One-Dimensional Integrable Systems of Interacting Particles

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Contents

1 Introduction

1.1 Constants of motion and integrability

In mechanics a constant of motion is called the quantity that is conserved throughout the motion. These quantities allow to describe properties of motion and solve equations of motion without doing it explicitly. Each constant of motion adds restriction to the motion and for each couple of constansts the following relation is true:

$$I_1, I_2 = const$$

Thus each constant of motion reduces degrees of freedom of the system by one.

In classical and quantum mechanics a system with N degrees of freedom is called completely integrable if N functionally independent constants of the motion can be obtained for the system. The system is superintegrable if more than N constants of the motion can be defined and maximally superintegrable if there exist $2N-1$ constants of the motion. These systems are very fundamental from the mathematical and physical perspective due to their many in teresting properties.

Another way of definig integrability is the Liouville integrability. In this sense integrability means that there exists a maximal set of Poisson commuting invariants: Liouville integrals. These are functions on the phase space whose Poisson brackets with the Hamiltonian of the system, and with each other, vanish. In quantum systems Poisson brackets shift to commutaition relations.

$$\{I_1, I_2\} = 0$$

1.2 Examples of integrable systems

The N-dimensional free particle, isotropic oscillator and particle moving in the Coulomb potential are the simplest maximally superintegrable systems. This means that apart from the Liouville integrals, such systems have $N-1$ additional constants of motion. Due to them, the orbits of the classical bounded motion are closed. For quantum systems, the superintegrability leads to a hight degeneracy of energy levels and an exact expression for the wavefunctions. The entire symmetry of the free-particle, oscillator and Coulomb systems constitute, correspondingly, the Euclidean $E(N)$, unitary $U(N)$ and (pseudo)orthogonal $SO(N+1)/SO(1,N)$ groups, which are responsible for the superintegrability. The described models lie at the origin of many (super)integrable models of classical and quantum mechanics.

Another example of a superintegrable system can be obtained by treating each coordinate as a separate particle in one dimension and consider an interaction between them. In particular, interaction with the inverse-square potential, introduced by Calogero, significantly complicates the above systems [1, 2, 3, 4, 6]. Nevertheless, it preserves the superintegrability [31, 7, 8, 9]. Apart from the Liouville integrals [2], the unbound Calogero system (Calogero-Moser model) possesses additional integrals of motion ensuring maximal superintegralility [31, 7, 32]. This property is retained in the presence of the oscillator [33] and Coulomb potentials including the spaces with constant curvature [8].

In the quantum case, the Calogero potential can be involved into a covariant derivative with flat connection, bringin closer to the original system [10, 11]. It was first introduced by Dunkl and contains particle-exchange operators [12]. As a result, the Calogero model (unbound or bound by oscillator or Coulomb potential) can be regarded as a Dunkl-operator deformation of the underlying system without particle interaction. The related symmetries are deformed as well. Their generators together with the exchange operators form a quadratic algebra [13, 14, 9, 15].

1.3 Truncated quantum Calogero model

A nonlocal gauge (similarity) transformation eliminates the connection together with the Calogero potential [16]. For indistinguishable bosons or fermions, it just shifts the ground-state energy level in the bound spectrum ensuring an equivalence with the related noninteracting system. Recently, a modified analog of this transformation have been applied in order to construct Calogero-type model with more general inverse-square interaction, containing also three-particle terms [17]. In contrast to the pure Calogero case, here most wavefunctions of the noninteracting oscillators are mapped to non-normalizable states. In the current talk, we introduce a simpler $U(1)$ gauge transform in N-dimensional space, which produce not only similar-type inverse-square potentials but also but also terms linear in momentum. It is equivalent to inclusion of the Aharonov-Both type magnetic potential which is inverse in coordinates. For quantum case, the momentum shift is reminiscent the truncated Dunkl operator without the particle exchanges. For an imaginary phase, the map produces non-Hermitian PT-invariant Hamiltonians.

2 N-dimensional hamonic oscillator

2.1 The solution of Schrodinger equation for the oscillator

In classical mechanics a harmonic oscillator is a particle or system that undergoes harmonic motion about an equilibrium position according to

$$x = a \sin \omega t \tag{1}$$

law, where $\omega = 2\pi\nu$. In quantum mechanics an oscillator is described via following Hamiltonian

$$\hat{H} = \frac{\hat{p_x^2}}{2} + \frac{\omega^2 x^2}{2} \tag{2}$$

where the first term represents the kinetic energy of the particle, and the second term the potential energy represented by ω frequency of the oscillator.

One may find eigenvalues and eigenstates of the system by directly solving time-independent Schrödinger equation.

$$\hat{H}\psi_n = E_n\hat{\psi}_n \tag{3}$$

Another approach called the 'ladder operator' method was developed by Paul Dirac which allows extraction of the energy eigenvalues without directly solving the differential equation by giving matrix representation of system operators. The energetic representat ion of the system has the following form:

$$\hat{H} = \frac{\hat{p}_i^2}{2} + \frac{\omega^2 \hat{x_i}^2}{2}. \tag{4}$$

where $\hat{p}_{ii}$ and $\hat{x}_{ii}$ are the matrices of impulse and coordinate. Which obey following relation:

$$\hat{x}_i\hat{p}_i - \hat{p}_i\hat{x}_i = i\hbar\hat{I} \tag{5}$$

Now let's introduce new $\hat{a}_i$, $\hat{a_i}^+$ lowering and rising operators that are related to impulse and coordinate with following expressions:

$$\hat{a}_i = \hat{p}_i - i\omega\hat{x}_i \tag{6}$$

$$\hat{a_i}^+ = \hat{p}_i + i\omega\hat{x}_i \tag{7}$$

Representations of $\hat{x}_i$ and $\hat{p}_i$ by ladder operators are:

$$\hat{p}_i = \frac{1}{2}\left(\hat{a}_i + \hat{a_i}^+\right) \tag{8}$$

$$\hat{x}_i = \frac{i}{2\omega}\left(\hat{a}_i - \hat{a_i}^+\right) \tag{9}$$

Following the (6) and (7) definitions it is easy to show that lowering-rising operators obey following commutative relations:

$$[\hat{a}_i, \hat{a}_j^+] = \delta_{ij} \tag{10}$$

Further (4) Hamiltonian can be exspressed via $\hat{a}_i, \hat{a}_i{}^+$ with following equation:

$$\hat{H} = \hbar\omega \sum_{i=1}^{N} \left(\hat{a}_i{}^+ \hat{a}_i + \frac{1}{2} \right) \tag{11}$$

Having two equations above it is easy to show that lowering-rising operators and oscillator Hamiltonian obey following relations:

$$[\hat{a}_i, \hat{H}] = \hbar\omega\hat{a}_i \tag{12}$$

$$[\hat{a}_i{}^+, \hat{H}] = -\hbar\omega\hat{a}_i \tag{13}$$

$$\hat{a}_i\hat{a}_i{}^+ = 2\hat{H} + \hbar\omega \tag{14}$$

$$\hat{a}_i{}^+\hat{a}_i = 2\hat{H} - \hbar\omega \tag{15}$$

Acting on both sides of (3) Schrodinger equation by $\hat{a}_i$ lowering operator one will get:

$$\hat{a}_i\hat{H}\psi_n = (\hat{H}\hat{a}_i + \hbar)\psi_n\hat{H}(\hat{a}_i\psi_n) = (E_n - \hbar\omega)(\hat{a}_i\psi_n) \tag{16}$$

Which means that $\hat{a}\psi_n$ is the eigenfunction of $\hat{H}$ for $E_n - \hbar\omega$ energetic level. Generalizing we will get that $\hat{a}_i\psi_n, \hat{a}_i\psi_{n-1}, ..., \hat{a}_i\psi_1$ are eigenfunctions for correspondingly $E_n - \hbar\omega, E_{n-1} - \hbar\omega, ..., E_1 - \hbar\omega$ eigenvalues. This means that $\hat{a}_i$ acts on ψ_n to produce ψ_{n-1} state. Thus $\hat{a}_i$ loweringg operator annihilates the ground state .

$$\hat{a}_i\psi_0 = 0 \tag{17}$$

$$\hat{a}_i\psi_n = a_n\psi_{n-1} \tag{18}$$

The spector will be $E_n = E_0 + n\hbar\omega_0$. The same way for the $\hat{a}_i{}^+$ raising operator

$$\hat{a}_i{}^+\psi_n = c_n\psi_{n+1} \tag{19}$$

It is easy to find the ground state E_0 energy and corresponding ψ_0 wave function having (11) and (17). Acting on a state by $\hat{a}_i{}^+\hat{a}_i$ operators one will get.

$$\hat{a}_i{}^+\hat{a}_i\psi_0 = 2\left(\hat{H} - \frac{\hbar\omega}{2}\right)\psi_0 = 2\left(E_0 - \frac{\hbar\omega}{2}\right)\psi_0 = 0 \tag{20}$$

From this the ground state energy will be:

$$E_0 = \frac{\hbar\omega}{2} \tag{21}$$

And for the spector we will get:

$$E_n = E_0 + n\hbar\omega = \hbar\omega\left(n + \frac{1}{2}\right) \tag{22}$$

where $n = 0, 1, 2, \ldots.$

The ground state wavefunction will be found via (17) annihilation relation:

$$\hat{a}_i\psi_0 = -i\hbar\left(\frac{d\psi_0}{dx} + \frac{\omega}{\hbar}\psi_0\right) = 0 \tag{23}$$

Which produces following differential equation:

$$\psi_0' + \frac{x}{a^2}\psi_0(x) = 0 \tag{24}$$

where $a^2 = h/\omega$.

The normalised solution for the above equation is [5]:

$$\psi_0(x) = \frac{1}{\sqrt{a\sqrt{\pi}}}\exp -\frac{x^2}{2a^2} \tag{25}$$

2.2 Superintegrability and symmetries of the N-dimensional oscillator

In the first subsection we have represented N-dimensional oscillator in matrix terms via $\hat{a}_i, \hat{a}_i{}^+$ ladder operators. Now it is easy to show that the system is invariant under a set of transformations isomorphic to the group $U(N)$.

One can define group generators as follows:

$$\hat{E}_{ij} = \hat{a}_i^+\hat{a}_j \tag{26}$$

The antisymmetric combinations of $\hat{E}_{ij}$ yield the angular momentum components:

$$\hat{L}_{ij} = \hat{E}_{ij} - \hat{E}_{ji} = \hat{x}_i\hat{p_j} - \hat{x}_j\hat{p_i} \tag{27}$$

To show that system is invariant to transformation under the angular momentum it is enought to show that (4) Hamiltonian commutes with $\hat{L}_{ij}$:

$$[\hat{H}, \hat{L}_{ij}] = 0 \tag{28}$$

Indeed following to matrix representations of (11) Hamiltonian and (27) angular momentum the commutation can be written as follows:

$$[\hat{H}, \hat{L}_{ij}] = \hbar\omega\sum_{i=1}^{N}\left[\left(\hat{a}_i^+\hat{a}_i + \frac{1}{2}\right), (\hat{a}_i^+\hat{a}_j - \hat{a}_j\hat{a}_i^+)\right] = \tag{29}$$

$$\hbar\omega\sum_{i=1}^{N}\left(\left[\hat{a}_i^+\hat{a}_i, \hat{a}_i^+\hat{a}_j\right] - \left[\hat{a}_i^+\hat{a}_i, \hat{a}_j\hat{a}_i^+\right]\right) = 0 \tag{30}$$

The symmetric combinations of the $U(N)$ generators produce another symmetry of the system, the Fradkin tensor:

$$\hat{I}_{ij} = \hat{E}_{ij} + \hat{E}_{ji} = \hat{p}_i \hat{p}_j - \hat{x}_i \hat{x}_j \tag{31}$$

Which also commutes with the oscillator Hamiltonian.

$$[\hat{H}, \hat{I}_{ij}] = 0 \tag{32}$$

Symmetries of N-dimensional isotropic oscillator make up complete $U(N)$ group which includes $SO(N)$ angular momentum and Fradkin's tensors. Thus the isotropic oscillator is maximally superintegrable since it possesses the maximal $(2N-1)$ number of functionally independent constants of motion.

3 Generalized Calogero model

The Calogero model describes one-dimensional identical particles interacting by an inverse-square potential.

$$\hat{H}_{\mathrm{C}} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2} + \sum_{i<j} \frac{g(g \mp \hbar)}{x_{ij}^2}. \tag{33}$$

Here $\hat{p}_i = -\imath\hbar\partial_i$ is the momentum operator, g is a coupling constant that characterizes interparticle interactions and the $\mp$ sign in the potential corresponds to the fermions (bosons). We use a conventional notation for the particle distance, $x_{ij} = x_i - x_j$.

Adding an external harmonic force to the system one gets the following Hamiltonian:

$$\hat{H}_{\mathrm{C}} = \frac{1}{2}\sum_{i=1}^{N}(\hat{p}_i^2 + x_i^2) + \sum_{i<j} \frac{g(g \mp \hbar)}{x_{ij}^2}. \tag{34}$$

This system is called bound Calogero model (the frequency of the external harmonic oscillator is set to unity $\omega = 1$).

Most properties of the Calogero model (and its various extensions), like (super)integrability, spectrum, wave functions, and conservation laws are conditioned by its slightly modified version known as a generalized Calogero Hamiltonian [10, 11] given by following Hamiltonian:

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N}\left(\hat{p}_i^2 + x_i^2\right) + \sum_{i<j} \frac{g(g - \hbar M_{ij})}{x_{ij}^2}. \tag{35}$$

It includes a permutation operator M_{ij}, which exchanges the coordinates of ith and jth particles. At a first glance, such a modification may look rather strange due to its nonlocality but for the identical particles the permutation operator becomes $M_{ij} = \pm 1$ and the system just reduces to the standard Calogero Hamiltonian (108). The advantage of the above Hamiltonian is the representation in terms of the deformed momentum operator with a derivative replaced with the Dunkl operator,

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N}\left(\hat{\pi}_i^2 + x_i^2\right), \tag{36}$$

$$\hat{\pi}_i = \hat{p}_i + \sum_{j\neq i}\frac{\imath g}{x_{ij}}M_{ij}. \tag{37}$$

Note that the inverse-square Calogero interaction is encapsulated into the Dunkl momentum operator. The latter can be considered as a kind of flat nonlocal covariant derivative with the following algebra [12]:

$$[\hat{\pi}_i, \hat{\pi}_j] = [x_i, x_j] = 0, \qquad [x_i, \hat{\pi}_j] = \imath\hat{S}_{ij}, \tag{38}$$

$$\hat{S}_{ij} = (\delta_{ij} - 1)gM_{ij} + \delta_{ij}\left(\hbar + g\sum_{k\neq i}M_{ik}\right). \tag{39}$$

At the $g \to 0$ limit when the Calogero inverse-square potential is absent, $\hat{\pi}_i$ is mapped to $\hat{p}_i$ and the matrix S_{ij} is reduced to the Kronecker delta, recovering the Heisenberg algebra commutation rules.

$$\hat{\pi}_i = \hat{p}_i, \qquad \hat{S}_{ij} = \hbar\delta_{ij}$$

Another set of quantities to describe the system, are the lowering-rising operators. A Dunkl-operator analog of lowering-rising operators are defined in the standard way [11, 10, 30],

$$\hat{a}_i^{\pm} = \frac{x_i \mp \imath\hat{\pi}_i}{\sqrt{2}}, \tag{40}$$

With this definition most of the canonical relations preserve. The lowering (and rising) operators mutually commute and the commutation between lowering and rising operators results in S_{ij} operator(39).

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0, \qquad [\hat{a}_i, \hat{a}_j^+] = \hat{S}_{ij}. \tag{41}$$

The generalized Hamiltonian can be expressed in terms of them,

$$\hat{H} = \frac{1}{2}\sum_i(\hat{a}_i^+\hat{a}_i + \hat{a}_i\hat{a}_i^+) = \sum_i \hat{a}_i^+\hat{a}_i + \frac{\hbar N}{2} - S. \tag{42}$$

Here, by S, the rescaled invariant of the permutation group algebra is denoted,

$$S = -g \sum_{i<j} M_{ij}, \qquad [S, M_{ij}] = 0. \tag{43}$$

4 Symmetries of the generalized Calogero model

As it is shown in section 2.1 symmetries of N-dimensional isotropic oscillator make up complete $U(N)$ group which includes $SO(N)$ angular momentum and Fradkin's tensors.

Using Dunkl representation we have defined the generalized Calogero system with a Hamiltonian quite similiar to N dimensional oscillator hamiltonian, here the inverse-square Calogero interaction is encapsulated into the Dunkl momentum operator. Thus the system symmetries as well can be expressed as Dunkl deformation of $U(N)$ group. It has been shown that as a result of (41) and (42) properties, the lowering-rising operators obey a standard spectrum generating relations [11, 30],

$$[\hat{H}, \hat{a}_i^{\pm}] = \pm\hbar\hat{a}_i^{\pm}. \tag{44}$$

Bilinear combinations of lowering-rising operators,

$$\hat{E}_{ij} = \hat{a}_i^{+}\hat{a}_j, \tag{45}$$

They are Dunkl-analogs of $U(N)$ group generators. Together with the permutations S_{ij}, they generate the symmetries of the nonlocal Calogero model.

As a result they commute with the Hamiltonian(satisfie the conservation law) and obey to following relation, which is a deformation of $U(N)$ generator commutation relations.

$$[\hat{H}, \hat{E}_{ij}] = 0. \qquad [\hat{H}, \hat{S}_{ij}] = 0. \tag{46}$$

The elements $\hat{E}_{ij}$ together with permutations M_{ij} provide entire algebra of symmetries for the generalized Calogero model. In addition, the following quadratic relation takes place among $\hat{E}_{ij}$ and $\hat{S}_{ij}$ [14]:

$$\hat{E}_{ij}(\hat{E}_{kl} + \hat{S}_{kl}) = \hat{E}_{il}(\hat{E}_{kj} + \hat{S}_{kj}). \tag{47}$$

The latter implies, in particular, the following commutation relation:

$$[\hat{E}_{ij}, \hat{E}_{kl} + \hat{S}_{kl}] = \hat{E}_{il}\hat{S}_{kj} - \hat{S}_{il}\hat{E}_{kj}. \tag{48}$$

As a consequence, the diagonal elements are closed under commutation. Unlike the Cartan algebra, they are not Abelian but obey a simple commutation,

$$[\hat{E}_{ii}, \hat{E}_{kk}] = (\hat{E}_{ii} - \hat{E}_{kk})\hat{S}_{ik}. \tag{49}$$

The above algebra ensures that the power sums form a system of Liouville integrals of the Calogero system [10],

$$\hat{\mathcal{E}}_k = \sum_i \hat{E}_{ii}^k, \qquad [\hat{\mathcal{E}}_k, \hat{\mathcal{E}}_l] = 0. \tag{50}$$

The generalized Hamiltonian itself is expressed in terms of the first member in this family,

$$\hat{H} = \hat{\mathcal{E}}_1 - S + \frac{N\hbar}{2}. \tag{51}$$

Moreover, it is a unique Casimir element (up to a nonessential constant term) of the Dunkl-deformed $u(N)$ algebra [14].

Remember that the same algebra (49) describes also the symmetries of the generalized Hamiltonian related to the Sutherland model [10], an analog of the Calogero-Moser system with trigonometric interactions [27].

The antisymmetric combinations of $\hat{E}_{ij}$ yield the Dunkl angular momentum components [40, 7],

$$\hat{L}_{ij} = \hat{E}_{ij} - \hat{E}_{ji} = x_i\hat{\pi}_j - x_j\hat{\pi}_i. \tag{52}$$

Together with permutations, they produce a deformation of $so(N)$ algebra with a unique Casimir element given by the Dunkl angular momentum square $\hat{L}^2 = \sum_{i<j} \hat{L}_{ij}^2$ shifted by a permutation invariant term [14],

$$\hat{\mathcal{L}}_2 = \hat{L}^2 + S^2 - \hbar(N-2)S, \qquad [\hat{L}_{ij}, \hat{\mathcal{L}}_2] = 0. \tag{53}$$

It can be considered as a generalized angular Calogero Hamiltonian, which is reduced to the angular part of the Calogero model for identical particles [41],

$$\begin{gathered} \hat{H} = -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{N-1}{r}\partial_r\right) + \frac{r^2}{2} + \frac{\hat{\mathcal{L}}_2}{2r^2} \\ \text{with} \quad r = \sqrt{x^2}. \end{gathered} \tag{54}$$

The symmetric combinations of the deformed $u(N)$ generators produce a Dunkl-operator deformation for the well-known Fradkin tensor [18],

$$\hat{I}_{ij} = \hat{E}_{ij} + \hat{E}_{ji} - \hat{S}_{ij} = \hat{\pi}_i\hat{\pi}_j + x_ix_j. \tag{55}$$

Remember that the angular momentum and Fradkin tensor describe, respectively, the dynamical and hidden symmetries of the N-dimensional isotropic oscillator [42].

The diagonal algebra (49) has an Abelian basis obtained by applying a shift to its elements. The shift is a tail composed of exchange operators [37],

$$\hat{D}_i = \hat{E}_{ii} - \hat{S}_i, \qquad \hat{S}_i = \sum_{j=1}^{i-1} \hat{S}_{ij}, \tag{56}$$

where $\hat{S}_1 = 0$ is supposed. Together with permutations, the elements $\hat{D}_i$ satisfy the defining relations of degenerate affine Hecke algebra,

$$\begin{gathered} [\hat{D}_i, \hat{D}_j] = 0, \qquad [\hat{D}_k, \hat{S}_{jj+1}] = 0 \quad \text{if} \;\; k \neq i, i+1, \\ \hat{D}_{i+1}\hat{S}_{jj+1} - \hat{S}_{jj+1}\hat{D}_i = g^2. \end{gathered} \tag{57}$$

Note that the tails S_i satisfy the same relations; i.e., the above equations remain true upon the substitution $\hat{D}_i \to \hat{S}_i$. As a result, the modified diagonal elements can be considered as an analog of Liouville integrals for the generalized Hamiltonian (35), which may be expressed via them using the representation (42),

$$\hat{H} = \sum_i \hat{D}_i + \frac{\hbar N}{2}. \tag{58}$$

The higher-order power sums define the higher Hamiltonians,

$$\hat{\mathcal{D}}_k = \sum_i \hat{D}_i^k. \tag{59}$$

The second member, $\hat{\mathcal{D}}_2$, corresponds to the generalized Calogero-Sutherland model. Note that in contrast to the previous integrals (50), the permutation invariance is not evident but can be verified. More familiar are the monomials given by the generating function $\prod_i (u - \hat{D}_i)$ [37]. In general, any symmetric polynomial in $\hat{\mathcal{D}}_i$ is permutation invariant and reduced to the constant of motion of the Calogero model (108) for indistinguishable particles.

The Calogero system is superintegrable, which means that we must be able to find integrals of motion for the system out of Liouville integral set. They an be constructed by taking the symmetric polynomials of the generators E_{ij}

To transition from generalized to bound Calogero models and preserve symmetry algebra we define integrals of motion as symmetric polynomials P_{sym} on the generators $\hat{E}_{ij}$, (or $\hat{L}_{ij}$ and $\hat{I}_{ij}$) and permutation S_{ij}.

$$P_{\text{sym}}(\hat{E}_{ij}, S_{ij}) = P'_{\text{sym}}(\hat{L}_{ij}, \hat{I}_{ij}, S_{ij})$$

Here are some simplest examples of simple integrals of motion describing the system.,

$$\sum_{i,j} \hat{I}^k_{ij}, \quad \sum_{i,j} \hat{L}^{2k}_{ij}, \quad \sum_{i<j} \hat{I}^k_{ij} M_{ij}, \quad \sum_{i,j} \hat{I}^k_{ii} \hat{L}^{2l}_{ij}. \tag{60}$$

5 Generalized Polychronakos-Frahm chain

Let us set the interaction constant to *unity*, $g = 1$. In the current section, we consider the discrete Calogero model and it's relation to Polychronakos-Frahm chain.

First let's describe the transition from the dynamical to discrete Calogero systems. Consider the generalized Calogero system at the equilibrium points, where the entire Calogero potential takes its minimal value,

$$\frac{\partial V}{\partial x_i} = 0, \qquad V(x) = \sum_{i=1}^{N} \frac{x_i^2}{2} + \sum_{i<j} \frac{1}{x_{ij}^2}. \tag{61}$$

This condition gives a differential equation, solving which we find that equilibrium coordinates for particles are roots of the N-th order Hermite polynomial.

This leaves us with N particles that are distributed along with the N equilibrium points, one per each. Since the momenta vanishes, $p_i = 0$, the permutations M_{ij} are the only allowed motion in the frozen system, so that it becomes discrete.[29].

All roots differ, so there are $N!$ equivalent minima connecting by the coordinate exchanges M_{ij}. These are only allowed evolutions in the frozen system.

Consider the expansion of the Hamiltonian (35) in powers of Planck's constant,

$$\hat{H} = V + \hbar H^{(1)} - \frac{\hbar^2 \partial^2}{2}. \tag{62}$$

The first-order term can be considered as a generalized Polychronakos-Frahm Hamiltonian, where the spin exchange operator is replaced by the coordinate exchange [28],

$$H^{(1)} = \sum_{i<j} \frac{1}{x_{ij}^2} M_{ij}. \tag{63}$$

The $SU(n)$ symmetric spin chain is recovered from the above Hamiltonian after the replacement of the coordinate permutations with spin exchange operators so that

$$H_{\mathrm{PF}} = \sum_{i<j} \frac{P_{ij}}{x_{ij}^2}. \tag{64}$$

Here, P_{ij} permutes the ith and jth spins, which take values in the fundamental representation of the $SU(n)$ group. Both Hamiltonians become identical on the bosonic (fermionic) states provided that the particles are endowed with additional spin degrees of freedom. Then the entire wave function must be symmetric (antisymmetric) under simultaneous exchanges of coordinates and spins for bosons (fermions). A permutation of spatial coordinates M_{ij} can be replaced by the spin exchange operators P_{ij} and $-P_{ij}$ in the bosonic and fermionic cases respectively. Note that the projection inverts the order of permutations so that the operator $M_{ij}M_{kl}$ must be substituted by $P_{kl}P_{ij}$.

Let us now construct the constants of motion of the chain Hamiltonian (63) by applying the $\hbar$ expansion to the integrals of the original dynamical model (35) at the equilibrium point. They do not provide directly the invariants of the $SU(n)$ spin chain (64). In order to get them, one needs to carry out symmetrization over all particles prior to the projection. For instance, $H^{(1)}$ stays invariant under a selected coordinate exchange, M_{ij}, but H_{PF} does not preserve its spin counterpart, P_{ij}. However, both Hamiltonians preserve the symmetrized version given by the element S (43).

We have established an analogy and the mapping mechanism between the dynamical and discrete generalized Calogero models. Let us apply the Plank's expansion for the dynamical operators as well. Accordingly the Dunkl momentum (37) and permutation matrix (39) can be presented as follows:

$$\hat{\pi}_i = \pi_i - \imath\hbar\partial_i \quad \text{with} \quad \pi_i = \sum_{j\neq i} \frac{\imath}{x_{ij}} M_{ij}, \tag{65}$$

$$\hat{S}_{ij} = S_{ij} + \hbar\delta_{ij} \quad \text{with} \tag{66}$$
$$S_{ij} = (\delta_{ij} - 1)M_{ij} + \delta_{ij}\sum_{k\neq i} M_{ik}.$$

Here and in the following, the superscript is omitted in the zeroth-order term of any operator, so that $\pi_i^0 = \pi_i$. In that limit, the canonical commutation relations resemble their original form (38),

$$[\pi_i, \pi_j] = 0, \qquad [x_i, \pi_j] = \imath S_{ij},$$
$$S_{ij} = \begin{cases} -M_{ij}, & \text{if } i \neq j, \\ \sum_{k\neq i} M_{ik}, & \text{otherwise.} \end{cases} \tag{67}$$

6 Symmetries of the Polychronakos-Frahm chain

Recall now that the particle coordinates are set by the roots of the Hermite polynomial, which imposes certain algebraic relations on them (see, for example, [43, 38]). As a result, the discrete Dunkl momenta are not independent any more but undergo additional algebraic constraints. In particular, the following relations hold among the fixed phase space variables:

$$\sum_i x_i = \sum_i \pi_i = \sum_i S_{ik} = 0, \tag{68}$$

$$x^2 = \pi^2 = \sum_{i<j} \frac{2}{x_{ij}^2} = {}^1\!/\!{}_2 N(N-1), \tag{69}$$

$$x \cdot \pi = -\pi \cdot x = -\imath S. \tag{70}$$

In general, all relations between the operators of the dynamical Calogero system are preserved at the $\hbar = 0$ limit. In particular, the frozen lowering-rising operators

$$a_i^{\pm} = \frac{x_i \mp \imath \pi_i}{\sqrt{2}}, \qquad \hat{a}_i^{\pm} = a_i^{\pm} \pm \frac{\hbar}{\sqrt{2}} \partial_i \tag{71}$$

obey a rule similar to the commutations of the deformed Heisenberg algebra (41) [28],

$$[a_i, a_j] = [a_i^+, a_j^+] = 0, \qquad [a_i, a_j^+] = S_{ij}. \tag{72}$$

Because of the minimum condition (61), the spectrum generating relation (44) remains valid for the generalized Polychronakos-Frahm chain too [28],

$$[H^{(1)}, a_i^{\pm}] = \pm a_i^{\pm}. \tag{73}$$

However, unlike the dynamical case, the discrete Hamiltonian is not expressed via lowering-rising operators [see Eq.(42)].

The constants of motion of the generalized Calogero model (45) have terms up to the second order in their expansion,

$$\hat{E}_{ij} = E_{ij} + \hbar E_{ij}^{(1)} - \frac{\hbar^2}{2} \partial_i \partial_j. \tag{74}$$

The relations (73) imply conservation of the constant terms,

$$E_{ij} = a_i^+ a_j, \qquad [H^{(1)}, E_{ij}] = 0. \tag{75}$$

For the Dunkl angular momentum (52), the $\hbar^2$ part vanishes, while the $\hbar$ term corresponds to the usual angular momentum operator in quantum mechanics,

$$\hat{L}_{ij} = L_{ij} - \imath\hbar(x_i \partial_j - x_j \partial_i), \qquad L_{ij} = x_i \pi_j - x_j \pi_i. \tag{76}$$

A similar expansion for the Fradkin tensor is more complex,

$$\hat{I}_{ij} = I_{ij} + \hbar I_{ij}^{(1)} - \hbar^2 \partial_i \partial_j, \qquad I_{ij} = x_i x_j + \pi_i \pi_j. \tag{77}$$

The first order operator-valued coefficient is given by

$$I_{ij}^{(1)} = \frac{1}{x_{ij}^2} M_{ij} + \sum_{k \neq i,j} \left(\frac{\partial_i}{x_{jk}} M_{jk} + \frac{\partial_j}{x_{ik}} M_{ik} \right) \tag{78}$$

for $i \neq j$ and

$$I_{ii}^{(1)} = \sum_{k \neq i} \frac{1}{x_{ik}} \left(\partial_i + \partial_k - \frac{1}{x_{ik}} \right) M_{ik}. \tag{79}$$

As was discussed above, the algebraic relations between the symmetry generators of the dynamical system remain true at the freezing limit. In particular, the most general relation (47) and its consequences (48), (49) are reduced, respectively, to the following equations:

$$E_{ij}(E_{kl} + S_{kl}) = E_{il}(E_{kj} + S_{kj}), \tag{80}$$

$$[E_{ij}, E_{kl} + S_{kl}] = E_{il}S_{kj} - S_{il}E_{kj}, \tag{81}$$

$$[E_i, E_k] = (E_i - E_k)S_{ik}. \tag{82}$$

The power sums of diagonal elements yield Liouville integrals of the Polychronakos-Frahm chain [28],

$$\mathcal{E}_k = \sum_i E_{ii}^k, \qquad [\mathcal{E}_k, \mathcal{E}_l] = 0, \qquad [H^{(1)}, \mathcal{E}_k] = 0. \tag{83}$$

The first element of this set is rather trivial, $\mathcal{E}_1 = S + \frac{N(N-1)}{2}$, as is easy to get using the equations (69) and (71). The higher rank $\mathcal{E}_k$ have more complicated expressions.

For the dynamical system, the quadratic relations (47) are the only constraints which the symmetry generators $\hat{E}_{ij}$ obey [14]. However, there are a lot of other restrictions on them at the equilibrium point. For example, the Eqs. (68) imply the sum vanishing rules,

$$\sum_i E_{ik} = \sum_i E_{ki} = \sum_i L_{ik} = \sum_i I_{ik} = 0. \tag{84}$$

In the dynamical case, the angular Calogero Hamiltonian (53) plays an important role among constants of motion. In the absence of an oscillator potential, it maps the Liouville set to additional integrals. However, in the equilibrium limit, the angular part does not produce a new integral but just is expressed via trivial ones. Using the relations (68), (69), (52), and (85), it is easy to verify that the operator $\hat{\mathcal{L}}_2$ is a scalar at the equilibrium. Its $\hbar$-linear coefficient reproduces the chain Hamiltonian as was argued earlier [41],

$$\mathcal{L}_2 = r^4, \qquad \mathcal{L}_2^{(1)} = 2r^2 H^{(1)} \tag{85}$$

with $r^2 = \frac{1}{2}N(N-1)$ (69).

Let us calculate also a chain analog of the Fradkin's tensor square. It corresponds to the second ($k = 2$) member of the first sequence presented in (60),

$$\sum_{i,j} I_{ij}^2 = -2S^2 + \sum_{i<j}(S_{ii} + S_{jj})M_{ij} + 2r^4 + N(N-1) = \tag{86}$$

$$\sum_{i\neq j\neq k} M_{ijk} - 2S^2 + {}^1\!/\!_2 N(N-1)(N^2 - N + 4). \tag{87}$$

Here, M_{ijk} is a cyclic permutation of the marked three coordinates. It also is expressed through the invariants of the permutation group algebra. Nevertheless, we expect that higher degree power sums from (60) at the equilibrium give rise to nontrivial integrals of motion for the chain Hamiltonians (63), (64).

Finally, consider the shifted diagonal elements (56). At the freezing level, they also commute,

$$[D_i, D_j] = 0, \qquad D_i = E_{ii} - S_i. \tag{88}$$

Together with permutations, they form also degenerate affine Hecke algebra (57) (with $g = 1$). Contrary to the nonshifted case (75), (83), the related symmetric polynomials, $\mathcal{D}_k = \sum_i D_i^k$, are scalars (multiples of identity) and do not lead to a conservation law. At the same time, the $\hbar$-linear terms of the dynamical integrals (59) form a family of commuting nontrivial integrals (83),

$$\mathcal{D}_k^{(1)} = \sum_i \sum_{l=0}^{k-1} D_i^l D_i^{(1)} D_i^{k-l-1}, \quad [\mathcal{D}_k^{(1)}, \mathcal{D}_l^{(1)}] = 0, \tag{89}$$

where $D_i^{(1)} = \frac{1}{2} I_{ii}^{(1)}$. The first element of the family describes the chain Hamiltonian: $\mathcal{D}_1^{(1)} = -2H^{(1)}$. This remarkable property was established first for the Haldane-Shastry chain using the Yangian represtation [39] and has been extended later to the Polychonakos-Frahm chain [38].

7 Two-dimensional system with dihedral symmetry

Consider the simple two-dimensional analog of the Calogero model (63), (108), which remains invariant with respect to the dihedral group D_n with *odd* n. The latter describes the symmetries of a regular polygon with n vertexes and contains $2n$ entries.

In the complex plane,

$$z = x_1 + \imath x_2, \qquad \bar{z} = x_1 - \imath x_2,$$

it consists on the n discrete rotations,

$$r_k(z) = w^{2k} z, \qquad w = e^{\frac{\imath\pi}{n}},$$

and n reflections with respect to the symmetry axes,

$$s_k(z) = w^{-2k}\bar{z}$$

with $k = 0, 1, \ldots, n-1$. Clearly, such operations obey the relations

$$r_k^n = s_k^2 = 1$$

and the commutation rules

$$s_k s_l = r_{k-l}, \qquad s_k r_l = s_{k-l}, \qquad r_k s_l = s_{k+l}, \qquad r_k r_l = r_{k+l}.$$

The Dunkl operators are defined by [12]

$$\nabla_z = \partial_z - g\sum_{k=0}^{n-1}\frac{w^{-k}}{f_k}s_k \qquad \text{with} \qquad f_k = zw^{-k} - \bar{z}w^k$$

and its complex conjugate. Note that $\bar{f}_k = -f_k$.

In two dimension, the Dunkl angular momentum (52) is described by a single operator [40]

$$L = z\nabla_z - \bar{z}\nabla_{\bar{z}} = \hbar(z\partial_z - \bar{z}\partial_{\bar{z}}) - g\sum_k \frac{zw^{-k} + \bar{z}w^k}{f_k}s_k. \tag{90}$$

The Fradkin tensor (55) has three components, and only two of them are independent. The mixed component is related to the Hamiltonian, while the diagonal one is related to an hidden integral of motion,

$$I_{z\bar{z}} = -\nabla_z\nabla_{\bar{z}} + {}^1\!/\!{}_2 z\bar{z} = -\hbar^2\partial_z\partial_{\bar{z}} + {}^1\!/\!{}_2 z\bar{z} + \frac{n^2 g^2 (z\bar{z})^{n-1}}{(z^n - \bar{z}^n)^2} - \hbar g\sum_{k=0}^{n-1}\frac{s_k}{f_k^2}, \tag{91}$$

$$\begin{aligned} I_{zz} = -\nabla_z^2 + z^2 = -\hbar^2\partial_z^2 + {}^1\!/\!{}_2 z^2 - \hbar g\sum_{k=0}^{n-1}\frac{w^{-2k}}{f_k}\left[w^k\partial_z + w^{-k}\partial_{\bar{z}} - \frac{w^k}{f_k}\right]s_k \\ + g^2\frac{nz^{n-2}}{z^n - \bar{z}^n}\left(\sum_{l=1}^{n-1} w^l r_l + \frac{n\bar{z}^n}{z^n - \bar{z}^n}\right). \end{aligned} \tag{92}$$

In the derivation of the above equations we have used the following identities:

$$\sum_{k=0}^{n-1}\frac{1}{f_k f_{k+l}} = \delta_{l,0}\frac{n^2(z\bar{z})^{n-1}}{(z^n - \bar{z}^n)^2}, \qquad \sum_{k=0}^{n-1}\frac{w^{-2k}}{f_k f_{k+l}} = \frac{w^l n z^{n-2}}{z^n - \bar{z}^n}\left[1 + \delta_{l,0}\frac{n\bar{z}^n}{z^n - \bar{z}^n}\right].$$

The complex Fradkin tensor (91), (92) is related to its Cartesian representation (55) by

$$I_{z\bar{z}} = \frac{1}{4}(I_{11} + I_{22}) = \frac{1}{2}H, \qquad I_{zz} = \frac{1}{4}(I_{11} - I_{22}) + \frac{\imath}{2}I_{12}.$$

Note that in the symmetric case when $r_k = 1$ and $s_k = \pm 1$, the radial and angular degrees of freedom separate from each other [?].

8 Canonical mapping of classical Hamiltonians

Consider the following map transformation of the phase space variables of an N-dimensional system, which mixes the coordinate and momentum,

$$p'_i = p_i + \sum_{1\le|i-j|\le r} \frac{f}{x_i - x_j}, \qquad x'_i = x_i, \tag{93}$$

parameterised by an integer $r \le N-1$.

It describes a canonical map with singularities at the hyperplanes $x_i = x_j$ with $|i-j| \le r$:

$$\sum_i p'_i dx'_i = \sum_i p_i dx_i + d\log F^f, \qquad F = \prod_{1\le i-j\le r} |x_i - x_j|, \tag{94}$$

so that the new variables obey the standard Poisson brackets,

$$\{p'_i, x_j\} = \delta_{ij}, \qquad \{p'_i, p'_j\} = 0.$$

Note that the primed momentum can be interpreted as a generalized momentum in the Aharonov-Bohm like magnetic potential

$$A_i = \partial_i \log F^f. \tag{95}$$

As a result, the free-particle system is equivalent to the following model with the two-particle and three-particle distant interactions as well as

$$\begin{aligned} H_0 &= \sum_i \frac{p'^2_i}{2} \\ &= \sum_i \frac{p_i^2}{2} + \sum_{1\le i-j\le r}\left(\frac{f^2}{(x_i-x_j)^2} + f\frac{p_i - p_j}{x_i - x_j}\right) - \sum_{\substack{i<j<k \\ r<k-i\le 2r}} \frac{f^2}{(x_i-x_j)(x_j-x_k)}. \end{aligned} \tag{96}$$

Of course, the above system is superintegrable with integrals of motion given by the momentum p'_i and angular momentum $L'_{ij} = x_i p'_j - x_j p'_i$ with the standard Poisson brackets between them.

The harmonic and Coulomb potentials preserve the superintegrability.

$$H_\omega = H_0 + \frac{\omega^2 r^2}{2}, \qquad H_\gamma = H_0 - \frac{\gamma}{r}, \qquad r^2 = \sum_i x_i^2. \tag{97}$$

The integrals of motion in the first case form the $u(N)$ algebra with the angular momentum and Fradkin tensor $T'_{ij} = x_i x_j + p'_i p'_j$. In the Coulomb case, the symmetry forms the $so(4)$ Lie algebra with the Runge-Lenz vector $A'_i = \sum_j x_j L'_{ij} - \gamma x_i / r$.

Of course, the canonical map (93) may be applied to other models. The above Hamiltonians look quite close to the Calogero-Moser model, which is also maximally supertintegrable [?]. The later, in its turn, is mapped to the following system:

$$
\begin{aligned}
H_g = \sum_i \frac{p_i'^2}{2} + \sum_{i<j} \frac{g^2}{(x_i - x_j)^2} = \sum_i \frac{p_i^2}{2} + \sum_{1 \le i-j \le r} \left(\frac{f^2 + g^2}{(x_i - x_j)^2} + f \frac{p_i - p_j}{x_i - x_j} \right) \\
+ \sum_{i-j>r} \frac{g^2}{(x_i - x_j)^2} - \sum_{\substack{i<j<k \\ r<k-i\le 2r}} \frac{f^2}{(x_i - x_j)(x_j - x_k)}.
\end{aligned}
\tag{98}
$$

Note that in $r = N - 1$ case, the last two terms in the Hamiltonian (98) disappear, and the system becomes invariant under particles exchanges.

Clearly, the constructed model is superintegrable too. The Lax matrix is inherited from the Calogero model [2]

$$
\mathcal{L}_{ij} = \delta_{ij} p_i' + (1 - \delta_{ij}) \frac{\imath g}{x_i - x_j},
$$

where the p_i' is defined by (93). It defines the Liouville integrals of motion $I_n = \mathrm{Tr} L^n$ for $n \le N$. The additional integrals may be taken as the Poisson brackets $\{I_n, H_g^\Omega\}$, where H_g^Ω is the angular part of the H_g []

Note that the Hamiltonian (98) is reminiscent of the classical version of the truncated Calogero model [17] without external harmonic potential, which contains both the two-body and three-body inverse-square interactions:

$$
H_g^{\mathrm{tr}} = \sum_i \frac{p_i^2}{2} + \sum_{1 \le i-j \le r} \frac{g^2}{(x_i - x_j)^2} - \sum_{\substack{i<j<k \\ r<k-i\le 2r}} \frac{g^2}{(x_i - x_j)(x_j - x_k)}.
\tag{99}
$$

9 Gauge transformation of quantum systems

In the quantum case the canonical map (93), (94) is generated by the following gauge transformation acting on any function G on the phase space variables,

$$
G(p_i, x_i) \to U^{-1} G(p_i, x_i) U = G(p_i', x_i)
\tag{100}
$$

with $p_i = -\imath \partial_i$. Here the local $U(1)$ phase

$$
U = F^{\imath f} = \exp\left(\imath f \sum_{1 \le i-j \le r} \ln |x_i - x_j| \right)
\tag{101}
$$

defines the unitary shift which does not contain any singularity.

In particular, the free one-dimensional particle Hamiltonian is mapped to the following system:

$$\begin{aligned} H_0 &= U^+ \left(\sum_{i=1}^{N} \frac{p_i^2}{2} \right) U = \sum_{i=1}^{N} \frac{p_i'^2}{2} \\ &= \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{1 \le i-j \le r} \left(\frac{f(f-\imath)}{(x_i - x_j)^2} + \frac{f}{x_i - x_j}(p_i - p_j) \right) - \sum_{\substack{i<j<k \\ r<k-i\le 2r}} \frac{f^2}{(x_j - x_i)(x_i - x_k)}. \end{aligned} \tag{102}$$

Note that the quantum corrections make the coefficient in front of the inverse-square potential term a complex number [compare with (96)]. The classical coefficient recovers upon applying the anticommutator in the momentum term:

$$H_0 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{1 \le i-j \le r} \left(\frac{f^2}{(x_i - x_j)^2} + \left\{ \frac{f}{x_i - x_j}, \frac{p_i - p_j}{2} \right\} \right) - \sum_{\substack{i<j<k \\ r<k-i\le 2r}} \frac{f^2}{(x_j - x_i)(x_i - x_k)}. \tag{103}$$

In this form the Hermiticity of the Hamiltonian becomes transparent.

The unitary map (100) does not change the spectrum. It just produces the local phase factor in front of the wavefunctions.

Consider the bound system with the oscillator potential H_ω (97) with unit frequency, $\omega = 1$. Therefore recalling the eigenfunctions of the usual isotropic oscillator, one can immediately write those for our model ,

$$\psi_{k_1 \ldots k_N}(x) = e^{-\frac{1}{2}r^2 - \imath f \sum_{1 \le i-j \le r} \ln |x_i - x_j|} \prod_{i=1}^{N} H_{k_i}(x_i), \tag{104}$$

where $k_i \ge 0$ are integral numbers, $H_k(x)$ is the k-th order Hermite polynomial. The normalization constant is omitted. The corresponding eigenvalues are the same as that of the free oscillators:

$$E_{k_1 \ldots k_N} = \sum_{i=1}^{N} k_i + \frac{N}{2}. \tag{105}$$

10 Non-Hermitian models

Consider now the transformation (100), (101) with an imaginary constant. Keeping the old notation, we just substitute $f \to \imath f$ anywhere,

$$U = F^{-f} = \prod_{1 \le i-j \le r} |x_i - x_j|^{-f}. \tag{106}$$

This is not unitary map any more so that the shifted Hamiltonians become non-Hermitian. But they remain invariant under the PT transformation: which inverts the sign of coordinates, so that $x_i \to -x_i$ and $\partial_i \to -\partial_i$.

For example, instead of the non-Hermitian version of the system (103) is given by

$$H_0 = -\sum_{i=1}^{N} \frac{\partial_i^2}{2} - \sum_{1\leq i-j\leq r} \left(\frac{f(f-1)}{(x_i-x_j)^2} - \frac{f}{x_i-x_j}(\partial_i - \partial_j) \right) + \sum_{\substack{i<j<k \\ r<k-i\leq 2r}} \frac{f^2}{(x_j-x_i)(x_i-x_k)}. \tag{107}$$

$$\begin{aligned} H_g = &\sum_i \frac{p_i^2}{2} + \sum_{1\leq i-j\leq r} \left(\frac{g(g-1)-f(f-1)}{(x_i-x_j)^2} + f\frac{\partial_i - \partial_j}{x_i-x_j} \right) + \sum_{i-j>r} \frac{g(g-1)}{(x_i-x_j)^2} \\ &+ \sum_{\substack{i<j<k \\ r<k-i\leq 2r}} \frac{f^2}{(x_i-x_j)(x_j-x_k)}. \end{aligned} \tag{108}$$

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