the Λ-constant at the d-dim generalization of ordinary Newtonian gravity,

$$\Phi(r) = -\frac{G_d M}{r^{d-2}} - \frac{\Lambda c^2 r^2}{2d}, \tag{2.10}$$

where G_d indicates the d-dimensional gravitational constant.

Note a remarkable fact: comparing the two constants - the gravitational constant G and the Λ - one can see their essential difference. Namely, the gravitational constant G is dimensional-dependent and couples to matter, while Λ is neither dimensional-dependent nor matter-coupled. Such universality of Λ can be considered as fitting its vacuum content noticed by Zeldovich from completely different principles [85].

So, the gravity has not one, but two fundamental constants - G and Λ - and the second one (cosmological constant) is more universal (dimensional-independent) than the gravitational constant! The two constants together are able to explain quantitatively the dark energy and the dark matter [39].

Then, the metric component of $d+1$ dimensional spacetime is

$$g_{00} = 1 + \frac{2\Phi}{c^2}. \tag{2.11}$$

From the d-dimensional Gauss's law

$$\Delta\Phi = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} G_d \rho - \Lambda c^2, \tag{2.12}$$

where ρ is the d-dimensional density of matter. Consequently one gets the Einstein constant

$$\kappa_d = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{G_d}{c^4}. \tag{2.13}$$

This completes the generalization of Newton's theorem to arbitrary dimension and its correspondence to classical and relativistic theories of gravity.

Then, for the 3 possible maximally symmetric $(d+1)$-dimensional spacetimes defined by the value of Λ one has the following geometries

$$dS_{d+1} = \frac{O(1,d+1)}{O(1,d)}, \quad M_{d+1} = \frac{IO(1,d)}{O(1,d)}, \quad AdS_{d+1} = \frac{O(2,d)}{O(1,d)}, \tag{2.14}$$

as the generalizations of Eq.(2.5); for $d = 3$ one easily recovers the 4-dimensional results. It is clear that in such case, irrespective which geometrical spacetime is considered, one has exact O(1,d) symmetry at each point, which in it's turn indicates the existence of spherical symmetry of Lorentzian geometry for all points. Fixing the relativistic geometries and symmetries one easily finds their non-relativistic limits

$$O(1,d+1) \to (O(d)\times O(1,1))\ltimes R^{2d}, \quad O(2,d) \to (O(d)\times O(2))\ltimes R^{2d}, \quad IO(1,d) \to (O(d)\times R)\ltimes R^{2d}. \tag{2.15}$$

As in section 3, one can find the non-relativistic background geometries for each case by quotienting O(d)$\ltimes$ R^d for all three symmetric groups. The resulting spacetimes are Gal(d+1), NH$^+$(d+1), NH$^-$(d+1), and clearly at $d = 3$ one obtains the classical spacetimes. As we have mentioned earlier, the interesting feature of these non-relativistic geometries is the fact that, in contrast to relativistic case they are not metric geometries because they do not admit single metric structure and their properties can be studied via corresponding affine connection. Furthermore from geometrical point of view, for all these three cases the spatial geometry seems to be Euclidean and the pure spatial algebra is equal to Euclidean algebra E(d). Then, since E(d)=R$^d\rtimes$ O(d), one easily concludes that in the spatial geometry the O(d) is the stabilizer group, which in it's turn means that all points can be considered as d-dimensional spheres S^{d-1}. This proves that for all these three geometries the Newton's theorem is hold. However, as mentioned above, the spatial part of all three geometries is equal to each other and the question is, how Λ affects these geometries. The answer becomes clear if one considers the temporal parts of Eq.(2.15). Indeed, the sign of Λ indicates that we are living either in oscillating NH(d+1)$^-$, flat Gal(d+1) or expanding NH(d+1)$^+$ universe. One can also check that for all these cases, depending on the sign and the value of Λ, the affine connection can be flat, for Gal(d+1) case, and either positive or negative for NH(d+1)$^+$ and NH(d+1)$^-$, respectively.

To conclude this brief but principal discussion, we write down the d-dimensional ($d \neq 2$) Schwarzschild metric for non-zero Λ

$$ds^2 = (1 - \frac{2G_d M}{r^{d-2}c^2} - \frac{\Lambda r^2}{3})c^2 dt^2 - (1 - \frac{2G_d M}{r^{d-2}c^2} - \frac{\Lambda r^2}{3})^{-1} dr^2 - r^2 d\Omega^2_{d-1}. \quad (2.16)$$

Although d-dimensional cases have been considered before, our approach to GR and its weak-field limit justifies the consideration of point-like dynamics for higher dimensional spheres based on Newton's original theorem.

Thus, according to our analysis:

1. Gravity has not one but two fundamental constants, the gravitational constant G and an additional one, Λ, which appears readily in General Relativity with weak-field limit as modified Newtonian gravity. Moreover, the Λ-constant (the cosmological constant) is dimensional-independent and matter-uncoupled and hence can be considered as even more universal than the gravitational constant G;

2. The Λ-constant of gravity emerges from Newton's theorem on the identity of the sphere's gravity and that of the point-mass located in its center;

3. Both constants, G and Λ, jointly are able to explain quantitatively the dark energy and the dark matter [39], which hence appear as gravity effects.

Also, the AdS spacetime of AdS/QFT emerges here readily from the genuine structure of classical and relativistic gravities. Positive Λ-constant is an essential condition in Conformal Cyclic Cosmology [86, 87].

The accuracy of the current tests of GR (e.g. [60]) are still far to probe the modified gravity discussed above, however, for example, the astronomical observations of galactic halos [62] can be efficient in testing the predictions regarding the dark matter nature.

2.3 Cosmological constant as a fundamental constant

The cosmological constant as a universal constant was mentioned by Einstein [5, 88] when he introduced it to describe a static cosmological model but later abandoned it. In the recent study [40] we have shown that the cosmological constant Λ does possess properties of a second physical constant, along with the gravitational one G, both for the General Relativity and the Newtonian gravity as its low-energy limit. That follows from the Newton theorem on the equivalency of gravity of a sphere and of a point-mass located in its center (see [39]), i.e. the motivation of emergence of Λ is entirely different from that of the static universe. The Λ-constant was shown to be dimension-independent and matter uncoupled and therefore even more universal than the gravitational constant [40]. That approach enables one to describe the dark matter and the dark energy as possessing common nature [38].

Here we discuss further consequences for Λ joining the set of fundamental constants $G, c, \hbar$, the gravitational (Newton) constant, speed of light and Planck constant, respectively; for detailed discussion of constants see [89].

The consideration of Λ together with the 3 constants affects the issue of Planck units. Planck [16] denoted the latter as natural units since they "retain their meaning for all times and for all cultures, even extraterrestrial and non-human ones". We show that Λ together with the Planck units leads to emergence of a dimensionless constant, also relevant for "all cultures", which in cosmological context acts as scaling for information. Among the consequences is that, a rescaling of values of the 3 fundamental physical constants will be allowed from one aeon to another aeon within the Conformal Cyclic Cosmology (CCC) [86, 87]; the rescaling satisfies a dimensionless relation and possess group symmetry.

2.3.1 The 4 units and information

The set of (3+1) constants and their units looks as

$$[c] = LT^{-1}, \quad [G] = M^{-1}L^3T^{-2}, \quad [\hbar] = ML^2T^{-1}, \quad [\Lambda] = L^{-2}, \tag{2.17}$$

where L, T and M stand for dimensionality of length, time and mass, respectively. The most general combinations of these constants can be represented in the form

$$[c^{n_1}\Lambda^{n_2}G^{n_3}\hbar^{n_4}] = L^{n_1-2n_2+3n_3+2n_4}T^{-n_1-2n_3-n_4}M^{-n_3+n_4}. \tag{2.18}$$

From here, two consequences follow readily.

First, for the set $(G, \Lambda, c, \hbar)$ the corresponding algebraic equation has no unique solution and hence no units can be defined by these 4 constants, as distinct of the case of $(G, c, \hbar)$ leading to Planck units.

Second, the following dimensionless quantity (constant) does emerge

$$I = \frac{c^{3a}}{\Lambda^a G^a \hbar^a}, \tag{2.19}$$

where a is a real number. In contrast, no dimensionless quantity was possible to construct from the 3-set $(G, c, \hbar)$. This difference, as we show below, can have consequences for the CCC.

Note, that for $a = 1$ in (2.19) one has $I \simeq 3.4 \times 10^{121}$, which obviously reflects the renown cosmological constant value problem.

The relation of the 4 constants in (2.19), except for a numerical factor, for $a = 1$ coincides with that of the information (or entropy, with the Boltzmann constant) of de Sitter event horizon [91, 92, 86]

$$I_{dS} = 3\pi\frac{c^3}{\Lambda G\hbar}. \tag{2.20}$$

This relation emerges also from the Bekenstein Bound [93] written for the information in de Sitter space

$$I_{BB} = \frac{3\pi c^3}{\Lambda G\hbar ln2}. \tag{2.21}$$

One may expect emergence in future of this same dimensionless relation of the 4 constants from other backgrounds or motivations.

The coincidence of I_{DS} and I_{BB} i.e. $\Delta I_{dS} = 0$, reflects that there is no information (entropy, thermodynamical) time evolution in de Sitter manifold. In the next section we will study the possible link between this statment and symmetries of manifolds with more details.

The importance of Newton theorem lies also on the fact that it enables one to generalize the "sphere-point" equivalence idea to higher dimensions. In those cases, of course we have hyperspheres $\mathbb{S}^{d-1}$, where d is the dimensionality of space. Then, according to [40], for gravitational potential (d≥3) we have

$$\Phi(r) = -\frac{G_d M}{r^{d-2}} - \frac{\Lambda c^2 r^2}{2d}. \tag{2.22}$$

As a consequence, the Newton gravitational constant becomes dimension-dependent and for four constants we have

$$[c] = LT^{-1}, \quad [G_d] = M^{-1}L^dT^{-2}, \quad [\hbar] = ML^2T^{-1}, \quad [\Lambda] = L^{-2}, \tag{2.23}$$

Then, the dimensionless quantity is obtained as

$$I_d = \frac{c^{3a}}{G_d^a \hbar^a \Lambda^{a\frac{d-1}{2}}}, \quad a \in \mathbb{R}, \tag{2.24}$$

obviously, for $d = 3$ we recover Eq.(2.19).

Since the information is related to the area of (d-1)-dimensional hypersurface, the Bekenstein's "elementary particle" has an area

$$\frac{d\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2}+1)}\frac{G_d\hbar}{c^3}, \tag{2.25}$$

and the evolution of the universe ends at de Sitter phase at corresponding maximum information

$$I_{dS} = \frac{c^3}{G_d\hbar\Lambda^{\frac{d-1}{2}}} d^{\frac{d-1}{2}}\pi \; bits. \tag{2.26}$$

2.3.2 Information, time evolution and Weyl principle

As shown above, upon introducing Λ as one of fundamental constants the notions of l_p, m_p and t_p as ordinary natural units, disappear. However, within Bekenstein's "elementary particle" [91] approach, one can consider Planck units as composing one bit of information. Namely, one bit of information is attributed to $4l_p^2$, so that in expanding universe upon the increase of the surface area more information is created. Creation of information continues until in de Sitter (dS) phase I_{dS}-th bit is created.

Thus the time evolution of the universe is reduced to discretized steps

$$T = \{1, 2, 3, ..., I_{dS}\} . \tag{2.27}$$

Such description based on creation of information, naturally, imposes a temporal order. Note, that for de Sitter (dS) universe where $\Delta I_{dS} = 0$, we have time-translational symmetry T(t) as the subgroup of isometry group O(1,4). Thus it seems that, there might be a link between the T(t) group and evolution of universe based on information.

In non-relativistic limit the geometry of universe is considered as Galilean spacetime, which has 10-parameter symmetry group Gal(4), where time translations T(t) make a subgroup of Gal(4). This is also true at non-zero cosmological constant case, where the symmetry groups are $\mathrm{NH}^{\pm}(4)$ as shown in Table 2.3. Meantime, for each case it is easy to show that

Table 2.3: Non-Relativistic Background Geometries

Sign	Geometry	Symmetry Group	T(t)	Relativistic limit
$\Lambda > 0$	Newton-Hooke	$\mathrm{NH}^{+}(4)$=(O(3) × O(1,1)) ⋉ R^6	O(1,1)	de Sitter
$\Lambda = 0$	Galileo	Gal(4)=(O(3) × R) ⋉ R^6	R	Minkowski
$\Lambda < 0$	Newton-Hooke	$\mathrm{NH}^{-}(4)$=(O(3) × O(2)) ⋉ R^6	O(2)	Anti de Sitter

Table 2.4: Spatial Geometries

Space	Symmetry Group	Curvature
Spherical: $\mathbb{S}^3$	O(4)	+
Euclidean: $\mathbb{R}^3$	E(3)	0
Hyperbolic: $\mathbb{H}^3$	$\mathrm{O}^{+}(1,3)$	-

there are non-relativistic limits of following groups

$$O(1,4) \to (O(3)\times O(1,1))\ltimes R^6, \quad O(2,3) \to (O(3)\times O(2))\ltimes R^6, \quad IO(1,3) \to (O(3)\times R)\ltimes R^6, \tag{2.28}$$

where clearly there is again time-translational symmetry. This implies that it is not possible to fix a preferred direction of time based only on symmetrical features of background geometries for both relativistic and non-relativistic ones.

At the same time, following [40], the sphere-point identity implies that, at each point of background geometry (spatial), we have O(3) symmetry. As in all of these non-relativistic geometries the spatial algebra is Euclidean E(3)=O(3)⋉ R^3, the Newton theorem is valid.

Thus, by considering Newton theorem and information theoretic evolution of universe simultaneously, it becomes clear that although the geometry initially during the creation of information does not posses T(t) symmetry group, O(3) is the stabilizer of the spatial geometry. All possible 3-geometries with O(3) as the stabilizer are listed in Table 2.4.

In relativistic cosmology the background geometry i.e. the Friedmann-Lemaître-Robertson-Walker (FLRW) metric is fixed by "Weyl principle" which assumes that at any moment of time the universe is homogeneous and isotropic

$$ds^2 = c^2 dt^2 - a(t)^2 d\Sigma^2, \tag{2.29}$$

where Σ is one of geometries listed in Table 2.4. Within the approach presented above, "Weyl principle" becomes not just a matter of simplification, but a condition to have gravity satisfying Newton theorem, on one side, and enabling information theoretical consideration, on the other side.

2.3.3 CCC: rescaling of physical constants

The key elements of CCC are the Second law of thermodynamics and the positive Λ [86]. That naturally implies the involvement of physical constants and Planck units through the concepts of entropy and information. Namely, within CCC the initial point of each aeon corresponds to vanishing of Weyl tensor, $\mathbf{C} = 0$, and then the evolution of each aeon is completed by de Sitter expansion. The re-set of entropy at the conformal boundary of aeons is reached by the loss of information in massive black holes situated in galactic centers and Hawking evaporation.

Since the expressions defined by 4 constants (2.19) are dimensionless numbers, they are transformed identically from one aeon to another (regarding the information transfer to the next aeon see [94]), as invariants with respect to conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{2.30}$$

Namely, the ratio

$$\frac{Q_{dS}}{Q_p} = m\left(\frac{c^3}{\hbar G \Lambda}\right)^n = mI^n, \quad m, n \in \mathbb{R} \tag{2.31}$$

of all physical quantities $\{Q\}$ in final (de Sitter) and initial (Planck) eras of an aeon will remain invariant under conformal transformations.

However, the invariance of mI^n does not imply the invariance of each of 4 constants involved. In other words, the constants can be rescaled from one aeon to another

$$c \to a_1 c, \quad \hbar \to a_2 \hbar, \quad G \to a_3 G, \quad \Lambda \to a_4 \Lambda, \quad a_i \in \mathbb{R}^+, \tag{2.32}$$

keeping satisfied the condition

$$\frac{a_1^3}{a_2 a_3 a_4} = 1, \tag{2.33}$$

From here we arrive at the conclusion that, the constants' transformations in an aeon are invariant under the following group

$$S = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \quad det|S| = 1, \quad a_{11} = a_1^3, \quad a_{22} = a_2^{-1}, \quad a_{33} = a_3^{-1}, \quad a_{44} = a_4^{-1} \right\}. \tag{2.34}$$

This means that, the subsequent aeons can possess rescaling of constants $c, \hbar, G, \Lambda$ and of Q_i keeping invariant the dynamics of an aeon.

Thus, the yet unknown "Master Equation" of the universe has to admit the S group's symmetry.

Then, in view of the relation (2.31) we see that there is a noted difference between the role of Λ and of other constants. In fact, since the Λ is absent in Planck era scales, by fixing Λ's value, the values of physical quantities (up to possible combination of other constants) Q_{dS} are fixed. This property is possessed only by Λ, since fixing any of the rest three constants does not define either the initial or final stages of an aeon. If so, one can rewrite the group

S as follows

$$S = \left\{ \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}, \quad q_{ii} \in \mathbb{R}^+, \quad det|S| = 1 \right\}, \tag{2.35}$$

where in this case, $q_{11} = a_{11}a_{22}a_{33}$ and $q_{22} = a_{44}$.

Thus, the expansion of an aeon starts at positive Λ and upon fixing its value, the values of 3 physical constants c, G and $\hbar$ are fixed according to formula (2.19), i.e. allowing several equivalent combinations satisfying the group S.

Note, a difference of the described information approach and the conventional one defining the dynamics of the universe with Friedmannian equations. Those equations are solved numerically for given input parameters with proper choice of time steps. Now, when Λ is considered a universal constant and the notion of natural units for time, length and mass disappear, we come to a dimensionless information and the dynamics of the universe is reduced to discrete steps $\{1, 2, 3, ..., I_{dS}\}$.

2.4 Conclusions

According to our analysis:

a) Gravity has not one but two fundamental constants, the gravitational constant G and an additional one, Λ, which appears readily in General Relativity with weak-field limit as modified Newtonian gravity. Moreover, the Λ-constant (the cosmological constant) is dimensional-independent and matter-uncoupled and hence can be considered as even more universal than the gravitational constant G;

b) The Λ-constant of gravity emerges from Newton's theorem on the identity of the sphere's gravity and that of the point-mass located in its center;

c) Both constants, G and Λ, jointly are able to explain quantitatively the dark energy and the dark matter [39], which hence appear as gravity effects.

Also, the AdS spacetime of AdS/QFT emerges here readily from the genuine structure of classical and relativistic gravities. Positive Λ-constant is an essential condition in Conformal Cyclic Cosmology [86, 87].

The accuracy of the current tests of GR (e.g. [60]) are still far to probe the modified gravity discussed above, however, for example, the astronomical observations of galactic halos [62] can be efficient in testing the predictions regarding the dark matter nature.

On the other hand, the cosmological constant Λ, which as shown in [40], acts as a physical constant defining the gravity, in combination with other fundamental constants leads to the following principal conclusions:

a) the 4 constants no longer define a unique scaling for length, time and mass, as were the Planck units for the 3 physical constants;

b) a dimensionless quantity (constant) is emerging composed of 4 constants $G, \Lambda, c, \hbar$ of a transformation group symmetry which was not possible with 3 constants $G, c, \hbar$.

Starting from 1970s the notion of information as of dimensionless quantity was attributed to event horizons [91, 92]. Now, as shown above, only together with Λ one can construct a natural dimensionless quantity, to which within Bekenstein's "elementary particle" approach one can attribute information content.

Thus, Λ as universal constant approach, enables one to consider dynamics of the universe as of (d+1)-dimensional Lorentzian geometry satisfying the following conditions:

• Newton theorem ensures O(d) symmetry at each point of d-dimensional spatial geometry;

• The evolution intrinsically imposes "time ordering" as described by group theoretical analysis;

• Evolution can be reduced to discrete increase of (dimensionless) information;

• Bekenstein's "elementary particle" corresponds to an area $\frac{d\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2}+1)}\frac{G_d\hbar}{c^3}$;

• Evolution tends to de Sitter phase with information $\frac{c^3}{G_d\hbar\Lambda^{\frac{d-1}{2}}}d^{\frac{d-1}{2}}\pi$ bits.

The group properties of transformations involving the physical constants within the Conformal Cyclic Cosmology imply that, at any positive value of Λ at initial state of each aeon the initial values of 3 physical constants will allow rescaling satisfying the dimensionless constraint. The rescaling of 4 fundamental constants will admit the same global cosmological dynamics but with rescaled internal physics. This opens an entire arena for modifications for physical processes and configurations from one aeon to another, since the values of physical constants define such basic concepts as e.g. the atomic physics, the Chandrasekhar limit, black hole collapse, etc.

The emergence of Λ as a physical constant in modified weak field limit of GR can be tested via gravity lensing observations [41].

Chapter 3

Λ-gravity and Observations

3.1 Introduction

After considering the results of previous chapter i.e. regarding Λ as one of the fundamental constants of Nature, we intend to investigate the current observational studies based on Λ-gravity. In this sense, we extend the analysis done in chapter 1.2, regarding the evaluation of Λ from different astrophysical configurations. Next, we check the universality and self-consistency of Λ-gravity by studying the extreme galaxies regarding the nature of DM. After that, we propose a possible solution for the recently sharpened discrepancy between the so-called "local" and "cosmological" values of Hubble constant which is known as H_0 tension. Finally, considering the dynamics of N-body systems, we study the problem of stability in these systems once the Λ-gravity is taken into account. The topics which we will discuss in this chapter have been published in [42], [44], [47],[48].

3.2 The cosmological constant derived via galaxy groups and clusters

A number of approaches are considered to reveal the nature of dark matter, including prediction of exotic particles and modified gravity models; for review see [51]. One of the recent approaches [39, 40] is based on the GR with a modified weak field limit following from the Newton theorem on equivalency of gravity of sphere and of point mass. That approach enables the common description of dark matter and dark energy, where Λ acts as a universal constant defining the GR and its weak field limit, along with the gravitational constant G [40]. An observational test for that approach is suggested involving the effect of gravity lensing [41].

The notable point following from the Newton theorem is that, on the one hand, Λ acts as a cosmological constant describing the expansion of the Universe, on the other hand, the same it defines the weak field gravity proper for the dynamics of galaxies, galaxy groups and clusters, i.e. on the distance scales where the Hubble flow cannot be valid. As shown in [39] the value of the cosmological constant indeed is supported by the parameters of galactic halos and of a sample of galaxy groups.

Here we continue to address the principal issue on the Λ-nature of dark matter considering the data on galaxy systems, from galaxy pairs to galaxy clusters. Among the used samples are data on galaxy groups of Local Supercluster, of galaxy clusters obtained both with gravity lensing and by Planck satellite. Assuming that the dynamical structure of the considered galaxy systems is governed by the modified Newton law [39], we obtain Λ which appear limited from below by the value of cosmological constant. In other words, the value of the cosmological constant is extracted from galaxy systems and not from cosmology.

3.2.1 The Λ constant and galaxy systems

The Newton theorem on "sphere-point" equivalency enables one to arrive at the GR metric [39, 40]

$$g_{00} = 1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}; \;\; g_{rr} = 1 + \frac{2Gm}{rc^2} + \frac{\Lambda r^2}{3}. \tag{3.1}$$

This implies that the weak field limit for GR as modified Newton law involves two constants (for the potential)

$$\phi(r) = C_1 r^{-1} + C_2 r^2, \tag{3.2}$$

where C_1 is assigned to G and C_2 to Λ (within numerical factors).

Eq.(3.1) was known before as Schwarzschild – de Sitter metric [68], where the constant Λ was introduced by Einstein to describe the static cosmological model.

Within our approach Λ is emerging from the general function satisfying the Newton's theorem and hence naturally emerges in weak field GR, and that correspondence can be represented via isometry groups. Namely, the Lorentz group O(1,3) acts as stabilizer subgroup of isometry group of 4D maximally symmetric Lorentzian geometries [40]. The results are shown in Table 3.1 and for all these geometries the Lorentzian spheres can be considered as points. Consequently, depending on the sign of Λ, we will have the following non-relativistic limits

$$O(1,4) \rightarrow (O(3)\times O(1,1))\ltimes R^6, \quad O(2,3) \rightarrow (O(3)\times O(2))\ltimes R^6, \quad IO(1,3) \rightarrow (O(3)\times R)\ltimes R^6, \tag{3.3}$$

for all three cases, since the spatial algebra is Euclidean i.e.

$$E(3) = R^3 \rtimes O(3). \tag{3.4}$$

Hence, for all three cases introduced above the O(3) is the stabilizer group for spatial geometry. This conclusion, in its turn, can be considered as the Newton's theorem in the language

Table 3.1: Background geometries for vacuum solutions

Sign	Spacetime	Isometry group
$\Lambda > 0$	de Sitter	O(1,4)
$\Lambda = 0$	Minkowski	IO(1,3)
$\Lambda < 0$	Anti de Sitter	O(2,3)

of group theory.

Then, as shown in [39], the weak field Λ extracted from galactic halo and galaxy group data quantitatively coincides with the Λ obtained from cosmological data.

The value of Λ as of the cosmological constant is currently obtained by several observational methods, the one obtained by Planck data yields [86]

$$\Lambda_{PL} = 1.11 \times 10^{-52} m^{-2}. \tag{3.5}$$

For Λ-modified Newtonian gravity Eq.(3.2) the following relation for virialized systems was derived [39]

$$\Lambda = \frac{3\sigma^2}{2c^2 R^2}, \tag{3.6}$$

where R is system's radius and σ is the velocity dispersion.

To test the conclusion on Λ-nature of dark matter in galaxy systems, below we use Eq.(3.6) for the analysis of data samples on galaxy pairs, galaxy groups and galaxy clusters.

Galaxy pairs

To probe Eqs.(3.1),(3.2) for galaxy systems we start with galaxy pairs. Here, using the weak lensing data by Gonzalez et al [95], we estimate the value of Λ from Eq.(3.6) as shown in Tables 3.10 and 3.11 for given classes of subsamples. Estimations are done both for Singular Isothermal sphere (SIS) and Navarro-Frenk-White (NFW) models as described in details in [95], M_{200} corresponds the mass (for the given model) within radius R_{200}, when the mean density equals 200 critical density values of the universe.

Table 3.2: SIS profile

Subsamples	σ (Km/s)	$M_{200}(M_\odot)$	Λ (m^{-2})
Total Sample	223 ±24	(6.9 ±2.2)E12	5.64E-51
Non Interacting	200 ±38	(5.0 ±3.0)E12	5.62E-51
Interacting Pairs	237 ±29	(8.3 ±3.0)E12	5.63E-51
Red Pairs	264 ±28	(11.4 ±3.7)E12	5.65E-51
Blue Pairs	167 ±45	(2.9 ±2.4)E12	5.64E-51
Higher Luminosity Pairs	278 ±27	(13.2 ±3.8)E12	5.69E-51
Lower Luminosity Pairs	149 ±50	(2.0 ±2.0)E12	5.75E-51
Average			5.66E-51
St. deviation			4.21E-53

Table 3.3: NFW profile

Subsamples	R_{200} (Mpc)	$M_{200}(M_\odot)$	Λ (m^{-2})
Total Sample	0.30 ±0.03	(7.1 ±2.1)E12	2.00E-50
Non Interacting	0.27 ±0.05	(5.2 ±2.8)E12	2.01E-50
Interacting Pairs	0.32 ±0.03	(8.0 ±2.9)E12	1.86E-50
Red Pairs	0.36 ±0.03	(12.1 ±3.6)E12	1.97E-50
Blue Pairs	0.22 ±0.06	(2.7 ±2.1)E12	1.93E-50
Higher Luminosity Pairs	0.36 ±0.03	(12.7 ±3.8)E12	2.07E-50
Lower Luminosity Pairs	0.19 ±0.07	(1.7 ±1.8)E12	1.89E-50
Average			1.96E-50
St. deviation			6.77E-52

Galaxy groups

Data on galaxy groups as of systems containing 3 and more galaxies situated within the Local Supercluster, namely, of the Leo/Cancer region and Bootes strip are obtained by Karachentsev et al [96, 97]; the analysis of their data on groups of the Hercules-Bootes region [59] is given in [39]. Tables 3.12 and 3.13 contain the observational data i.e. the velocity dispersion of galaxies σ and the harmonic average radius R_h of the groups denoted by their brightest galaxy listed in the first column. The last column contains Λ estimated by Eq.(3.6).

Table 3.4: Galaxy groups of Leo/Cancer region

Group	$\sigma_V(km/s^{-1})$	$R_h(kpc)$	$\Lambda(m^{-2})$
NGC2648	55	128	3.24E-51
NGC2775	89	296	1.59E-51
NGC2894	50	458	2.09E-52
NGC2962	53	161	1.90E-51
NGC2967	62	507	2.63E-52
UGC5228	40	188	7.95E-52
NGC3023	21	35	6.32E-51
NGC3020	45	44	1.84E-50
NGC3049	15	144	1.91E-52
UGC5376	66	253	1.20E-51
NGC3166	44	126	2.14E-51
NGC3227	74	128	5.87E-51
NGC3338	50	112	3.50E-51
NGC3379	193	191	1.79E-50
NGC3423	21	570	2.38E-53
NGC3521	37	132	1.38E-51
NGC3596	42	41	1.84E-50
NGC3607	115	471	1.05E-51
NGC3626	86	187	3.72E-51
NGC3627	136	201	8.04E-51
NGC3640	134	252	4.97E-51
NGC3686	91	175	4.75E-51
NGC3810	43	360	2.51E-52
Average			4.62E-51
St. deviation			5.70E-51

Galaxy clusters

We now turn to galaxy clusters, the higher scale structure in the hierarchy of galaxy systems. We use the data both on cluster gravity lensing and supernova survey with HST (CLASH) [98]. In the Table 3.14 we exhibit the data for all 19 clusters of CLASH survey along with the results of estimation of Λ using Eq.(3.6).

Besides the CLASH data we use also data of 3 Planck clusters from [99] and represent the estimated Λ (Table 3.7).

Finally, we estimate the Λ for a sample of Local Cluster Substructure Survey (LoCuSS) clusters [100], as represented in Table 3.8.

Table 3.5: Galaxy groups of Bootes strip region

Group	$\sigma_V(km/s^{-1})$	$R_h(kpc)$	$\Lambda(m^{-2})$
N4900	36	116	1.69E-51
N5248	38	151	1.11E-51
N5363	114	165	8.39E-51
N5506	23	35	7.59E-51
N5566	103	196	4.85E-51
N5638	74	203	2.33E-51
P51971	10	100	1.76E-52
IC1048	83	150	5.38E-51
N5746	107	296	2.30E-51
N5775	87	120	9.23E-51
N5792	48	290	4.81E-52
N5838	53	210	1.12E-51
N5846	228	415	5.30E-51
Average			3.84E-51
St. deviation			3.01E-51

The advantages of using CLASH data is that they provide the values for virial masses M_{vir} of clusters. Although the degree of virialization of a given cluster varies from one cluster to another and hence is a separate issue for analysis, those masses provide the upper limit of error for the value of Λ.

Namely, by considering the Λ-modified potential from one side, and the error limit for M_{vir}, on the other side, we have an upper limit for the value of Λ

$$\frac{\mathbb{E}(\sigma^2)}{\sigma^2} = \frac{\Lambda R_{vir}^3 c^2}{6GM_{vir}}, \tag{3.7}$$

where $\mathbb{E}(\sigma^2)$ is the error limit for σ^2. The results for the CLASH clusters are shown in Table 3.9. Note that, for all these clusters the error limit covers the value of Λ as of cosmological constant.

Meantime, let us note that the virial mass and radius can be defined as [101]

$$M = \frac{4}{3}\pi R^3 \Delta_c \rho_{crit}, \tag{3.8}$$

Table 3.6: CLASH Survey

Cluster	R_{vir} (Mpc)	M_{vir} ($M_\odot$)	Λ (m^{-2})
Abell 383	1.86	(1.04 ±0.07)E15	1.23E-50
Abell 209	1.95	(1.17 ±0.07)E15	1.20E-50
Abell 2261	2.26	(1.76 ±0.18)E15	1.16E-50
RXJ2129+0005	1.65	(0.73 ±0.07)E15	1.24E-50
Abell 611	1.79	(1.03 ±0.07)E15	1.37E-50
MS2137-2353	1.89	(1.26 ±0.06)E15	1.42E-50
RXCJ2248-4431	1.92	(1.40 ±0.12)E15	1.51E-50
MACSJ1115+0129	1.78	(1.13 ±0.10)E15	1.52E-50
MACSJ1931-26	1.61	(0.83 ±0.06)E15	1.51E-50
RXJ1532.8+3021	1.47	(0.64 ±0.09)E15	1.53E-50
MACSJ1720+3536	1.61	(0.88 ±0.08)E15	1.60E-50
MACSJ0429-02	1.65	(0.96 ±0.14)E15	1.63E-50
MACSJ1206-08	1.63	(1.00 ±0.11)E15	1.76E-50
MACSJ0329-02	1.54	(0.86 ±0.11)E15	1.79E-50
RXJ1347-1145	1.80	(1.35 ±0.19)E15	1.76E-50
MACSJ1311-03	1.28	(0.53 ±0.04)E15	1.92E-50
MACSJ1423+24	1.34	(0.65 ±0.11)E15	2.06E-50
MACSJ0744+39	1.33	(0.79 ±0.04)E15	2.56E-50
CLJ1226+3332	1.57	(1.72 ±0.11)E15	3.38E-50
Average			1.69E-50
St. deviation			5.16E-51

Table 3.7: Planck Clusters

Clusters	σ (Km/s)	M_{200}($M_\odot$)	Λ (m^{-2})
PSZ1 G109.88+27.94	1800 ±200	$(44\ ^{+16}_{-13})$E14	4.96E-51
PSZ1 G139.61+24.20	800 ±100	$(6.3\ ^{+2.7}_{-2.1})$E14	3.58E-51
PSZ1 G186.98+38.66	1100 ±200	$(14.5\ ^{+9.4}_{-6.5})$E14	3.88E-51
Average			4.14E-51
St. deviation			5.93E-52

where Δ_c is the overdensity parameter and ρ_{crit} is the critical density of the universe. The Δ_c, in its turn is obtained for each era of the universe as [102]

$$\Delta_c = 18\pi^2 + 82x - 39x^2, \quad x = \Omega(z) - 1, \quad \Omega(z) = \frac{H_0^2}{H^2(z)}\Omega_0(1+z)^3, \tag{3.9}$$

Table 3.8: LoCuSS Clusters

Cluster	R_{200} ($h_{100}^{-1} Mpc$)	M_{200} ($M_\odot$)	Λ (m^{-2})
Abell 586	1.2 $\pm$0.2	(5.1 $\pm$2.1)E14	2.25E-50
Abell 611	1.1 $\pm$0.1	(4.0 $^{+0.7}_{-0.8}$)E14	2.29E-50
Abell 621	1.2 $^{+0.2}_{-0.1}$	(4.8 $^{+1.7}_{-1.8}$)E14	2.11E-50
Abell 773	1.1 $\pm$0.1	(3.6 $\pm$1.2)E14	2.06E-50
Abell 781	1.1 $\pm$0.1	(4.1 $\pm$0.8)E14	2.34E-50
Abell 990	0.9 $\pm$0.1	(2.0 $^{+0.4}_{-0.1}$)E14	2.09E-50
Abell 1413	1.1 $\pm$0.1	(4.0 $\pm$1.0)E14	2.29E-50
Abell 1423	0.9 $\pm$0.1	(2.2 $\pm$0.8)E14	2.30E-50
Abell 1758a	1.1 $\pm$0.1	(4.1 $^{+0.7}_{-08}$)E14	2.34E-50
Abell 1758b	1.1 $\pm$0.2	(4.4 $\pm$1.9)E14	2.52E-50
Abell 2009	1.2 $\pm$0.1	(4.6 $\pm$1.5)E14	2.03E-50
Abell 2111	1.1 $\pm$0.1	(4.2 $\pm$0.9)E14	2.40E-50
Abell 2146	1.2$\pm$0.1	(5.0 $\pm$0.7)E14	2.20E-50
Abell 2218	1.3$\pm$0.1	(6.1 $\pm$0.9)E14	2.11E-50
RXJ0142+2131	1.0$\pm$0.1	(3.7 $^{+1.1}_{-1.2}$)E14	2.82E-50
RXJ1720+2638	0.9$\pm$0.1	(2.0 $\pm$0.4) E14	2.09E-50
Average			2.27E-50
St. deviation			1.96E-51

where H is the Hubble constant.

Thus it turns out that considering the Newtonian virial theorem $\sigma^2 = \frac{GM_{vir}}{R_{vir}}$ and Eq.(3.8) for virialized systems one can calculate the value of Λ based on Eq.(3.6)

$$\Lambda = \frac{3}{4}\frac{\Delta_c}{c^2}H^2. \tag{3.10}$$

Eq.(3.9) enables one to calculate the value of Δ_c at each era of Universe. For pure de Sitter universe $\Delta_c = 18\pi^2 \approx 178$, while within the "spherical collapse" model its value yields Δ_c =200 [103]. However, within the ΛCDM cosmology $\Delta_c \approx 100$ ([104] and references therein). Considering all above, for $H = 70$ (km/s)/(Mpc) one will obtain $\Lambda \approx 4.30 \times 10^{-51} m^{-2}$.

Table 3.9: Error limit of Λ for CLASH Survey

Cluster	Radius (Mpc)	M_{vir}($M_\odot$)	Λ (m^{-2}) ≤
Abell 383	1.86	(1.04 ±0.07)E15	3.31E-51
Abell 209	1.95	(1.17 ±0.07)E15	2.87E-51
Abell 2261	2.26	(1.76 ±0.18)E15	4.74E-51
RXJ2129+0005	1.65	(0.73 ±0.07)E15	4.74E-51
Abell 611	1.79	(1.03 ±0.07)E15	3.71E-51
MS2137-2353	1.89	(1.26 ±0.06)E15	2.70E-51
RXCJ2248-4431	1.92	(1.40 ±0.12)E15	5.16E-51
MACSJ1115+0129	1.78	(1.13 ±0.10)E15	5.39E-51
MACSJ1931-26	1.61	(0.83 ±0.06)E15	4.37E-51
RXJ1532.8+3021	1.47	(0.64 ±0.09)E15	8.62E-51
MACSJ1720+3536	1.61	(0.88 ±0.08)E15	5.83E-51
MACSJ0429-02	1.65	(0.96 ±0.14)E15	9.48E-51
MACSJ1206-08	1.63	(1.00 ±0.11)E15	7.73E-51
MACSJ0329-02	1.54	(0.86 ±0.11) E15	9.16E-51
RXJ1347-1145	1.80	(1.35 ±0.19)E15	9.91E-51
MACSJ1311-03	1.28	(0.53 ±0.04)E15	5.80E-51
MACSJ1423+24	1.34	(0.65 ±0.11)E15	1.39E-50
MACSJ0744+39	1.33	(0.79 ±0.04)E15	5.17E-51
CLJ1226+3332	1.57	(1.72 ±0.11)E15	8.65E-51
Average			6.38E-51
St. deviation			2.85E-51

3.3 Dark matter extremal galaxies probing Lambda-gravity

The DM continues to remain one of key mysteries of cosmology and fundamental physics. A number of models are proposed to explain the available observational data or provide tests for ongoing surveys and experimental programs. In conditions of current entire uncertainty regarding the nature of DM, the extreme observational data i.e. astrophysical objects either with apparent lack of DM or those formed mainly of DM, can be of particular interest especially for testing modified gravity models. In general, extremal experimental data are known to be instrumental, at least in ruling out certain theoretical models or approaches.

The recently discovered galaxies NGC1052-DF2 [110, 111] and NGC1052-DF4 [112] are claimed to show no indication for DM in their structures. Among the other extreme cases one can consider the ultra diffuse galaxy (UDG) Dragonfly 44 [113] as made up almost mostly (98%) of DM.

These DM-extremal galaxies we use below to test one of recent approaches to describe the dark sector based on a modification of GR when the cosmological constant Λ enters its weak-field limit [39, 40, 41]. That approach follows directly from Newton theorem on the equivalency of sphere's gravity and that of a point mass situated in its center. Within that approach both the dark matter and dark energy are determined by cosmological constant Λ which acts as a second fundamental constant of gravity along with G [42]. Then, the DM is defined by weak-field limit of GR [40, 43]. That Λ-gravity approach enables to explain the dynamical properties of groups and clusters of galaxies [39, 43] and also the H-tension [44]. We also mention the recent study [114], where the non-particle nature of the DM is concluded. Note that, while the role of cosmological constant has been considered in context of phenomenological (isothermal) galactic models [115], the analysis based on Newton theorem offers deeper insights mentioned above, also now far more advanced observational data are available.

The DM extreme galaxies can therefore provide an informative testing option for modified gravity theories, thus complementing the possibilities of gravity lensing [66, 41], celestial mechanics [64], galaxy cluster dynamics [147, 148, 47], cosmological perturbation evolution [109] and dedicated GR experimental programs [118, 60].

3.3.1 Newton theorem and Λ-gravity

The Newton theorem on "sphere-point" equivalency enables to arrive to the weak-field modification of General Relativity, i.e. given by the metric [40] ($c = 1$)

$$g_{00} = 1 - \frac{2Gm}{r} - \frac{\Lambda r^2}{3}; \qquad g_{rr} = \left(1 - \frac{2Gm}{r} - \frac{\Lambda r^2}{3}\right)^{-1}. \tag{3.11}$$

Table 3.10: Background geometries for GR solutions

Sign	Spacetime	Isometry group	Curvature
$\Lambda > 0$	de Sitter (dS)	O(1,4)	+
$\Lambda = 0$	Minkowski (M)	IO(1,3)	0
$\Lambda < 0$	Anti de Sitter (AdS)	O(2,3)	-

This metric was known before (Schwarzschild - de Sitter metric), however when deduced based on Newton theorem it provides a description of astrophysical structures (e.g. of galaxy clusters) in weak-field limit of GR [41].

The general function for force $\mathbf{F}(r)$ satisfying Newton's theorem has the form (see [38, 39, 40])

$$\mathbf{F}(r) = \left(-\frac{A}{r^2} + Br\right)\hat{\mathbf{r}}\ . \tag{3.12}$$

The second term here leads to the cosmological term in the solutions of Einstein equations and the cosmological constant Λ appears in weak-field GR [42].

The Λs entering in Eq.(3.11) and Eq.(3.12) have clear group theory background. Namely, depending on the sign of Λ - positive, negative or zero - one has three different vacuum solutions for Einstein equations corresponding to isometry groups, as shown in Table 3.10.

These maximally symmetric Lorentzian 4D-geometries have Lorentz group O(1,3) as their isometry stabilizer group. The group O(1,3) of orthogonal transformations in these Lorentzian geometries implies spherical symmetry (in Lorentzian sense) at each point of spacetime, i.e.

$$dS = \frac{O(1,4)}{O(1,3)}, \quad M = \frac{IO(1,3)}{O(1,3)}, \quad AdS = \frac{O(2,3)}{O(1,3)}. \tag{3.13}$$

In non-relativistic limit the full Poincare group IO(1,3) is reduced to the so-called Galilei group Gal(4)=(O(3)$\times$R)$\ltimes$R^6, which can be considered as the action of O(3)$\times$R on group of boosts and spatial translations R^6. Similarly, for non-relativistic limit of O(1,4) and O(2,3) groups one has

$$O(1,4) \to (O(3) \times O(1,1)) \ltimes R^6, \quad O(2,3) \to (O(3) \times O(2)) \ltimes R^6. \tag{3.14}$$

Consequently, the Galilei spacetime is achieved via quotienting Gal(4) by O(3)×R^3, while the Newton-Hooke $NH^{\pm}$ spacetimes are given by the same quotient group but for groups achieved in Eq.(3.14). For all these cases O(3) is the stabilizer group of spatial geometry, that is each point (in spatial geometry) admits O(3) symmetry. This statement can be regarded as group theory formulation of Newton theorem.

It is important that, the force of Eq.(3.12) defines non-force-free field inside a spherical shell, thus drastically contrasting with Newton's gravity when the shell has force-free field in its interior. The non-force-free field agrees with the observational indications that galactic halos do determine features of galactic disks [55]. The weak-field GR is able to describe the observational features of galactic halos [39, 62], of groups and clusters of galaxies [41].

3.3.2 DM-extremal galaxies

The weak-field GR given by Eqs.(3.12) and (3.11) has been applied to describe the dynamics of galactic halos, galaxy groups and clusters [39, 43] by means of the virial theorem for the gravitational potential containing besides the Newtonian term also the one with the cosmological constant Λ. So, as in [43] at comparison with observational data, the current numerical value of the cosmological constant Λ has to be smaller than the error of velocity dispersion

$$|1 + \Lambda \frac{c^2 R_{vir}^3}{6GM_{vir}}| \leq (\frac{\sigma \pm \mathbb{E}(\sigma)}{\sigma})^2. \tag{3.15}$$

We will now extend such an analysis to two categories of extremal cases i.e. to galaxies with no DM and galaxies made up of DM only. Indeed, the analysis of extreme cases can pose strict constraints over various theories of gravity and even rule them out [119].

DM-missing galaxies

For the galaxies without DM we study the recently discovered NGC 1052-DF2 [110, 111] and NGC 1052-DF4 [112]. Then, the upper limit over the Λ for first case will be

$$\Lambda \leq 6.41 \times 10^{-49}, \tag{3.16}$$

while for the second case

$$\Lambda \leq 7.05 \times 10^{-50}. \tag{3.17}$$

DM-rich galaxy

For the other extremal category we check the structure of Dragonfly 44 as one of best known ultra diffuse galaxies (UDG) [113]. Here by considering the total dynamical mass M_{dyn} within the half-light radius i.e. $r = 4.3\ kpc$ equal to $0.7^{+0.3}_{-0.2} \times 10^{10} M_{\odot}$ we have

$$\Lambda \leq 3.82 \times 10^{-48}. \tag{3.18}$$

Thus by considering the results of both categories of objects - galaxies lacking DM and the one made almost entirely of DM - it turns out that the modification of gravity according to Eq.(3.11) not only is able to describe these structures, but fits the considered weak-field GR with the numerical value of Λ not contradicting the observational data on these extremal astrophysical structures.

3.3.3 DM deficient dwarf galaxies

Besides the above two extremal categories of galaxies, a new group denoted as *DM deficient dwarf galaxies* has been studied in [120]. For them it has been reported that the matter content consists mainly of baryons. We start our discussion by checking the velocity of galaxies according to Eq.(3.11) i.e.

$$V_{cir}^2 = \frac{GM_{dyn}}{r} - \frac{\Lambda c^2 r^2}{3}, \tag{3.19}$$

where M_{dyn} is the total dynamical mass. Thus, by taking the reported values of these galaxies we find the error limits of Λ. The results are shown in Table 3.11. The $w20$ denotes the 20% of the HI line width which has been considered as indicator of the gas velocity. Considering the results of Table 3.11, it becomes clear that again there is no contradiction between Λ-modified gravity and the observed parameters of the galaxies.

Table 3.11: Constraints on Λ for DM deficient dwarf galaxies

Galaxy	$\log M_{\rm dyn}(M_\odot)$	$w20$ (km/s)	$w20_{er}$ (km/s)	Λ (m^{-2}) $\leq$
AGC 6438	9.444	80.36	2.03	9.57×10^{-50}
AGC 6980	9.592	56.63	1.54	6.37×10^{-51}
AGC 7817	9.061	82.37	4.45	1.36×10^{-48}
AGC 7920	8.981	79.03	2.6	9.47×10^{-49}
AGC 7983	9.046	46.12	0.83	1.52×10^{-50}
AGC 9500	9.092	39.08	0.31	2.02×10^{-51}
AGC 191707	9.08	49.27	1.21	2.64×10^{-50}
AGC 205215	9.706	72.5	4.41	3.65×10^{-50}
AGC 213086	9.8	78.35	4.33	3.43×10^{-50}
AGC 220901	8.864	45.38	0.74	2.91×10^{-50}
AGC 241266	9.547	52.82	1.98	7.08×10^{-51}
AGC 242440	9.467	42.47	1.18	2.06×10^{-51}
AGC 258421	10.124	87.79	8.53	2.63×10^{-50}
AGC 321435	9.204	56.83	4.41	1.08×10^{-49}
AGC 331776	8.503	29.59	2.9	6.79×10^{-50}
AGC 733302	9.042	48.36	0.99	2.37×10^{-50}
AGC 749244	9.778	70.87	4.91	2.59×10^{-50}
AGC 749445	9.264	54.51	3.06	4.67×10^{-50}
AGC 749457	9.445	58.68	5.49	5.16×10^{-50}

3.3.4 DM-free dwarf spheroidals in the Local Group

In addition to above extreme cases, 62 dwarf spheroidals (dSphs) in the Local Group (LG) are considered as another sample to analyze the validity of different modified theories of gravity and the paradigm of DM. Namely, the study of dSphs surrounding the Milky Way has suggested those are DM-free structures [121]. Here, by considering Eq.(3.15) we have obtained error limits of Λ for them. The results are exhibited in Table 3.12.

Table 3.12: Constraints on Λ for 24 dwarf galaxies surrounding the Milky Way

Galaxy	$\sigma(km/s^2)$	r (pc)	Λ (m^{-2}) $\leq$
Aquarius2	5.4 ± 3.4	160.0 ± 24.0	1.32×10^{-46}
Bootes1	2.4 ± 0.9	192.5 ± 5.039	9.72×10^{-48}
Carina	6.6 ± 1.2	303.1 ± 2.952	1.32×10^{-47}
Coma	4.6 ± 0.8	68.59 ± 3.615	1.18×10^{-46}
CraterII	2.7 ± 0.3	1066 ± 86	1.05×10^{-49}
CVenI	7.6 ± 0.4	437.9 ± 12.59	2.28×10^{-48}
CVenII	4.6 ± 1.0	70.83 ± 11.22	1.42×10^{-46}
Draco	9.1 ± 1.2	222.4 ± 2.079	3.30×10^{-47}
Draco2	2.9 ± 2.1	20.73 ± 7.639	2.71×10^{-45}
Fornax	11.7 ± 0.9	792.5 ± 2.837	2.44×10^{-48}
Hercules	3.7 ± 0.9	221.1 ± 17.4	1.07×10^{-47}
LeoI	9.2 ± 1.4	287.9 ± 2.133	2.35×10^{-47}
LeoII	6.6 ± 0.7	164.7 ± 1.926	2.52×10^{-47}
LeoIV	3.3 ± 1.7	114.3 ± 12.03	7.59×10^{-47}
LeoV	2.3 ± 3.2	50.41 ± 16.15	6.90×10^{-46}
Sagittarius	11.4 ± 0.7	1636.0 ± 52.78	4.31×10^{-49}
Sculptor	9.2 ± 1.4	276.4 ± 0.9872	2.54×10^{-47}
Segue1	3.9 ± 0.8	24.11 ± 2.79	8.31×10^{-46}
Sextans	7.9 ± 1.3	412.1 ± 2.993	9.19×10^{-48}
TucanaII	8.6 ± 3.5	156.3 ± 23.68	2.08×10^{-46}
UMaI	7.6 ± 1.0	234.2 ± 10.01	2.07×10^{-47}
UMaII	6.7 ± 1.4	136.3 ± 5.325	7.83×10^{-47}
UMi	9.5 ± 1.2	407.0 ± 2.0	1.02×10^{-47}
Willman1	4.3 ± 2.3	27.7 ± 2.4	2.29×10^{-45}

Moreover, considering Eq.(3.11), the radial acceleration will be written as

$$a(r) = -\nabla\Phi = \frac{GM_{tot}}{r^2} - \frac{\Lambda c^2 r}{3} \tag{3.20}$$

where M_{tot} is the total mass (both ordinary and DM) of the configuration according to [122]. Consequently, the constrains over Λ will be obtained. For 20 of them these limits are shown in Table 3.13.

Table 3.13: Constraints on Λ for 20 dwarf spheroidals of LG

Galaxy	$\log a(r)(m/s^2)$	r (pc)	Λ $(m^{-2}) \leq$
Bootes I	-11.14 $\pm$ 0.15	283 $\pm$ 7	8.09×10^{-48}
Bootes II	-9.75 $\pm$ 0.63	61 $\pm$ 24	2.41×10^{-45}
Canes Venatici I	-11.06 $\pm$ 0.05	647 $\pm$ 27	1.58×10^{-48}
Canes Venatici II	-10.69 $\pm$ 0.19	101 $\pm$ 5	7.75×10^{-47}
Carina	-10.81 $\pm$ 0.18	273 $\pm$ 45	2.08×10^{-47}
Coma Berenices	-10.59 $\pm$ 0.16	79 $\pm$ 6	1.08×10^{-46}
Draco	-10.48 $\pm$ 0.12	244 $\pm$ 9	3.54×10^{-47}
Fornax	-10.77 $\pm$ 0.08	792 $\pm$ 58	3.90×10^{-48}
Hercules	-11.11 $\pm$ 0.22	175 $\pm$ 22	1.90×10^{-47}
Hydra II	-10.65 $\pm$ 0.12	88 $\pm$ 17	6.64×10^{-47}
Leo I	-10.56 $\pm$ 0.06	298 $\pm$ 29	1.29×10^{-47}
Leo II	-10.71 $\pm$ 0.14	219 $\pm$ 52	2.65×10^{-47}
Leo IV	-11.15 $\pm$ 0.47	149 $\pm$ 47	3.39×10^{-47}
Leo V	-11.35 $\pm$ 0.88	125 $\pm$ 47	3.35×10^{-47}
Leo T	-10.47 $\pm$ 0.19	160 $\pm$ 10	8.12×10^{-47}
Sculptor	-10.58 $\pm$ 0.13	311 $\pm$ 46	2.36×10^{-47}
Sextans	-11.09 $\pm$ 0.15	748 $\pm$ 66	3.43×10^{-48}
Ursa Minor	-10.66 $\pm$ 0.12	398 $\pm$ 44	1.43×10^{-47}
Ursa Major I	-11.35 $\pm$ 0.88	125 $\pm$ 47	3.35×10^{-47}
Ursa Major II	-10.47 $\pm$ 0.19	160 $\pm$ 10	8.12×10^{-47}

3.3.5 Early type galaxies

In this section we extend our analysis of Λ-modified gravity regarding the nature of DM, on the data of early type galaxies [123, 124]. Namely, by taking the HI circular velocity according to Eq.(3.19), we obtain the error limits of Λ as it is shown in Table 3.14.

3.4 H_0 tension: clue to common nature of dark sector?

Recent measurements [125] increase the existing tension between the Hubble constant determinations from Planck satellite data [16] and lower redshift observations; the earlier studies and various approaches for resolving the tension are discussed in [125].

We will consider the H_0 tension within the approach of weak-field modified GR which enabled the common description of the dark matter and dark energy by means of the same

Table 3.14: Constraints on Λ for early type galaxies

Galaxy	$V_{circ}(HI)(km/s)$	R_{HI} (arcsec)	Λ (m^{-2}) $\leq$
NGC 2685	144 ± 10	320	1.56×10^{-49}
NGC 2824	162 ± 10	40	1.79×10^{-48}
NGC 2859	215 ± 41	115	1.41×10^{-48}
NGC 2974	310 ± 10	130	2.18×10^{-49}
NGC 3522	121 ± 8	85	6.00×10^{-49}
NGC 3626	169 ± 8	120	7.25×10^{-49}
NGC 3838	159 ± 14	150	5.15×10^{-49}
NGC 3941	148 ± 8	195	7.25×10^{-49}
NGC 3945	237 ± 13	130	9.90×10^{-49}
NGC 3998	246 ± 20	195	1.98×10^{-48}
NGC 4203	197 ± 35	195	1.29×10^{-47}
NGC 4262	169 ± 10	120	1.69×10^{-48}
NGC 4278	256 ± 26	150	3.46×10^{-48}
NGC 5582	258 ± 10	210	2.25×10^{-49}
NGC 6798	190 ± 8	150	1.41×10^{-49}
UGC 06176	144 ± 14	60	9.99×10^{-49}

value of the cosmological constant [40, 39, 42]. That approach is based on the Newton's theorem on the equivalency of the gravity of the sphere and of a point situated in its center and provides a natural way for the weak-field modification of GR, so that dark energy is described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) equations while the dark matter in galaxy groups and clusters is described by the weak-field GR.

It is a principal fact that by now both the strong field GR has been tested by the discovery of gravitational waves, while the weak-field effects such as at the frame-dragging are traced by measurements of laser ranging satellites [60]. The weak-field modifications we are discussing below are by now far from being tested at satellite measurements and therefore the dynamical features of the local universe including of the galactic dark halos [62], galaxy groups [59, 43], can serve as unique probes for such weak-field modifications of GR. Among other modified gravity tests are the accurate measurements of gravitational lenses [41], along with the effects in the Solar system [64] or traced from large scale matter distribution [109].

Thus, we show that if the cosmological constant Λ describes both the accelerated expansion and dark matter at galaxy cluster scales, then it will lead to the intrinsic discrepancy in the global and local values of the Hubble constant.

3.4.1 Newton's theorem and Λ

In [40] it is shown that the weak-field GR can involve the cosmological constant Λ, so that the metric tensor components have the form

$$g_{00} = 1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3}; \; g_{rr} = (1 - \frac{2Gm}{rc^2} - \frac{\Lambda r^2}{3})^{-1}. \tag{3.21}$$

This follows from the consideration of the general function for the force satisfying Newton's theorem on the identity of sphere's gravity and that of a point situated in its center and crucially, then shell's internal gravity is no more force-free [38]. Namely, the most general form of the function for the gravitational force which satisfies that theorem is

$$F(r) = C_1 r^{-2} + C_2 r, \tag{3.22}$$

where C_1 and C_2 are constants of integration; for derivation and discussion see [38, 39]. The first term in Eq.(3.22) corresponds to the ordinary Newtonian law, and once the modified Newtonian law (for the potential) is taken as weak-field GR, one has Eq.(3.21), where the second constant C_2 corresponds to Λ (up to a numerical coefficient and c^2) [40, 39]. Namely, the second constant Λ, on the one hand, acts as the cosmological constant in the cosmological solutions of Einstein equations, on the other hand, enters in the low-energy limit of GR which hence is attributed to the Hamiltonian dynamics of galaxy groups and clusters [42], instead of commonly used Newtonian potential.

Within isometry group representation the Lorentz group O(1,3) acts as stabilizer subgroup of isometry group of 4D maximally symmetric Lorentzian geometries and depending on the sign of Λ (+,-, 0) one has the non-relativistic limits [40]

$$\begin{aligned}
&\Lambda > 0 : O(1,4) \to (O(3) \times O(1,1)) \ltimes R^6, \\
&\Lambda = 0 : IO(1,3) \to (O(3) \times R) \ltimes R^6, \\
&\Lambda < 0 : O(2,3) \to (O(3) \times O(2)) \ltimes R^6.
\end{aligned} \tag{3.23}$$

The O(3) is the stabilizer group for the spatial geometry since for all three cases the spatial algebra is Euclidean

$$E(3) = R^3 \rtimes O(3). \tag{3.24}$$

Thus, the Newton's theorem in the language of group theory can be formulated as each point of spatial geometry admitting the O(3) symmetry.

An important consequence of Eq.(3.22) is that the linear term (related to C_2 constant) can produce a non-zero force inside the shell. This is a unique feature since the pure Newtonian gravity according to Gauss' law cannot influence anything inside the shell. Furthermore, this mathematical feature of Eq.(3.22) can be considered as agreeing with the observational indications that the properties of galactic disks are determined by halos, see [39].

3.4.2 Local and global Hubble flows with Λ

The Hubble-Lemaitre law as one of established pillars of modern cosmology is characterized by the Hubble constant H_0 which can be derived by various ways depending on the observational dataset. Namely, the Planck satellite provided the data on CMB which within the ΛCDM model led to the following *global* value $H_0 = 67,66 \pm 0.42 kms^{-1}Mpc^{-1}$, as well as $\Lambda = 1.11 \times 10^{-52} m^{-2}$ [86]. The recent analysis of Cepheid variables in Large Magellanic Cloud (LMC) by Hubble Space Telescope (HST) [125] led to the *local* value $H = 74.03 \pm 1.42 kms^{-1}Mpc^{-1}$. This discrepancy between the global and local values of the Hubble constant is the above mentioned tension.

Our Universe is considered to be described by FLRW metric

$$ds^2 = -c^2 dt^2 + a^2(t)(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2), \tag{3.25}$$

where depending on the sign of sectional curvature k, the spatial geometry can be spherical $k = 1$, Euclidean $k = 0$ or hyperbolic $k = -1$. Consequently, the 00-component of Einstein equations for this metric is written as

$$H^2 = -\frac{k^2 c^2}{a^2(t)} + \frac{\Lambda c^2}{3} + \frac{8\pi G\rho}{3}, \tag{3.26}$$

where $H = \dot{a}(t)/a(t)$ is the Hubble constant.

Here an important point is the following. The Hubble-Lemaitre law originally was established for a sample of nearby galaxies, which are members of the Local Group. For them the empirical Hubble-Lemaitre law seemed to confirm the FLRW equations, however, later it became clear that not only the galaxies have their peculiar velocities but the Local Group itself is gravitationally bounded to a larger configuration, see [126]. In other words, that law was observed at scales for which it should not be observed. Nevertheless, in spite of this apparent contradiction the local flow has been confirmed by observations: the detailed analysis of the nearby galaxy surveys reveal the local Hubble flow with $H_{loc} = 78 \pm 2kms^{-1}Mpc^{-1}$ [127].

We will now show that considering Eq.(3.21) as the weak-field limit of GR, it is possible to solve this tension. Namely, the global Hubble flow will be described by the cosmological constant of FLRW metric, while the local flow by the weak-field GR given by Eq.(3.21).

So, we are not allowed to use FLRW metric in local scales since the Local Supercluster galaxies do not move by FLRW geodesics. On the other hand due to attractive nature of pure Newtonian gravity one cannot produce a repulsive force to cause the local Hubble flow. However, if we consider the additional linear term of Eq.(3.22) the Λ-term can cause

Table 3.15: Critical distance for different objects

Central Object	Mass (Kg)	Radius (m)
Earth	5.97×10^{24}	4.92×10^{16}
Sun	$2 \times 10^{30} = M_{\odot}$	3.42×10^{18}
Sgr A*	$4.3 \times 10^{6} M_{\odot}$	5.56×10^{20}
Milky Way	$1.5 \times 10^{12} M_{\odot}$	3.91×10^{22}
Local Group	$2 \times 10^{12} M_{\odot}$	4.31×10^{22}

a repulsive acceleration as

$$a = -\frac{GM}{r^2} + \frac{\Lambda c^2 r}{3}. \tag{3.27}$$

It is simple to find out the distance at which the acceleration of ordinary Newtonian term becomes subdominant with respect to the second term. In Table 3.15 the values for such distances are listed for different mass scales. For objects less massive than the Local Group (LG), that *critical distance* is located outside the object's boundary, which means that it cannot be observed. For LG, the critical distance is around 1.4 Mpc. Here, it is worth to mention that, since we have used Eq.(3.21) according to Newton's theorem, this distance can be considered as the radius of a sphere which the whole mass of LG is concentrated at its center. Thus, we conclude that, for those objects located outside this radius we will be able to observe an outward acceleration. These results obtained based on Newton's theorem are in agreement with other analysis [130].

Meantime considering the weak-field limit according to Eq.(3.21), one can obtain the analogue of Eq.(3.26) for the non-relativistic case

$$H^2 = \frac{\Lambda c^2}{3} + \frac{8\pi G\rho}{3}. \tag{3.28}$$

In spite of apparent similarity of Eq.(3.28) and Eq.(3.26), there is an important difference between them. Indeed, in Eq.(3.28) $k = 0$ and ρ stands only for matter density (baryonic and non-baryonic), while ρ in Eq.(3.26) includes the contribution also of radiation density;

in this context see the comparative discussion on FLRW and McCrea-Milne cosmologies in [131]. Thus, one can conclude that the H observed by HST in local scales is not the one obtained via Eq.(3.26) by considering the FLRW metric. It is a local effect which can be described by Eq.(3.28). However, before considering the weak-field limit equations for local flow, first let us take a look at the Eq.(3.21) itself. According to principles of GR, the weak-field limit is defined when $\phi/c^2 \ll 1$, where ϕ is the weak-field potential. Now, by taking this into consideration, besides the Newtonian term a new limit is defined at large distances

$$\frac{\Lambda r^2}{3} << 1,\, r \simeq 1.46\, 10^{26} m = 5.33 Gpc. \tag{3.29}$$

Considering the fact that, the local Hubble flow is observed in few Mpc scales, we are allowed to use the Eq.(3.28) to describe that flow. By taking cosmological parameters [86], Eq.(3.26) confirms that the total matter density in our Universe is $\rho = 2.68\, 10^{-27} kgm^{-3}$. However, by substituting $H = 74.03 \pm 1.42 kms^{-1}Mpc^{-1}$, the matter density which causes the observed local Hubble flow will be $\rho_{loc} = 4.37^{+0.40}_{-0.39}\, 10^{-27} kgm^{-3}$.

Now, in order to complete our justification we need to check the mean density of the local astrophysical structures. From hierarchical point of view the LG is located about 20 Mpc away from Virgo cluster[128]. The, Virgo cluster itself together with LG is in a larger Virgo supercluster [132], which itself is the part of Laniakea supercluster [129]. Considering the mass and their distances from LG, it is possible to find the distance where the density of these objects become exactly equal to ρ_{loc}. These results are exhibited in Table 3.16.

From these results it becomes clear that not only the error bars fully cover each other, but also the whole range of the local flow is covered by these values i.e. from 1.70 to 7.07 Mpc. Meantime, according to Eq.(3.27) the critical distance of Virgo supercluster from LG roughly is 7.27 Mpc, which means that the objects beyond that distance are gravitationally bounded to the supercluster. Considering the upper limit of Table 3.16 it turns out that there is no overlapping between the bounded objects and those who move away according

Table 3.16: Distances of objects where the density is ρ_{loc}

Object	Mass (Kg)	Distance from LG (Mpc)
Local Group	$2\times10^{12}M_\odot$	1.95 ± 0.06
Virgo cluster	$1.2\times10^{15}M_\odot$	$3.45^{0.48}_{0.52}$
Virgo supercluster	$1.48\times10^{15}M_\odot$	$2.26^{0.51}_{0.56}$
Laniakea	$10^{17}M_\odot$	$5.00^{2.07}_{2.29}$

to Eq.(3.28). Furthermore, these values exactly coincide with the density of Virgo cluster at distances in which Virgocentric flow changes to the FLRW linear Hubble-Lemaitre law [130].

Thus, the H_0 tension is not a calibration discrepancy but is a natural consequence of presence of Λ in GR as well as weak-field limit equations. While for global value we have to consider the Eq.(3.26) as the immediate consequence of FLRW metric and the cosmological parameters defined as

$$\Omega_k=-\frac{k^2c^2}{a^2(t)H^2},\ \Omega_\Lambda=\frac{\Lambda c^2}{3H^2},\ \Omega_m=\frac{8\pi G\rho}{3H^2}. \tag{3.30}$$

The local value of H is obtained by weak-field limit equations and depends strictly on the local density of matter distribution.

Note that, besides the above mentioned two evaluations of H, other independent measurements also confirm this discrepancy. Among such measurements are those of the Dark Energy Survey (DES) Collaboration, where the so-called inverse distance ladder method based on baryon acoustic oscillations (BAO) is used [133]. Considering the BAO as a standard ruler in cosmology, it turns out that its scale is roughly equal to 150 Mpc which clearly exceeds the typical distance of our local structures (the Virgo cluster etc). Namely, the relevant SNe Ia are located at redshifts $0.018<z<0.85$ [133], which means that according to the Planck data [16] such objects are located at distances $80Mpc<r<3Gpc$. Thus, by comparing these scales with the typical distance to our local structures, one concludes that the measured H for these observations should mainly be induced by cosmological parameters. This statement is justified by their measured value $H = 67.77 \pm 1.30 kms^{-1}Mpc^{-1}$.

Other measurements, again using BAO, are those of [134], where like the DES survey, the distances are $1.8Gpc < r < 6.2Gpc$ and yield $H = 67.6^{+0.91}_{-0.87} kms^{-1}Mpc^{-1}$.

Thus, one can conclude that there are two different Hs of two different scales, *local* and *global* ones. Consequently, the measurement of these two quantities will depend on scales attributed by the observations. Namely, for observations of local scales it is expected to get the local H, while moving to cosmological scales i.e. beyond the Virgo cluster, the measurements should yield the global H.

Note one more important point: although currently the numerical values of these two different Hs are close to each other, their physical content is totally different. Namely, this semi-coincidence is due to the fact that, for the global case the density in Eq.(3.26) is the current mean density in the Universe. At earlier phases of the Universe the radiation density had a major contribution to the mean density

$$\Omega_\rho = \Omega_m + \Omega_r. \tag{3.31}$$

Also, current observations [86] indicate close to zero curvature of the Universe, k=0, and hence Eq.(3.26) will be similar to the weak-field equation. In other words, while for the local flow - no matter in which era - the contribution of matter density would have been the dominant one, for the global flow the contribution to the density in Eq.(3.26) was different for other cosmological eras where the radiation and k were not negligible.

Considering the FLRW metric's Hubble constant i.e. the global H for different eras, one has

$$H(t) = H_0[\Omega_m a^{-3}(t) + \Omega_r a^{-4}(t) + \Omega_k a^{-2}(t) + \Omega_\Lambda]^{1/2}, \tag{3.32}$$

where H_0 is the current value of the global Hubble constant. In this sense, the above statement about the differences between Hs will be also true as the Universe tends to de Sitter phase. In that case, all Ωs except Ω_Λ will gradually tend to zero. But again, for the local measures one still will have the same non-zero matter density.

3.5 On the Lambda-evolution of galaxy clusters

The nature of the dark sector - dark matter and dark energy - remains a major puzzle for fundamental physics in spite of intense observational, experimental and theoretical investigations of the last decades. The recently sharpened H-tension, i.e. the discrepancy between the Hubble constant determinations from *Planck's* data and observations at lower redshift [125, 135], activated the discussions regarding beyond ΛCDM and new physics [136, 137].

The modified weak-field GR provides one of recent approaches to describing the dark sector [39, 40, 41]. That modification is based on Newton's theorem on the identity of gravity of a sphere and of a point mass situated in its center and enables to conclude on the common nature of the dark matter and dark energy, both being described by the cosmological constant [40, 41]. That approach also offers a solution to the H-tension [42]. The cosmological constant Λ within that approach acts as a fundamental constant along with the gravitational constant G [43], with consequences also for the Conformal Cyclic cosmology [87]. The non-particle nature of the dark matter is concluded in [**?**].

If the gravitational interaction at the galaxy cluster scales is defined not only by an attracting force but also by a repulsive force due to the cosmological constant, then the latter can influence the evolution of galaxy clusters. Analogous modified gravity effects are among the discussed ones with respect to various astrophysical systems, from celestial mechanical scales [64] to those of cosmological structure formation [109], whereas the dedicated experimental tests of General Relativity (e.g. [118, 60]) are still not reachable to those effects. Other testing opportunities are provided by the lensing of galaxies [66, 44].

Here we will try to reveal a possible difference in the dynamical evolution of two types of galaxy clusters, those determined by usual Newtonian gravity (standard N-body problem) and those by modified gravity with the repulsion term. We use geometrical methods of the theory of dynamical systems [138, 139] first applied to gravitational N-body dynamics in [140] to describe their chaos and relaxation; for further application of those methods in General Relativity see [141, 142]. The Ricci curvature criterion of relative instability, that we use here, was introduced in [143] and has been applied to different types of gravitational systems (e.g. [144]). Our aim is not the study of entire evolution of galaxy clusters affected by modified gravity, which will need extensive strategy of N-body simulations but to verify if the Λ term is able to influence the cluster dynamics and the evolution. Such an approach appears informative for nonlinear systems, as known since the renown Fermi-Pasta-Ulam study [145].

The results of our analysis indicate that the Λ-gravity does affect the instability features of galaxy clusters. We note that previously it was shown that the cosmological constant is able to introduce a time arrow for the system [63].

3.5.1 Newton's theorem and Λ

Proceeding from the Newton's theorem on the "sphere-point" identity and the resulting weak-field modification of General Relativity, one arrives to the metric [40] ($c = 1$)

$$g_{00} = 1 - \frac{2Gm}{r} - \frac{\Lambda r^2}{3}; \qquad g_{rr} = \left(1 - \frac{2Gm}{r} - \frac{\Lambda r^2}{3}\right)^{-1}. \tag{3.33}$$

The general functions for force and potential i.e. $U(r)$ and $\mathbf{F}(r)$ satisfying Newton's theorem used for the above weak-field limit of GR have the form (for derivation and discussion see [38, 39, 40])

$$U(r) = -\frac{A}{r} - \frac{B}{2}r^2; \qquad \mathbf{F}(r) = -\nabla_r U(r) = \left(-\frac{A}{r^2} + Br\right)\hat{\mathbf{r}}\ . \tag{3.34}$$

Here the second term leads to the cosmological term in the solutions of Einstein equations, so that the cosmological constant Λ enters also the weak-field GR regimes, e.g. in the Hamiltonian dynamics of galaxy clusters [42].

Crucial feature of the force law of Eq.(3.34) is that it defines non-force-free field inside a spherical shell, contrary to Newton's gravity law when the shell has no influence in its interior. In this regard we mention the observational indications that the galactic halos do determine the properties of galactic disks [55]. The weak-field GR thus can be used to describe the observational features of galactic halos [39, 62], of groups and clusters of galaxies [41].

3.5.2 Ricci curvature

The Lagrangian for N-body system interacting by the Λ-potential (from Eq.(3.34)) is

$$L(r,v) = \tfrac{1}{2}\sum_{a=1}^{N} m_a v_a^2 - U(r), \tag{3.35}$$

$$U(r) = -\sum_{a=1}^{N}\sum_{b=1}^{a-1}\frac{G m_a m_b}{|r_a - r_b|} - \frac{\Lambda}{6}\sum_{a=1}^{N} m_a |r_a|^2. \tag{3.36}$$

According to the criterion of relative instability defined in [143], among two systems the more unstable is the one with smaller negative Ricci curvature

$$\mathfrak{r} = \tfrac{1}{3N}\inf_{0\le s\le s_*}\mathfrak{r}_u(s), \qquad \mathfrak{r} < 0, \tag{3.37}$$

within $0 \le s \le s_*$ interval of geodesic in the configurational space.

This criterion follows from the equation of geodesic deviation (Jacobi-Levi-Civita) equation [138, 139] averaged via the deviation vector

$$\frac{d^2 z}{ds^2} = -\frac{1}{3N}\mathfrak{r}_u(s) + \langle \| \nabla_u n \|^2 \rangle, \tag{3.38}$$

where

$$n = z\hat{n}, \qquad \| \hat{n} \|^2 = 1,$$

and $\mathfrak{r}_u(s)$ is the Ricci curvature in the direction of the velocity of the geodesic u, and

$$\mathfrak{r}_u(s) = \frac{\mathrm{Ric}(u,u)}{u^2} = \sum_{\mu=1}^{3N-1} K_{\mathbf{e}_\mu, u}(s), \qquad (\mathbf{e}_\mu \perp u,\ \mathbf{e}_\mu \perp \mathbf{e}_\nu,\ \mu \neq \nu). \tag{3.39}$$

The Ricci tensor for N-body system yields [140, 143]

$$\mathrm{Ric}_{\alpha\beta} = -\frac{1}{2}\frac{\Delta W}{W} g_{\alpha\beta} - \frac{(3N-2)}{2}\frac{W_{\alpha\beta}}{W} + \frac{3(3N-2)}{4}\frac{W_\alpha W_\beta}{W^2} - \frac{(3N-4)}{4}\frac{\|dW\|^2}{W^2} g_{\alpha\beta}, \tag{3.40}$$

where $g_{\alpha\beta} = m_a \delta_{\alpha\beta}$, and

$$W = E - U = E + \sum_{a=1}^{N}\sum_{b=1}^{a-1} \frac{G m_a m_b}{|r_a - r_b|} + \frac{\Lambda}{6}\sum_{a=1}^{N} m_a |r_a|^2 = \tfrac{1}{2}\sum_{a=1}^{N} m_a v_a^2. \tag{3.41}$$

For the Lagrangian of Eq.(3.35) the latter is

$$\begin{aligned}\mathrm{Ric}(v,v) = {} & \frac{(3N-2)}{2W} \sum_{c=1}^{N}\sum_{\substack{a=1\\a\neq c}}^{N} \frac{G m_c m_a}{\rho_{ca}^3}\left(v_c \cdot v_{ca} - 3\frac{(r_{ca}\cdot v_c)(r_{ca}\cdot v_{ca})}{\rho_{ca}^2}\right) \\ & + \frac{3(3N-2)}{4W^2}\left(-\sum_{c=1}^{N}\sum_{\substack{a=1\\a\neq c}}^{N} G m_c m_a \frac{r_{ca}\cdot v_c}{\rho_{ca}^3} + \frac{\Lambda}{3}\sum_{c=1}^{N} m_c r_c \cdot v_c\right)^2 \\ & - \frac{(3N-4)}{2W}\sum_{c=1}^{N} m_c \left| -\sum_{\substack{a=1\\a\neq c}}^{N} G m_a \frac{r_{ca}}{\rho_{ca}^3} + \frac{\Lambda}{3} r_c \right|^2 - \frac{2(3N-1)}{3}\Lambda\ ,\end{aligned} \tag{3.42}$$

where $r_{ab} = r_a - r_b$, $v_a = \dot{r}_a$, $v_{ab} = v_a - v_b$, $\rho_a = |r_a|$, $\rho_{ab} = |r_{ab}|$, for any two vectors $\mathbf{e}_1$ and

$\mathbf{e}_2$ we have $\mathbf{e}_1 \cdot \mathbf{e}_2 = \delta_{ij} e_1^i e_2^j$.

3.5.3 Results

We simulated the dynamics of two types of N-body spherical systems of typical galaxy cluster parameters, one defined by Newtonian gravity, the other defined by an additional Λ-potential i.e. by Lagrangian Eq.(3.35). The Ricci curvature was estimated for both, to see if the instability properties of the both systems according to the criterion Eq.(3.37) do reveal differences during their evolution over cosmological time scale.

To simulate systems with typical parameters of galaxy clusters we used a spherical distribution of $N = 1,000$ particles (galaxies), each of mass $m = 10^{11} M_\odot$, inside a sphere of $R = 1.5$ Mpc. The velocities were defined by considering such galaxy clusters (i.e. of $1,000$ members) as semi-virialized configurations, i.e.

$$\sigma^2 = \frac{GNm}{R} = \frac{GM}{R}, \tag{3.43}$$

where σ^2 is the velocity dispersion of galaxies of the cluster and M is the total mass of the cluster. Consequently, the dynamical time scale for a typical cluster in Newtonian and Λ-modified regime will be

$$t_G = \left(\frac{2R^3}{GM}\right)^{1/2} = 3.86\,\mathrm{Gyr}, \quad t_{G\Lambda} = \left(\frac{2R}{\frac{GM}{R^2} - \frac{\Lambda R}{3}}\right)^{1/2} = 3.90\,\mathrm{Gyr}\,. \tag{3.44}$$

The results of computations using Eqs.(3.37)-(3.42) are shown in Fig.3.1. Comparing both Newtonian and Λ-modified gravity it turns out that the behavior of Ricci curvature $\mathfrak{r}$ for both cases are similar to each other (Fig.3.1). However, while the values of Ricci curvatures practically coincide at time scales up to around 2 Gyrs, at later phases those values for pure Newtonian case are systematically larger than for Λ-modified gravity. According to the criterion Eq.(3.37) in view of the fact that the value $\mathfrak{r}_{G\Lambda}$ is smaller than $\mathfrak{r}_G$ - both having negative *infimum* according to criterion Eq.(3.37) within the cosmological time interval [2.4 Gyr - 17.4 Gyr] - one can conclude that spherical Newtonian systems which at large N limit are known to be exponentially unstable (chaotic) [140], become even more unstable with the Λ-term in the gravity force Eq.(3.34).

Then we performed the same analysis for systems of parameters of superclusters, using the data of the Virgo Supercluster. Note, that there is a principal difference between this case and those of galaxy clusters for Λ-modified gravity. Namely, from Eq.(3.33) and Eq.(3.34) one can define a critical distance scale for a system, where the repulsive term of Λ becomes dominant [42] over the Newtonian gravity

$$r_{crit}^3 = \frac{3GM}{\Lambda}\,. \tag{3.45}$$

For structures of smaller than superclusters' scale this radius lies outside the configuration which means that the role of the Λ term in its properties is suppressed. But for superclusters of scales larger than r_{crit} the role of Λ-term can be felt in the dynamics of galaxies [42]. For Virgo Supercluster that critical radius yields around 12.66 Mpc. In this regard, we checked the behavior of Ricci curvature for three different cases. First, we analysed a system of parameters of the Virgo Supercluster, i.e. R= 16.5 Mpc, $N = 1,480$, and $M = 1.48 \times 10^{15}\ M_\odot$. The results i.e. the difference of Ricci curvature for Newtonian and Λ-modified gravity, are given in Fig.3.1. Then we studied two different cases, i.e. with the same mass and number of particles but for different radii i.e. $R = 18$ (>12.66) Mpc and $R = 10$ Mpc (<12.66). For latter two cases the results are shown in Figs.3.4-3.5. It is interesting that for both, 18 Mpc and 16.5 Mpc (both exceeding the critical distance 12.66 Mpc), as time goes on the difference of Ricci is increasing and even for $R = 18$ Mpc, it becomes positive, which can be interpreted as tending to free particle system. While for the bound structure $R = 10$ Mpc the Ricci curve shows a tendency to decrease, i.e. indicating the unstable

N-body system.

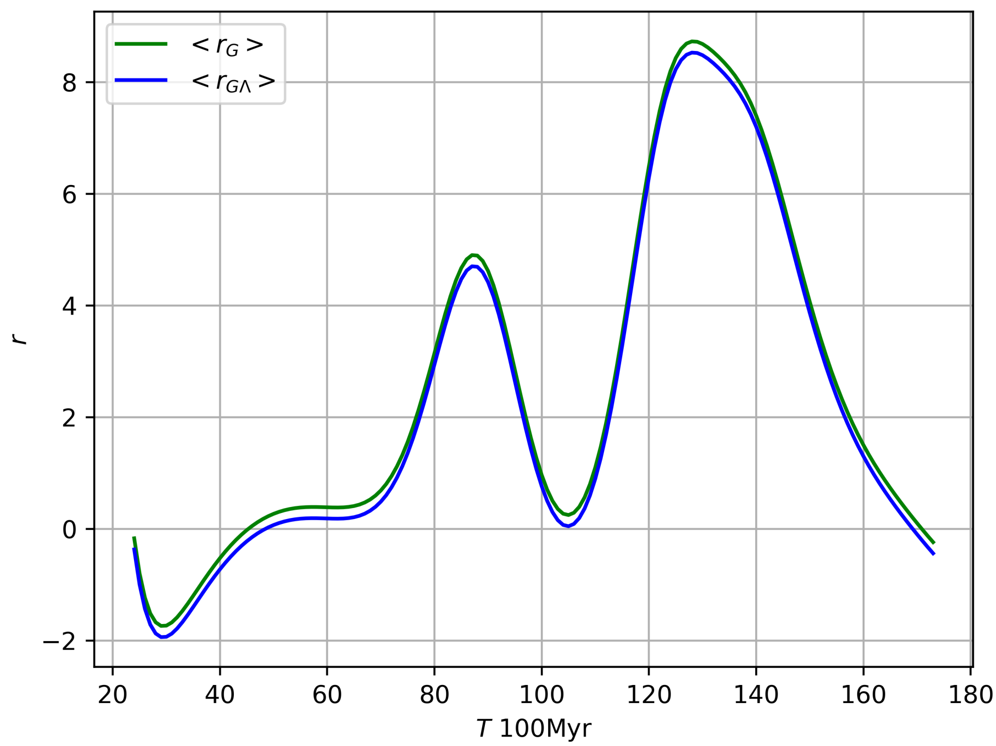

Figure 3.1: The Ricci curvature variation vs cosmological time for galaxy cluster parameters in Newtonian (green) and Λ-modified gravity (blue) regimes.

3.6 Conclusions

In this chapter, we studied the relevance of the weak field GR with modified potential Eq.(3.2) to the description of the dynamics of galaxy systems, tracing galaxy pairs, groups and clusters. It is important that the used data are not of the same origin but were obtained at galactic surveys, gravity lensing studies and by Planck satellite. For the analysed hierarchy of systems of galaxies the numerical value of Λ obtained as a weak field GR without any cosmological considerations is in visible agreement with Λ obtained from relativistic cosmology.

There are at least two aspects to be outlined at the interpretation of the obtained Λs

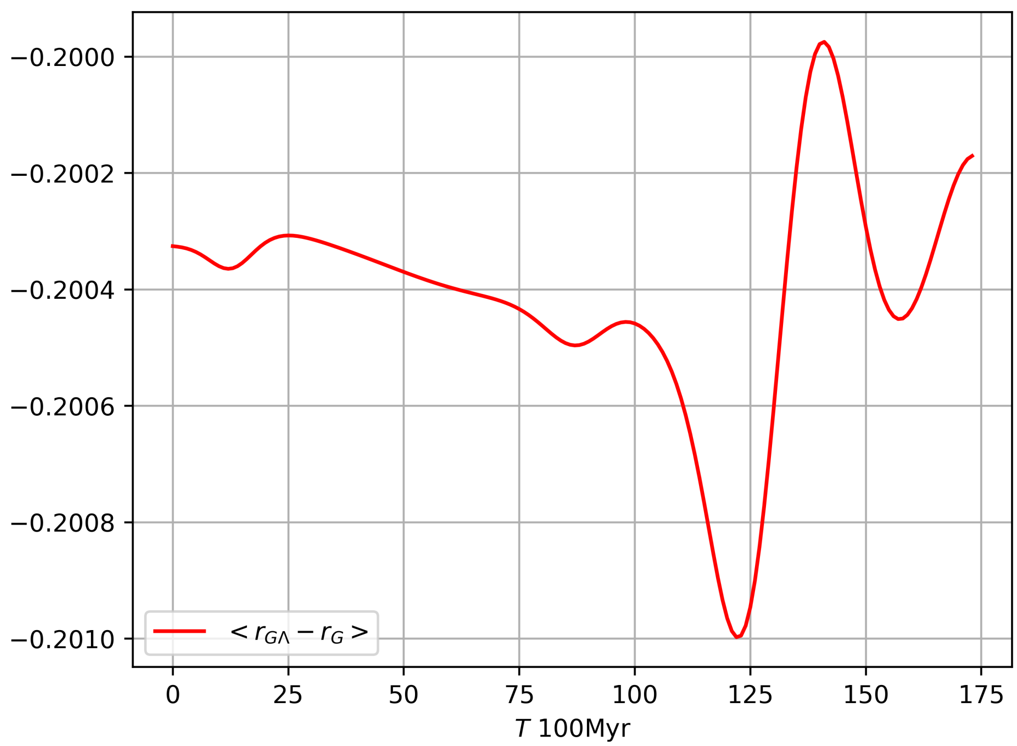

Figure 3.2: The difference of Ricci curvature values of curves in Fig.3.1, i.e. at Newtonian and Λ-modified gravity laws.

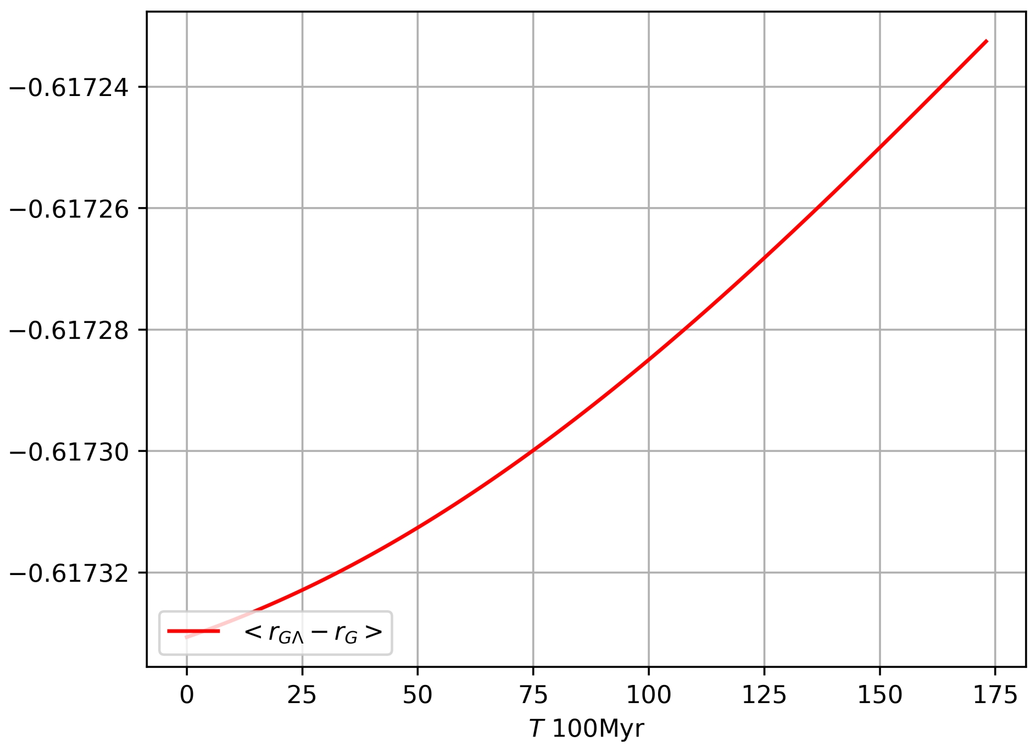

Figure 3.3: The same as in Fig.3.2 but for supercluster (Virgo) parameters.

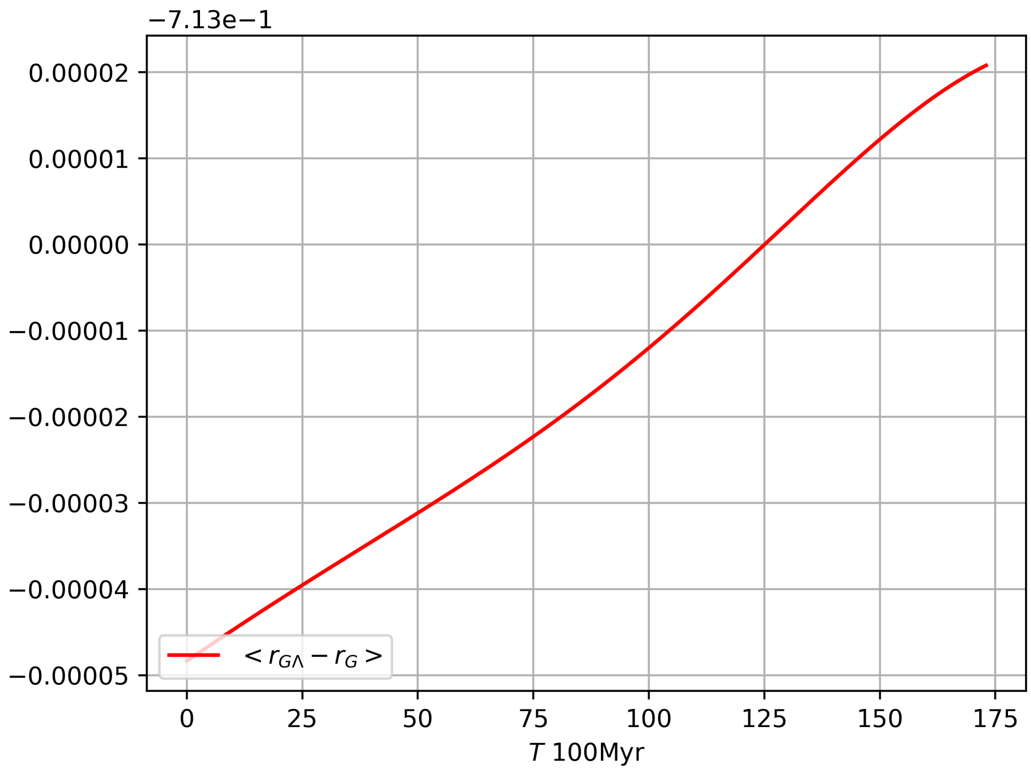

Figure 3.4: The same as in Fig.3.3 but for $R = 18$ Mpc.

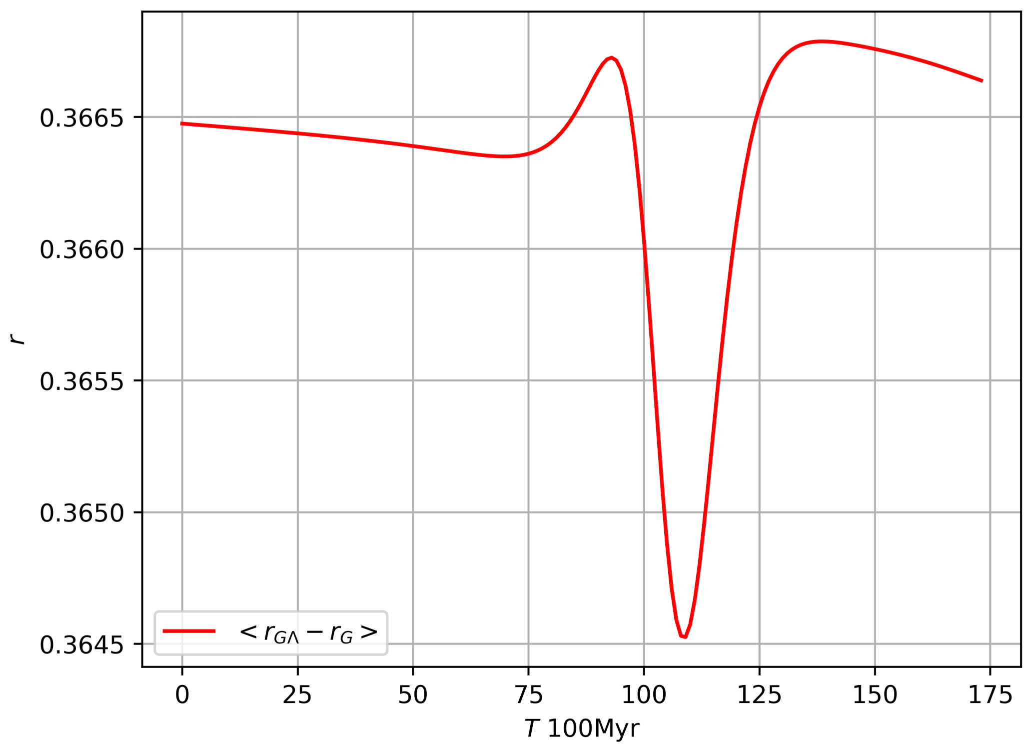

Figure 3.5: The same as in Fig.3.3 but for $R = 10$ Mpc.

regarding both the values and their scatter:

a) Eq.(3.6) used for the estimation of Λ assumes that the virial theorem, i.e. the equipartition is determined purely due to the Λ term, while obviously the conventional Newtonian term's contribution has to be there as well. Therefore, the cosmological constant Eq.(3.5) has to be a lower limit for the Λ values obtained for galaxy systems. Namely, for fully virialized systems the empirical values of Λ have to be close to the cosmological constant, Eq.(3.5), while for others the empirical values have to be higher. Exactly such a behavior is visible in the exhibited Tables;

b) the scatter in values of Λs in Tables is certainly expected due to the various virialization degree reached in each given system (galaxy pair, group, cluster). Decrease of that scatter will be possible when more refined dynamical structure of any individual system can be available.

Thus, while for certain systems e.g. galaxy groups of NGC 2894, NGC 3049, NGC 3810, P 51971 the obtained Λs are close to the cosmological constant value, Eq.(3.5), in their absolute majority they exceed the latter.

The obtained values of Λ are also influenced by the inhomogeneity of the data sources, e.g. by the inevitable difference between the lensing and dynamical masses of galaxy systems or the masses of CMB-SZ clusters (see [105]). The structure and the extension of the dark halos of galaxies [62] are especially relevant for pair galaxies and galaxy groups studies. Note that, a crucial difference of the modified potential Eq.(3.2) from the Newtonian one is that, the second term in Eq.(3.2) defines non-force free field inside a shell [38] and hence e.g. the properties of galactic disks have to be determined by dark halos [39, 62]. The Λ-constant can be associated also to the time arrow [63]. Regarding the Λ-constant, including in the context of the local Universe within equilibria concepts, see [106, 107, 54, 108]. Among other means for probing weak-field modified gravity are the large scale matter distribution (see [109]), the effects in Solar system [64, 60].

Thus, the Λ-constant enables the description both of the accelerated universe and the dynamics of galactic systems. The cosmology is described by GR, while the galaxy system dynamics is described by weak field GR, both containing the same Λ. One can therefore conclude that, the non-zero cosmological constant might be discovered from galactic systems even before high redshift SN surveys and CMB.

Then, by considering the fact that extremal data are known to be sensitive indicators for ruling out certain theoretical models, we studied DM extreme galaxies, i.e. either of anomalously high or low DM content, for testing various models.

Here we used samples of DM-deficient, of a DM-rich galaxy, to see if there are any contradictions regarding the Λ-gravity, i.e. a modified weak-field General Relativity [39, 40]. Using the available observational data we obtained the upper limits for the Λ for dozens of objects claimed as DM-extreme ones. Certainly, if the extreme data do not rule out a model, that still by no means proves the validity of the model, but merely claims the need of even more tight data to narrow the ambiguity windows.

The fact that by now, in spite of intense observational surveys, as well as of experimental studies including on accelerators, there is no even basic judgment on the DM nature, identification and analysis of even more DM-extreme objects can be among informative goals.

On the other hand we proposed a possible solution for H_0 tension. In this sense, H_0 tension can be not a result of data calibration/systematic but a genuine indication for the common nature of the dark matter and dark energy. This conclusion is argued above based on the Newton's theorem and resulting weak-field limit of General Relativity which includes the Λ constant. Within that approach while the Friedmannian equations with the Λ term are describing the accelerated Universe, the same Λ is responsible for the dynamics of galaxy groups and clusters. Correspondingly, the *global* Hubble constant derived from the CMB and the *local* one devised from the galaxy surveys, including within the Local Supercluster, have to differ.

Then, the long known so-called Local Hubble flow [59] i.e. when the galaxies within the Local Supercluster are fitting the Hubble-Lemaître law while the galaxies themselves are not moving via geodesics of FLRW metric, finds its natural explanation within the metric Eq.(3.21). In other words, the local value of H_0 has to take into account the contribution of the cosmological constant (as entering the weak-field GR) in the kinematics of the galaxies along with the observed value of the mean density of matter.

Accurate studies of the dynamics of galactic halos, groups and galaxy clusters, the gravitational lensing, can be decisive for further probing of described weak-field GR and the common nature of the dark sector.

In this sense, the weak-field GR modified based on the Newton theorem enabled one the common description of the dark matter and dark energy [39, 40], as weak-field and cosmological manifestation of GR, respectively, both determined by the cosmological constant.

As further step to probe that GR modification, we analysed its possible role in the evolution of the galaxy clusters, i.e. at spatial scales where the repulsive gravity term becomes non-negligible. We used the Ricci curvature criterion to follow the comparative instability of two type of spherical systems, i.e. those evolving according to modified Λ-gravity of Eqs.(3.33) and (3.34) with respect to usual Newtonian systems.

Our main conclusions can be formulated as follows:

(a) the studied types of systems of galaxy cluster parameters do reveal discrepancy in their instability properties during the evolution, namely, the Λ-modified gravity systems tend to become more unstable with respect to those described by Newtonian law. The discrepancy starts to be visible at cosmological times i.e. at time scales exceeding roughly 2 Gyr.

(b) the supercluster (Virgo) parameter systems reveal differences in their instability properties depending on their spatial scales. Namely, at distance scales where the Λ-term dominates over the Newtonian gravity, the systems tend to free particle systems at cosmological time scales, while at smaller distances their behavior remains unstable as of the galaxy clusters, as expected.

We note that $f(R)$-gravity, "beyond Horndeski" covariant Galileon models have been already used to describe the observable features of clusters of galaxies [146, 147, 148], thus indicating the suitability of the latter for testing of modified gravity theories.

The study of the evolutionary effects of galaxy clusters at dedicated numerical simulations (including using advanced methods of the theory of dynamical systems [149]) can provide additional tests to Λ-gravity as the weak-field limit for General Relativity.

Conclusions

The main results of this thesis are as follows:

1- A critical test of gravitational lensing is proposed for the parameter of the parametrized post-Newtonian (PPN) formalism $\gamma = 0.998$ (normalized to given lens mass and light impact distance), which if observed at gravity lenses with proper significance will reveal the weak-field modification of General Relativity.

2- The weak-field limit of General Relativity based on the Newton's theorem on sphere-point identity reveals that the gravity is defined by two fundamental constants, the gravitational constant G and the cosmological constant Λ. The higher dimensional analysis of Newton theorem shows that in contrast to G, Λ preserves its dimensionality and remains matter-uncoupled and hence can be considered as even more universal than the gravitational constant G.

3- Consideration of Λ as one of the fundamental constants of Nature enables one to construct a whole class of dimensionless quantities and eventually describe the "Information" evolution of the Universe.

4- The unified picture of the Dark Sector - dark matter and dark energy - based on the weak-field modified General Ralativity is shown to be valid for hierarchical astrophysical configurations from galaxy binaries, galaxy groups up to galaxy clusters. Besides the so-called "standard" galaxies, the "extreme" ones regarding the nature of dark matter, are shown to also fit the Λ-gravity.

5- It is shown that Λ-gravity provides a natural explanation for the so-called "H-tension" problem, in agreement with both “local” and “global” measurements of the Hubble constant.

6- Considering the stability problem of N-body gravitating systems of the parameters of galaxy clusters, it is shown that the Λ-gravity makes the configurations more unstable as compared to those governed by Newtonian gravity.

Bibliography

[1] A. Einstein, Preussische Akademie der Wissenschaften, 315 (1915)

[2] A. Einstein, Preussische Akademie der Wissenschaften, 778, 799 (1915)

[3] A. Einstein, Preussische Akademie der Wissenschaften, 831 (1915)

[4] A. Einstein, Preussische Akademie der Wissenschaften, 844 (1915)

[5] A. Einstein, Königlich-Preußische Akademie der Wissenschaften, 142 (1917)

[6] A. Friedmann, Zeitschrift fur Physik A, 10, 377 (1922)

[7] A. Friedmann, Zeitschrift fur Physik A, 21, 326 (1924)

[8] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman, 1973.

[9] P.J.E. Peebles, *Principles of Physical Cosmology*, Princeton University Press, 1993.

[10] V.G. Gurzadyan, A.A. Kocharyan, *Paradigms of the Large-Scale Universe*, Gordon and Breach, 1994.

[11] S. Dodelson, *Modern Cosmology*, Academic Press, 2003.

[12] S. Weinberg, *Cosmology*, Oxford University Press, 2008.

[13] E. Hubble, Proceedings of the National Academy of Sciences, 15, 3, 168 (1929)

[14] E. Zwicky, Helvetica Physica Acta., 6, 110 (1933)

[15] V. Rubin, W. K. Jr. Ford, ApJ. 159, 379 (1970)

[16] P.A.R. Ade, et al, A&A, 594, A13 (2016)

[17] J. Gunn, J.R. Gott, ApJ., 176, 1 (1972)

[18] G. Bertone, *Particle Dark Matter*, Cambridge University Press, (2010)

[19] J. F. Navarro, C. S. Frenk, S. D. M. White, ApJ., 490, 493 (1997)

[20] A. Burkert, ApJL, 447 L25 (1995)

[21] B. Moore et al, MNRAS, 310, 1147 (1999)

[22] E. Tempel, A. Tamm, P. Tenjes, arXiv:0707.4374 (2007)

[23] A. Amekhyan, Int. J. Mod. Phys. D, 28, 2040016 (2019)

[24] D. Merritt, Astron.J., 132, 2685 (2006)

[25] J. Einasto, Trudy Inst. Astrofiz. Alma-Ata, 5, 87 (1965)

[26] H. C. Brans, H. R. Dicke, Physical Review, 270, 365, 124, 3, 925 (1961)

[27] T. P. Sotiriou, V. Faraoni, Rev. Mod. Phys., 82, 451 (2010)

[28] S. Capozziello, M. De Laurentis, Physics Reports, 509, 4–5, 167 (2011)

[29] M. Milgrom, ApJ., 270, 365 (1983)

[30] M. Milgrom, ApJ., 270, 371 (1983)

[31] M. Milgrom, ApJ., 270, 384 (1983)

[32] E. Corbelli, P. Salucci, MNRAS, 374, 1051 (2007)

[33] R. Bottema, J. L. G. Pestana, B. Rothberg, R. H. Sanders, A&A, 393, 453 (2002)

[34] F. J. Sanchez-Salcedo, V. Lora, in *Progress in Dark Matter Research*, ed. J. Val Blain, Nova Publications: New York, (2005)

[35] R. A. Swaters, R. H. Sanders, S. S. McGaugh, ApJ, 718, 380 (2010)

[36] R. H. Sanders, E. Noordermeer, MNRAS, 379, 702 (2007)

[37] I. Newton, *The Principia: Mathematical Principles of Natural Philosophy* (1687)

[38] V.G. Gurzadyan, Observatory, 105, 42 (1985)

[39] V.G. Gurzadyan, Eur. Phys. J. Plus, 134, 98 (2019)

[40] V.G. Gurzadyan, A. Stepanian, Eur. Phys. J. C, 78, 632 (2018)

[41] V.G. Gurzadyan, A. Stepanian, Eur. Phys. J. C, 78, 869 (2018)

[42] V.G. Gurzadyan, A. Stepanian, Eur. Phys. J. C, 79, 169 (2019)

[43] V.G. Gurzadyan, A. Stepanian, Eur. Phys. J. Plus, 134, 98 (2019)

[44] V.G. Gurzadyan, A. Stepanian, Eur. Phys. J. C, 79, 568 (2019)

[45] A. Stepanian, Mod. Phys. Lett. A, 34, 1975002 (2019)

[46] A. Amekhyan, S. Sargsyan, A. Stepanian, Mod. Phys.Lett. A, 34, 1950308 (2019)

[47] V.G. Gurzadyan, A.A. Kocharyan, A. Stepanian, Eur. Phys. J. C, 80, 24 (2020)

[48] V.G. Gurzadyan, A. Stepanian, Eur. Phys. J. C, (2020)

[49] R.H. Sanders, *The dark matter problem*, Cambridge Univ. Press, 2010

[50] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rept. 513, 1 (2012)

[51] G. Bertone and D. Hooper, Rev. Mod. Phys. 90, 045002 (2018)

[52] S. Bahamonde et al, arXiv:1712.03107

[53] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, 1972

[54] M. Nowakowski, Int. J. Mod. Phys. D10, 649 (2001)

[55] A.V. Kravtsov, Ap J Lett, 764, L31 (2013)

[56] L.D. Landau, E.M. Lifshitz, *Mechanics*, Pergamon Press, 2000

[57] Ya.B. Zeldovich, Soviet Physics Uspekhi, 11, 381 (1968)

[58] S. Capozziello et al, JCAP, 06, 044 (2017)

[59] I.D. Karachentsev, O.G. Kashibadze, V.E. Karachentseva, Astrophys. Bull., 72, 111 (2017)

[60] I. Ciufolini et al., Eur. Phys. J. C, 76, 120 (2016)

[61] L. Heisenberg, S. Tsujikawa, JCAP, 1801, 044 (2018)

[62] V.G. Gurzadyan et al, A & A, 609, A131 (2018)

[63] A.E. Allahverdyan, V.G. Gurzadyan, Phys. Rev. E, 93, 052125 (2016)

[64] S. Kopeikin, M. Efroimsky, G. Kaplan, *Relativistic celestial mechanics of the solar system*, (Wiley, 2001)

[65] C.M. Will, Living Rev. Relativ., 17, 4 (2014)

[66] T.E. Collett, et al, Science, 360, 1342 (2018)

[67] S. Cao, et al., ApJ , 835, 92 (2017)

[68] W. Rindler, *Relativity, Special, General, and Cosmological* (Oxford University Press, 2006)

[69] M. Ishak, W. Rindler, J. Dossett, J. Moldenhauer, C. Allison, MNRAS, 388, 1279 (2008)

[70] R.J. Smith, J.R. Lucey, MNRAS, 434, 1964 (2013)

[71] S. Kopeikin, MNRAS, 399, 1539 (2009)

[72] A.S. Bolton, S.A. Rappaport, S. Burles, Phys. Rev. D, 74, 061501 (2006)

[73] J. Schwab, A.S. Bolton, S.A. Rappaport, ApJ, 708, 750 (2010)

[74] A. Einstein, Science, 84, 506 (1936)

[75] J. S. Farnes, A& A 620, A92 (2018)

[76] H. Socas-Navarro, A &A, A5, 626, (2019)

[77] G.'tHooft, http://www.staff.science.uu.nl/ hooft101/gravitating_misconceptions.html

[78] B.P. Abbott , et al, Phys. Rev. Lett., 061102, 116, (2016)

[79] T. M. Nieuwenhuizen, Fortsch. Phys., 65, 6-8, 1600050 (2017)

[80] M. Gasperini, G. Veneziano, Phys. Rep., 1, 373, (2003)

[81] S. B. Whitehouse, G. V. Kraniotis, arXiv:astro-ph/9911485 (1999)

[82] H. Bacry, J-M. Levy-Leblond, J. Math.Phys. 9, 1605 (1967)

[83] R. Aldrovandi, A. L. Barbosa, L. C. B. Crispino, J. G. Pereira, Class. Quant. Grav. 16, 495 (1999)

[84] G.W. Gibbons, C.E. Patricot, Class. Quant. Grav. 20, 5225 (2003)

[85] Ya.B. Zeldovich, JETP Lett, 6, 316 (1967)

[86] R. Penrose, *Cycles of Time: An Extraordinary New View of the Universe*, Bodley Head, London, 2010.

[87] V.G. Gurzadyan, R. Penrose, Eur.Phys. J. Plus, 128, 22 (2013)

[88] A. Einstein, Phys. Z., 19, 165 (1918)

[89] J-P Uzan, Living Rev. Relativ. 14, 2 (2011)

[90] M. Planck, Ann. Phys. (Berlin), 1, 69 (1900)

[91] J.D. Bekenstein, Phys. Rev. D, 7, 2333 (1973)

[92] G.W. Gibbons, S.W. Hawking, Phys. Rev. D, 15, 2738 (1977)

[93] J.D. Bekenstein, Phys. Rev. D, 23, 287 (1981)

[94] V.G. Gurzadyan, R. Penrose, Eur. Phys. J. Plus, 131, 11 (2016)

[95] E.J. Gonzalez, et al, A & A, 621, A90 (2019)

[96] I.D. Karachentsev, O.G. Nasonova, V.E. Karachentseva, Astrophys. Bull. 70, 1 (2015)

[97] I.D. Karachentsev, V.E. Karachentseva, O.G. Nasonova, Astrophysics, 57, 457 (2014)

[98] J. Merten et al. Astrophys.J., 806, 4 (2015)

[99] R.L. Rehmann, et al, MNRAS, 486, 77 (2019)

[100] AMI Consortium, MNRAS, 425, 162 (2012)

[101] L.S. Sparke, J.S. Gallagher, *Galaxies and the Universe* (Cambridge University Press, 2007)

[102] H. Mo, F. van den Bosch, S. White, *Galaxy Formation and Evolution* (Cambridge University Press, 2011)

[103] S. White, A & A, 367, 1 (2001)

[104] F. Pace, S. Meyer, M. Bartelmann, JCAP 10, 040 (2017)

[105] M. Douspis, L. Salvati, N. Aghanim, arXiv:1901.05289 (2019)

[106] J.D.Barrow, F.J. Tipler, *The Anthropic Cosmological Principle* (Oxford University Press, Oxford, 1986).

[107] J. D Barrow, A. B. Burd, D. Lancaster, Class. Quant. Grav. 3, 551 (1986)

[108] M. Nowakowski, in: *Cosmology on Small Scales 2016*; Local Hubble Expansion and Selected Controversies in Cosmology (Eds. M. Krizek and Y.V. Dumin), p.95, (Publ. Czech Acad. Sci., Prague, 2016)

[109] M. Eingorn, ApJ, 825, 84 (2016)

[110] P. van Dokkum, et al., Nature, 555, 629 (2018)

[111] P. van Dokkum, et al., Research Notes AAS, 2, 54 (2018)

[112] P. van Dokkum, et al., ApJ Letters, 874, L5 (2019)

[113] P. van Dokkum, et al., ApJ Letters, 826, L6 (2016)

[114] N.J. Poplawski, Eur. Phys. J. C, 79, 734 (2019)

[115] R.A. Sussman, X. Hernandez, MNRAS, 345, 871 (2003)

[116] V. Salzano et al, JCAP, 10, 033 (2016)

[117] S. Capozziello et al, MNRAS, 474, 2430 (2018)

[118] S. Turyshev, Phys. Uspekhi 52, 1 (2009)

[119] T. Islam, K. Dutta, Phys. Rev. D, 100, 104049 (2019)

[120] Q. Guo, et al., Nat. Astron. (2019)

[121] F. Hammer, et al., ApJ, 883, 171 (2019)

[122] F. Lelli, et al., ApJ, 836, 152 (2017)

[123] M. Cappellari, et al., MNRAS, 413, 2 (2011)

[124] P. Serra, et al., MNRAS, 460, 2 (2016)

[125] A.G. Riess et al, ApJ, 876 85 (2019)

[126] O'Raifeartaigh C., O'Keeffe M., Nahm W., Mitton S., Eur.Phys. J. H 43, 73 (2018)

[127] Karachentsev I.D., et al, MNRAS, 393, 1265 (2009)

[128] Fouque P. et al, A&A, 375, 3p (2001)

[129] Tully R.B. et al, Nature, 513, 71 (2014)

[130] Chernin A.D. et al, A&A, 520, A104 (2010)

[131] Christodoulou D.M., Kazanas D., arXiv:1905.04296 (2019)

[132] Einasto M. et al, A&A, 476, 2 (2007)

[133] DES Collaboration, arXiv:1811.02376 (2018)

[134] Ryan J., Chen Y., Ratra B., arXiv:1902.03196, Submitted to MNRAS, (2019)

[135] A.G. Riess et al, ApJ, 855, 13 (2018)

[136] L. Verde, T. Treu, A.G. Riess, arXiv:1907.10625

[137] E. Di Valentino et al, arXiv:1910.09853

[138] D.V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, Proceed. Steklov Math. Inst., 90, 1 (1967)

[139] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, (Springer, 1989)

[140] V.G. Gurzadyan, G.K. Savvidy, A & A, 160, 203 (1986)

[141] A.A. Kocharyan, Commun. Math. Phys. 143, 27 (1991)

[142] A.V. Gurzadyan, A.A. Kocharyan, Int.J.Mod.Phys. D, 26, 1741019 (2017)

[143] V.G. Gurzadyan, A.A. Kocharyan, Ap. Space Sci., 135, 307 (1987)

[144] A.A. El-Zant, V.G. Gurzadyan, Physica, D, 122, 241 (1998)

[145] E. Fermi, J. Pasta, S. Ulam, Los Alamos National Laboratory, LA-1940 (1955)

[146] S. Capozziello, E. De Filippis, V. Salzano, MNRAS, 394, 947 (2009)

[147] V. Salzano et al, JCAP, 10, 033 (2016)

[148] S. Capozziello et al, MNRAS, 474, 2430 (2018)

[149] A.V. Gurzadyan, A.A. Kocharyan, Eur. Phys. J. C, 76, 685 (2016)