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Interacting Higher Spin theories in flat and AdS spaces

Thesis for Acquiring the Degree of a Candidate of Physical and Mathematical Sciences in Division 01.04.02, Theoretical Physics

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Abstract

This thesis is devoted to the investigation of the interacting higher spin gauge theories in flat and AdS spaces. First of all we start with the introduction to the Fronsdal formalism for Higher spin fields. We calculate first the equations of motion for lower spin 1,2,3 cases and then turn to it's arbitrary spin s generalization. We show that *in De Donder gauge the equation of motion becomes harmonic*. Then to simplify our work with HS objects we introduce polynomial notation, which is widely used in HS theories and leads to working with polynomials instead of symmetric tensors with many indices. Next, we present the Fronsdal Lagrangian and equation of motion using the polynomial notation. After that we introduce the Neother's procedure, which is standard method for perturbative derivation of interaction using deformations of free equations and symmetries in self consistent way.

After this, we concentrate on the computation of cubic interaction for a higher spin on AdS space in explicit covariant form. We succeeded in solving all necessary *recurrence relations* to finalize full radial pullback of the main term of cubic self-interaction for higher spin gauge fields in Fronsdal's formulation from flat to one dimension less AdS_{d+1} space. As a result, non-trivial solutions of recurrence relations lead to the possibility to obtain the complete set of AdS_{d+1} dimensional interacting terms with all curvature corrections, including trace and divergence terms from any interaction term in d+2 dimensional flat space.

Next, we solved the non-trivial task of construction of interacting Lagrangian for the higher spin field in physical gauge, using the full power of Noether's procedure. As a result, the linear on-field gauge transformation is obtained, and the corresponding commutator of transformation is analyzed. To understand the closure of this algebra, the right-hand side of

this commutator is classified with respect to gauge transformations coming from cubic interactions with different higher spin symmetric tensor fields and mixed symmetry tensor fields transformations.

During the all research process, we have intensively used the Wolfram Mathematica along with $xAct$, $xTensor$, $xTras$ packages to model complex problems and solve them programmatically. We have developed methods and tools for working with higher spin fields in Wolfram Mathematica, and these methods are generic enough to be used on other use-cases as well. We introduce some of the modeling approaches in Wolfram Mathematica language and include the developed methods and functions as codding snippets in the separate section on each chapter.

AIM OF THE DISSERTATION

The main goal of this thesis is to analyse the Higher spin interactions in flat and AdS spaces. In this work we focused on two problems. One of which was the construction of cubic interaction in AdS_{d+1} space in explicit covariant form.

- Construction of the main term of the cubic interaction in flat space.
- The structure of the polynomial coefficients and the solution of the recurrence relations.
- Problem modelling in Wolfram Mathematica
- Pullback of the main term of the cubic self-interaction

The second problem was construction of the some $s$pecial case of local quartic interaction for the higher spin field in physical gauge.

- Computing the commutator of δ_1 transformations for spin four
- Expressing the commutator and variations using generalized Christopher symbols

- Classification of the commutator and interpretation of the right hand side.

PRACTICAL VALUE

The result of this thesis can be used for development of new approaches in important and contemporary area of theoretical physics such as investigations of quantum gravity, dark matter, string theory and any other modification of previously known theories with aims to go beyond standard knowledge. The results, specifically the modelling part in Wolfram Mathematica is very generic and can be used for solving other problems in Higher Spin Gauge Theory area.

RESULTS AND PUBLISHED ARTICLES

Results submitted for defense

- We were able to successfully finalize the radial pullback procedure for the main term of cubic self-interaction by solving all necessary recurrence relations. The solutions of these equations lead to the possibility of obtaining the full set of AdS_{d+1} dimensional interacting terms with all curvature corrections, including trace and divergence terms from any interaction term in $d+2$ dimensional flat space.

- Using Wolfram Mathematica programming language, we developed general methods for solving recurrence relations which we faced during the radial pullback procedure. These methods are also generic enough to be used for solving different recurrence relations in other problems.

- We have considered local quartic interaction between higher-spin gauge field and scalar field. Using the full power of Neother's procedure, we successfully constructed interacting Lagrangian for this special case in a physical gauge.

- We successfully constructed and analyzed the commutator of the linearized higher spin gauge field transformation and successfully classified the right-hand side of the commutator. As an important result, we discovered that on the right-hand side of the commutator, there are terms with mixed symmetry.

- Using Wolfram Mathematica programming language, we have developed generic functions and methods which introduced the possibility of working with symmetric tensors with higher rank using the polynomial notation, which significantly makes computation much easier and human-readable.

The current work is based on the following articles:

- M. Karapetyan. "Solving Recurrence Relations for Radial Pullback of Cubic Interaction in AdSd+1". Phys. Part. Nucl. Lett., 17(5):760–762, 2020.

- M. Karapetyan. "Commutator of higher spin gauge transformation." PoS, Regio2021:043, 2022.

- M. Karapetyan, R. Manvelyan, and R. Poghossian. "Cubic interaction for higher spins in AdSd+1 space in the explicit covariant form." Nucl. Phys. B, 950:114876, 2020.

- M. Karapetyan, R. Manvelyan, and R. Poghossian. "On Cubic Interaction for Higher Spins in AdSd+1." Phys. Part. Nucl. Lett., 17(5):696–700, 2020.

- M. Karapetyan, R. Manvelyan, and G. Poghosyan. "On special quartic interaction of higher spin gauge fields with scalars and gauge symmetry commutator in the linear approximation." Nucl. Phys. B, 971:115512, 2021.

- R. Manvelyan and M. Karapetyan. "Local quartic interaction of scalars with higher spin gauge fields and commutator of linear gauge transformations." PoS, Regio2020:012, 2021

This thesis is organized as follows:

The first chapter is introduction, where we present some well-known results from various papers and literature regarding Higher spin theories, Fronsdal formalism and Neother's procedure.

In *Section 1.1* we present Fronsdal equation for trivial spin $1, 2, 3$ cases and then the generalization for the arbitrary spin s case.

In *Section 1.2* we introduce polynomial notation of working with HS object, which is widely used in HS theories and helps to work with polynomials instead of symmetric tensors with a lot of indices.

In *Section 1.3* we present the Fronsdal Lagrangian and equation of motion using the polynomial notation.

In *Section 1.4* we focus on the Neother's procedure, which is widely used as a perturbative method for computing the interactions.

The second chapter is devoted to the computation of cubic interaction for a higher spin on AdS space in explicit covariant form and it is based on the following articles [1–3] written with Ruben Manvelyan and Ruben Poghossian. We present a slightly modified version of the radial pullback formalism proposed previously by R. Manvelyan, R. Mkrtchyan, and W. Rühl in 2012, where authors investigated the possibility to connect the main term of higher spin interaction in flat d+2 dimensional space to the main term of interaction in AdS_{d+1} space ignoring all trace and divergent terms but expressed directly through the AdS covariant derivatives and including some curvature corrections. In this work, we succeeded in solving all necessary *recurrence relations* to finalize full radial pullback of the main term of cubic self-interaction for higher spin gauge fields in Fronsdal's formulation from flat to one dimension less AdS_{d+1} space. As a result, non-trivial solutions of recurrence relations lead to the possibility to obtain the complete set of AdS_{d+1} dimensional interacting terms with all curvature corrections, including trace and divergence terms from any interaction term in d+2 dimensional flat space.

The third chapter is based on [4–6] written with Ruben Manvelyan and Gabriel Poghosyan, where we solved the non-trivial task of construction of interacting Lagrangian for the higher spin field in physical gauge, using the full power of Noether's procedure. As a result, the linear on-field gauge transformation is obtained, and the corresponding commutator of transformation is analyzed. To understand the closure of this algebra, the right-hand side of this commutator is classified with respect to gauge transformations coming from cubic interactions with different higher spin symmetric tensor fields and mixed symmetry tensor fields transformations.

Chapter 1

Introduction

Higher spin gauge theories are an essential part of modern theoretical physics. The spectrum of these theories includes graviton as a massless spin-two field. These theories are supposed to be consistent quantum theories, hence they are supposed to give examples of quantum gravity. Higher Spin Theories are interesting polygon for checking the AdS/CFT correspondence, where there are many conjectures related weakly coupled higher spin theories living in the bulk to strong coupled conformal field theories on boundary of Anti de Sitter space. There are a lot of interesting problems for research, such as the construction of interacting Higher Spin theories or the construction of interacting Lagrangians.

Construction of an interacting Higher Spin (HS) gauge theory is a problem with some permanent background interest for more than the last thirty years, starting from early work [7]. Periodically, one can observe a growing interest in this object of investigation mainly realized as some success in constructing cubic interaction in AdS or flat background and in connection with AdS/CFT and HS gravity in various dimensions. These attempts were always attractive as one more way to relate quantum theory with General Relativity and investigate HS gauge fields on the same shelf with gravity or understand the uniqueness of gravity (spin 2 field) compared to other members of the HS hierarchy. It is worth recalling that even though consistent equations of motion [8] for *interacting higher spin* fields have been known for many years, the action principle for these theories remains problematic.

The usual method to construct this interacting Lagrangian was to develop Fronsdal metric formalism for free fields [9]. The essential point here is that during perturbative (Noether method) construction of interaction for HS models, one can see perturbative deformation of the free field's gauge transformation and certain difficulties connected with the locality of the theory beyond cubic order [10–14]. Whilist, the cubic interaction is the main building object of HS interaction, not all problems are solved in a fast way, even on a cubic level.For example, the light-cone gauge construction and classification started from the eighties of the last century for four dimensions [15] and continued and finished by Metsaev [16] during the first decade of current century for arbitrary dimension and even with some interesting results during last years [17] . The covariant approach went even slowly: after seminal work of Berends, Baurgers and Van Dam in 1985 [18] and then Fradkin and Vasiliev in 1987 [19] the cubic interaction and classification of vertices came to the center of interest again in 2006-2012 [20–33]. This development in particularly brought to interesting and elegant formulation through the generating function [31, 33] and connection with String Theory [31, 32]. It is worth to mention also that all these activities supplemented with the parallel development of Vasiliev's frame like formalism to cubic interaction in AdS space [34–36]. It is interesting also in this aspects that covariant classification of cubic vertices was done for parity even dimensions $d \geq 4$ in [29] but classification including parity odd vertices for four and three dimensions was completed only recently in [37–39]. The last point we want to mention here is that although cubic interaction in AdS space has formulation developed in ambient space some years ago [40–44] the direct formulation on the language of AdS_{d+1} covariant derivatives was still unknown and realized before in [44] for some simplest part of interaction only. From other side realization of the Noether program directly in AdS space [45] is also extremely difficult due to noncommutativity of covariant derivatives in space with constant curvature. Therefore at the moment, the only way to see this interaction in AdS space directly is to continue the approach defined in [44], which we did in [1] and in this thesis.

The next important problem, which has a lot of increased interest, is the construction of complete Higher Spin (HS) interaction Lagrangian [8, 46–51]. Because HS theories play a role for the development of other theories such as AdS/CFT, the interest in this field increased even more. Construction of HS interaction Lagrangian in itself is a very interesting task due to its complexity and necessity to develop non-trivial computing techniques for even small achievements. During the previous 10-12 years, there was significant progress in this area, especially in the understanding of the construction and structure of cubic interaction in different approaches, dimensions, and backgrounds ([1, 7, 16–19, 25, 27–40, 43, 52]) yet, our knowledge is far from being complete and seems to be bounded to the idea that quartic interaction should be non-local([10, 12–14, 53–57]). In parallel with these activities, the questions of possible non-locality beyond cubic level have been discussed in Vasilev's nonlinear theory of interacting HS fields equations in AdS background (see [58], [59] and ref. there). In some exceptional cases, it seems like it is possible to construct local interactions between fields with different spins, at least as a part of a more complicated covering theory (maybe non-local), including other gauge fields and symmetries.

1.1 FRONSDAL EQUATION

In this section, the equation of motion for bosonic higher spin fields is considered for the special spin 1,2,3 case. The spin s generalization of the equation is derived along with the special field conditions for gauge invariance. Various gauge fixing conditions are considered, and it was shown that in the De Donder gauge, the equation of motion becomes diagonal. This results are well known and can be found in literature of Higher Spins.

SPIN 1 CASE, PHOTON

From electrodynamics, it is well known that photon has two polarization, hence two degrees of freedom. On the other hand, the vector potential has four components, meaning that there are only two physical components, and we should get rid of the other two components.

If vector potential has the following symmetry

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \alpha \tag{1.1}$$

We can pick α such that we can reduce its degree of freedom by one.

$$\partial^\mu A'_\mu = 0 \tag{1.2}$$

We can always put such condition, because

$$\partial^\mu \Big(A_\mu + \partial_\mu \alpha \Big) = 0 \tag{1.3}$$

This equations always has local solution for the α

$$\partial^\mu A_\mu + \Box \alpha = 0 \tag{1.4}$$

$$\alpha = -\Box^{-1} \partial^\mu A_\mu \tag{1.5}$$

where $\Box^{-1}$ is non-local operator. Placing the new A'_μ into the (1.2), we see that the condition

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \lambda \tag{1.6}$$

$$\partial^\mu \Big(A_\mu + \partial_\mu \lambda \Big) = \partial^\mu A_\mu + \Box \lambda = 0 \tag{1.7}$$

Will be fulfilled if λ is a harmonic function. This condition reduces the degree of freedom by one more, leaving two physical components.

Now we construct the equation of motion, taking into account these symmetry considerations. For that, we take a generic ansatz for the second-order differential operator for A_μ.

$$\Box A_\mu + C_1 \partial_\mu \partial^\nu A_\nu = 0 \tag{1.8}$$

If A_μ is a solution than $A'_\mu = A_\mu + \partial_\mu \alpha$ is also a solution which leads

$$\delta\Big(\Box A_\mu + C_1 \partial_\mu \partial^\nu A_\nu\Big) = 0 \tag{1.9}$$

$$\delta A_\mu = \partial_\mu \alpha \tag{1.10}$$

$$\Box \partial_\mu \alpha + C_1 \partial_\mu \Box \alpha = 0 \tag{1.11}$$

$$(1 + C_1)\Box \partial_\mu \alpha = 0 \tag{1.12}$$

We can fix $C_1 = -1$ than for the equation of motion we will get

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \tag{1.13}$$

And by fixing the gauge using Lorentz condition, we will get

$$\Box A_\mu = 0 \tag{1.14}$$

This equation is also possible to get from the lagrangian

$$\mathcal{L} = F_{\mu\nu} F^{\mu\nu} \tag{1.15}$$

SPIN 2 CASE, GRAVITON

We use the same approach which we used of spin 1 case for spin 2 case. This time our field is a symmetric tensor with rank 2 with the following symmetry:

$$\delta h_{\mu_1 \mu_2} = \partial_{(\mu_1} \alpha_{\mu_2)}, \tag{1.16}$$

where $\partial_{(\mu_1} \alpha_{\mu_2)}$ is symmetrization operation without normalization. Let's construct the most generic second-order differential operator for $h_{\mu_1 \mu_2}$ field and find the equation of motion considering into account the gauge symmetry.

$$\Box h_{\mu_1 \mu_2} + C_1 \partial_{(\mu_1} \partial^\lambda h_{\mu_2)\lambda} + C_2 \partial_{\mu_1} \partial_{\mu_2} h^\lambda_\lambda = 0$$

$$\delta\big(\Box h_{\mu_1\mu_2} + C_1\partial_{(\mu_1}\partial^\lambda h_{\mu_2)\lambda} + C_2\partial_{\mu_1}\partial_{\mu_2}h^\lambda_\lambda\big) = 0$$

Fixing $C_1 = -1$ and $C_2 = -C_1 = 1$ we get

$$\Box h_{\mu_1\mu_2} - \partial_{(\mu_1}\partial^\lambda h_{\mu_2)\lambda} + \partial_{\mu_1}\partial_{\mu_2}h^\lambda_\lambda = 0. \tag{1.17}$$

This equation is also possible to get from the Einstein equation by linearization.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{1.18}$$

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \tag{1.19}$$

Writing the equation of motion in the following form

$$\Box h_{\mu_1\mu_2} - \partial_{(\mu_1}D_{\mu_2)} = 0 \tag{1.20}$$

$$D_{\mu_2} = (\partial\cdot h)_{\mu_2} - \frac{1}{2}\partial_{\mu_2}\bar{h} \tag{1.21}$$

and fixing the gauge by

$$D_{\mu_2} = 0 \quad \texttt{(De Donder gauge)} \tag{1.22}$$

we get diagonal equation of motion for spin 2 case.

$$\Box h_{\mu_1\mu_2} = 0 \tag{1.23}$$

SPIN 3 CASE

In spin 3 case our field is symmetric tensor with rank 3 and the gauge parameter is symmetric tensor of rank 2.

$$\alpha_{\mu_1\mu_2} = \alpha_{\mu_2\mu_1} \tag{1.24}$$

$$\delta h_{\mu_1\mu_2\mu_3} = \partial_{(\mu_1}\alpha_{\mu_2\mu_3)} \tag{1.25}$$

The ansatz for the $h_{\mu_1\mu_2\mu_3}$ is

$$\Box h_{\mu_1\mu_2\mu_3} + C_1\partial_{(\mu_1}(\partial\cdot h)_{\mu_2\mu_3)} + C_2\partial_{(\mu_1}\partial_{\mu_2}\bar{h}_{\mu_3)} = 0, \tag{1.26}$$

where

$$(\partial\cdot h)_{\mu_2\mu_3} = \partial_{\mu_1}h^{\mu_1}_{\mu_2\mu_3}$$

$$\bar{h}_\mu = h^\lambda_{\lambda\mu} = \big(Tr(h)\big)_\mu$$

By taking the gauge variation of the (1.26) we get

$$\delta\big(\Box h_{\mu_1\mu_2\mu_3} + C_1\partial_{(\mu_1}(\partial\cdot h)_{\mu_2\mu_3)} + C_2\partial_{(\mu_1}\partial_{\mu_2}\bar{h}_{\mu_3)}\big) = 0 \tag{1.27}$$

$$\Box\delta h_{\mu_1\mu_2\mu_3} + C_1\partial_{(\mu_1}(\partial\cdot\delta h)_{\mu_2\mu_3)} + C_2\partial_{(\mu_1}\partial_{\mu_2}\delta\bar{h}_{\mu_3)} = 0 \tag{1.28}$$

This time fixing $C_1 = -1$ and $C_2 = 1$ is not enough, and we are required to put a condition on the gauge parameter requiring it to be traceless.

$$\bar{\alpha} = 0$$

Now we can write the equation of motion in the following form

$$\Box h_{\mu_1\mu_2\mu_3} - \partial_{(\mu_1}(\partial\cdot h)_{\mu_2\mu_3)} + \partial_{(\mu_1}\partial_{\mu_2}\bar{h}_{\mu_3)} = 0 \tag{1.29}$$

and by introducing the De Donder tensor

$$\Box h_{\mu_1\mu_2\mu_3} - \partial_{(\mu_1}D_{\mu_2\mu_3)} = 0 \tag{1.30}$$

$$D_{\mu_2\mu_3} = \partial^\lambda h_{\lambda\mu_2\mu_3} - \frac{1}{2}\partial_{(\mu_2}\bar{h}_{\mu_3)} \tag{1.31}$$

Here we can see that by putting the De Donder condition we can successfully diagonalize the equation.

$$D_{\mu_2\mu_3} = 0 \tag{1.32}$$

$$\delta D_{\mu_2\mu_3} = \partial^\lambda\partial_{(\lambda}\alpha_{\mu_2\mu_3)} - \frac{1}{2}\partial_{(\mu_2}\Big(2\partial\cdot_{\mu_3)} + \partial_{\mu_3}\bar{\alpha}\Big) = \Box\alpha_{\mu_2\mu_3} \tag{1.33}$$

$$\Box h_{\mu_1\mu_2\mu_3} = 0 \tag{1.34}$$

SPIN S CASE

In spin s case our gauge field is symmetric tensor of rank s and gauge parameter is symmetric tensor of rank $s-1$.

$$\delta h_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\alpha_{\mu_2\cdots\mu_s)} \tag{1.35}$$

By computing the gauge variation of our equation ansatz we get

$$\delta\Big(\Box h_{\mu_1\cdots\mu_s} + C_1\partial_{(\mu_1}(\partial\cdot h)_{\mu_2\cdots\mu_s)} + C_2\partial_{(\mu_1}\partial_{\mu_2}\bar{h}_{\mu_3\cdots\mu_s)}\Big) = 0, \tag{1.36}$$

which simplifies to

$$(1+C_1)\Box\partial_{(\mu_1}\alpha_{\mu_2\cdots\mu_s)} + 2(C_1+C_2)\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3\cdots\mu_s)} + \tag{1.37}$$

$$+ (s-2)C_2\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\bar{\alpha}_{\mu_4\cdots\mu_s)} = 0. \tag{1.38}$$

Here we get the following solutions for the coefficients $C_1 = -1; C_2 = 1$ and $\bar{\alpha} = 0$ which means that our gauge field should be double traceless

$$TrTrh = 0. \tag{1.39}$$

For the equation of motion we get

$$\Box h_{\mu_1\cdots\mu_s} - \partial_{(\mu_1}(\partial\cdot h)_{\mu_2\cdots\mu_s)} + \partial_{(\mu_1}\partial_{\mu_2}\bar{h}_{\mu_3\cdots\mu_s)} = 0 \tag{1.40}$$

and after the introduction of De Donder tensor

$$\Box h_{\mu_1\cdots\mu_s} - \partial_{(\mu_1}D_{\mu_2\cdots\mu_s)} = 0 \tag{1.41}$$

$$D_{\mu_2\cdots\mu_s} = \partial^\lambda h_{\lambda\mu_2\cdots\mu_s} - \frac{1}{2}\partial_{(\mu_2}\bar{h}_{\mu_3\cdots\mu_s)} \tag{1.42}$$

and putting the De Donder condition we diagonalize the equation of motion

$$\Box h_{\mu_1\cdots\mu_s} = 0 \tag{1.43}$$

Manual investigation of the special spin 1,2,3 case shows that it is possible to generalize the equation of motion for higher spin gauge fields and write in for the arbitrary spin s case. Moreover, there is special gauge condition called De Donder condition, which when applied diagonalizes the equation of motion.

1.2 POLYNOMIAL NOTATION OF HS FIELDS

In order to work with Higher spin fields and do not mess with a lot of tensorial indices we will use standard notation coming from papers about HSF [26, 27, 30, 60, 61]. We utilize instead of symmetric tensors such as $h^{(s)}_{\mu_1\mu_2\ldots\mu_s}(z)$ the homogeneous polynomials in the vector a^μ of degree s at the base point z

$$h^{(s)}(z;a) = \sum_{\mu_i}(\prod_{i=1}^{s} a^{\mu_i})h^{(s)}_{\mu_1\mu_2\ldots\mu_s}(z). \tag{1.44}$$

Then we can write the symmetrized gradient, trace and divergence [1]

$$Grad : h^{(s)}(z;a) \Rightarrow Gradh^{(s+1)}(z;a) = (a\nabla)h^{(s)}(z;a), \tag{1.45}$$

$$Tr : h^{(s)}(z;a) \Rightarrow Trh^{(s-2)}(z;a) = \frac{1}{s(s-1)}\Box_a h^{(s)}(z;a), \tag{1.46}$$

$$Div : h^{(s)}(z;a) \Rightarrow Divh^{(s-1)}(z;a) = \frac{1}{s}(\nabla\partial_a)h^{(s)}(z;a). \tag{1.47}$$

Moreover we introduce the notation $*_a, *_b, \ldots$ for a contraction in the symmetric spaces of indices a or b

$$*_a \;=\; \frac{1}{(s!)^2}\prod_{i=1}^{s}\overleftarrow{\partial}^{\mu_i}_{a}\overrightarrow{\partial}^{a}_{\mu_i}. \tag{1.48}$$

Then we see that the operators $(a\partial_b), a^2, b^2$ are dual (or adjoint) to $(b\partial_a), \Box_a, \Box_b$ with respect to the "star" product of tensors with two sets of symmetrized indices (1.48)

$$\frac{1}{n}(a\partial_b)f^{(m-1,n)}(a,b) *_{a,b} g^{(m,n-1)}(a,b) = f^{(m-1,n)}(a,b) *_{a,b} \frac{1}{m}(b\partial_a)g^{(m,n-1)}(a,b), \tag{1.49}$$

$$a^2 f^{(m-2,n)}(a,b) *_{a,b} g^{(m,n)}(a,b) = f^{(m-2,n)}(a,b) *_{a,b} \frac{1}{m(m-1)}\Box_a g^{(m,n)}(a,b). \tag{1.50}$$

[1] To distinguish easily between "a" and "z" spaces we introduce the notation ∇_μ for space-time derivatives $\frac{\partial}{\partial z^\mu}$.

In the same fashion gradients and divergences are dual with respect to the full scalar product in the space (z, a, b)

$$(a\nabla)f^{(m-1,n)}(z;a,b) *_{a,b} g^{(m,n)}(z;a,b) \quad = \quad -f^{(m-1,n)}(z;a,b) *_{a,b} \frac{1}{m}(\nabla\partial_a)g^{(m,n)}(z;a,b). \tag{1.51}$$

REALIZATION ON WOLFRAM MATHEMATICA IN FLAT SPACE

We will demonstrate how polynomial notation can be realized in Wolfram Mathematica. This realization is very useful, because formulas on the computer and on the paper are similar and there is no need to convert them to Tensor notation to perform the computation on the machine, moreover, the results are also in polynomial notation which helps when working on the paper. First of all we need to import the package, define the manifold, metric and other required objects.

```
<< xAct`xTras`;
DefManifold[M, dim, IndexRange[a, m]]
AddIndices[TangentM, {n, o, p, q, s, t}]
DefMetric[
 1, metric[-n, -o], PD, PrintAs -> "η",
 FlatMetric -> True, SymbolOfCovD -> {",", "∂"}
 ]
SetOptions[SymmetryOf, ConstantMetric -> True];
DefTensor[Cron[a, b], M, Symmetric[{a, b}], PrintAs -> "δ"]
```

Listing 1.1: Setup

Next we define rules and custom simplification function which will apply rules to the expressions.

```
rules = {
  MakeRule[{PD[e][A[b]], delta[e, b]}, MetricOn -> All, ContractMetrics -> True]
};
(* Simplification  *)
 FulSim[exp_, rules_:  rules] := Module[{Simpl},
  Simpl[expp_, rule_] := Module[{ expS, Simpliator},
     expS[ex_] := Module[{}, ex // ExpandAll];
     expS @ expS[expp]  /. rule // ToCanonical // ContractMetric //
      FullSimplify
     ];
  Fold[Simpl[#1, #2] &, exp, rules]
  ];
 (* Ordinary derivative which treats the elements in second argument as constants *)
 Der[a_, consts_: {A, B}] := Module[{rules, Deriv, e, b},
    rules =
    Table[MakeRule[{PD[e][i[b]], 0}, MetricOn -> All,
      ContractMetrics -> True] // ReleaseHold , {i, consts}];
    Deriv[exp_] := Module[{der},
    der = PD[a]@exp ;
    Fold[#1 /. #2 &, der, rules ]
    ];
    Deriv
]
```

Listing 1.2: Rules of simplification

Here in listing (1.2) we created a function which will apply rules to the expression defined in the rules object. If there are other conditions which should be taken into consideration during simplification one can simply add in the **rules** object.

Now we are all set for defining $(a\nabla)$, ∂_a, ∇_μ, $\Box_a$, $*_a$ operators. Below we present implementaton of all this operators.

```
xGrad[A_, exp_] := Module[{f},
 A[-f]*Der[f]@exp
 ];
```

Listing 1.3: Definition of symmetrized gradient $(a\nabla)$

```
DerV[A_, e_] := Module[{DAA, finn},
 DAA[x_] :=
 Module[{der, derr, HOnly, res, b, rule3, rule2, rule4, rule5, der1, der2,
  rule6, a, rule7,rule8, rule9, diff, simm, fin},
   der = PD[e][x] // ExpandAll;
   rule3 = MakeRule[{PD[e][A[b]], Cron[e, b]}, MetricOn -> All,
  ContractMetrics -> True];
   rule4 = MakeRule[{metric[-b, -e]*Cron[b, e], 1}, MetricOn -> All,
  ContractMetrics -> True];
   rule5 = MakeRule[{Cron[b, -b], 1}, MetricOn -> All, ContractMetrics ->
  True];
   rule6 = MakeRule[{A[-e]*Cron[e, b], A[b]}, MetricOn -> All,
  ContractMetrics -> True];
   rule7 = MakeRule[{Cron[a, b]*Cron[e, -b], Cron[a, e]}, MetricOn -> All,
  ContractMetrics ->True];
   rule8 = MakeRule[{PD[a]@Cron[e, -b], 0}, MetricOn -> All,
  ContractMetrics -> True];
   der1 = der /. rule3 /. rule4 /. rule7  /. rule8;
   rule2 = MakeRule[{Cron[e, b], 0}, MetricOn -> All, ContractMetrics ->
  True];
   der2 = der1 /. rule2;
   diff = der1 - der2;
   res = diff    /. rule6 /. rule7 /. rule8 ;
   rule9 = MakeRule[{Cron[e, b], metric[e, b]}, MetricOn -> All,
  ContractMetrics -> True];
   simm = FulSim@res ;
   fin = simm /. rule9 // ContractMetric;
```

```
  DAA[x] = fin;
  fin
];
DerV[A, e] =  DAA;
DAA
];
```

Listing 1.4: Definition of ∂_a

```
(* Rank operator: Returns number of constant vectors of the given type *)
xRank[A_ , exp_] := Module[{a, rank, t, dif},
rank = 0;
t = exp;
dif[t_] := Module[{b}, DerV[A, b]@t];
While[Not[dif[t] === 0] , rank = rank + 1; t = dif[t] ];
xRank[A, exp] = rank;
rank
];
```

Listing 1.5: Definition of Rank operator

```
(* Trace operator: Computes the trace of the expression by differentiating
  two times with the given constant vector. *)
xTrace[A_, exp_] := Module[{f, s},
s = xRank[A, exp];
(1 / (s (s - 1)) )* DerV[A, f]@DerV[A, -f]@exp
];
```

Listing 1.6: Definition of trace $\Box_a$ operator

```
(* Divergence: Computes the divergence of an expression by removing one
 constant vector and replacing it with a derivative*)
xDiv[A_, exp_] := Module[{f, b, s},
s = xRank[A, exp];
1/s * Der[-b]@DerV[A, b]@exp
```

```
  ];
```

Listing 1.7: Definition of divergence $\nabla_\mu \partial_a$ operator

```
(* Star operator: Computes a star contraction of two expressions using
   given constant vector *)
xStar[A_, s_, exp1_ , exp2_] := Module[{rng, dif, difd, res, ret},
 rng = Range[s];
 dif[t_, c_] := Module[{b},
     { DerV[A, b]@ReplaceDummies[t[[1]]] ,
     DerV[A, -b]@ReplaceDummies[t[[2]]] }
    ];
 res = Fold[dif[#1, #2] &, {exp1, exp2}, rng ];
 ret = (1 / (s!)^2)*res[[1]] * res[[2]];
 xStar[A, s, exp1, exp2] = ret;
 ret
];
```

Listing 1.8: Definition of star $*_a$ operator

```
(* Power operator: Raises the expression to the given power *)
xPow[exp_, n_] := Module[{res, a, b},
 res = 1;
 Fold[#1 * ReplaceDummies[exp] &, res, Range[n] ]
]
```

Listing 1.9: Definition of power operator

We were able to successfully model the polynomial notation in Wolfram Mathematica which allows to work inside Mathematica using polynomial notation and speed up a big part of the manual work which otherwise one should have done by hand on paper.

1.3 FRONSDAL LAGRANGIAN

Here we present Fronsdal Lagrangian in this notation and suppose integration everywhere when it is necessary and we are treating Lagrangian as an action, hence we will neglect all d dimensional space-time total derivatives when making a partial integration. There are several papers where this calculations are presented such as [27, 30].

$$\mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2}h^{(s)}(a) *_a \mathcal{F}^{(s)}(a) + \frac{1}{8s(s-1)}\Box_a h^{(s)}(a) *_a \Box_a \mathcal{F}^{(s)}(a), \tag{1.52}$$

where $\mathcal{F}^{(s)}(z; a)$ is the Fronsdal tensor

$$\mathcal{F}^{(s)}(z; a) = \Box h^{(s)}(z; a) - s(a\nabla)D^{(s-1)}(z; a), \tag{1.53}$$

and $D^{(s-1)}(z; a)$ is the deDonder tensor or traceless divergence of the higher spin gauge field

$$D^{(s-1)}(z; a) = Divh^{(s-1)}(z; a) - \frac{s-1}{2}(a\nabla)Trh^{(s-2)}(z; a), \tag{1.54}$$

$$\Box_a D^{(s-1)}(z; a) = 0. \tag{1.55}$$

The initial gauge variation of zeroth order in the spin s field is

$$\delta_{(0)}h^{(s)}(z; a) = s(a\nabla)\epsilon^{(s-1)}(z; a), \tag{1.56}$$

with the traceless gauge parameter for the double traceless gauge field

$$\Box_a \epsilon^{(s-1)}(z; a) = 0, \tag{1.57}$$

$$\Box_a^2 h^{(s)}(z; a) = 0. \tag{1.58}$$

Therefore at this point we can see from (1.56) and (1.57) that the de Donder gauge condition

$$D^{(s-1)}(z; a) = 0. \tag{1.59}$$

is a correct generalization of the Lorentz gauge condition in the case of spin $s > 2$. Finally we note that in deDonder gauge (1.59) $\mathcal{F}^{(s)}(z;a) = \Box h^{(s)}(z;a)$ and the field $h^{(s)}$ decouples from it's trace in Fronsdal's Lagrangian (1.52).

The equation of motion following from (1.52) is

$$\delta\mathcal{L}_0(h^{(s)}(a)) = -(\mathcal{F}^{(s)}(a) - \frac{a^2}{4}\Box_a\mathcal{F}^{(s)}(a)) *_a \delta h^{(s)}(a), \tag{1.60}$$

1.4 NEOTHER'S PRODECURE

Noether procedure is a perturbative method to introduce interactions. The idea is to begin with a sum of free actions S_2 and linearised gauge symmetry δ_0, which are given by Fronsdal Lagrangian and by the gauge transformations. The procedure assumes the addition of all possible corrections that are cubic in the fields S_3 and, at the same time, allow for field-dependent deformations δ_1 of the gauge transformations. Then it is required for full action to be gauge invariant.

$$0 = \delta S = \delta_0 S_2 + \delta_0 S_3 + \delta_1 S_2 + ... \tag{1.61}$$

Because free action is gauge invariant $\delta_0 S_2 = 0$, hence the first condition becomes

$$\delta_0 S_3 + \delta_1 S_2 = 0 \tag{1.62}$$

One may stop procedure at this level or add the next order quartic terms S_4 and more.

Chapter 2

Cubic interaction for higher spins in AdS_{d+1} space in the explicit covariant form

2.1 INTRODUCTION

This chapter is based on several articles [1–3] written with Ruben Manvelyan and Ruben Poghossian.

Here we present a slightly modified prescription of the radial pullback formalism proposed previously by R. Manvelyan, R. Mkrtchyan and W. Rühl in 2012, where authors investigated possibility to connect the main term of higher spin interaction in flat $d+2$ dimensional space to the main term of interaction in AdS_{d+1} space ignoring all trace and divergent terms but expressed directly through the AdS covariant derivatives and including some curvature corrections. We were able to successfully solve all necessary *recurrence relations* and finalize full radial pullback of the main term of cubic self-interaction for higher spin gauge fields in Fronsdal's formulation from flat to one dimension less AdS_{d+1} space. Nontrivial solutions of recurrence relations lead to the possibility to obtain the full set of AdS_{d+1} dimensional interacting terms with all curvature corrections including trace and divergence terms from any interaction term in $d+2$ dimensional flat space.

2.2 PRESCRIPTION FOR RADIAL PULLBACK AND FREE HS GAUGE FIELDS IN ADS

In this section, we present a short review of the radial pullback technique developed in [62,63] and applied in detail to the free higher spin case in [44]. We start from $d+2$ dimensional flat space with coordinates X^A and flat $SO(1,d+1)$ invariant metric

$$X^A \quad A = 1, 2, \dots . d+2, \tag{2.1}$$

$$ds^2 = \eta_{AB} dX^A dX^B = -(dX^{d+2})^2 + (dX^{d+1})^2 + dX^i dX^j \eta_{ij}, \tag{2.2}$$

To recognize Euclidian AdS_{d+1} hypersphere inside of this Ambient space we should define the following coordinate transformation to a curvilinear coordinate system $X^A \to (u, r, x^i)$:

$$\begin{aligned} X^{d+2} &= \frac{1}{2} e^u [r + \frac{1}{r}(L^2 + x^i x^j \eta_{ij})], \\ X^{d+1} &= \frac{1}{2} e^u [r - \frac{1}{r}(L^2 - x^i x^j \eta_{ij})], \\ X^i &= e^u L \frac{x^i}{r}, \end{aligned} \tag{2.3}$$

$$-e^{2u} L^2 = -(X^{d+2})^2 + (X^{d+1})^2 + X^i X^j \eta_{ij}, \tag{2.4}$$

$$ds^2 = L^2 e^{2u} [-du^2 + \frac{1}{r^2}(dr^2 + dx^i dx^j \eta_{ij})]. \tag{2.5}$$

The restriction $e^u = 1$ leads instead of coordinate transformations to the usual embedding of the Euclidian AdS_{d+1} hypersphere with local coordinates $x^\mu = (x^0, x^i) = (r, x^i)$ into $d+2$

dimensional flat space.

In other words, we can define the Jacobian matrix for transformation (2.3) in the following compact form:

$$E^A_\mu(u,x^\nu) = \frac{\partial X^A}{\partial x^\mu} = e^u e^A_\mu(x^\nu), \tag{2.6}$$
$$E^A_u(u,x^\nu) = \frac{\partial X^A}{\partial u} = X^A(u,x^\nu) = e^u L n^A(x^\nu), \tag{2.7}$$

where due to (2.4) the $d+1$ tangent vectors $\{e^A_\mu(x)\}^d_{\mu=0}$ and one normal vector $n^A(x)$

$$n^A(x)e^B_\mu(x)\eta_{AB} = 0 \tag{2.8}$$

$$n^A(x)n^B(x)\eta_{AB} = -1 \tag{2.9}$$

for embedded AdS_{d+1} space define the standard induced metric $g_{\mu\nu}(x)$ and extrinsic curvature $K_{\mu\nu}(x)$ for our embedded AdS_{d+1} space:

$$g_{\mu\nu}(x) = e^A_\mu(x)e^B_\nu(x)\eta_{AB} = \left(\frac{L}{x^0}\right)^2 \delta_{\mu\nu} \tag{2.10}$$

and

$$\partial_\mu e^A_\nu(x) = \Gamma^\lambda_{\mu\nu}(g)e^A_\nu(x) + K_{\mu\nu}(x)n^A(x) \tag{2.11}$$

where

$$\Gamma^\lambda_{\mu\nu}(g) = \Gamma^{\lambda(AdS)}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right), \tag{2.12}$$
$$K_{\mu\nu} = \frac{g_{\mu\nu}}{L} \tag{2.13}$$

So we see that $\Gamma^\lambda_{\mu\nu}(g)$ is usual Christoffel symbol constructed from induced AdS_{d+1} metric and therefore we can introduce AdS_{d+1} covariant derivative ∇_μ and rewrite (2.10) in convenient form:

$$\nabla_\mu e^A_\nu(x) = K_{\mu\nu}(x)n^A(x) \tag{2.14}$$

$$K_{\mu\nu}(x) = e^A_\nu(x)\partial_\mu n_A = -n_A\nabla_\mu e^A_\nu(x) \tag{2.15}$$

Therefor to restrict our flat theory to AdS hypersphere we should first formulate $d+2$ dimensional field theory in the curvilinear coordinates with flat $e^{2u}(AdS_{d+1}\times\mathcal{R}_u)$ metric

$$ds^2 = e^{2u}[-L^2du^2 + g_{\mu\nu}(x)dx^\mu dx^\nu] = G_{uu}(u)du^2 + G_{\mu\nu}(u,x)dx^\mu dx^\nu, \tag{2.16}$$

where

$$G_{uu}(u) = E^A_u(u,x^\nu)E^B_u(u,x^\nu)\eta_{AB} = X^A X_A = -L^2 e^{2u} \tag{2.17}$$

$$G_{\mu\nu} = E^A_\mu(u,x^\nu)E^B_\nu(u,x^\nu)\eta_{AB} = e^{2u} g_{\mu\nu}(x) \tag{2.18}$$

and then define the correct prescription to go from theory in flat curvilinear space defined by Jacobian matrix E^A_u, E^A_μ to the theory with negative constant curvature on the level of $d+2 \times d+1$ embedding matrix e^A_μ or induced metric $g_{\mu\nu}(x)$ getting rid off normal components along of n^A. The most simple check of this statement we can obtain calculating Riemann curvature of the embedded hypersphere. To perform this we should first derive differentiation rules for Frenet basis using (2.13)-(2.15):

$$\nabla_\mu e^A_\nu(x) = \frac{g_{\mu\nu}(x)}{L} n^A(x) \tag{2.19}$$

$$\partial_\mu n^A(x) = \frac{1}{L} e^A_\mu(x), \tag{2.20}$$

and then taking commutator :

$$[\nabla_\mu, \nabla_\nu] e^A_\lambda = R_{\mu\nu,\lambda}{}^\rho e^A_\rho = K_{\lambda[\nu} K^\rho_{\mu]} e^A_\rho \tag{2.21}$$

we get the standard expression for AdS_{d+1} Riemann curvature and Ricci tensors

$$R_{\mu\nu,\lambda}{}^\rho = -\frac{1}{L^2}(g_{\mu\lambda}\delta^\rho_\nu - g_{\nu\lambda}\delta^\rho_\mu) \tag{2.22}$$

$$R_{\mu,\lambda} = -\frac{d}{L^2} g_{\mu\nu}, \quad R = g^{\mu\lambda} R_{\mu\lambda} = -\frac{d(d+1)}{L^2} \tag{2.23}$$

Turning to higher spins in flat ambient space we will use polynomial notation for the higher spins which we have introduced on Chapter 1, Section 2.

To perform correct pullback of higher spin theory from flat ambient to one dimensional less AdS space, we should fix two important points :

- We should fix the ansatz for $d+2$ dimensional HS field in a way to get from one spin s field exactly one spin s field in AdS_{d+1}. The natural condition here send to zero all components normal to the embedded hypersphere

$$n^A h^{(s)}_{AA_2\dots A_s}(u,x^\nu) \sim X^A(u,x^\nu) h^{(s)}_{AA_2\dots A_s}(u,x^\nu) = 0 \tag{2.24}$$

- Our auxiliary vector a^A is constant in flat space

$$a^A = E^A_u(u,x)a^u(u,x^\nu) + E^A_\mu(u,x)a^\mu(u,x^\nu)$$
$$= e^u\left(Ln^A(x)a^u(u,x) + e^A_\mu(x)a^\mu(u,x)\right) \tag{2.25}$$
$$\partial_B a^A = 0, \tag{2.26}$$

  but in curve AdS_{d+1} space there is no possibility to get covariantly constant vectors.

This means that ansatz for HS field itself is not enough for getting correct pullback for objects with derivatives contracted with constant vector a^A. From the other side we have in hand curvilinear metric (2.16)-(2.18) which we can invert and then easily invert the Jacobian matrix (2.6)-(2.7)

$$G^{uu}(u,x) = -\frac{e^{-2u}}{L^2} \tag{2.27}$$
$$G^{\mu\nu}(u,x) = e^{-2u}g^{\mu\nu}(x) \tag{2.28}$$
$$E^u_A(u,x) = E^B_u(u,x)\eta_{AB}G^{uu}(u,x) = -\frac{e^{-u}}{L}n_A(x) \tag{2.29}$$
$$E^\mu_A(u,x) = E^B_\nu(u,x)\eta_{AB}G^{\mu\nu}(u,x) = e^{-u}e^\mu_A(x), \tag{2.30}$$

where $g^{\mu\nu}(x)$ is inverse AdS_{d+1} metric and $e^\mu_A(x) = e^B_\nu(x)\eta_{AB}g^{\mu\nu}(x)$.

Then our flat-space derivative in (2.26) after coordinate transformation is:

$$\partial_A = E^u_A(u,x)\partial_u + E^\mu_A(u,x)\partial_{x^\mu} = -\frac{e^{-u}}{L}n_A(x)\partial_u + e^{-u}e^\mu_A(x)\partial_{x^\mu} \tag{2.31}$$

Substituting this in (2.26) and taking into account (2.25), (2.19) and (2.20) we obtain the following four relations for derivatives of components $a^u(u,x), a^\mu(u,x)$:

$$\partial_u a^u(u,x) + a^u(u,x) = 0 \tag{2.32}$$
$$\partial_u a^\mu(u,x) + a^\mu(u,x) = 0 \tag{2.33}$$
$$\partial_\mu a^u(u,x) + \frac{1}{L^2}a_\mu(u,x) = 0 \tag{2.34}$$
$$\nabla_\mu a^\nu(u,x) + \delta^\nu_\mu a^u(u,x) = 0 \tag{2.35}$$

First two equations we can solve directly:

$$a^u(u,x) = e^{-u}a^u(x) \tag{2.36}$$

$$a^\mu(u,x) = e^{-u}a^\mu(x) \tag{2.37}$$

Substituting these solutions in (2.25) and using restriction (2.24) we see that in curvilinear coordinates our ansatz leads to the following relation:

$$\begin{aligned} h^{(s)}(X,a^B) &= h^{(s)}_{A_1A_2\dots A_s}(X)a^{A_1}a^{A_2}\dots a^{A_s}|_{X^A=(u,x^\mu),n^A h^{(s)}_{A\dots}=0} \\ &= h^{(s)}_{\mu_1\mu_2\dots\mu_s}(u,x)a^{\mu_1}(x)a^{\mu_2}(x)\dots a^{\mu_s}(x) = h^{(s)}(u,x,a^\mu(x)) \end{aligned} \tag{2.38}$$

where:

$$h^{(s)}_{\mu_1\mu_2\dots\mu_s}(u,x) = h^{(s)}_{A_1A_2\dots A_s}(u,x)e^{A_1}_{\mu_1}(x)e^{A_2}_{\mu_1}(x)\dots e^{A_s}_{\mu_s}(x) \tag{2.39}$$

This is correct pullback of spin s tensor field from $d+2$ dimensional flat space to AdS_{d+1} space. The only reminder about flat space we have here is u-dependance of $d+1$ dimensional field components in (2.39)

The initial gauge variation of order zero in the spin s field is

$$\delta_{(0)}h^{(s)}(X^A;a^A) = s(a^A\partial_A)\epsilon^{(s-1)}(X^A;a^A), \tag{2.40}$$

with the traceless gauge parameter for the double traceless gauge field

$$\Box_{a^A}\epsilon^{(s-1)}(X^A;a^A) = 0, \tag{2.41}$$

$$\Box^2_{a^A}h^{(s)}(X^A;a^A) = 0 \tag{2.42}$$

Then combining (2.25) and (2.31) we obtain due to (2.37)

$$a^A\partial_A\epsilon^{(s-1)}(X^A;a^A) = e^{-u}\left(a^u(x)\partial_u + a^\mu(x)\partial_{x^\mu}\right)\epsilon^{(s-1)}(u,x;a^\mu(x)) \tag{2.43}$$

where parameter $\epsilon^{(s-1)}(X^A;a^A)$ obeys to the same type ansatz rule as the $h^{(s)}(X^A;a^A)$ in (2.38)

$$\epsilon^{(s-1)}(X^A;a^A) = \epsilon^{(s-1)}(u,x;a^\mu(x)) \tag{2.44}$$

The next important observation is about derivatives $\partial_{x^\mu} \equiv \partial_\mu$ in respect to AdS_{d+1} coordinates x^μ:

- First note that we mapped scalar object in flat space constructed from X - dependent tensor contracted with constant vectors a^A to the scalar object in curve space constructed from x-dependent tensor contracted with x-dependent vectors $a^\mu(x)$. So as a result we obtain in r.h.s of (2.43) ordinary derivative ∂_{x^μ}

- To see appearance of the AdS_{d+1} covariant derivatives we should use Leibnitz rule in curve space and conditions (2.34), (2.35):

$$\partial_{x^\mu}(T_\nu(x)a^\nu(x)) = \nabla_\mu T_\nu(x)a^\nu(x) + T_\nu(x)\nabla_\mu a^\nu(x)$$
$$= (\nabla_\mu T_\nu(x))a^\nu(x) - T_\mu(x)a^u(x) = (\nabla_\mu T_\nu(x))a^\nu(x) - a^u(x)\frac{\partial}{\partial a^\mu}(T_\nu(x)a^\nu) \qquad (2.45)$$

From this example we see that instead of x-dependent vectors we can use formally x-independent vectors a^μ (and component a^u also) and split AdS space from formal a^μ space inserted only for shortening symmetric tensor contractions and symmetrizing procedures just like in the Cartesian case. But at the same time according to (2.45) we should replace the usual derivative with the following operators in Frenet basis:

$$\partial_A \;=>\; (e^{-u}\partial_u, e^{-u}\partial_\mu), \qquad (2.46)$$

$$\partial_\mu \;=>\; D_\mu = \nabla_\mu - a^u\partial_{a^\mu} - \frac{a_\mu}{L^2}\partial_{a^u}, \qquad (2.47)$$

where ∇_μ is AdS covariant derivative constructed from the Christoffel symbols (2.12) with the following action rule:

$$\nabla_\mu h^{(s)}(u,x;a) = \nabla_\mu h_{\mu_1\mu_2\dots\mu_s}(u,x)a^{\mu_1}a^{\mu_2}\dots a^{\mu_s}. \qquad (2.48)$$

So from now on we have instead of usual differential operator and coordinate dependent auxiliary vector components "constant" objects a^u and a^μ and covariant derivative operator (2.47) working on rank s symmetric tensors as operators working in both x and a spaces.

Then we can write (2.43) in the form:

$$a^A\partial_A\epsilon^{(s-1)}(X^A;a^A) = e^{-u}\left(a^u\partial_u + a^\mu D_\mu\right)\epsilon^{(s-1)}(u,x;a^\mu)$$
$$= e^{-u}\left[a^u(\partial_u - s + 1) + a^\mu\nabla_\mu\right]\epsilon^{(s-1)}(u,x;a^\mu) \qquad (2.49)$$

Using this and restricting the dependence on additional "u" coordinates for all fields and gauge parameters in the following (exponential) way

$$h^{(s)}(u, x^\mu; a^\mu) = e^{\Delta_h u} h^{(s)}(x^\mu; a^\mu), \tag{2.50}$$

$$\epsilon^{(s-1)}(u, x^\mu; a^\mu) = e^{\Delta_\epsilon u} \epsilon^{(s-1)}(x^\mu; a^\mu), \tag{2.51}$$

we obtain from the (2.40) the following relation

$$e^{\Delta_h u} \delta h^{(s)}(x^\mu; a^\mu) = e^{(\Delta_\epsilon - 1)u} s \left[a^u(\Delta_\epsilon - s + 1) + a^\mu \nabla_\mu\right] \epsilon^{(s-1)}(x; a^\mu). \tag{2.52}$$

So we see that for getting from gauge transformation in $d+2$ dimensional flat space (2.40) the correct AdS_{d+1} gauge transformation

$$\delta h^{(s)}(x^\mu; a^\mu) = s a^\mu \nabla_\mu \epsilon^{(s-1)}(x; a^\mu) \tag{2.53}$$

we should fix the last freedom in our ansatz in unique form

$$\Delta_\epsilon = s - 1 \tag{2.54}$$

$$\Delta_h = \Delta_\epsilon - 1 = s - 2 \tag{2.55}$$

which is in agreement with consideration in [40–43].

After all, we can formulate our final prescription for radial pullback in the massless AdS case slightly differs from reduction formulated in [44] and can be summarized by the following three points.

1. Expand auxiliary vectors a^A using Frenet basis for embedded AdS space (2.25) and take into account u dependents (2.36),(2.37) for components normal and tangential to the embedded hypersphere coming from condition (2.26) and formal x^μ independence explained above. Finally, we have the following embedding rule

$$a^A => L n^A(x) a^u + e^A_\mu(x) a^\mu \tag{2.56}$$

2. Replace all derivatives in the following way:

$$\partial_A => e^{-u}\left(-\frac{n_A(x)}{L}\partial_u + e^\mu_A(x) D_\mu\right) \tag{2.57}$$

where D_μ defined in (2.47)

3. Restrict the dependence on additional "u" coordinates for all fields and gauge parameters in an exponential way with corresponding weights (2.54) (2.55) to preserve gauge invariants during pullback.

Note also that our reduction rules here slightly different from rules, formulated in [44], especially in the area of "u" dependance. This happened because we used direct solutions (2.36), (2.37) and keep derivative ∂_u unchanged. In [44] authors removed exponential factor e^{-u} a front of derivatives and all a^u and a^μvector components, replacing radial derivatives also with operator $\partial_u - a^u\partial_{a^u} - a^\mu\partial_{a^\mu}$ working in both u and a spaces. In that case scaling behaviour of field components and parameters are different from our here and in [44] [1]

In any case, the final result is the same: After some straightforward calculation using our reduction rules we can prove that $d+2$ dimensional gauge invariant Fronsdal tensor

$$\begin{aligned}\mathcal{F}^{(s)}(X^A;a^A) \quad &= \Box_{d+2}h^{(s)}(X^A;a^A) - a^A\partial_A\Big(\partial^B\partial_{a^B}h^{(s)}(X^A;a^A)\\ &-\frac{1}{2}(a^B\partial_B)\Box_{a^A}h^{(s)}(X^A;a^A)\Big),\end{aligned} \tag{2.58}$$

reduces to the AdS_{d+1} gauge invariant Fronsdal tensor

$$\begin{aligned}&\mathcal{F}^{(s)}(x;a^\mu) = \Box_{d+1}h^{(s)}(x^\mu;a^\mu)\\ &-(a^\mu\nabla_\mu)\Big[(\nabla^\nu\partial_{a^\nu})h^{(s)}(x;a^\mu) - \frac{1}{2}(a^\nu\nabla_\nu)\Box_{a^\mu}h^{(s)}(x;a^\mu)\Big]\\ &-\frac{1}{L^2}[s^2+s(d-5)-2(d-2)]h^{(s)}(x^\mu;a^\mu)) - \frac{1}{L^2}a^\mu a_\mu\Box_{a^\mu}h^{(s)}(x^\mu;a^\mu).\end{aligned} \tag{2.59}$$

in the following way

$$\mathcal{F}^{(s)}(X^A;a^A) \quad = \quad e^{(s-4)u}\mathcal{F}^{(s)}(x;a^\mu), \tag{2.60}$$

Supplementing this with the reductions for field (2.50), (2.55) and for integration volume:

$$\int d^{d+2}X = \int dud^{d+1}x\sqrt{-G} = L\int dud^{d+1}x\sqrt{g}e^{(d+2)u} \tag{2.61}$$

we obtain the following reduction rule for Fronsdal actions :

$$S_0[h^{(s)}(X^A;a^A)] \quad = \quad \left[L\int due^{(d+2s-4)u}\right] \times S_0[h^{(s)}(x^\mu;a^\mu)], \tag{2.62}$$

[1]in [44] we had $\Delta_h = \Delta_\epsilon = 2(s-1)$.

where

$$
\begin{aligned}
S_0[h^{(s)}(X^A;a^A)] &= \int d^{d+2}X\Big[-\frac{1}{2}h^{(s)}(X^A;a^A)*_{a^A}\mathcal{F}^{(s)}(X^A;a^A) \\
&+ \frac{1}{8s(s-1)}\Box_{a^A}h^{(s)}(X^A;a^A)*_{a^A}\Box_{a^A}\mathcal{F}^{(s)}(X^A;a^A)\Big] \qquad (2.63)\\
S_0[h^{(s)}(x^\mu;a^\mu)] &= \int d^{d+1}x\sqrt{g}\Big[-\frac{1}{2}h^{(s)}(x;a^\mu)*_{a^\mu}\mathcal{F}^{(s)}(x;a^\mu) \\
&+ \frac{1}{8s(s-1)}\Box_{a^\mu}h^{(s)}(x;a^\mu)*_{a^\mu}\Box_{a^\mu}\mathcal{F}^{(s)}(x;a^\mu)\Big], \qquad (2.64)
\end{aligned}
$$

The overall infinite factor

$$
\left[L\int due^{(d+2s-4)u}\right], \qquad (2.65)
$$

here the same as in [44] , where we described prescription to get correct additional AdS correction terms from the full "u" derivative part of interaction terms. This additional terms can be found with insertion of the dimensionless delta function in measure (2.61) [40–43]

$$
\int d^{d+2}X\delta\left(\frac{\sqrt{-X^2}}{L}-1\right) \qquad (2.66)
$$

then full derivative terms will survive only for normal u derivatives:

$$
\begin{aligned}
&\int d^{d+2}X\delta\left(\frac{\sqrt{-X^2}}{L}-1\right)\partial^A\mathfrak{L}_A = \int d^{d+2}X\delta^{(1)}\left(\frac{\sqrt{-X^2}}{L}-1\right)\frac{X^A}{L^2}E^u_A\mathfrak{L}_u \\
&= \int dud^{d+1}x\sqrt{g}e^{(d+2)u}\frac{\delta^{(1)}(e^u-1)}{L}\mathfrak{L}_u \qquad (2.67)
\end{aligned}
$$

So we see that both approaches produce the same additional corrections coming from the differentiation of overall "u" phase a front of full derivatives in the normal direction. Finally, we note that this reduction procedure is more useful for investigation of interaction terms due to the very simple form of the pullback of fields and auxiliary vectors a^A and star contractions:

$$
\begin{aligned}
*^s_{a^A} &= \frac{1}{(s!)^2}\prod_{i=1}^{s}\big(-\overleftarrow{\partial}_{a^{u_i}}\overrightarrow{\partial}_{a^{u_i}}+\overleftarrow{\partial}_{a_{\mu_i}}\overrightarrow{\partial}_{a^{\mu_i}}\big) \\
&= \sum_{n=0}^{s}\frac{(-1)^n}{\binom{s}{n}}*^n_{a^u}*^{s-n}_{a^\mu}. \qquad (2.68)
\end{aligned}
$$

2.3 MAIN TERM OF CUBIC INTERACTION IN FLAT SPACE

In this section we repeat the general formula for a covariant cubic interaction of higher spin gauge fields in a flat background as presented in [29] and [30]. The main result of [29,30] is the following. The gauge invariance fixes in a unique way the cubic interaction if the main cyclic ansatz term without divergences and traces is given. Accordingly in this thesis we consider only the main term of the cubic interaction postponing the proof for all other terms to a future publication, and understanding intuitively that gauge invariance is going to regulate in a correct fashion the radial reduction for all other terms presented in [29, 30] and classified in corresponding tables there.

In [29,30] authors considered three potentials $h^{(s_1)}(X_1; a^A), h^{(s_2)}(X_2; b^A), h^{(s_3)}(X_3; c^A)$ of $d+2$ dimensional flat theory with ordered spins s_i

$$s_1 \geq s_2 \geq s_3, \tag{2.69}$$

and with the cyclic ansatz for the interaction

$$\begin{aligned}
&\mathcal{L}_I^{main}(h^{(s_1)}(X_1, a^A), h^{(s_2)}(X_2, b^A), h^{(s_3)}(X_3 c^A)) \\
&\qquad = \sum_{n_i} C^{s_1,s_2,s_3}_{n_1,n_2,n_3} \int d^{d+2}X_1 d^{d+2}X_2 d^{d+2}X_3 \delta(X_3 - X_1)\delta(X_2 - X_1) \\
&\qquad \times \tilde{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2, n_3) h^{(s_1)}(X_1; a^A) h^{(s_2)}(X_2; b^B) h^{(s_3)}(X_3; c^C),
\end{aligned} \tag{2.70}$$

where

$$\tilde{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2, n_3)$$
$$= (\partial_{a^A}\partial_{b_A})^{Q_{12}}(\partial_{b^B}\partial_{c_B})^{Q_{23}}(\partial_{c^C}\partial_{a_C})^{Q_{31}}(\partial_{a^D}\tilde{\nabla}_2^D)^{n_1}(\partial_{b^E}\tilde{\nabla}_3^E)^{n_2}(\partial_{c^F}\tilde{\nabla}_1^F)^{n_3}, \tag{2.71}$$

and the notation "*main*" as a superscript means that it is an ansatz for terms without $Divh^{(s_i-1)}$ and $Trh^{(s_i-2)}$. Denoting the number of derivatives by Δ we have

$$n_1 + n_2 + n_3 = \Delta. \tag{2.72}$$

We shall later determine and then use the minimal possible Δ. As balance equations we have

$$n_1 + Q_{12} + Q_{31} = s_1,$$
$$n_2 + Q_{23} + Q_{12} = s_2,$$
$$n_3 + Q_{31} + Q_{23} = s_3. \tag{2.73}$$

These equations are solved by

$$Q_{12} = n_3 - \nu_3,$$
$$Q_{23} = n_1 - \nu_1,$$
$$Q_{31} = n_2 - \nu_2. \tag{2.74}$$

Since the l.h.s. cannot be negative, we have

$$n_i \geq \nu_i.$$

The ν_i are determined to be

$$\nu_i = 1/2(\Delta + s_i - s_j - s_k), \quad i, j, k \quad \text{are all different.} \tag{2.75}$$

It follows that the minimally possible Δ is expressed by Metsaev's [16] (using the ordering of the s_i).

$$\Delta_{min} = \max\,[s_i + s_j - s_k] = s_1 + s_2 - s_3. \tag{2.76}$$

Another result of [29, 30] is the trinomial expression for the coefficients in (2.70) fixed by Noether's procedure. Taking into account (2.73)-(2.76) we can write it in the following elegant form

$$C^{s_1,s_2,s_3}_{n_1,n_2,n_3} = C^{s_1,s_2,s_3}_{Q_{12},Q_{23},Q_{31}} = const \begin{pmatrix} s_{min} \\ Q_{12}, Q_{23}, Q_{31} \end{pmatrix}. \tag{2.77}$$

In the next section we will reformulate the main term of cubic interaction as described [44] which is more convenient for the radial pullback.

2.4 PULLBACK FOR POWER OF DERIVATIVES OF HS FIELDS FROM FLAT TO EMBEDDED *ADS* SPACE

In this section, we discuss radial pullback for Cubic interaction for higher spins in a covariant off-shell formulation derived In [29, 30] . This result for flat space is in full agreement with light cone gauge results of Metsaev [16]. Moreover this agreement shows that all interactions of higher spin gauge fields with any spin s_1, s_2, s_3 both in flat space and in dS or AdS are unique up to partial integration and field redefinition[2] The formulation of the cubic interactions for higher spin fields in ambient space was considered in several papers [40–45], In [44] we investigated the possibility to connect the main term of interaction in flat $d+2$ dimensional space to the main term of interaction in AdS_{d+1} space one dimension lower ignoring all trace and divergent terms but expressed directly through the AdS covariant derivatives and including some curvature corrections. In this thesis, we perform one important step forward solving task for flat main term completely and presenting full reduction or pullback including all trace and other related

[2]This was already proven for some low spin cases of both the Fradkin-Vasiliev vertex for $2, s, s$ and the nonabelian vertex for $1, s, s$ in [25].

terms coming from main term of cubic interaction in direct AdS_{d+1} covariant form. We start here from the more convenient for radial pullback form described in [44] where we reformulated the main term of cubic interaction (2.70), (2.71) in the following way

$$\begin{aligned}
&\mathcal{L}_I^{main}(h^{(s_1)}(X,a^A),h^{(s_2)}(X,b^A),h^{(s_3)}(X,c^A)) = \\
&\sum_{Q_{ij}} C^{s_1,s_2,s_3}_{Q_{12},Q_{23},Q_{31}} \int d^{d+2}X *_{c^A}^{Q_{31}+n_3} K^{(s_1)}(Q_{31},n_3;c^A,a^A;X) \\
&*_{a^A}^{Q_{12}+n_1} K^{(s_2)}(Q_{12},n_1;a^A,b^A;X) *_{b^A}^{Q_{23}+n_2} K^{(s_3)}(Q_{23},n_2;b^A,c^A;X),
\end{aligned} \tag{2.78}$$

where

$$K^{(s_1)}(Q_{12},n_1;a^A,b^A;X) = (a^A\partial_{b^A})^{Q_{12}}(a^B\partial_B)^{n_1}h^{(s_1)}(X;b^C). \tag{2.79}$$

The most important advantage of this form that here we can express our cubic interaction as a cube of above bitensor function with cyclic index contraction. From now on we put AdS radius $L=1$ and use for shortness the brackets $(\dots,\dots)$ for AdS_{d+1} index summation. In other words

$$(a,\partial_b) = a^\mu \partial_{b^\mu}, \tag{2.80}$$

$$(a,\nabla) = a^\mu \nabla_\mu, \tag{2.81}$$

and

$$(a,D) = a^\mu D_\mu. \tag{2.82}$$

Another important point here is the difference in the definition of the covariant differentiation operator (2.47) in the case of interaction. The minimal object here is a bitensor (2.79) which has two sets of symmetrized indices. In this case, we should define covariant differentiation operators for both sets of indices:

$$D_\mu = \nabla_\mu - a^u\partial_{a^\mu} - a_\mu\partial_{a^u} - b^u\partial_{b^\mu} - b_\mu\partial_{b^u}. \tag{2.83}$$

and in a similar way for other sets of indices. Now we have all ingredients to start analyzing the "u"- dependence of interaction Lagrangian (2.78) in curvilinear coordinates (2.3). First of all we note that in the new frame only the measure and derivatives create additional u phase (2.61) and (2.57) in addition to the three similar phase (2.55) coming from reduced fields. Finally, we get

$$d+2+\sum_{i=1}^{3}(\Delta_{h^{(s_i)}}-n_i)=\sum_{i=1}^{3}(s_i)-\Delta+d-4 \tag{2.84}$$

where Δ is the number of derivatives in interaction. Then inserting minimal number of derivatives from (2.76) we see that our interaction rescales as[3]

$$\sum_{i=1}^{3}s_i-\Delta_{min}+d-4 \;=\; d+2s_3-4 \tag{2.85}$$

with the obvious limit $d+2s-4$ in the self-interacting case $s_1=s_2=s_3=s$. So we see that the cubic interaction in the case of the minimal number of derivatives is relevant for the radial reduction procedure described in the previous section. Therefore it should produce the right curvature corrections for the main term of the cubic interaction in AdS_{d+1}.

2.4.1 NONCOMMUTATIVE ALGEBRA AND A^U STRIPPING

In this subsection, we consider a possible radial pullback scheme for the main object of cubic interaction (2.78): the bitensorial function

$$K^{(s)}(Q,n;a^A,b^A;X)=(a^A\partial_{b^A})^Q(a^B\partial_B)^n h^{(s)}(X;b^C). \tag{2.86}$$

This term should generate all AdS curvature corrections coming from main term. For that we study these operators in a representation that act on pullback HS field

$$h^{(s)}(X;b^A)|_{X=X(u,x)}=h^{(s)}(u,x^\mu;b^\mu)=e^{(s-2)u}h^{(s)}(x^\mu;b^\mu). \tag{2.87}$$

Then we can obtain these AdS corrections expanding all flat $d+2$ dimensional objects in Frenet basis or in other words in term of $d+1$ dimensional AdS space derivatives and vectors and

[3]In the case of three spins ordered as $s_1\geq s_2\geq s_3$

normal components surviving after applying our ansatz rules:

$$(a^B\partial_B)^n|_{X=X(u,x)} = \left[e^{-u}(a^u\partial_u + a^\mu D_\mu)\right]^n \tag{2.88}$$

$$a^\mu D_\mu = (a, D) = (a, \nabla) - a^u(a, \partial_a) - b^u(a, \partial_b) - a^2\partial_{a^u} - (a, b)\partial_{b^u} \tag{2.89}$$

$$\text{where} \qquad a^2 = (a, a) = a^\mu a^\nu g_{\mu\nu}(x)$$

and contracting over all a^u, b^u, c^u.

So we must deal with the $d+1$ dimensional expansion for the n'th power of $d+2$ dimensional derivatives (2.88), where the operator

$$a^u\partial_u + a^\mu D_\mu = a^\mu\hat{\nabla}_\mu(g) - R, \tag{2.90}$$

$$\hat{\nabla}_\mu = \nabla_\mu - b^u\partial_{b^\mu} - b_\mu\partial_{b^u}, \tag{2.91}$$

$$R = a^u[(a\partial_a) - \partial_u] + a^2\partial_{a^u}, \tag{2.92}$$

act on ground states (2.87). These ground states can be characterized by the total symmetry in the argument and by the fact that they are annihilated by the following operators:

$$| 0 >= e^{(s-2)u}h^{(s)}(x^\mu; b^\mu) \tag{2.93}$$

$$\partial_{a^\mu} | 0 >= \partial_{a^u} | 0 >= \partial_{b^u} | 0 >= 0, \tag{2.94}$$

$$R | 0 >= (2-s)a^u | 0 > . \tag{2.95}$$

The operator of interest is

$$\left[e^{-u}(a, \hat{\nabla}) - e^{-u}R\right]^n, \tag{2.96}$$

where in the sequel it is advantageous to write the operator R in the following way

$$R = a^u[(a\partial_a) + a^u\partial_{a^u} - \partial_u] + (a^2 - (a^u)^2)\partial_{a^u} \tag{2.97}$$

with the following important algebraic relations:

$$[(a\partial_a) + a^u\partial_{a^u}, R] = R, \tag{2.98}$$

$$[(a\partial_a) + a^u\partial_{a^u}, (a, \hat{\nabla})] = (a, \hat{\nabla}), \tag{2.99}$$

$$[R, e^{-u}(a, \hat{\nabla})] = 2e^{-u}a^u(a, \hat{\nabla}). \tag{2.100}$$

We have to evaluate (2.96) on the ground state (2.93). For that Expanding this operator power (2.96) into a non-commutative binomial series we get

$$
\begin{aligned}
[(a,e^{-u}\hat{\nabla}) \quad & -e^{-u}R]^n \mid 0 >= \sum_{p=0}^{n}(-1)^p \sum_{n-p\geq i_p\geq i_{p-1}\geq i_{p-2}...\geq i_1\geq 0} \\
& (a,e^{-u}\hat{\nabla})^{n-p-i_p} e^{-u}R(a,e^{-u}\hat{\nabla})^{i_p-i_{p-1}} \dots e^{-u}R(a,e^{-u}\hat{\nabla})^{i_1} \mid 0 > .
\end{aligned}
\tag{2.101}
$$

Then using relation

$$
[R,(a,e^{-u}\hat{\nabla})^{i_k}] = 2i_k e^{-i_k u} a^u (a,\hat{\nabla})^{i_k}, \tag{2.102}
$$

we can rewrite (2.101) in the following form

$$
\begin{aligned}
[(a,e^{-u}\hat{\nabla}) - e^{-u}R]^n \mid 0 >= & \sum_{p=0}^{n}(-1)^p (a,\hat{\nabla})^{n-p} e^{(p-n)u} \\
\sum_{n-p\geq i_p\geq i_{p-1}\geq i_{p-2}...\geq i_1\geq 0} & e^{-u}(2i_p a^u + R)e^{-u}(2i_{p-1}a^u + R)...e^{-u}(2i_1 a^u + R)e^{(s-2)u} h^{(s)}(x^\mu; b^\mu).
\end{aligned}
\tag{2.103}
$$

Then introducing the new objects

$$
\phi_{i_k} = 2i_k a^u + R = a^u[2i_k + (a,\partial_a) + a^u\partial_{a^u} - \partial_u] + [a^2 - (a^u)^2]\partial_{a^u}. \tag{2.104}
$$

and taking into account that

$$
[(a,\partial_a) + a^u\partial_{a^u} - \partial_u]e^{-nu} f^{(m)}(a^\mu, a^u) = (m+n)e^{-nu} f^{(m)}(a^\mu, a^u), \tag{2.105}
$$

we obtain

$$
\begin{aligned}
[(a,e^{-u}\hat{\nabla}) - e^{-u}R]^n \mid 0 >= e^{(s-2-n)u} & \sum_{p=0}^{n}(-1)^p (a,\hat{\nabla})^{n-p} \\
\sum_{n-p\geq i_p\geq i_{p-1}\geq i_{p-2}...\geq i_1\geq 0} & \phi_{i_p}\phi_{i_{p-1}}\dots\phi_{i_2}\phi_{i_1} h^{(s)}(x^\mu; b^\mu),
\end{aligned}
\tag{2.106}
$$

where we have ϕ_{i_k} as a very simple "creation" operators

$$
\phi_{i_k} = a^u[2(i_k + k) - s] + [a^2 - (a^u)^2]\partial_{a^u}. \tag{2.107}
$$

Now we show how to perform summation in (2.106) and obtain wanted expansion on the power of a^u to contract after. Introducing notation

$$
V^{p+1}(i_{p+1})h^{(s)}(x^\mu; b^\mu) = \sum_{i_{p+1}\geq i_p\geq i_{p-1}\geq i_{p-2}...\geq i_1\geq 0} \phi_{i_p}\phi_{i_{p-1}}\dots\phi_{i_2}\phi_{i_1} h^{(s)}(x^\mu; b^\mu), \tag{2.108}
$$

and performing summation over the labels $\{i_k\}|_{k=1}^p$ we should obtain a polynomial in a^u and (a^2) of the form [4]

$$V^{p+1}(i_{p+1}) = \sum_{k=0}^{[\frac{p}{2}]} \xi_k^{p+1}(i_{p+1})(a^2)^k (a^u)^{p-2k}. \qquad (2.109)$$

Considering the last expression as an ansatz for equation

$$V^{p+1}(i_{p+1}) = \sum_{i_p=0}^{i_{p+1}} \phi_{i_p} V^p(i_p) \qquad (2.110)$$

and using (2.107) we obtain the following recurrence relation for $2p-k$ order polynomials coefficients $\xi_k^{p+1}(i_{p+1}) \sim (i_{p+1})^{2p-k} + \ldots$

$$\xi_k^{p+1}(j) = \sum_{i=0}^{j}(2i+p+1+2k-s)\xi_k^p(i) + \sum_{i=0}^{j}(p+1-2k)\xi_{k-1}^p(i) \qquad (2.111)$$

This equation is easier to consider in "differential" form

$$\xi_k^{p+1}(i) - \xi_k^{p+1}(i-1) = (2i+p+1+2k-s)\xi_k^p(i) + (p+1-2k)\xi_{k-1}^p(i) \qquad (2.112)$$

In the next section we presented solutions of latter equation obtained by direct calculation of V^{p+1} using (2.108) for $p = 1, 2, 3, 4, \ldots$. Investigating these we arrive to the following important ansatz for $\xi_k^{p+1}(i)$

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(i+1)_p(2k+2+i-s)_{p-2k}P_k(i) \qquad (2.113)$$

where $P_k(i) \sim i^k + \ldots$ is now p- *independent polynomial of order* k and we introduced Pochhammer symbols[5]

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1) \qquad (2.114)$$

Inserting (2.113) in equation (2.112) we obtain equation for $P_k(i)$:

$$(i+2k)P_k(i) - iP_k(i-1) = (i+2k-s)P_{k-1}(i) \qquad (2.115)$$

Then after more convenient normalization of our polynomials with additional $2k$ order factor:

$$\mathcal{P}_k(i) \equiv (i+1)_{2k}P_k(i) \qquad (2.116)$$

[4] Note that $[p/2]$ is integer part of $p/2$ and at the end we have to insert $i_{p+1} = n - p$

[5] for falling factorial we use in this paper another notation $[s]_n = s(s-1)\ldots(s-n+1)$

we arrive to the following simple equation with boundary condition:

$$\mathcal{P}_k(i) - \mathcal{P}_k(i-1) = (i+2k-1)(i+2k-s)\mathcal{P}_{k-1}(i) \tag{2.117}$$

$$\mathcal{P}_0(i) = P_0(i) = 1 \tag{2.118}$$

This we can solve in two way: first in the form of multiple sums:

$$\mathcal{P}_k(i) = \sum_{i\geq i_k\geq i_{k-1}\geq i_{k-2}\dots\geq i_1\geq 0} \prod_{n=1}^{k} (i_n+2n-1)(i_n+2n-s) \tag{2.119}$$

or solving differential equation for generating function

$$\mathcal{P}_k(y) \equiv \sum_{i=0}^{\infty} \mathcal{P}_k(i) y^i \tag{2.120}$$

where we introduced formal variable y with $|y| < 1$ for production of the boundary condition:

$$\mathcal{P}_0(y) = \sum_{i=0}^{\infty} y^i = \frac{1}{1-y} \tag{2.121}$$

For this generation function, we obtain from recurrence relation (2.117) the equation

$$(1-y)\mathcal{P}_k(y) = (y\frac{d}{dy} + 2k - 1)(y\frac{d}{dy} + 2k - s)\mathcal{P}_{k-1}(y) \tag{2.122}$$

Solving recursively and using (2.121) we can write the solution in the form:

$$\mathcal{P}_k(y) = y^{-(2k+1)} \left[\frac{y^4}{1-y}\frac{d}{dy}y^s\frac{d}{dy}y^{-s}\right]^k \frac{y^2}{1-y} \tag{2.123}$$

Finally, we can write (2.113) in term of $\mathcal{P}_k(i)$

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(2k+i+1)_{p-2k}(2k+2+i-s)_{p-2k}\mathcal{P}_k(i) \tag{2.124}$$

2.4.2 NONCOMMUTATIVE ALGEBRA AND B^U STRIPPING

To extract exact dependence from b^u and obtain final expressions written directly through the AdS_{d+1} covariant derivatives ∇ we have to evaluate the remaining factors

$$(a,\hat{\nabla})^{n-p} = [(a,\nabla) - b^u(a,\partial_b) - (a,b)\partial_{b^u}]^{n-p}$$
$$= \sum_{\tilde{p}=0}^{n-p}(-1)^{\tilde{p}}\binom{n-p}{\tilde{p}}(a,\nabla)^{n-p-\tilde{p}}(L^+ + L^-)^{\tilde{p}}, \tag{2.125}$$

where L^+, L^- generate a Lie algebra

$$L^+ = b^u(a,\partial_b), \quad L^- = (a,b)\partial_{b^u}, \tag{2.126}$$

$$[L^+, L^-] = H = a^2 b^u \partial_{b^u} - (a,b)(a,\partial_b), \tag{2.127}$$

$$[H, L^\pm] = \pm 2a^2 L^\pm. \tag{2.128}$$

Representations of this Lie algebra are created from an $(s+1)$-dimensional vector space of "null vectors" $\{\Phi_n(a;b)\}|_{n=0}^s$ of "level" n

$$\Phi_n(a;b) = h^{(s)}_{\mu_1,\mu_2,\ldots\mu_s} a^{\mu_1}a^{\mu_2}\ldots a^{\mu_n} b^{\mu_{n+1}} b^{\mu_{n+2}}\ldots b^{\mu_s}, \qquad L^-\Phi_n(a;b) = 0, \tag{2.129}$$

for any fixed tensor function h^s. From (2.126)-(2.128) follows that starting from $\Phi_0(a;b)$ all $\Phi_n(a;b)$ can be produced by application of H

$$H\Phi_0(a;b) = -s(a,b)\Phi_1(a,b), \tag{2.130}$$

$$H^2\Phi_0(a;b) = [s]_2(a,b)^2\Phi_2(a;b) + sa^2(a,b)\Phi_1(a;b), \tag{2.131}$$

$$H^3\Phi_0(a;b) = -\{[s]_3(a,b)^3\Phi_3(a;b) + 3[s]_2 a^2(a,b)^2\Phi_2(a;b) + s(a^2)^2(a,b)\Phi_1(a;b)\}. \tag{2.132}$$

The ansatz

$$H^n\Phi_0(a;b) = (-1)^n \sum_{r=1}^{n} A_r^{(n)}[s]_r (a^2)^{n-r}(a,b)^r \Phi_r(a;b), \tag{2.133}$$

leads to the recurrence relation

$$A_{r-1}^{(n)} + rA_r^{(n)} = A_r^{(n+1)}, \tag{2.134}$$

$$A_r^{(n)} = 0 \quad \text{for} \quad r > n. \tag{2.135}$$

The boundary conditions $A_{-1}^{(n)} = 0$ and $A_0^{(0)} = 1$ are assumed.

Multiplying by x^r and introducing

$$P_n(x) = \sum_{r=0}^{\infty} A_r^{(n)} x^r \tag{2.136}$$

we obtain simple differential equation

$$x \frac{d}{dx} \left(e^x P_n(x) \right) = e^x P_{n+1}(x) \,. \tag{2.137}$$

which we can easily solve since $P_0(x) = 1$. Iterating n times we find

$$e^x P_n(x) = \left(x \frac{d}{dx} \right)^n e^x \,, \tag{2.138}$$

or

$$P_n(x) = e^{-x} \left(x \frac{d}{dx} \right)^n e^x \,. \tag{2.139}$$

Evidently, $P_n(x)$ is a polynomial of order n, which means that $A_r^{(n)} = 0$ for $r > n$.

Finally, we can find a "double" generating function. Introducing

$$Q(x,t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \tag{2.140}$$

we see that

$$Q(x,t) = e^{-x} e^{tx\frac{d}{dx}} e^x = e^{x(e^t - 1)} \tag{2.141}$$

where we have explored the fact that the operator $e^{tx\frac{d}{dx}}$ rescales the variable x by the factor e^t.

It is not difficult to get a simple combinatorial formula for $A_r^{(n)}$. Let us denote by $\mathcal{P}(n,r)$ the set of partitions of n into r nonzero parts. The partitions are in one to one correspondence with Young diagrams with n boxes and r rows. An arbitrary partition λ may be represented as $\lambda = 1^{k_1} 2^{k_2} 3^{k_2} \cdots$, where the nonnegative integer k_i indicates the number of rows with length i. For example the partition $8 = 1 + 1 + 3 + 3$ is represented as $1^2 2^0 3^2$, hence $\{k_1, k_2, k_3\} = \{2, 0, 2\}$ and $k_4 = k_5 = \cdots = 0$. The corresponding Young diagram consists of two rows of length 3 and two rows of length 1. For a diagram $\lambda \in \mathcal{P}(n,r)$ let us arbitrarily distribute the integers $1, 2, \cdots n$ among boxes. Let us identify two configurations which differ from each other by permutations of numbers along rows or by permutation of entire rows of same lengths. Evidently, the number of non-equivalent distributions is given by

$$S(\lambda) = \frac{n!}{\prod_{i\geq 1} k_i!(i!)^{k_i}} \tag{2.142}$$

Expanding (2.141) in x and t we get

$$e^{x(e^t-1)} = \prod_{i=1}^{\infty}\sum_{k_i=0}^{\infty} \frac{x^{k_i} t^{ik_i}}{k_i!(i!)^{k_i}} \tag{2.143}$$

Now comparing (2.143) with (2.142) one easily gets

$$A_n^{(r)} = \sum_{\lambda\in\mathcal{P}(n,r)} S(\lambda) \tag{2.144}$$

With the help of the basis $\{\Phi_n(a;b)\}_{n=0}^{s}$ of null vectors the representation of the Lie algebra can be constructed as follows. We start from

$$(L^+ + L^-)^{\tilde{p}}\Phi_0(b) = \sum_{\tilde{k}=0}^{\tilde{p}} \sum_{\tilde{p}-\tilde{k}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\dots\geq i_1\geq 1}$$
$$(L^+)^{\tilde{p}-\tilde{k}-i_{\tilde{k}}} L^-(L^+)^{i_{\tilde{k}}-i_{\tilde{k}-1}} L^-(L^+)^{i_{\tilde{k}-1}-i_{\tilde{k}-2}} L^- \dots (L^+)^{i_2-i_1} L^-(L^+)^{i_1}\Phi_0(b). \tag{2.145}$$

Only commutators of L^- with powers of L^+ arise

$$[L^-,(L^+)^i] = -\sum_{j=0}^{i-1}(L^+)^{i-j-1}H(L^+)^j =$$
$$-\sum_{j=0}^{i-1}(L^+)^{i-1}(H+2ja^2) = -(L^+)^{i-1}(iH+[i]_2\, a^2). \tag{2.146}$$

Here we recognize that the whole basis $\{\Phi_n(a;b)\}$ of null vectors is produced from $\Phi_0(b)$ by the action of H. With the shorthand

$$\psi_i = iH + [i]_2\, a^2, \tag{2.147}$$

the result is

$$\sum_{\tilde{k}=1}^{[\frac{\tilde{p}}{2}]}(-1)^{\tilde{k}}(L^+)^{\tilde{p}-2\tilde{k}} W^{\tilde{k}}(a^2,H)\Phi_0(b) = \sum_{\tilde{k}=1}^{[\frac{\tilde{p}}{2}]}(b^u)^{\tilde{p}-2\tilde{k}}(-1)^{\tilde{k}}(a,\partial_b)^{\tilde{p}-2\tilde{k}} W^{\tilde{k}}(a^2,H)\Phi_0(b) \tag{2.148}$$

where

$$W^{\tilde{k}}(a^2,H,i_{\tilde{k}+1})\Phi_0(b) = \sum_{i_{\tilde{k}+1}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\dots\geq i_2\geq i_1\geq 1} \psi_{i_{\tilde{k}}-\tilde{k}+1}\psi_{i_{\tilde{k}-1}-\tilde{k}+2}\psi_{i_{\tilde{k}-2}-\tilde{k}+3}\dots\psi_{i_2-1}\psi_{i_1}\Phi_0(b). \tag{2.149}$$

The sum is a homogeneous polynomial of H and a^2 of degree $\tilde{k}$, [6]:

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) = \sum_{m=0}^{\tilde{k}} \eta_{\tilde{k}}^{m}(i_{\tilde{k}+1})(a^2)^m H^{\tilde{k}-m} \tag{2.150}$$

Using this ansatz and doing in the way similar to (2.109) we derive from

$$W^{\tilde{k}+1}(a^2, H, i_{\tilde{k}+2}) = \sum_{i_{\tilde{k}+1}=1}^{i_{\tilde{k}+2}} \psi_{i_{\tilde{k}+1}-\tilde{k}} W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) \tag{2.151}$$

the following recurrence relation

$$\eta_{\tilde{k}+1}^{m}(j) = \sum_{i=1}^{j} \left[(i-\tilde{k})\eta_{\tilde{k}}^{m}(i) + (i-\tilde{k})(i-\tilde{k}-1)\eta_{\tilde{k}}^{m-1}(i) \right] \tag{2.152}$$

or without summation:

$$\eta_{\tilde{k}+1}^{m}(i) - \eta_{\tilde{k}+1}^{m}(i-1) = (i-\tilde{k})\eta_{\tilde{k}}^{m}(i) + (i-\tilde{k})(i-\tilde{k}-1)\eta_{\tilde{k}}^{m-1}(i) \tag{2.153}$$

Investigating the structure of this polynomial coefficients (See Section 2.5) we can factorize again $i^{2\tilde{k}}$ terms and write in this form

$$\eta_{\tilde{k}}^{m}(i) = \frac{2^{m-\tilde{k}}3^{-m}}{(\tilde{k}-m)!m!}(i-\tilde{k}+1)_{2\tilde{k}} P_m(i,\tilde{k}), \quad P_0(i,\tilde{k}) = 1 \tag{2.154}$$

where the polynomials $P_m(i,\tilde{k}) \sim (i-\frac{\tilde{k}}{2})^m + \dots$ is mth orders in i and $\tilde{k}$ with binomial leading term and satisfy the equation

$$(i+\tilde{k}+1)P_m(i,\tilde{k}+1) - (i-\tilde{k}-1)P_m(i-1,\tilde{k}+1) = 2(\tilde{k}-m+1)P_m(i,\tilde{k}) + 3m(i-\tilde{k}-1)P_{m-1}(i,\tilde{k}) \tag{2.155}$$

with the same level of difficulty to solve as (2.153). From the other hand representation (2.148) extract b^u dependence and we can calculate coefficients $\eta_{\tilde{k}}^{m}(i_{\tilde{k}+1})$ from (2.149) directly. comparing (2.150) with (2.149) and taking into account (2.147) we see that it is possible to write

$$\eta_{\tilde{k}}^{m}(\tilde{p}-\tilde{k}) = \eta_{\tilde{k}}^{m}(i_{\tilde{k}+1})|_{i_{\tilde{k}+1}=\tilde{p}-\tilde{k}} \tag{2.156}$$

[6]Remember that H is second order in a as well.

in the following form:

$$\eta^m_{\tilde{k}}(\tilde{p}-\tilde{k}) = \sum_{\tilde{p}-\tilde{k}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\dots\geq i_2\geq i_1\geq 1} \sum_{\tilde{k}\geq n_m\geq n_{m-1}\geq n_{m-2}\dots\geq n_2\geq n_1\geq 1}$$
$$\prod_{l_m=n_m+1}^{\tilde{k}} (i_{l_m}-l_k+1)[i_{n_m}-n_m+1]_2 \prod_{l_{m-1}=n_{m-1}+1}^{n_m-1} (i_{l_{m-1}}-l_{m-1}+1)[i_{n_{m-1}}-n_{m-1}+1]_2 \dots$$
$$\dots \prod_{l_2=n_2+1}^{n_3-1} (i_{l_2}-l_2+1)[i_{n_2}-n_2+1]_2 \prod_{l_1=n_1+1}^{n_2-1} (i_{l_1}-l_1+1)[i_{n_1}-n_1+1]_2 \prod_{l=1}^{n_1-1} (i_l-l+1) \tag{2.157}$$

This formula means that we should inside of expression for $\eta^0_{\tilde{k}}(\tilde{p}-\tilde{k})$:

$$\eta^0_{\tilde{k}}(\tilde{p}-\tilde{k}) = \sum_{\tilde{p}-\tilde{k}\geq i_{\tilde{k}}\geq i_{\tilde{k}-1}\geq i_{\tilde{k}-2}\dots\geq i_2\geq i_1\geq 1} \prod_{l=1}^{\tilde{k}} (i_l-l+1) \tag{2.158}$$

replace m brackets $(i_{n_r}-n_r+1)|_{r=1}^{m}$ with the m Pochhammers $\{[i_{n_r}-n_r+1]_2\}|_{r=1}^{m}$ in all possible ways and then take sums.

2.5 THE STRUCTURE OF THE POLYNOMIAL COEFFICIENTS AND THE ITERATIVE APPROACH OF FINDING SOLUTIONS

2.5.1 PROBLEM MODELLING IN WOLFRAM MATHEMATICA

In this section we present the general approach for defining recursive objects in Wolfram Mathematica. The approach is fairly generic and can be applied for different use-cases with a little bit of modification. Successful modelling of recursive objects and recurrence relation into the Wolfram Mathematica significantly reduced the research time and helped a lot in hard and tedious computations.

THE RECURRENCE RELATION

When considering a possible radial pullback scheme for the main object of cubic interaction objects we obtain (2.109) and (2.110) objects defined recursively:

$$V^{p+1}(i_{p+1}) = \sum_{i_p=0}^{i_{p+1}} \phi_{i_p} V^p(i_p), \qquad V^{p+1}(i_{p+1}) = \sum_{k=0}^{[\frac{p}{2}]} \xi_k^{p+1}(i_{p+1})(a^2)^k(a^u)^{p-2k}. \tag{2.159}$$

where

$$\phi_{i_k} = a^u[2(i_k + k) - s] + [a^2 - (a^u)^2]\partial_{a^u} \tag{2.160}$$

Using (2.159)-(2.160) we see that $\xi_k^{p+1}(i)$ satisfies the following recurrence relation

$$\xi_k^{p+1}(i) - \xi_k^{p+1}(i-1) = (2i + p + 1 + 2k - s)\xi_k^p(i) + (p + 1 - 2k)\xi_{k-1}^p(i) \tag{2.161}$$

MODELLING RECURRENCE RELATION

To solve the (2.161) the (2.159) is modeled using Wolfram Matematica by creating function that will effectively compute (2.159) terms.

```
a0 = Superscript[a, 0];
psi[ps_, {i_, k_Integer}] :=
        a0 *( 2 * (i + k) - s)*ps + (a^2 - a0^2) D[ps, a0]
```

Listing 2.1: Definition of (2.160) creation operators

The **psi** is a Mathematica module which acts as an operator defined in (2.160). It acts on an object which is given as a first argument. This means that we can use this module to model the (2.108) summation by plugging the result of each **psi** module execution as an argument for the next. To do so we will also need the summation indexes with their corresponding ranges, as we have multiple nested sums in the expression.

```
summInds[len_Integer, bound_] := Module[{inds},
  Subscript[i, len + 1] = bound;
  inds = Table[{Subscript[i, k], 0, Subscript[i, k + 1]}, {k, 1, len}];
  Subscript[i, len + 1] =.;
  inds
  ]
```

Listing 2.2: Definition of summation indexes

The **summInds** function takes as an argument the number of nested sums along with the upper bound of the last sum and generates the list of summation indexes for the whole nested sum. Below are some examples of the function usage when the upper bound is equal to i_{p+1}

```
summInds[2, Subscript[i, p + 1]]
summInds[3, Subscript[i, p + 1]]
summInds[4, Subscript[i, p + 1]]
```

Listing 2.3: Example usage of **summInds** function when upper bound is i_{p+1}

The first line of the listing (2.3) produces the (2.162), the second line the (2.163) and the last third line the (2.164)

$$\begin{pmatrix} i_1 & 0 & i_2 \\ i_2 & 0 & i_{p+1} \end{pmatrix} \tag{2.162}$$

$$\begin{pmatrix} i_1 & 0 & i_2 \\ i_2 & 0 & i_3 \\ i_3 & 0 & i_{p+1} \end{pmatrix} \tag{2.163}$$

$$\begin{pmatrix} i_1 & 0 & i_2 \\ i_2 & 0 & i_3 \\ i_3 & 0 & i_4 \\ i_4 & 0 & i_{p+1} \end{pmatrix} \tag{2.164}$$

Next we define several helper functions which are mainly for working with Wolfram Mathematica list objects such as slicing them or generating substitution indexes which will be used in summations.

```
(* Slices the list from given index *)
slice[l_List, k_Integer] := l[[Table[n + k, {n, Length[l] - k}]]]

(* Returns list of indexes such as i_1, i_2 ... and so on *)
inds[j_Integer] := Table[Subscript[i, k], {k, j}]

(* Returns list of indexes from specific range such as {i_5,i_6} *)
inds[j_Integer, q_Integer] := slice[inds[q], j - 1]

(* Returns list of indexes along side with their subscripts such as \
{{i_1,1},{i_2,2}} *)
indsSub[j_Integer] := Table[{Subscript[i, k], k}, {k, j}]

(* Returns list of indexes along side with their indexes from \
specific range *)
indsSub[j_Integer, q_Integer] := slice[indsSub[q], j - 1]
```

Listing 2.4: Helper functions

Now we are all set for modeling the (2.159) using the above defined models.

First of all we need to define the product of (2.160) operators under the sum starting from the vacuum $|0>$

```
psiProd[1] = psi[1, indsSub[1][[1]]];
psiProd[end_Integer] := FoldList[psi, psiProd[1], indsSub[2, end]][[end]]
```

Listing 2.5: ϕ_{i_k} operator product

We explicitly defined the first element and then defined function which can recursively compute the i-th element. And finally we define the summation of this product.

```
(* Recursively computes all diagonal sums of ϕ_i  ϕ_{i-1} ...  functions *)
summPsiProd[psiCount_Integer] :=
 FoldList[Sum, psiProd[psiCount],
   summInds[psiCount, Subscript[i, psiCount + 1]]][[psiCount + 1]];
```

Listing 2.6: $V^{p+1}(i_{p+1})h^{(s)}(x^\mu; b^\mu) = \sum_{i_{p+1}\geq i_p\geq i_{p-1}\geq i_{p-2}\ldots\geq i_1\geq 0} \phi_{i_p}\phi_{i_{p-1}}\ldots\phi_{i_2}\phi_{i_1}h^{(s)}(x^\mu; b^\mu)$

Here $summPsiProd(p)$ function takes as argument p and computes the expression for $V^{p+1}(i_{p+1})$ using

$$V^{p+1}(i_{p+1}) = \sum_{i_{p=0}}^{i_{p+1}} \phi_{i_p} V^p(i_p) \tag{2.165}$$

To demonstrate the results we present couple of examples computed using this module

$$sumPsiProd(1) = V^2(i_2) = (i_2+1)\, a^0\, (i_2 - s + 2) \tag{2.166}$$

$$sumPsiProd(2) = V^3(i_3) = \frac{1}{6}(i_3+1)(i_3+2) \left(a^2 (2i_3 - 3s + 6) + 3\left(a^0\right)^2 (i_3 - s + 2)(i_3 - s + 3)\right) \tag{2.167}$$

$$sumPsiProd(3) = V^4(i_4) = \frac{1}{6}(i_4+1)(i_4+2)(i_4+3)\, a^0\, (i_4 - s + 4) \left(a^2 (2i_4 - 3s + 6) + \left(a^0\right)^2 (-i_4 + s - 3)(-i_4 + s - 2)\right) \tag{2.168}$$

As one can see things become complicated very quickly and we need to find a way to represent it in a more convenient and simple form. For doing so a new function is introduced which will take as argument an expression and return equivalent expression with all coefficients of given variables factorized.

```
factorPolynom[polynom_, vars_List] :=
 Module[{coefList, dims, powsInds, generator, powIter, i, varMatrix,
   polyMatrix},

  (*factorize coefficients of polynom*)
  coefList = CoefficientList[polynom, vars] // Factor ;

  (*generate variable matrix for polynom*)
  dims = Dimensions[coefList];
  powsInds = Table[Subscript[i, n], {n, Length[vars]}];
  generator = Times @@ (vars^powsInds);
  powIter[{k_, b_}] := {k, 0, b};

  (* This line generates all possible combinations of vars in given dims *)
```

```
varMatrix = Table @@ Prepend[
   Map[powIter, Transpose[{powsInds, dims - 1}]],
    generator] ;

polyMatrix = varMatrix * coefList;

(* returns the resulting polynom with factorized coefficients*)
Total[polyMatrix, Length[vars]]
]
```

Listing 2.7: Polinom factorization function

The final computing function for (2.159) is

```
V[p_] := factorPolynom[#, {a, a0}] &  @
  Collect[summPsiProd[p - 1], {a0, a}, Simplify] // Evaluate // HoldForm
```

Listing 2.8: Polinom factorization function

The $V[p]$ function above computes $V^p(i_p)$. Now we can effectively compute the values by simply plugging values for different p.

$$V[2] = V^2(i_2) = (1 + i_2)(2 - s + i_2)\, a^0 \tag{2.169}$$

$$V[3] = V^3(i_3) = \frac{1}{6}a^2 (1 + i_3)(2 + i_3)(6 - 3s + 2i_3) + \frac{1}{2}(1 + i_3)(2 + i_3)(2 - s + i_3)(3 - s + i_3)\left(a^0\right)^2 \tag{2.170}$$

$$V[4] = V^4(i_4) = \frac{1}{6}a^2 (1 + i_4)(2 + i_4)(3 + i_4)(4 - s + i_4)(6 - 3s + 2i_4)\, a^0 + \frac{1}{6}(1 + i_4)(2 + i_4)(3 + i_4)(2 - s + i_4)(3 - s + i_4)(4 - s + i_4)\left(a^0\right)^3 \tag{2.171}$$

We use this function extensively to compute all the formulas in the next subsection.

2.5.2 THE STRUCTURE OF ξ_K^{P+1} AND THE CONSTRUCTION OF THE ANSATZ

We start the computation of the following expression using iterative approach for different values of p

$$V^{p+1}(i_{p+1}) = \sum_{i_{p=0}}^{i_{p+1}} \phi_{i_p} V^p(i_p), \tag{2.172}$$

where

$$\phi_{i_k} = a^u[2(i_k + k) - s] + [a^2 - (a^u)^2]\partial_{a^u}. \tag{2.173}$$

After the computation, we can expand the result using the following formula

$$V^{p+1}(i_{p+1}) = \sum_{k=0}^{[\frac{p}{2}]} \xi_k^{p+1}(i_{p+1})(a^2)^k(a^u)^{p-2k}. \tag{2.174}$$

and obtain ξ_k^{p+1} coefficients.

$$V^2(i_2) = \sum_{i_1=0}^{i_2} \phi_{i_1}|0>= (1+i_2)(2-s+i_2)\, a^u|0> \tag{2.175}$$

$$V^3(i_3) = \sum_{i_1=0}^{i_2} \phi_{i_2} V^2(i_2) = \frac{1}{6} a^2 (1+i_3)(2+i_3)(6-3s+2i_3)\,|0> \tag{2.176}$$

$$+\frac{1}{2}(1+i_3)(2+i_3)(2-s+i_3)(3-s+i_3)(a^u)^2\,|0>$$

$$V^4(i_4) = \sum_{i_3=0}^{i_4} \phi_{i_3} V^3(i_3) = \frac{1}{6} a^2 (1+i_4)(2+i_4)(3+i_4)(4-s+i_4)(6-3s+2i_4)\, a^u|0> \tag{2.177}$$

$$+\frac{1}{6}(1+i_4)(2+i_4)(3+i_4)(2-s+i_4)(3-s+i_4)(4-s+i_4)(a^u)^3\,|0>$$

$$V^5(i_5) = \sum_{i_4=0}^{i_5} \phi_{i_4} V^4(i_4) = \frac{1}{360} a^4 (1+i_5)(2+i_5)(3+i_5)(4+i_5) \tag{2.178}$$

$$\left(360 - 270s + 45s^2 + 172i_5 - 60si_5 + 20i_5^2\right)|0>$$

$$+\frac{1}{12}a^2\left(1+i_5\right)\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)\left(4-s+i_5\right)\left(5-s+i_5\right)\left(6-3s+2i_5\right)\left(a^u\right)^2|0>$$

$$+\frac{1}{24}\left(1+i_5\right)\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)\left(2-s+i_5\right)\left(3-s+i_5\right)\left(4-s+i_5\right)\left(5-s+i_5\right)\left(a^u\right)^4|0>$$

After investigating the structures of ξ_k^{p+1} coefficients we notice that they all have the following general factor

$$\frac{1}{(p-2k)!}(i+1)_p(2k+2+i-s)_{p-2k} \tag{2.179}$$

Using this information we can write the following ansatz for ξ_k^{p+1}

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(i+1)_p(2k+2+i-s)_{p-2k}P_k(i) \tag{2.180}$$

Where $P_k(i)$ is p independent polynomial.

2.5.3 THE STRUCTURE OF THE $\eta_{\tilde{K}}^M(I)$ POLYNOMIAL COEFFICIENTS

To gain more information about the structure of polynomial coefficients we compute the following expression for initial values of $\tilde{k} = 1, 2, 3, 4\ldots$

$$W^{\tilde{k}+1}(a^2, H, i_{\tilde{k}+2}) = \sum_{i_{\tilde{k}+1}=1}^{i_{\tilde{k}+2}} \psi_{i_{\tilde{k}+1}-\tilde{k}} W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) \tag{2.181}$$

where

$$\psi_i = iH + [i]_2\, a^2 \tag{2.182}$$

and factorize the resulting polynomial of variables H and a^2. Then using the following expansion

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) = \sum_{m=0}^{\tilde{k}} \eta_{\tilde{k}}^m(i_{\tilde{k}+1})(a^2)^m H^{\tilde{k}-m} \tag{2.183}$$

we can get the $\eta_{\tilde{k}}^m(i_{\tilde{k}+1})$ coefficients.

To do so we need to model the essential objects in the Mathematica.

First we define (2.182)

```
psi[k_] = (k * H + FactorialPower[k, 2] * a^2) // FunctionExpand;
```

Listing 2.9: Definition of (2.182)

Next we need to compute the product of ψ_i operators under the sum, just like we did in the previous section.

```
(* List of indices of each ψ*)
inds[k_Integer] := Reverse[Table[Subscript[i, p] - p + 1, {p, 1, k}]];

(* Product of ψ-s where each ψ  has its corresponding correct index. *)
psiProd[k_] := (Times @@ psi[inds[k]]) * Subscript[\[CapitalPhi], 0]
```

Listing 2.10: Product of ψ-s

The last thing required for building the summation is the summation indices, which we define in a following way:

```

  (* Returns the summation indexes which will be used later on as
  arguments to Sum function, len_Integer is the number of summations,
  bound_ is the upper bound of the last summation *)
  summInds[len_Integer, bound_] := Module[{inds},
    Subscript[i, len + 1] = bound;
    inds = Table[{Subscript[i, k], 1, Subscript[i, k + 1]}, {k, 1, len}];
    Subscript[i, len + 1] =.;
    inds
  ]
```

Listing 2.11: Summation indices

We demonstrate some of the examples of this function output for different number of sums when the upper bound of the last summation is $i_{\tilde{k}+2}$ which is the case in (2.181)

$$summInds[2, i_{\tilde{k}+2}] = \begin{pmatrix} i_1 & 1 & i_2 \\ i_2 & 1 & i_{\tilde{k}+2} \end{pmatrix} \quad (2.184)$$

$$summInds[3, i_{\tilde{k}+2}] = \begin{pmatrix} i_1 & 1 & i_2 \\ i_2 & 1 & i_3 \\ i_3 & 1 & i_{\tilde{k}+2} \end{pmatrix} \quad (2.185)$$

Now we have everything defined for the definition of (2.181).

```
(* Explicit definition of the first element *)
summPsiProd[1] = FoldList[Sum, psiProd[1] , summInds[1, Subscript[i, 1 +
 1]]][[1 + 1]];

summPsiProd[k_] := Module[{ ps, oldSum, func},
    ps = psi[inds[k]][[1]];
    oldSum = summPsiProd[k - 1];
    func = ps * oldSum;
    summPsiProd[k] = Sum[func, {Subscript[i, k], 1, Subscript[i, k + 1]}]]
```

Listing 2.12: Definition of (2.181)

And finally to make expression more readable we will use the same **factorPolynom** function defined in the previous section.

```
W[k_] := factorPolynom[summPsiProd[k], {H, a^2}]    // Evaluate //
HoldForm
```

Listing 2.13: Factorized definition of $W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1})$

Having the modelled function for (2.181) we plug different values for $\tilde{k} = 1, 2, 3, 4$ and are able to investigate the resulting polynomial expressions.

$$W^1(a^2, H, i_2) = \sum_{i_1=1}^{i_2} \psi_{i_1} \Phi_0(b) = \frac{1}{2} H i_2 (1 + i_2) \Phi_0 + \frac{1}{3} a^2 (-1 + i_2) i_2 (1 + i_2) \Phi_0 \quad (2.186)$$

$$W^2(a^2,H,i_3)=\sum_{i_2=1}^{i_3}\psi_{i_2-1}W^1(a^2,H,i_2)=\frac{1}{8}H^2\left(-1+i_3\right)i_3\left(1+i_3\right)\left(2+i_3\right)\Phi_0 \tag{2.187}$$

$$+\frac{1}{12}a^2H\left(-1+i_3\right)i_3\left(1+i_3\right)\left(2+i_3\right)\left(-3+2i_3\right)\Phi_0$$

$$+\frac{1}{90}a^4\left(-2+i_3\right)\left(-1+i_3\right)i_3\left(1+i_3\right)\left(2+i_3\right)\left(-3+5i_3\right)\Phi_0$$

$$W^3(a^2,H,i_4)=\sum_{i_3=1}^{i_4}\psi_{i_3-2}W^2(a^2,H,i_3)= \tag{2.188}$$

$$\frac{1}{48}H^3\left(-2+i_4\right)\left(-1+i_4\right)i_4\left(1+i_4\right)\left(2+i_4\right)\left(3+i_4\right)\Phi_0$$

$$+\frac{1}{24}a^2H^2\left(-2+i_4\right)^2\left(-1+i_4\right)i_4\left(1+i_4\right)\left(2+i_4\right)\left(3+i_4\right)\Phi_0$$

$$+\frac{1}{180}a^4H\left(-2+i_4\right)\left(-1+i_4\right)^2i_4\left(1+i_4\right)\left(2+i_4\right)\left(3+i_4\right)\left(-13+5i_4\right)\Phi_0$$

$$+\frac{a^6\left(-3+i_4\right)\left(-2+i_4\right)\left(-1+i_4\right)i_4\left(1+i_4\right)\left(2+i_4\right)\left(3+i_4\right)\left(-2-63i_4+35i_4^2\right)\Phi_0}{5670}$$

$$W^4(a^2,H,i_5)=\sum_{i_4=1}^{i_5}\psi_{i_4-3}W^3(a^2,H,i_4)= \tag{2.189}$$

$$\frac{1}{384}H^4\left(-3+i_5\right)\left(-2+i_5\right)\left(-1+i_5\right)i_5\left(1+i_5\right)$$

$$\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)\Phi_0$$

$$+\frac{1}{288}a^2H^3\left(-3+i_5\right)\left(-2+i_5\right)\left(-1+i_5\right)i_5\left(1+i_5\right)$$

$$\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)\left(-5+2i_5\right)\Phi_0$$

$$+\frac{1}{1440}a^4H^2\left(-3+i_5\right)\left(-2+i_5\right)\left(-1+i_5\right)i_5\left(1+i_5\right)\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)$$

$$\left(45-46i_5+10i_5^2\right)\Phi_0$$

$$+\frac{1}{22680}a^6H\left(-3+i_5\right)\left(-2+i_5\right)\left(-1+i_5\right)i_5\left(1+i_5\right)\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)$$

$$\left(-195+731i_5-441i_5^2+70i_5^3\right)\Phi_0$$

$$+\frac{1}{340200}a^8\left(-4+i_5\right)\left(-3+i_5\right)\left(-2+i_5\right)\left(-1+i_5\right)i_5\left(1+i_5\right)\left(2+i_5\right)\left(3+i_5\right)\left(4+i_5\right)$$

$$\left(570 + 149 i_5 - 630 i_5^2 + 175 i_5^3\right) \Phi_0$$

Examining the $\eta_{\tilde{k}}^m(i)$ coefficients we can see that they have the following form

$$\eta_{\tilde{k}}^m(i) = \frac{2^{m-\tilde{k}} 3^{-m}}{(\tilde{k}-m)!m!}(i-\tilde{k}+1)_{2\tilde{k}} P_m(i,\tilde{k}), \quad P_0(i,\tilde{k}) = 1 \tag{2.190}$$

It can be shown that $P_k(i,p)$ polynomials satisfy the following recurrent relation

$$(i+p+1)P_k(i,p+1) - (i-p-1)P_k(i-1,p+1) = \tag{2.191}$$

$$2(p-k+1)P_k(i,p) + 3k(i-p-1)P_{k-1}(i,p) \tag{2.192}$$

The solutions of this recurrent equation can be calculated step by step from the (2.186)-(2.189) for each k

$$P_0(i,p) = 1 \tag{2.193}$$

$$P_1(i,p) = i - \left(\frac{p}{2} + \frac{1}{2}\right) \tag{2.194}$$

$$P_2(i,p) = i^2 - 2i\left(\frac{p}{2} + \frac{3}{10}\right) + \left(\frac{p^2}{4} + \frac{3p}{20} - \frac{1}{10}\right) \tag{2.195}$$

$$P_3(i,p) = i^3 - 3i^2\left(\frac{p}{2} + \frac{1}{10}\right) + 3i\left(\frac{p^2}{4} - \frac{p}{20} - \frac{67}{210}\right) - \left(\frac{p^3}{8} - \frac{3p^2}{20} - \frac{173p}{280} - \frac{12}{35}\right) \tag{2.196}$$

$$P_4(i,p) = i^4 - 4i^3\left(\frac{p}{2} - \frac{1}{10}\right) + 6i^2\left(\frac{p^2}{4} - \frac{p}{4} - \frac{481}{1050}\right) \tag{2.197}$$

$$-4i\left(\frac{p^3}{8} - \frac{3p^2}{10} - \frac{1031p}{1400} - \frac{38}{175}\right) + \frac{p^4}{16} - \frac{11p^3}{40} - \frac{2011p^2}{2800} - \frac{89p}{1400} + \frac{111}{350}$$

$$P_5(i,p) = i^5 - 5i^4\left(\frac{p}{2} - \frac{3}{10}\right) + 10i^3\left(\frac{p^2}{4} - \frac{9p}{20} - \frac{181}{350}\right) \tag{2.198}$$
$$-10i^2\left(\frac{p^3}{8} - \frac{9p^2}{20} - \frac{147p}{200} + \frac{9}{175}\right) + 5i\left(\frac{p^4}{16} - \frac{3p^3}{8} - \frac{351p^2}{560} + \frac{843p}{1400} + \frac{3131}{3850}\right)$$
$$-\frac{p^5}{32} + \frac{9p^4}{32} + \frac{421p^3}{1120} - \frac{1587p^2}{1120} - \frac{15839p}{6160} - \frac{12}{11}$$

From the solutions above we can see that the general ansatz for $P_k(i,p)$ has the following form

$$P_k(i,p) = \sum_{n=0}^{k} i^{k-n}(-1)^n \binom{k}{n} B_k^n(p)$$

From the solutions above for different $P_k(i,p)$ it is possible to find the solutions for $B_k^n(p)$ as follows [7]

$$B^1{}_k(p) = \frac{p}{2} - \frac{k}{5} + \frac{7}{10} \tag{2.199}$$

$$B^2{}_k(p) = \frac{p^2}{4} + p\left(\frac{11}{20} - \frac{k}{5}\right) + \frac{k^2}{25} - \frac{44k}{105} + \frac{607}{1050} \tag{2.200}$$

$$B^3{}_k(p) = \frac{p^3}{8} + p^2\left(\frac{3}{10} - \frac{3k}{20}\right) + p\left(\frac{3k^2}{50} - \frac{377k}{700} + \frac{641}{1400}\right) \tag{2.201}$$

$$-\frac{k^3}{125} + \frac{293k^2}{1750} - \frac{1313k}{1750} + \frac{108}{175}$$

$$B^4{}_k(p) = \frac{p^4}{16} + p^3\left(\frac{1}{8} - \frac{k}{10}\right) + p^2\left(\frac{3k^2}{50} - \frac{157k}{350} + \frac{13}{112}\right) \tag{2.202}$$

$$+p\left(-\frac{2k^3}{125} + \frac{523k^2}{1750} - \frac{131k}{125} + \frac{519}{1400}\right) + \frac{k^4}{625} - \frac{244k^3}{4375} + \frac{47728k^2}{91875} - \frac{460722k}{336875} + \frac{256957}{404250}$$

[7] In order to compute the $B_k^n(p)$ using this iterative approach one should compute and know the expressions of $P_m(i,p)$ for up to $m = 2k$

$$B^5{}_k(p) = \frac{p^5}{32} + p^4\left(\frac{1}{32} - \frac{k}{16}\right) + p^3\left(\frac{k^2}{20} - \frac{251k}{840} - \frac{443}{3360}\right) \tag{2.203}$$

$$+p^2\left(-\frac{k^3}{50} + \frac{23k^2}{70} - \frac{2273k}{2800} - \frac{267}{1120}\right)$$
$$+p\left(\frac{k^4}{250} - \frac{223k^3}{1750} + \frac{2969k^2}{2940} - \frac{980587k}{539000} - \frac{97283}{646800}\right)$$
$$-\frac{k^5}{3125} + \frac{439k^4}{26250} - \frac{24404k^3}{91875} + \frac{23911k^2}{16170} - \frac{52309518k}{21896875} - \frac{72612}{398125}$$

The final form of $\eta^m_{\tilde{k}}(i)$ coefficients will be

$$\eta^m_{\tilde{k}}(i) = \frac{2^{m-\tilde{k}}3^{-m}}{(\tilde{k}-m)!m!}(i-\tilde{k}+1)_{2\tilde{k}}\sum_{n=0}^{m} i^{m-n}(-1)^n\binom{m}{n}B^n_m(\tilde{k}) \tag{2.204}$$

We end up with an equation with has the same difficulty as the original equation and that's why we used the direct approach of solving this specific recurrence relation.

2.6 MAPPING OPERATOR $(A, \partial_B)^P$ TO THE PRODUCT OF H AND A^2

This is the final exercise to get more freedom in writing of our cubic interaction after our "stripping" for u components of auxiliary vectors. Investigating (2.148) and first operator in (2.86):

$$(a^A\partial_{b^A})^Q = \sum_{q=0}^{Q}\binom{Q}{q}(a^u\partial_{b^u})^{Q-q}(a,\partial_b)^q \tag{2.205}$$

we see that last thing to do is transform the power of (a, ∂_b) to H and a^2 to write interaction without (a, ∂_b), hiding them then in $\Phi(a, b)$. Note that starting from (2.130) operator H effectively worked with only its second part:

$$H \equiv H(a, b) = -(a, b)(a, \partial_b) \tag{2.206}$$

due to the separation of all b^u dependence to the left from H dependent part in (2.148). Therefore we can write simple relation

$$(a,\partial_b)^p = (-1)^p \left(\frac{1}{(a,b)}H\right)^p \tag{2.207}$$

Then introducing ansatz for ordered power:

$$\left(\frac{1}{(a,b)}H\right)^p = \frac{1}{(a,b)^p}\sum_{k=0}^{p-1}\rho_k(p)H^{p-k}(a^2)^k \tag{2.208}$$

and taking into account commutator

$$[H,(a,b)^{-k}] = \frac{ka^2}{(a,b)^k} \tag{2.209}$$

we arrive to the following simple triangular recurrence relation for polynomials $\rho_k(p)$

$$\rho_k(p+1) = \rho_k(p) + p\rho_{k-1}(p) \tag{2.210}$$

with boundary conditions:

$$\rho_0(p) = 1, \qquad \rho_{p-1}(p) = (p-1)! \tag{2.211}$$

Recurrence relation (2.210) we can easily solve using generation function. Introducing formal variable z with $|z| < 1$

$$\rho_k(z) = \sum_{p=0}^{\infty} z^p \rho_k(p) \tag{2.212}$$

we obtain recursive equation:

$$\rho_k(z) = \frac{z^2}{1-z}\frac{d}{dz}\rho_{k-1}(z) \tag{2.213}$$

with the simple solution due to boundary value $\rho_0(z) = (1-z)^{-1}$:

$$\rho_k(z) = \left[\frac{z^2}{1-z}\frac{d}{dz}\right]^k \frac{1}{1-z} \tag{2.214}$$

In another way we can write the same solution of (2.210) in the form of multiple sums:

$$\rho_k(p) = \sum_{i_k=k}^{p-1} i_k \sum_{i_{k-1}=k-1}^{i_k-1} i_{k-1}\cdots\sum_{i_2=2}^{i_3-1} i_2 \sum_{i_1=1}^{i_2-1} i_1 \tag{2.215}$$

2.7 PULLBACK OF THE MAIN TERM OF CUBIC SELF-INTERACTION

Now we start to collect things together and present all terms of cubic interaction produced from the main term in one dimension more flat space. First, we look at the main term in the case of a cubic self-interaction. This can be obtained from the general expressions (2.72)-(2.74) taking

$$s_1 = s_2 = s_3 = s, \tag{2.216}$$

$$\nu_1 = \nu_2 = \nu_3 = 0, \tag{2.217}$$

$$Q_{23} = n_1 = \alpha, \tag{2.218}$$

$$Q_{31} = n_2 = \beta, \tag{2.219}$$

$$Q_{12} = n_3 = \gamma. \tag{2.220}$$

Then (2.78), (2.79) transform to the following nice cyclic (in $(a)\alpha, (b)\beta, (c)\gamma$) expression with trinomial coefficients :

$$\begin{aligned}\mathcal{L}_I^{main} = & \sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha,\beta,\gamma} \int d^{d+2}X \\ & *_a^{\gamma+\alpha}(a^A\partial_{b^A})^\gamma (a^B\partial_B)^\alpha h^{(s)}(X;b^C) \\ & *_b^{\alpha+\beta}(b^D\partial_{c^D})^\alpha (b^E\partial_E)^\beta h^{(s)}(X;c^F) \\ & *_c^{\beta+\gamma}(c^G\partial_{a^G})^\beta (c^H\partial_H)^\gamma h^{(s)}(X;a^K),\end{aligned} \tag{2.221}$$

The main result of the previous section is that we can expand each line of (2.221) and extract a^u, b^u, c^u dependence to contract with expansion of star product and write exact expression in the term of AdS_{d+1} dimensional covariant derivatives and curvature corrections. Combining

(2.106)-(2.109) and (2.125)-(2.150) we can write[8]

$$(a^B\partial_B)^\alpha h^{(s)}(X;b^C) = e^{(s-2-\alpha)u}\sum_{p_1=0}^{\alpha}\sum_{k_1=0}^{[\frac{p_1}{2}]}\sum_{\tilde{p_1}=0}^{\alpha-p_1}\sum_{\tilde{k_1}=1}^{[\frac{\tilde{p}_1}{2}]}(-1)^{p_1+\tilde{p_1}+\tilde{k_1}}(a^u)^{p_1-2k_1}(b^u)^{\tilde{p_1}-2\tilde{k_1}}$$
$$(a,\nabla)^{\alpha-p_1-\tilde{p_1}}\xi_{k_1}^{p_1+1}(\alpha-p_1)\binom{\alpha-p_1}{\tilde{p_1}}(a^2)^{k_1}(a,\partial_b)^{\tilde{p_1}-2\tilde{k_1}}W^{\tilde{k_1}}(a^2,H_1)h^{(s)}(x^\mu;b^\mu) \quad (2.222)$$

Then expanding :

$$(a^A\partial_{b^A})^\gamma = \sum_{m=0}^{\gamma}\binom{\gamma}{m_1}(a^u\partial_{b^u})^{m_1}(a,\partial_b)^{\gamma-m_1} \quad (2.223)$$

we obtain

$$(a^A\partial_{b^A})^\gamma(a^B\partial_B)^\alpha h^{(s)}(X;b^C) = e^{(s-2-\alpha)u}\sum_{m_1=0}^{\gamma}\sum_{m_1,p_1,k_1,\tilde{p_1},\tilde{k_1}}^{\gamma,\alpha,[\frac{p_1}{2}],\alpha-p_1,[\frac{\tilde{p}_1}{2}]}$$
$$(a^u)^{p_1-2k_1+m_1}(b^u)^{\tilde{p_1}-2\tilde{k_1}-m_1}(a,\partial_b)^{\gamma+\tilde{p_1}-2\tilde{k_1}-m_1}(a,\nabla)^{\alpha-p_1-\tilde{p_1}}$$
$$\Theta[\gamma,\alpha,m_1,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1]h^{(s)}(b^\mu). \quad (2.224)$$

where:

$$\sum_{m_1,p_1,k_1,\tilde{p_1},\tilde{k_1}}^{\gamma,\alpha,[\frac{p_1}{2}],\alpha-p_1,[\frac{\tilde{p}_1}{2}]} = \sum_{m_1=0}^{\gamma}\sum_{p_1=0}^{\alpha}\sum_{k_1=0}^{[\frac{p_1}{2}]}\sum_{\tilde{p_1}=0}^{\alpha-p_1}\sum_{\tilde{k_1}=1}^{[\frac{\tilde{p}_1}{2}]} \quad (2.225)$$

and

$$\Theta[\gamma,\alpha,m_1,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1]$$
$$= (-1)^{p_1+\tilde{p_1}+\tilde{k_1}}[\tilde{p_1}-2\tilde{k_1}]_{m_1}\binom{\gamma}{m_1}\xi_{k_1}^{p_1+1}(\alpha-p_1)\binom{\alpha-p_1}{\tilde{p_1}}(a^2)^{k_1}W^{\tilde{k_1}}(a^2,H_1)$$
$$(2.226)$$

[8]For shortening notation we introduce instead of $H(a,b)$ from (2.127) H_1 and then $H_2 = H(b,c)$ and $H_3 = H(c,a)$ correspondingly.

Then we can write expression for the whole main interaction term

$$\mathcal{L}_I^{main} = \int due^{(d+2s-4)u}d^{d+1}x\sqrt{g}\sum_{\substack{\alpha,\beta,\gamma\\ \alpha+\beta+\gamma=s}}\binom{s}{\alpha,\beta,\gamma}$$
$$\sum_{m_1,p_1,k_1,\tilde{p_1},\tilde{k_1}}^{\gamma,\alpha,[\frac{p_1}{2}],\alpha-p_1,[\frac{\tilde{p_1}}{2}]}\ \sum_{m_2,p_2,k_2,\tilde{p_2},\tilde{k_2}}^{\alpha,\beta,[\frac{p_2}{2}],\beta-p_2,[\frac{\tilde{p_2}}{2}]}\ \sum_{m_3,p_3,k_3,\tilde{p_3},\tilde{k_3}}^{\beta,\gamma,[\frac{p_3}{2}],\gamma-p_3,[\frac{\tilde{p_3}}{2}]}$$
$$\sum_{n_1,n_2,n_3=0}^{\gamma+\alpha,\alpha+\beta,\beta+\gamma}\frac{(-1)^{n_1+n_2+n_3}}{\binom{\gamma+\alpha}{n_1}\binom{\alpha+\beta}{n_2}\binom{\beta+\gamma}{n_3}}*_{a^u}^{n_1}*_{b^u}^{n_2}*_{c^u}^{n_3}*_{a^\mu}^{\gamma+\alpha-n_1}*_{b^\mu}^{\alpha+\beta-n_2}*_{c^\mu}^{\beta+\gamma-n_3}$$
$$(a^u)^{p_1-2k_1+m_1}(b^u)^{\tilde{p_1}-2\tilde{k_1}-m_1}(a,\partial_b)^{\gamma+\tilde{p_1}-2\tilde{k_1}-m_1}$$
$$(a,\nabla)^{\alpha-p_1-\tilde{p_1}}\Theta[\gamma,\alpha,m_1,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1]h^{(s)}(b^\mu)$$
$$(b^u)^{p_2-2k_2+m_2}(c^u)^{\tilde{p_2}-2\tilde{k_2}-m_2}(b,\partial_c)^{\alpha+\tilde{p_2}-2\tilde{k_2}-m_2}$$
$$(b,\nabla)^{\beta-p_2-\tilde{p_2}}\Theta[\alpha,\beta,m_2,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2]h^{(s)}(c^\mu)$$
$$(c^u)^{p_3-2k_3+m_3}(a^u)^{\tilde{p_3}-2\tilde{k_3}-m_3}(c,\partial_a)^{\beta+\tilde{p_3}-2\tilde{k_3}-m_3}$$
$$(c,\nabla)^{\gamma-p_3-\tilde{p_3}}\Theta[\beta,\gamma,m_3,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3]h^{(s)}(a^\mu)$$
$$(2.227)$$

Now we can contract all non AdS_{d+1} components a^u, b^u, c^u using corresponding "u"-stars from second line of (2.227). This leads to the following constraints for summation indices:

$$p_1 - 2k_1 + m_1 = \tilde{p_3} - 2\tilde{k_3} - m_3 = n_1 \qquad (2.228)$$

$$p_2 - 2k_2 + m_2 = \tilde{p_1} - 2\tilde{k_1} - m_1 = n_2 \qquad (2.229)$$

$$p_3 - 2k_3 + m_3 = \tilde{p_2} - 2\tilde{k_2} - m_2 = n_3 \qquad (2.230)$$

So we can take summation over $m_i, i = 1, 2, 3$ with remaining constraints on other variables :

$$p_1 + \tilde{p_1} = n_1 + n_2 + 2(k_1 + \tilde{k_1}) \qquad (2.231)$$

$$p_2 + \tilde{p_2} = n_2 + n_3 + 2(k_2 + \tilde{k_2}) \qquad (2.232)$$

$$p_3 + \tilde{p_3} = n_3 + n_1 + 2(k_3 + \tilde{k_3}) \qquad (2.233)$$

Relations (2.228)-(2.230) restrict also summation ranges for n_1, n_2, n_3 from zero to α, β, γ. Then we have

$$
\begin{aligned}
\mathcal{L}_I^{main} = \int due^{(d+2s-4)u} d^{d+1}x\sqrt{g} \sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha,\beta,\gamma} \sum_{p_1,k_1,\tilde{p_1},\tilde{k_1}}^{\alpha,[\frac{p_1}{2}],\alpha-p_1,[\frac{\tilde{p_1}}{2}]} \sum_{p_2,k_2,\tilde{p_2},\tilde{k_2}}^{\beta,[\frac{p_2}{2}],\beta-p_2,[\frac{\tilde{p_2}}{2}]} \sum_{p_3,k_3,\tilde{p_3},\tilde{k_3}}^{\gamma,[\frac{p_3}{2}],\gamma-p_3,[\frac{\tilde{p_3}}{2}]} \\
\sum_{n_1,n_2,n_3=0}^{\alpha,\beta,\gamma} \frac{(-1)^{n_1+n_2+n_3}}{\binom{\gamma+\alpha}{n_1}\binom{\alpha+\beta}{n_2}\binom{\beta+\gamma}{n_3}} *_{a^\mu}^{\gamma+\alpha-n_1} *_{b^\mu}^{\alpha+\beta-n_2} *_{c^\mu}^{\beta+\gamma-n_3} \\
(a,\partial_b)^{\gamma+n_2}(a,\nabla)^{\alpha-n_1-n_2-2(k_1+\tilde{k_1})}\tilde{\Theta}[\gamma,\alpha,n_2,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1]h^{(s)}(b^\mu) \\
(b,\partial_c)^{\alpha+n_3}(b,\nabla)^{\beta-n_2-n_3-2(k_2+\tilde{k_2})}\tilde{\Theta}[\alpha,\beta,n_3,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2]h^{(s)}(c^\mu) \\
(c,\partial_a)^{\beta+n_1}(c,\nabla)^{\gamma-n_3-n_1-2(k_3+\tilde{k_3})}\tilde{\Theta}[\beta,\gamma,n_1,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3]h^{(s)}(a^\mu)
\end{aligned}
\tag{2.234}
$$

where

$$\tilde{\Theta}[\gamma,\alpha,n_2,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1] = \Theta[\gamma,\alpha,m_1=\tilde{p_1}-2\tilde{k_1}-n_2,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1] \tag{2.235}$$

$$\tilde{\Theta}[\alpha,\beta,n_3,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2] = \Theta[\alpha,\beta,m_2=\tilde{p_2}-2\tilde{k_2}-n_3,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2] \tag{2.236}$$

$$\tilde{\Theta}[\beta,\gamma,n_1,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3] = \Theta[\beta,\gamma,m_3=\tilde{p_3}-2\tilde{k_3}-n_1,\tilde{p_3},p_3,k_3,\tilde{k_3},c^2,H_3] \tag{2.237}$$

Taking into account that $\Theta[\dots,a^2,H_1] \sim (a^2)^{k_1+\tilde{k_1}}$ we see that our star products in (2.234) contract correctly all auxiliary vectors a^μ, b^ν, c^λ.

Then to understand better the structure of the derivativesof interaction we can take into account constraints (2.231)-(2.233) and rearrange the summations coming from (2.234) in the following way

$$\sum_{n_3\geq 0}\sum_{n_2\geq 0}\sum_{n_1\geq 0}(-1)^{n_1+n_2+n_3} = \sum_{N\geq 0}(-1)^N \sum_{\substack{n_1,n_2,n_3 \\ \sum n_i=N}}, \tag{2.238}$$

$$\sum_{\substack{\{p_i,k_i,\tilde{p_i},\tilde{k_i}\}_{i=1,2,3} \\ p_i+\tilde{p_i}=n_i+n_{i+1}+2(k_i+\tilde{k_i})}} = \sum_{K\geq 0} \sum_{\substack{\{P_i,K_i\}_{i=1,2,3} \\ P_i=n_i+n_{i+1}+2K_i \\ \sum K_i=K}} \sum_{\substack{\{p_i,k_i,\tilde{p_i},\tilde{k_i}\}_{i=1,2,3} \\ p_i+\tilde{p_i}=P_i;k_i+\tilde{k_i}=K_i}} \tag{2.239}$$

where in last equation $\{n_i\} = n_1,n_2,n_3$ with cyclic property $n_4 = n_1$

After that we should introduce instead of α,β,γ new summation variables

$$\tilde{\alpha} = \alpha - n_1 - n_2 - 2K_1 = \alpha - P_1, \tag{2.240}$$

$$\tilde{\beta} = \beta - n_2 - n_3 - 2K_2 = \beta - P_2, \tag{2.241}$$

$$\tilde{\gamma} = \gamma - n_3 - n_1 - 2K_3 = \gamma - P_3. \tag{2.242}$$

with corresponding summation limits and constraints

$$0 \leq \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \leq s - 2(N+K), \tag{2.243}$$

$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = s - 2(N+K), \tag{2.244}$$

$$N = \sum_i n_i; \quad K = \sum_i K_i = \sum_i (k_i + \tilde{k}_i). \tag{2.245}$$

These transformations lead to the following formula:

$$\mathcal{L}_I^{main} = \int du e^{(d+2s-4)u} d^{d+1}x \sqrt{g} \sum_{N\geq 0} \sum_{K \geq 0} \frac{(-1)^N s!}{(s-2(N+K))!} \sum_{\substack{\tilde{\alpha},\tilde{\beta},\tilde{\gamma} \\ \tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}=s-2(N+K)}} \binom{s-2(N+K)}{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$$

$$\sum_{\substack{\{n_i\}_{i=1,2,3} \\ \sum n_i = N}} \sum_{\substack{\{P_i,K_i\}_{i=1,2,3} \\ P_i = n_i + n_{i+1} + 2K_i \\ \sum K_i = K}} \sum_{\substack{\{p_i,k_i,\tilde{p_i},\tilde{k_i}\}_{i=1,2,3} \\ p_i+\tilde{p_i}=P_i; k_i+\tilde{k_i}=K_i}} \tag{2.246}$$

$$\frac{*_{a^\mu}^{\tilde{\gamma}+\tilde{\alpha}+N+2(K_3+K_1)} *_{b^\mu}^{\tilde{\alpha}+\tilde{\beta}+N+2(K_1+K_2)} *_{c^\mu}^{\tilde{\beta}+\tilde{\gamma}+N+2(K_2+K_3)}}{\binom{\tilde{\gamma}+\tilde{\alpha}+N+2(K_3+K_1)+n_1}{n_1}\binom{\tilde{\alpha}+\tilde{\beta}+N+2(K_1+K_2)+n_2}{n_2}\binom{\tilde{\beta}+\tilde{\gamma}+N+2(K_2+K_3)+n_3}{n_3}}$$
$$(a,\partial_b)^{\tilde{\gamma}+N+2K_3}(a,\nabla)^{\tilde{\alpha}}\, \Xi^{2K_1}[\tilde{\gamma},\tilde{\alpha},n_2,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1] h^{(s)}(b^\mu)$$
$$(b,\partial_c)^{\tilde{\alpha}+N+2K_1}(b,\nabla)^{\tilde{\beta}}\, \Xi^{2K_2}[\tilde{\alpha},\tilde{\beta},n_3,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2] h^{(s)}(c^\mu)$$
$$(c,\partial_a)^{\tilde{\beta}+N+2K_2}(c,\nabla)^{\tilde{\gamma}}\, \Xi^{2K_3}[\tilde{\beta},\tilde{\gamma},n_1,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3] h^{(s)}(a^\mu) \tag{2.247}$$

where

$$\tilde{\Theta}[\gamma,\alpha,n_2,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1] = \frac{\gamma!}{\tilde{\alpha}!}\, \Xi^{2K_1}[\tilde{\gamma},\tilde{\alpha},n_2,P_3,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1] \tag{2.248}$$

$$\tilde{\Theta}[\alpha,\beta,n_3,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2] = \frac{\alpha!}{\tilde{\beta}!}\, \Xi^{2K_2}[\tilde{\alpha},\tilde{\beta},n_3,P_1,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2] \tag{2.249}$$

$$\tilde{\Theta}[\beta,\gamma,n_1,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3] = \frac{\beta!}{\tilde{\gamma}!}\, \Xi^{2K_3}[\tilde{\beta},\tilde{\gamma},n_1,P_2,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3] \tag{2.250}$$

and

$$\Xi^{2K_1}[\tilde{\gamma},\tilde{\alpha},n_2,P_3,p_1,k_1,\tilde{p_1},\tilde{k_1},a^2,H_1]$$
$$=\frac{(\tilde{\alpha}+\tilde{p_1})!(a^2)^{k_1}}{(\tilde{\gamma}+P_3-\tilde{p_1}+2\tilde{k_1}+n_2)!}\binom{\tilde{p_1}-2\tilde{k_1}}{n_2}\xi^{p_1+1}_{k_1}(\tilde{\alpha}+\tilde{p_1})W^{\tilde{k_1}}(a^2,H_1)\,, \tag{2.251}$$
$$\Xi^{2K_2}[\tilde{\alpha},\tilde{\beta},n_3,P_1,p_2,k_2,\tilde{p_2},\tilde{k_2},b^2,H_2]$$
$$=\frac{(\tilde{\beta}+\tilde{p_2})!(a^2)^{k_2}}{(\tilde{\alpha}+P_1-\tilde{p_2}+2\tilde{k_2}+n_3)!}\binom{\tilde{p_2}-2\tilde{k_2}}{n_3}\xi^{p_2+1}_{k_2}(\tilde{\beta}+\tilde{p_2})W^{\tilde{k_2}}(b^2,H_2)\,, \tag{2.252}$$
$$\Xi^{2K_3}[\tilde{\beta},\tilde{\gamma},n_1,P_2,p_3,k_3,\tilde{p_3},\tilde{k_3},c^2,H_3]$$
$$=\frac{(\tilde{\gamma}+\tilde{p_3})!(a^2)^{k_3}}{(\tilde{\beta}+P_2-\tilde{p_3}+2\tilde{k_3}+n_1)!}\binom{\tilde{p_3}-2\tilde{k_3}}{n_1}\xi^{p_3+1}_{k_3}(\tilde{\gamma}+\tilde{p_3})W^{\tilde{k_3}}(c^2,H_3)\,. \tag{2.253}$$

Finalizing our consideration we can write direct $(a^2),(b)^2,(c)^2$ expansion of corresponding Ξ^{2K_i} terms using (2.133) and (2.150)

$$(a^2)^{k_1}W^{\tilde{k_1}}(a^2,H_1)h^{(s)}(b^\mu)=$$
$$\sum_{t_1=0}^{\tilde{k_1}}(-1)^{t_1}\sum_{r_1=1}^{\tilde{k_1}-t_1}\eta^{t_1}_{\tilde{k_1}}(\tilde{p_1}-\tilde{k_1})A^{\tilde{k_1}-t_1}_{r_1}[s]_{r_1}(a^2)^{K_1-r_1}(a,b)^{r_1}\Phi_{r_1}(a,b) \tag{2.254}$$
$$(b^2)^{k_2}W^{\tilde{k_2}}(b^2,H_2)h^{(s)}(c^\mu)=$$
$$\sum_{t_2=0}^{\tilde{k_2}}(-1)^{t_2}\sum_{r_2=1}^{\tilde{k_2}-t_2}\eta^{t_2}_{\tilde{k_2}}(\tilde{p_2}-\tilde{k_2})A^{\tilde{k_2}-t_2}_{r_2}[s]_{r_2}(b^2)^{K_2-r_2}(b,c)^{r_2}\Phi_{r_2}(b,c) \tag{2.255}$$
$$(c^2)^{k_3}W^{\tilde{k_3}}(a^3,H_3)h^{(s)}(c^\mu)=$$
$$\sum_{t_3=0}^{\tilde{k_3}}(-1)^{t_3}\sum_{r_3=1}^{\tilde{k_3}-t_3}\eta^{t_3}_{\tilde{k_3}}(\tilde{p_3}-\tilde{k_3})A^{\tilde{k_3}-t_3}_{r_3}[s]_{r_3}(c^2)^{K_3-r_3}(c,a)^{r_3}\Phi_{r_3}(c,a) \tag{2.256}$$

So we see that Ξ^{2K_i} in (2.248)-(2.250) really behave like $a^{2K_1},b^{2K_2},c^{2K_3}$ as they should for correct contractions of indices.

2.8 CONCLUSION

We have constructed all AdS corrections including trace and divergence terms to the main term of the cubic self-interaction by a slightly modified method of radial pullback (reduction) proposed in [44] where all quantum fields are carried by a real AdS space and corresponding interaction terms expressed through the covariant AdS derivatives. For given spin s and $\Delta_{min} = s$ we derived all curvature correction terms (2.247) in the form of series of terms with numbers $s - 2(N + K)$ of derivatives, where $0 \leq N + K \leq \frac{s}{2}$. The latter is the number of seized pair of derivatives replaced by corresponding power of $1/L^2$ and K is the sum of power of a^2, b^2, c^2 terms connected with trace and divergent correction terms produced from the main term of interaction after pullback. Correction terms appear with coefficients that are polynomials in the dimension $d + 1$ and spin number s with rational coefficients. Now we can expect that the same method can be used for the derivation of the AdS corrections to traces and deDonder terms connected with the main term by Noether's procedure derived for the flat case in [29,30]

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Chapter 3

Special quartic interaction of higher spin gauge fields with scalars and gauge symmetry commutator in the linear approximation

3.1 INTRODUCTION

This chapter is based on three articles [4–6] written with Ruben Manvelyan and Gabriel Poghosyan.

We consider a Local quartic interaction of higher-spin gauge field with a scalar field. In this special case, the nontrivial task of construction of interacting Lagrangian for the higher spin field in physical gauge was solved using the full power of Noether's procedure. There are two interesting points worth highlighting:

- We see that we are required to add additional cubic interaction of scalar with other spin gauge fields and corresponding HS gauge symmetries to close the Neother's procedure

- While constructing quartic vertex we derive fixed linear in gauge field gauge transformation of our HS field $\delta_1^{(\epsilon)}$ and then we are able to investigate the closure of commutators of two such a transformation

$$[\delta_1^{(\eta)}\delta_1^{(\epsilon)}] \sim \delta_1^{([\eta,\epsilon])} + \text{additional terms}$$

and understand whether it leads to non-locality or not.

As a result, the linear on-field gauge transformation is obtained and the corresponding commutator of transformation is analyzed. To understand the closure of this algebra the right-hand side of this commutator is classified in respect to gauge transformations coming from cubic interactions with different higher spin symmetric tensor fields and with mixed symmetry tensor fields transformations.

In the next section we show spin 2 exercise for construction similar quartic interaction for spin 2 case. Then the essential Noether's construction is shown in section two with the final derivation of interaction Lagrangian and first order on HS gauge field gauge transformation. The last section is devoted to the investigation of the commutator of two δ_1 transformations and classification of terms different from the same δ_1 with the composed parameter on the right-hand side.

The technique and notation for the calculations have been developed in [64], [26, 60, 65].

3.2 ILLUSTRATION: SPIN TWO CASE

For better understanding how we should construct special quartic interaction in higher spin case we start first from the following lagrangian for the interaction of scalar with spin 2 gauge field in the flat background:

$$S^{\Phi\Phi h^{(2)}} = S_0(\Phi) + S_1(\Phi, h^{(2)}), \tag{3.1}$$

where

$$\begin{aligned} S_0(\Phi) &= \frac{1}{2}\int d^d x \partial_\mu \Phi \partial^\mu \Phi, & (3.2) \\ S_1(\Phi, h^{(2)}) &= \frac{1}{2}\int d^d x h^{(2)\mu\nu}\Big[-\partial_\mu \Phi \partial_\nu \Phi + \frac{\eta_{\mu\nu}}{2}\partial_\lambda \Phi \partial^\lambda \Phi\Big]. & (3.3) \end{aligned}$$

This is a well known minimal coupling of scalar with gravity, linearized in flat background and the bracket in (3.3) is the usual energy-momentum tensor for massless scalar field. The spliting of (3.1) into the quadratic and cubic parts allows us to formulate the Noether's equations

$$\delta_1 S_0(\Phi) + \delta_0 S_1(\Phi, h^{(2)}) = 0, \tag{3.4}$$

where:

$$\begin{aligned} \delta_0 h^{(2)}_{\mu\nu} &= \partial_{(\mu}\varepsilon^{(1)}_{\nu)} = \partial_\mu \varepsilon^{(1)}_\nu + \partial_\nu \varepsilon^{(1)}_\mu, & (3.5) \\ \delta_0 \Phi &= 0, & (3.6) \\ \delta_1 \Phi &= \varepsilon^{(1)\lambda}\partial_\lambda \Phi. & (3.7) \end{aligned}$$

The crucial point here that we can discover interacting part (3.3) solving functional equation (3.4) variating known free part (3.2) in respect to admitting first order diffeomorphism of scalar (3.7) and using a zero-order variation of gauge field as an integration rule. Note also that the scalar field has no zero-order variation being matter field here.

Then we can formulate task for construction of the next order interaction of the two scalars with two spin two fields in the similar way:

$$\delta_2 S_0(\Phi) + \delta_1 S_1(\Phi, h^{(2)}) + \delta_0 S_2(\Phi, h^{(2)}) = 0 \tag{3.8}$$

admitting that: $\delta_2 \Phi = 0$ we see that in this case we need to solve again two terms functional equation

$$\delta_1 S_1(\Phi, h^{(2)}) + \delta_0 S_2(\Phi, h^{(2)}) = 0, \tag{3.9}$$

using the same transformations (3.5)-(3.7) and introducing an assumption about the form of first order transformation of spin 2 gauge field:

$$\delta_1 h^{(1)}_{\mu\nu} \;=\; \varepsilon^{(1)\lambda}\partial_\lambda h^{(2)}_{\mu\nu} + \bar{\delta}_1 h^{(2)}_{\mu\nu}, \tag{3.10}$$

where $\bar{\delta}_1 h_{\mu\nu}$ we should find from equation (3.9) together with $S_2(\Phi, h^{(2)})$. To show technology of solution we present variation of $S_1(\Phi, h^{(2)})$ in the following form (after some algebra and partial integrations)

$$\begin{aligned}\delta_1 S_1(\Phi, h^{(2)}) \;=\; & \int d^d x\Big\{-\frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi[\bar{\delta}_1 h^{(2)}_{\mu\nu} - \partial_{(\mu}\varepsilon^{(1)\lambda}h^{(2)}_{\nu)\lambda} + 2h^{(2)\lambda}_{\mu}\partial_{(\nu}\varepsilon^{(1)}_{\lambda)}] \\ + \;& \frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi[\partial_\lambda\varepsilon^{(1)\lambda}h^{(2)}_{\mu\nu} + \frac{1}{2}\partial_{(\mu}\varepsilon^{(1)}_{\nu)}h^{(2)\alpha}_{\alpha}] \\ + \;& \frac{1}{4}\partial^\lambda\Phi\partial_\lambda\Phi[\bar{\delta}_1 h^{(2)\alpha}_{\alpha} - \partial_\beta\varepsilon^{(1)\beta}h^{(2)\alpha}_{\alpha}]\Big\}.\end{aligned} \tag{3.11}$$

From first line of (3.11) we can derive that if we define

$$\bar{\delta}_1 h^{(2)}_{\mu\nu} = \partial_{(\mu}\varepsilon^{(1)\lambda}h^{(2)}_{\nu)\lambda}, \tag{3.12}$$

then last term of first line can be integrated to $\delta_0[-\frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi h^{\lambda}_{\mu}h_{\nu\lambda}]$

Then taking into account that

$$\bar{\delta}_0 h^{(2)\alpha}_{\alpha} \;=\; 2\partial_\lambda\varepsilon^{(1)\lambda}, \tag{3.13}$$

$$\bar{\delta}_1 h^{(2)\alpha}_{\alpha} \;=\; 2\partial^\lambda\varepsilon^{(1)\alpha}h^{(2)}_{\lambda\alpha} = \frac{1}{2}\delta_0[h^{(2)\lambda\alpha}h^{(2)}_{\lambda\alpha}], \tag{3.14}$$

we see that we can immediately integrate second and third line of (3.11) and arrive to the following quartic action:

$$\begin{aligned}S_2(\Phi, h^{(2)}) \;=\; & \int d^d x\Big\{\frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi h^{(2)\lambda}_{\mu}h^{(2)}_{\nu\lambda} - \frac{1}{4}\partial^\mu\Phi\partial^\nu\Phi h^{(2)}_{\mu\nu}h^{(2)\alpha}_{\alpha} \\ - \;& \frac{1}{8}\partial^\lambda\Phi\partial_\lambda\Phi h^{(2)\alpha\beta}h^{(2)}_{\alpha\beta} + \frac{1}{16}\partial^\lambda\Phi\partial_\lambda\Phi h^{(2)\alpha}_{\alpha}h^{(2)\beta}_{\beta}\Big\},\end{aligned} \tag{3.15}$$

with expected Lie derivative as a solution for first order variation of spin two fluctuation:

$$\delta^{(\varepsilon)}_1 h^{(2)}_{\mu\nu} = \varepsilon^{(1)\lambda}\partial_\lambda h^{(2)}_{\mu\nu} + \partial_\mu\varepsilon^{(1)\lambda}h^{(2)}_{\nu\lambda} + \partial_\nu\varepsilon^{(1)\lambda}h^{(2)}_{\mu\lambda} = \mathfrak{L}_{\varepsilon^{(1)\lambda}}h^{(2)}_{\mu\nu}, \tag{3.16}$$

with standard algebra

$$[\delta^{(\eta)}_1, \delta^{(\varepsilon)}_1]h^{(2)}_{\mu\nu} = \delta^{([\varepsilon,\eta])}_1 h^{(2)}_{\mu\nu}, \tag{3.17}$$

where composite parameter is the usual Lie commutator of vectors:

$$[\eta, \varepsilon] = [\eta^{(1)}, \varepsilon^{(1)}]^{\lambda} = \eta^{(1)\nu}\partial_{\nu}\varepsilon^{(1)\lambda} - \varepsilon^{(1)\nu}\partial_{\nu}\eta^{(1)\lambda}. \tag{3.18}$$

Note that one can work with restricted external field also. As an example we can choose at once traceless $h^{(2)\mu\nu}$ supplemented with the corresponding constraint on gauge parameter $\partial_{\mu}\varepsilon^{(1)\mu} = 0$. This means that in expression (3.11) survive only first line and first term of the last line. Then we see from (3.12) and (3.14) that $\bar{\delta}_1 h^{(2)}_{\mu\nu}$ is not traceless but can be integrated. So we arrive to the first and third terms of interaction (3.15) describing interaction of $h^{(2)\mu\lambda}h^{(2)\nu}_{\lambda}$ with traceless part of current $J^{(2)}_{\mu\nu} = \partial_{\mu}\Phi\partial_{\nu}\Phi$.

3.3 SPIN FOUR CASE

SETUP

Our main task is to construct similar quartic interaction for spin 4 using prescriptions developed in the previous simple spin 2 case. In our previous articles [60], [26] we prove that in both AdS and flat backgrounds after corresponding field redefinition interaction of even spin s gauge field with spin s current constructed from scalar and derivatives could be written in the form supplemented by the whole tower of invariant actions for couplings of the same scalar with all gauge fields of smaller even spin. So the starting lagrangian for our task we take from [26] rewriting all terms in the flat background:

$$S^{\Phi\Phi h^{(4)}}(\Phi, h^{(2)}, h^{(4)}) \quad = \quad S_0(\Phi) + S_1(\Phi, h^{(2)}) + S_1(\Phi, h^{(4)}), \tag{3.1}$$

where $S_0(\Phi)$, $S_1(\Phi, h^{(2)})$ are defined in (3.1)-(3.3) and

$$S_1(\Phi, h^{(4)}) = \frac{1}{4}\int d^d x h^{(4)\mu\nu\alpha\beta}[\partial_{\mu}\partial_{\nu}\Phi\partial_{\alpha}\partial_{\beta}\Phi - \eta_{\mu\nu}\partial_{\alpha}\partial^{\gamma}\Phi\partial_{\beta}\partial_{\gamma}\Phi]. \tag{3.2}$$

From now on to avoid cumbersome notation and overlapping with symmetrization brackets we reserve notation h and ε for gauge field for spin four $h^{(4)}$ and corresponding gauge parameter $\varepsilon^{(3)}$ except the cases when we do not explicitly write out indices. In the case of other spin (rank) fields and parameters, we use these letters with an exact indication of rank.

The action (3.1) is invariant with respect to the gauge transformations of the spin four field with an additional spin two field gauge transformation inspired by the second divergence of the spin four gauge parameter[1]

$$\delta_1\Phi(x) = \varepsilon^{\mu\nu\lambda}(x)\partial_\mu\partial_\nu\partial_\lambda\Phi(x), \tag{3.3}$$

$$\delta_0 h^{\mu\nu\lambda\rho} = \partial^{(\mu}\varepsilon^{\nu\lambda\rho)} = \partial^\mu\varepsilon^{\nu\lambda\rho} + \partial^\nu\varepsilon^{\mu\lambda\rho} + \partial^\lambda\varepsilon^{\mu\nu\rho} + \partial^\rho\varepsilon^{\mu\nu\lambda}, \tag{3.4}$$

$$\delta_0 h^{\mu\nu}_{(2)} = \partial^{(\mu}\epsilon^{\nu)}, \tag{3.5}$$

$$\epsilon^\nu = \partial_\alpha\partial_\beta\varepsilon^{\nu\alpha\beta}. \tag{3.6}$$

For further simplification in calculation of quartic terms in spin four case we will use physical traceless and transverse gauge for our external spin four field:

$$\partial_\mu h^{\mu\nu\lambda\rho} = 0, \tag{3.7}$$

$$h^{\mu\lambda\rho}_{\mu} = 0, \tag{3.8}$$

which leads to the corresponding restrictions on already traceless spin four gauge parameter:

$$\partial_\alpha\varepsilon^{\alpha\beta\gamma} = 0, \tag{3.9}$$

$$\partial_\mu\partial^\mu\varepsilon^{\alpha\beta\gamma} = \Box\varepsilon^{\alpha\beta\gamma} = 0. \tag{3.10}$$

Note that because in our gauge the gauge parameter is transverse, we should get decoupling of spin two mode from spin four due to degeneration of the additional gauge transformation (3.5). Another convention is that from now on we will admit integration everywhere where it is necessary. So we work with a Lagrangian as with action and therefore we neglect all d dimensional space-time total derivatives when making a partial integration.

[1]Note that the spin two part of our action continues to be invariant in respect of usual linearized reparametrization

VARIATION OF CUBIC TERM

So we arrive to the following simplified task: Starting from a single cubic term due to (3.7)-(3.10)[2]

$$L_1 \sim h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi, \tag{3.11}$$

and using known variation:

$$\begin{aligned} \delta_1\Phi &= \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma\Phi, & (3.12)\\ \delta_0 h^{\mu\nu\lambda\rho} &= \partial^{(\mu}\varepsilon^{\nu\lambda\rho)}, & (3.13) \end{aligned}$$

we try to solve functional equation:

$$\delta_1 L_1(\Phi, h^{(4)}) + \delta_0 L_2(\Phi, h^{(4)}) = 0, \tag{3.14}$$

and construct unknown quartic interaction and first order gauge variation of spin four field $\delta_1 h^{\mu\nu\lambda\rho}$. Doing that and taking into account that according to (3.7) and (3.9) α, β, γ derivatives commute with ε and μ, ν, λ, ρ derivatives commute with h and after long manipulations and multiple partial integrations we arrive to the following important variation:

$$\begin{aligned} \delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) &= \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho}J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi \\ &+ \frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\ &+ \frac{1}{5}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma} \\ &+ \frac{2}{15}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho} \\ &+ \frac{1}{5}[\partial_\mu\partial_\nu\varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho} - \partial_\mu\partial_\nu\partial_\gamma\varepsilon^{\alpha\beta(\lambda}\partial_\alpha\partial_\beta h^{\rho)\mu\nu\gamma}]J^{(2)}_{\lambda\rho} \\ &+ \frac{1}{5}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(2)}_{\beta\gamma}. \end{aligned} \tag{3.15}$$

[2]As usual in our articles we widely use Lagrangian instead of Action performing Noether procedure admitting possibility for partial integration

Here $J^{(6)}, J^{(4)}, J^{(2)}$ are symmetrized currents:

$$J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} = \partial_{(\nu}\partial_\lambda\partial_\rho\Phi\partial_\alpha\partial_\beta\partial_{\gamma)}\Phi = \partial_\nu\partial_{(\lambda}\partial_\rho\Phi\partial_\alpha\partial_\beta\partial_{\gamma)}\Phi, \quad (3.16)$$

$$J^{(4)}_{\mu\nu\lambda\rho} = \partial_{(\mu}\partial_\nu\Phi\partial_\lambda\partial_{\rho)}\Phi = \partial_\mu\partial_{(\nu}\Phi\partial_\lambda\partial_{\rho)}\Phi, \quad (3.17)$$

$$J^{(2)}_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi. \quad (3.18)$$

From (3.15) we see several differences from spin two case:

- From second line of (3.15) follows that *we cannot integrate Noether's equation without introduction of the cubic interaction with a gauge field of spin 6 coupled to the spin 6 current:*

$$h^{\nu\lambda\rho\alpha\beta\gamma}_{(6)} J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}. \quad (3.19)$$

- From third and fourth lines we see that $J^{(4)}$ terms arose with different weight $\frac{1}{5}$ and $\frac{2}{15}$. But we will see below that they should come with same weight to complete integration for interaction terms.

- In last two lines we have three unwanted $J^{(2)}$ terms. We should discover way to get rid of them.

To remove these three obstructions we note that there are several connections between our parts in (3.15) leading to a redefinition of the initial cubic interactions. In another words we can modify our initial interaction with higher spin currents adding gradients of lower spin currents with some coefficients:

$$J^{(6)}_{\alpha\beta\mu\nu\lambda\rho} \Rightarrow J^{(6)}_{\alpha\beta\mu\nu\lambda\rho} + A\partial_{(\alpha}\partial_\beta J^{(4)}_{\mu\nu\lambda\rho)} + B\partial_{(\alpha}\partial_\beta\partial_\mu\partial_\nu J^{(2)}_{\lambda\rho)}, \quad (3.20)$$

$$J^{(4)}_{\mu\nu\lambda\rho} \Rightarrow J^{(4)}_{\mu\nu\lambda\rho} + C\partial_{(\mu}\partial_\nu J^{(2)}_{\lambda\rho)}. \quad (3.21)$$

And it works! Hiding all details of derivations in Appendix A we present final variation we obtained by tuning procedure (3.20) instead of (3.15)

$$
\begin{aligned}
\delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) &= \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho}J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi \\
&+ \frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\
&+ \frac{1}{6}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma} \\
&+ \frac{1}{6}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda,\rho}
\end{aligned}
\tag{3.22}
$$

where modified $\tilde{J}^{(6)}$ is

$$
\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} = J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} + \frac{1}{9}\partial_{(\alpha}\partial_\beta J^{(4)}_{\gamma\nu\lambda\rho)} + \frac{1}{3}\partial_{(\nu}\partial_\lambda\partial_\rho\partial_\alpha J^{(2)}_{\beta\gamma)}. \tag{3.23}
$$

Supplemented by traceless Stueckelberg like transformation (3.61) of the spin two gauge field from linear coupling with $J^{(2)}$ current:

$$
\delta_1 h^{\beta\gamma}_{(2)} \sim \partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}. \tag{3.24}
$$

INTEGRATION AND INTERACTION

Now we can start to integrate the last three lines of expression (3.22). Doing that in the corresponding subsection of Appendix A we finally obtain quartic interactions:

$$
\begin{aligned}
S_2(\Phi, h^{(4)}) = \int d^d x \Big\{ &\frac{1}{10} h^{\alpha\beta\gamma}_\mu h^{\nu\lambda\rho\mu}\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\
-\frac{2}{3} h^{\alpha\beta\gamma}_\mu \partial_\alpha\partial_\beta h^{\mu\nu\lambda\rho}J^{(4)}_{\nu\lambda\rho\gamma} &+ \frac{1}{2}\partial_\nu h^{\alpha\beta\gamma}_\mu \partial_\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\lambda\rho\beta\gamma} - \frac{1}{4}\partial^\alpha h^{\beta\gamma}_{\mu\nu}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\lambda\rho\beta\gamma} \\
-\partial^\beta h^{\alpha\gamma}_{\mu\nu}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\lambda\rho\beta\gamma} &+ \frac{1}{3}\partial^\beta h^{\gamma}_{\mu\nu\lambda}\partial^\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\rho\alpha\beta\gamma} - \frac{1}{4} h^{\beta\gamma}_{\mu\nu} h^{\lambda\rho\mu\nu}\Box J^{(4)}_{\lambda\rho\beta\gamma}\Big\},
\end{aligned}
\tag{3.25}
$$

and linear on spin four gauge field transformations fixed by Noether's procedure:

$$\delta_1 h^{\mu\nu\lambda\alpha\beta\gamma}_{(6)} = \varepsilon^{\rho(\alpha\beta}\partial_\rho h^{\gamma\mu\nu\lambda)} + \partial^{(\alpha}\varepsilon^{\beta\gamma}_{\rho} h^{\mu\nu\lambda)\rho}, \tag{3.26}$$

$$\delta_1 h^{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho} + \partial^{(\mu}\varepsilon^{|\alpha\beta|}_{\gamma}\partial_\alpha\partial_\beta h^{\nu\lambda\rho)\gamma} + \partial^{(\mu}\partial^{\nu}\varepsilon^{|\alpha|}_{\beta\gamma}\partial_\alpha h^{\lambda\rho)\beta\gamma}$$
$$+\partial^{(\mu}\partial^{\nu}\partial^{\lambda}\varepsilon_{\alpha\beta\gamma}h^{\rho)\alpha\beta\gamma}, \tag{3.27}$$

$$\delta_1 h^{\beta\gamma}_{(2)} = \partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}. \tag{3.28}$$

So we prove that Noether's procedure in this particular case can be done for the construction of the local quartic interaction of the scalar and higher spin fields restricted by transverse and traceless gauge conditions (3.7) and (3.8).

3.4 COMMUTATOR OF δ_1 TRANSFORMATIONS FOR SPIN FOUR

In this section we investigate algebra of linear in gauge field transformation (3.27) obtained from Noether's procedure in previous section:

$$\delta^{(\varepsilon)}_1 h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h_{\mu\nu\lambda\rho} + \partial_{(\mu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_\beta h_{\gamma|\nu\lambda\rho)} + \partial_{(\mu}\partial_\nu\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}h_{\beta\gamma|\lambda\rho)}$$
$$+\partial_{(\mu}\partial_\nu\partial_\lambda\varepsilon^{\alpha\beta\gamma}h_{\rho)\alpha\beta\gamma}. \tag{3.1}$$

The structure of this expression is similar to linear transformation obtained in [64] where nonlinear curvature for general higher spin and in particular for spin three case is considered. First of all for understanding of corresponding gauge algebra we can derive commutator of this linear δ_1 transformation (3.1) with zero order gauge transformation δ_0 (3.13) from which our investigation of Noether equation (3.14) began. Straightforward calculations leads to the following expression:

$$\big[\delta_0^{(\omega)}\delta_1^{(\varepsilon)} - \delta_0^{(\varepsilon)}\delta_1^{(\omega)}\big]h_{\mu\nu\lambda\rho} = \partial_{(\mu}\big[\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_\beta\partial_{\gamma|}\omega_{\nu\lambda\rho)} + t_{\nu\lambda\rho)}(\varepsilon,\omega) - (\varepsilon\leftrightarrow\omega)\big], \tag{3.2}$$

$$t_{\nu\lambda\rho}(\varepsilon,\omega) = \partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_{\beta|}\omega_{\lambda\rho)\gamma} + \partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha|}\omega_{\rho)\beta\gamma} + \frac{1}{3}\partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma}\partial_{\rho)}\omega_{\alpha\beta\gamma}. \tag{3.3}$$

Here we should make two important comments:

First, we see that in (3.1) the form of the last three terms is ambiguously defined due to the freedom in the definition of the δ_1. This transformation can be modified by adding zero-order (full gradient) transformation with field-dependent parameter. Ruffly speaking we can add δ_0 transformation with linear on gauge field parameter to (3.1) modifying the last three terms and getting corresponding modification for tensor $t_{\nu\lambda\rho}(\varepsilon,\omega)$ in definition of the commutator (3.2).

Second following the ideas of [64] and extracting the same type δ_0 terms described above we can rewrite (3.1) in the following form:

$$\delta_1^{(\varepsilon)}h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon,h), \tag{3.4}$$

$$\begin{aligned}\Lambda_{\nu\lambda\rho}(\varepsilon,h) &= \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta h_{\gamma\nu\lambda\rho} + \frac{1}{2}\left[\partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}h_{\beta\gamma|\lambda\rho)} - \varepsilon^{\alpha\beta\gamma}\partial_{(\nu}\partial_{|\alpha}h_{\beta\gamma|\lambda\rho)}\right]\\ &+ \frac{1}{3}\left[\partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma}h_{\rho)\alpha\beta\gamma} + \varepsilon^{\alpha\beta\gamma}\partial_{(\nu}\partial_\lambda h_{\rho)\alpha\beta\gamma} - \frac{1}{2}\partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_\lambda h_{\rho)\alpha\beta\gamma}\right],\end{aligned} \tag{3.5}$$

where[3]

$$\begin{aligned}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) &= \partial_\alpha\partial_\beta\partial_\gamma h_{\mu\nu\lambda\rho} - \frac{1}{3}\partial_{<\alpha}\partial_\beta\partial_{(\mu}h_{\nu\lambda\rho)\gamma>} + \frac{1}{3}\partial_{<\alpha}\partial_{(\mu}\partial_\nu h_{\lambda\rho)\beta\gamma>}\\ &- \partial_{(\mu}\partial_\nu\partial_\lambda h_{\rho)\alpha\beta\gamma},\end{aligned} \tag{3.6}$$

is the third for spin four gauge field (last before Curvature) Christoffel Symbol in deWit-Freedman hierarchy of connections defined in [65][4]. The key point of the splitting (3.4) is

[3]We use $(\dots)$ and $<\cdots>$ brackets for symmetrization of the different set of indices, reserving $[\dots]$ and $\Big[\dots\Big]$ for antisymmetrization. All other necessary for our case formulas connected with this hierarchy can be found in Appendix B.

[4]It is worth noting here that the transformation of the additional spin 6 gauge field through the spin four gauge field (3.26) can also be written in a similar to (3.4) form:

$$\delta_1 h^{(6)}_{\mu\nu\lambda\alpha\beta\gamma} = \varepsilon^{\rho}_{(\alpha\beta}\Gamma^{(1)}_{|\rho|;\gamma\mu\nu\lambda)}(h) + \partial_{(\alpha}[\varepsilon^{\rho}_{\beta\gamma}h_{\mu\nu\lambda)\rho}],$$

where $\Gamma^{(1)}_{\rho;\gamma\mu\nu\lambda)}(h)$ is first generalized Christoffel symbol for spin four field defined in (3.79).

the simple form of zero order on field gauge transformation of connection (3.6) (see formulas (3.79)-(3.86) for details):

$$\delta_0^{(\varepsilon)}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) = -4\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon_{\alpha\beta\gamma}, \tag{3.7}$$

and possibility in the future calculations *to identify in r.h.s of commutator different symmetries by existence of the terms in the form of Christoffel symbols or Generalized Curvatures (in some case with symmetrized derivatives) contracted with composite parameters like in (3.4) but with different rank and symmetry structure of the indices for composite parameters. So from now on we call such a type of terms as a "regular"* . In this way we see that expressions (3.1)-(3.7) is really looks like higher spin generalization of the gauge transformation (3.16) (Lie derivative) and usual Christoffel symbol for linearized gravity[5]

$$\delta_1^{(\varepsilon)}h_{\mu\nu} = \mathfrak{L}_{\varepsilon^\lambda}h_{\mu\nu} = \varepsilon^\alpha\Gamma^{(1)}_{\alpha;\mu\nu} + \partial_{(\mu}\left(\varepsilon^\alpha h_{\nu)\alpha}\right), \tag{3.8}$$

$$\Gamma^{(1)}_{\alpha;\mu\nu} = \partial_\alpha h_{\mu\nu} - \partial_{(\mu}h_{\nu)\alpha}, \tag{3.9}$$

$$\delta_0^{(\varepsilon)}\Gamma^{(1)}_{\alpha;\mu\nu}(h) = -2\partial_\mu\partial_\nu\varepsilon_\alpha. \tag{3.10}$$

Using representation (3.4) and transformation rule (3.7) we can derive the following expression for commutator:

$$[\delta_1^{(\omega)},\delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)}h) - 4\varepsilon^{\alpha\beta\gamma}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\Lambda_{\alpha\beta\gamma}(\omega,h)$$
$$+ \partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon,\delta_1^{(\omega)}h) - (\varepsilon\leftrightarrow\omega). \tag{3.11}$$

Then taking int account that all symmetrized full gradients in r.h.s we can drop as a trivial δ_0 contribution from composite symmetric third rank gauge parameter linear in gauge field, we can first of all drop second line in (3.11). Then we can put four μ,ν,λ,ρ, derivatives in second term of first line from $\Lambda_{\alpha\beta\gamma}$ to parameter $\varepsilon^{\alpha\beta\gamma}$ and integrate using formula (3.7) and came to the following expression

$$[\delta_1^{(\omega)},\delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho} \sim \varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)}h) + \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)\delta_0^{(\omega)}\Lambda^{\alpha\beta\gamma}(\varepsilon,h) - (\varepsilon\leftrightarrow\omega), \tag{3.12}$$

[5]Note that most common definition of Christoffel symbol $\Gamma^\beta_{\mu\nu}(g) = \frac{1}{2}g^{\beta\alpha}(\partial_{(\mu}g_{\nu)\alpha} - \partial_\alpha g_{\mu\nu})$ for general metric $g_{\mu\nu}$ relates with our definition after linearization in the flat background in the following way $\Gamma^\beta_{\mu\nu}(\eta_{\mu\nu}+h_{\mu\nu}) = -\frac{1}{2}\eta^{\beta\alpha}\Gamma^{(1)}_{\alpha;\mu\nu}(h)$.

where $\sim$ means an equality up to any δ_0 variations with composed field dependent parameter described above or delta zero variation with usual parameter ε or ω from any second order on gauge field tensor. At this point it is worth to note that considering perturbative on linearized gauge field deformation of the initial gauge transformation regulated by Noether's procedure

$$\delta^{(\epsilon)} h_{\mu\nu\lambda\rho} = (\delta_0^{(\epsilon)} + \delta_1^{(\epsilon)} + \delta_2^{(\epsilon)} + \dots) h_{\mu\nu\lambda\rho}, \tag{3.13}$$

for commutator on the linear level on gauge field we obtain:

$$\Big\{[\delta^{(\omega)}, \delta^{(\epsilon)}] h_{\mu\nu\lambda\rho}\Big\}_1 = ([\delta_1^{(\omega)}, \delta_1^{(\epsilon)}] + \delta_0^{(\omega)}\delta_2^{(\varepsilon)} - \delta_0^{(\varepsilon)}\delta_2^{(\omega)}) h_{\mu\nu\lambda\rho}. \tag{3.14}$$

So we see that we can factorize in right hand side of our commutator of the first order gauge transformation two type of trivial terms:

- Symmetrized full derivatives from composed gauge parameter linear in gauge fields $\partial_{(\mu}\tilde{\Lambda}_{\nu\lambda\rho)}(\varepsilon, \omega, h) - (\varepsilon \leftrightarrow \omega)$.
- The terms which can be classified as a second part of r.h.s of (3.14): $\delta_0^{(\omega)}\delta_2^{(\varepsilon)} h_{\mu\nu\lambda\rho} - (\varepsilon \leftrightarrow \omega)$, and we can throw them out also to understand algebra of two δ_1 transformations.

Now following this simple methodology we can present final result for commutator moving long and tedious calculation to the next subsection:

$$\begin{aligned}
&[\delta_1^{(\omega)}, \delta_1^{(\varepsilon)}] h_{\mu\nu\lambda\rho} \sim \left[\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma} + T^{\alpha\beta\gamma}(\partial, \varepsilon, \omega)\right] \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) \\
&+ 3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma} R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \frac{9}{20}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\sigma} R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h) \\
&+ [Rem]_{\mu\nu\lambda\rho}(\varepsilon, \omega, h) - (\varepsilon \leftrightarrow \omega),
\end{aligned} \tag{3.15}$$

where:

$$\begin{aligned}
T^{\alpha\beta\gamma}(\partial, \varepsilon, \omega) &= \frac{1}{4}\partial^{(\alpha}\partial^{\beta}\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)} h^{\gamma)}_{\ \delta\sigma\eta} - \frac{5}{48}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial^{\beta}\delta_0^{(\omega)} h^{\gamma)}_{\delta\sigma\eta} + \frac{7}{16}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial_\delta\delta_0^{(\omega)} h_{\sigma\eta}^{\ \ \beta\gamma)} \\
&- \frac{1}{16}\partial^{\delta}\varepsilon^{\sigma\eta(\alpha}\partial^{\beta}\delta_0^{(\omega)} h^{\gamma)}_{\delta\sigma\eta} + \frac{1}{16}\partial^{\delta}\varepsilon^{\sigma\eta(\alpha}\partial_\delta\delta_0^{(\omega)} h^{\beta\gamma)}_{\sigma\eta},
\end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
&[Rem]_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) = \\
&\frac{9}{20}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) + \frac{3}{2}\partial_{(\mu}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) \\
&-\frac{9}{40}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\alpha\gamma}(H^{(3)}_{[\beta\sigma]}) \\
&+\frac{3}{8}\varepsilon^{\eta}_{\sigma\delta}\partial^{[\sigma}\partial^{\left[\delta\right.}\omega^{\left.\alpha\right]\,\beta]\gamma}\partial_{(\mu}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) + \frac{1}{2}\partial_{(\mu}\varepsilon^{\eta}_{\delta\sigma}\partial^{[\sigma}\partial^{\left[\delta\right.}\omega^{\left.\alpha\right]\,\beta]\gamma}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) \\
&+\frac{3}{8}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) + \frac{1}{2}\partial_{(\mu}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) \\
&+\frac{3}{4}\partial_{(\mu}\partial_{\nu}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) \qquad (3.17)
\end{aligned}
$$

is remaining part of commutator contained transformation described by composed gauge parameter with mixed symmetry of indices in the form of one or two antisymmetrized pairs.

To be more precise when classifying terms on the right side of (3.15) let us consider each line separately:

1. The first line describes spin four gauge transformation with composite *symmetric rank 3 tensor parameter* in the form

$$[\omega,\varepsilon]^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h), \qquad (3.18)$$

where

$$[\omega,\varepsilon]^{\alpha\beta\gamma} = \varepsilon^{\delta\sigma\eta}\partial_{\delta}\partial_{\sigma}\partial_{\eta}\omega^{\alpha\beta\gamma} + T^{\alpha\beta\gamma}(\partial,\varepsilon,\omega) - (\varepsilon \leftrightarrow \omega). \qquad (3.19)$$

2. The second line also corresponds to the transformation of the spin four gauge field in respect to gauge transformation with symmetric tensor parameter. But in this case we have *symmetric tensor parameters of rank 4 and 5*, which means that it is transformation coming from gauge field with spin 5 and 6 and our spin four gauge field participates in these transformations through the spin four gauge invariant (in zero order on field transformations) curvature. In other words we have here regular terms in the form

$$\Omega^{\eta\alpha\beta\gamma\delta}_{(4)}(\varepsilon,\omega)R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h), \qquad (3.20)$$

$$\Omega^{\sigma\eta\alpha\beta\gamma\delta}_{(5)}(\varepsilon,\omega)\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h), \qquad (3.21)$$

where

$$\Omega^{\eta\alpha\beta\gamma\delta}_{(4)}(\varepsilon,\omega) = \frac{3}{4}\varepsilon^{\delta\sigma(\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma)} - (\varepsilon \leftrightarrow \omega), \tag{3.22}$$

$$\Omega^{\sigma\eta\alpha\beta\gamma\delta}_{(5)}(\varepsilon,\omega) = \frac{9}{200}\varepsilon^{\delta(\sigma\eta}\partial_\delta\omega^{\alpha\beta\gamma)} - \frac{3}{200}\varepsilon^{(\sigma\eta}_{\delta}\partial^\alpha\omega^{\beta\gamma)\delta} - (\varepsilon \leftrightarrow \omega). \tag{3.23}$$

3. Now we analyze the third line of (3.15) or eight terms in expression (3.17). First of all we see that in this remaining part of commutator our spin four field expressed through the reduced curvatures and Christoffel symbols defined in Appendix B (see (3.97)-(3.105) and (3.112), (3.113)). All such a objects possess one (first two lines of (3.17)) ore two (remaining two lines of (3.17)) pair of antisymmetrized indices contracted with composed gauge parameter. Therefore they could describe some mixed symmetry field gauge transformation acting on spin four symmetric gauge field. For example first term in (3.17) we can rewrite in the form:

$$\Omega^{[\alpha\sigma],\eta\beta\gamma}_{[2],(3)}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) \tag{3.24}$$

where

$$\Omega^{[\alpha\sigma],\eta\beta\gamma}_{[2],(3)} = \frac{3}{40}(\varepsilon^{\delta(\eta[\sigma}\partial_\delta\omega^{\alpha]\beta\gamma)} - \varepsilon^{(\eta[\sigma}_{\delta}\partial^{\alpha]}\omega^{\beta\gamma)\delta}) \tag{3.25}$$

and in the same way the sixth term with two pair of antisymmetrized indices we can express as

$$\Omega^{[\eta\alpha],[\sigma\beta],\gamma}_{[2],[2],(1)}\partial_{(\mu}\partial_\nu\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) \tag{3.26}$$

where composit parameter is

$$\Omega^{[\eta\alpha],[\sigma\beta],\gamma}_{[2],[2],(1)} = \frac{3}{32}(\varepsilon^{\delta\left[\sigma[\eta}\partial_\delta\omega^{\alpha]\beta\right]\gamma} - \varepsilon^{\left[\sigma[\eta}_{\delta}\partial^{\alpha]}\omega^{\beta\right]\delta\gamma}) \tag{3.27}$$

This type of terms (first, third, fourth and sixth in (3.17)) with mixed symmetry composed parameters we can still call "regular". But four remaining terms of (3.17) (second, fifth, seventh and eighth) we cannot transform to regular form because they all have non contracted derivatives from one (non composed) gauge parameter ($\partial_\mu\varepsilon$ or $\partial_\mu\partial_\nu\varepsilon$) and we call these terms irregular because do not have at the moment interpretation of them in means of additional symmetries or equation of motion of theory under construction. But at least we can clime that all irregular terms are in the mixed symmetry parameter sector.

So we see that our *commutator of spin four linear on gauge field transformations produce regular terms coming from gauge transformation of symmetric tensors with spin* $s < 6$ and remaining irregular transformation with mixed symmetry gauge field parameters.

DERIVATION OF (3.15)-(3.17)

So now we can analyze remaining two terms in (3.12). Taking δ_0 variation of the (3.5) we see easily that

$$\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)\delta_0^{(\omega)}\Lambda^{\alpha\beta\gamma}(\varepsilon,h) = \Big[\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma} + \varepsilon^{\delta\sigma\eta}\partial^\alpha\partial^\beta\partial^\gamma\omega_{\delta\sigma\eta} + \frac{3}{2}\partial^\alpha\varepsilon^{\delta\sigma\eta}\partial_\delta\delta_0^{(\omega)}h_{\sigma\eta}^{\ \ \beta\gamma}$$
$$-\partial^\alpha\varepsilon^{\delta\sigma\eta}\partial^\beta\delta_0^{(\omega)}h^{\gamma}_{\delta\sigma\eta} + \partial^\alpha\partial^\beta\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)}h^{\gamma}_{\ \delta\sigma\eta}\Big]\,\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) - (\varepsilon\leftrightarrow\omega). \tag{3.28}$$

The first term in r.h.s of (3.12) can be split into the sum of the following terms:

$$\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)}h) \sim \varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + I^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) + I^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h)$$
$$4\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\omega^{\delta\sigma\eta}(\partial_\delta\partial_\sigma h_{\eta\alpha\beta\gamma} + \partial_\alpha\partial_\beta h_{\gamma\delta\sigma\eta} - \frac{3}{2}\partial_\eta\partial_\alpha h_{\delta\sigma\beta\gamma}) - (\varepsilon\leftrightarrow\omega), \tag{3.29}$$

where

$$I^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) \quad \sim 3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}\partial_\eta\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + 4\partial_\mu\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\eta\nu\lambda\rho}(h), \tag{3.30}$$

$$I^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) \quad \sim 3\varepsilon^{\delta\sigma\eta}\partial_\delta\omega^{\alpha\beta\gamma}\partial_\sigma\partial_\eta\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + 8\partial_\mu\varepsilon^{\delta\sigma\eta}\partial_\delta\omega^{\alpha\beta\gamma}\partial_\sigma\Gamma^{(3)}_{\alpha\beta\gamma;\eta\nu\lambda\rho}(h)$$
$$+6\partial_\mu\partial_\nu\varepsilon^{\delta\sigma\eta}\partial_\delta\omega^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\sigma\eta\lambda\rho}(h). \tag{3.31}$$

Then using transformation rule (3.85) to integrate $\Gamma^{(3)\alpha\beta\gamma}_{\qquad;\mu\nu\lambda\rho}(h)$ in r.h.s of (3.29) we observe cancelation of the second line of (3.29) with first line of (3.28). So we arrive to the following preliminary result:

$$[\delta_1^{(\omega)},\delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho} \sim \left[\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma}+T_1^{\alpha\beta\gamma}(\partial,\varepsilon,\omega)\right]\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$$
$$+I^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h)+I^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h)-(\varepsilon\leftrightarrow\omega), \quad (3.32)$$

where

$$T_1^{\alpha\beta\gamma}=\frac{1}{2}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial_\delta\delta_0^{(\omega)}h_{\sigma\eta}^{\ \ \beta\gamma)}-\frac{1}{6}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial^\beta\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta}+\frac{1}{3}\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)}h^{\gamma)}_{\ \ \delta\sigma\eta}$$
$$=\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega_\eta^{\beta\gamma)}+\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\partial_\delta\omega^{\gamma)}_{\sigma\eta}+\frac{1}{3}\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\partial^{\gamma)}\omega_{\delta\sigma\eta}-\frac{1}{3}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial^\beta\partial^{\gamma)}\omega_{\delta\sigma\eta}$$
$$(3.33)$$

is some additional input to the symmetric rang three generalized tensorial bracket of two parameters ε and ω with three derivatives:

$$[\omega,\varepsilon]_1^{\alpha\beta\gamma}=\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma}+T_1^{\alpha\beta\gamma}(\partial,\varepsilon,\omega). \quad (3.34)$$

But it is not the full story about commutator because we still have unclassified transformations in two objects (3.30) and (3.31) of the second line of (3.32). Our goal is to extract all transformation terms including more than three rang symmetric tensors as a combined parameter, correcting in parallel the third rank bracket (3.34). Indeed using integration relations:

$$\partial_\delta\omega_{\alpha\beta\gamma}=\delta_0^{(\omega)}h_{\delta\alpha\beta\gamma}-\partial_{(\alpha}\omega_{\beta\gamma)\delta}, \quad (3.35)$$
$$\partial_\delta\omega_{\alpha\beta\gamma}\dots\Gamma^{(3)\alpha\beta\gamma}_{\qquad\ ;\mu\nu\lambda\rho}(h)=\big(\frac{3}{4}\partial_{[\delta}\omega_{\alpha]\beta\gamma}+\frac{1}{4}\delta_0^{(\omega)}h_{\delta\alpha\beta\gamma}\big)\dots\Gamma^{(3)\alpha\beta\gamma}_{\qquad\ ;\mu\nu\lambda\rho}(h), \quad (3.36)$$

and after long calculation we can transform the sum of $I^{(1)}$ and $I^{(2)}$ in the following way:

$$I^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h)+I^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,h)\sim\tilde{I}^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)})+\tilde{I}^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)})$$
$$-\frac{1}{4}\partial^\alpha\partial^\beta\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)}h^{\gamma}_{\ \ \delta\sigma\eta}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)+3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$$
$$+\frac{9}{20}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Big[\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h)+\partial_{[\sigma}R^{(4)}_{\alpha]\beta\gamma\eta);\mu\nu\lambda\rho}(h)+2\partial_{[\sigma}R^{(4)}_{\beta]\gamma\alpha\eta);\mu\nu\lambda\rho}(h)\Big], \quad (3.37)$$

where $R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$ is gauge invariant spin four linearized curvature (3.82) with two set of four symmetrized indices[6] and:

$$\tilde{I}^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) = \frac{3}{4}\Big[\frac{3}{4}\varepsilon^{\eta\sigma}_{\delta}\partial_\sigma\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) + \partial_{(\mu}\varepsilon^{\eta\sigma}_{\delta}\partial_\sigma\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]})\Big], \tag{3.38}$$

$$\begin{aligned}\tilde{I}^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) = &\frac{3}{4}\Big[\frac{3}{4}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_\sigma\partial_{(\mu}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) + 2\partial_{(\mu}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_\sigma\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]})\\ &+ \frac{3}{2}\partial_{(\mu}\partial_\nu\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Gamma^{(2)}_{\beta\gamma;\sigma\lambda\rho)}(H^{(3)}_{[\eta\alpha]})\Big],\end{aligned} \tag{3.39}$$

where $\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho}(H^{(3)}_{[\eta\alpha]})$ is defined in (3.97) as second Christoffel Symbol for third rank symmetric tensor $H^{(3)}_{[\eta\alpha];\beta\gamma\rho}$ with additional antisymmetric pair of indices. This tensor is actually curl of our spin four field (3.100) and second Christoffel Symbol is constructed as usual for spin three field:

$$\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho}(H^{(3)}_{[\eta\alpha]}) = \partial_\beta\partial_\gamma H^{(3)}_{[\eta\alpha];\beta\gamma\rho} - \frac{1}{2}\partial_{<\beta}\partial_{(\nu}H^{(3)}_{[\eta\alpha];\gamma\rho)\gamma>} + \partial_{(\nu}\partial_\lambda H^{(3)}_{[\eta\alpha];\rho)\beta\gamma}, \tag{3.40}$$

$$\delta^{\varepsilon}_0\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) = 3\partial_\nu\partial_\lambda\partial_\rho E^{(2)}_{[\eta\alpha];\beta\gamma}(\varepsilon), \tag{3.41}$$

$$E^{(2)}_{[\eta\alpha];\beta\gamma}(\varepsilon) = \partial_{[\eta}\varepsilon_{\alpha]\beta\gamma}. \tag{3.42}$$

And we see that on this stage our commutator can be written in the following form:

$$\begin{aligned}[\delta^{(\omega)}_1,\delta^{(\varepsilon)}_1]h_{\mu\nu\lambda\rho} \sim\ & [\omega,\varepsilon]^{\alpha\beta\gamma}_2\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta^{(\omega)}_1 h) + 3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h)\\ &+ \frac{9}{20}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Big[\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h) + \frac{1}{5}\partial_{(\mu}R^{(4)}_{\nu\lambda\rho)[\sigma;\alpha]\beta\gamma\eta}(h) + \frac{2}{5}\partial_{(\mu}R^{(4)}_{\nu\lambda\rho)[\sigma;\beta]\gamma\alpha\eta}(h)\Big]\\ &+ \tilde{I}^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) + \tilde{I}^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) - (\varepsilon\leftrightarrow\omega),\end{aligned} \tag{3.43}$$

where in first line we define next deformation of spin four gauge symmetry parameter:

$$[\omega,\varepsilon]^{\alpha\beta\gamma}_2 = [\omega,\varepsilon]^{\alpha\beta\gamma}_1 - \frac{1}{12}\partial^{(\alpha}\partial^{\beta}\varepsilon^{\delta\sigma\eta}\delta^{(\omega)}_0 h^{\gamma)}_{\delta\sigma\eta}, \tag{3.44}$$

[6]Here in last line of (3.37) we separated symmetric and antisymmetric part of derivative of Generalized Curvature

and in second line of (3.43) we used Bianchi identity (3.116). Then we can perform one more step and extract dependence from symmetry parameter with two pair of antisymmetrized indices. To do that we use another cycle of gauge field integration from derivative of gauge parameter:

$$\partial_\sigma \partial_{[\delta}\omega_{\alpha]\beta\gamma} = \delta_0^{(\omega)} \partial_{[\delta} h_{\alpha]\beta\gamma\sigma} - \partial_{(\sigma}\partial_{[\delta}\omega_{\alpha]\beta\gamma)}, \tag{3.45}$$

$$\partial_\sigma \partial_{[\delta}\omega_{\alpha]\beta\gamma} \dots \Gamma^{(2)\beta\gamma}{}_{;\nu\lambda\rho}(H^{(3)}) = \big(\frac{2}{3}\partial_{\left[\sigma\right.}\partial_{[\delta}\omega_{\left.\alpha]\beta\right]\gamma} + \frac{1}{3}\delta_0^{(\omega)}\partial_{[\delta}h_{\alpha]\beta\gamma\sigma}\big) \dots \Gamma^{(2)\beta\gamma}{}_{;\nu\lambda\rho}(H^{(3)}). \tag{3.46}$$

This leads to the following transformation:

$$\begin{aligned}
&\tilde{I}^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) + \tilde{I}^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) \sim \tilde{\tilde{I}}^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(2)}) + \tilde{\tilde{I}}^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) \\
&- \frac{3}{16}\partial^{[\alpha}\varepsilon^{\delta]\sigma\eta}\delta_0^{(\omega)}\partial_{[\delta}h^{\gamma}_{\beta]\sigma\eta}\Gamma^{(3)\beta}_{\ \alpha\ \gamma;\mu\nu\lambda\rho}(h) \\
&- \frac{3}{2}\partial_{(\mu}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}R^{(3)}_{\nu\lambda\rho);\sigma\beta\gamma}(H^3_{[\eta\alpha]}) - \frac{3}{16}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\sigma\beta\gamma}(H^3_{[\eta\alpha]}),
\end{aligned} \tag{3.47}$$

where

$$\begin{aligned}
&\tilde{\tilde{I}}^{(2)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(3)}) = \\
&\quad \frac{1}{2}\Big[\frac{3}{4}\varepsilon^{\eta}_{\delta\sigma}\partial^{\left[\sigma\right.}\partial^{[\delta}\omega^{\left.\alpha]\beta\right]\gamma}\partial_{(\mu}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) + \partial_{(\mu}\varepsilon^{\eta}_{\delta\sigma}\partial^{\left[\sigma\right.}\partial^{[\delta}\omega^{\left.\alpha]\beta\right]\gamma}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]})\Big],
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
&\tilde{\tilde{I}}^{(1)}_{\mu\nu\lambda\rho}(\varepsilon,\omega,H^{(2)}) = \\
&\quad \frac{3}{4}\Big[\frac{1}{2}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) + \frac{2}{3}\partial_{(\mu}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) \\
&\quad + \partial_{(\mu}\partial_{\nu}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]})\Big].
\end{aligned} \tag{3.49}$$

Here in (3.49) $\Gamma^{(1)}_{\gamma;\lambda\rho}(H^{(2)}_{[\eta\alpha][\sigma\beta]})$ is defined in (3.104) as first Christoffel Symbol for second rank symmetric tensor $H^{(2)}_{[\eta\alpha][\sigma\beta];\gamma\rho}(h)$ with additional two antisymmetric pair of indices. This tensor is second curl of our spin four field:

$$H^{(2)}_{[\eta\alpha][\sigma\beta];\gamma\rho}(h) = \partial_{\left[\sigma\right.}\partial_{[\eta}h_{\left.\alpha]\beta\right]\gamma\rho}, \tag{3.50}$$

and Christoffel Symbol is constructed as usual for spin two field ignoring both antisymmetric pairs:

$$\Gamma^{(1)}_{\gamma;\lambda\rho}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) = \partial_\gamma H^{(2)}_{[\eta\alpha][\sigma\beta];\lambda\rho} - \partial_{(\lambda}H^{(2)}_{[\eta\alpha][\sigma\beta];\rho)\gamma}, \tag{3.51}$$

$$\delta_0^{\varepsilon}\Gamma^{(1)}_{\gamma;\lambda\rho}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) = -2\partial_\lambda\partial_\rho E^{(1)}_{[\eta\alpha][\sigma\beta];\gamma}(\varepsilon), \tag{3.52}$$

$$E^{(1)}_{[\eta\alpha][\sigma\beta];\gamma}(\varepsilon) = \partial_{\left[\sigma\right.}\partial_{[\eta}\varepsilon_{\left.\alpha]\beta\right]\gamma}. \tag{3.53}$$

Now we can finish our proof in three steps:

1. From second term in r.h.s. of (3.47) we can extract final deformation of the spin four gauge symmetry parameter

$$
\begin{aligned}
&[\omega, \varepsilon]_3^{\alpha\beta\gamma} = [\omega, \varepsilon]_2^{\alpha\beta\gamma} - \frac{3}{16}\partial^{[\alpha}\varepsilon^{\delta]\sigma\eta}\delta_0^{(\omega)}(\partial_\delta h^{\beta\gamma}_{\sigma\eta} - \partial^\beta h^{\gamma}_{\delta\sigma\eta}) \\
&+ \textit{symmetrization in } (\alpha\beta\gamma)\,, \qquad (3.54)
\end{aligned}
$$

   and this expression after some algebra produces (3.16) and therefore first line of (3.15).

2. Combining first two line of (3.43), last line of (3.47) and using relation (3.112) we obtain all curvature dependent terms in (3.15) and (3.17) .

3. All other terms with mixed symmetry parameters in (3.17) are just sum of (3.48) and (3.49).

So we prove our final result for commutator (3.15)-(3.17).

3.5 DETAILS OF THE NOETHER'S PROCEDURE

TUNING OF CUBIC INTERACTIONS

To prove (3.25)-(3.28) we should first derive special relations. After some partial integration and symmetrization of indices we can prove for our constrained field and parameters (3.7)-(3.10) the following relations

$$
\begin{aligned}
&\frac{1}{15}\left[\partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu} - \varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)}\right]\partial_{(\nu}\partial_\lambda J^{(4)}_{\rho\alpha\beta\gamma)} = \\
&-\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma} \\
&+\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho}, \qquad (3.55)
\end{aligned}
$$

$$\begin{aligned}
& \frac{1}{30}\left[\partial_\mu \varepsilon^{(\alpha\beta\gamma} h^{\nu\lambda\rho)\mu} - \varepsilon^{\mu(\alpha\beta} \partial_\mu h^{\gamma\nu\lambda\rho)}\right] \partial_{(\nu}\partial_\lambda\partial_\rho\partial_\alpha J^{(2)}_{\beta\gamma)} = \\
+ \quad & [\partial_\mu\partial_\nu\partial_\gamma \varepsilon^{\alpha\beta(\lambda}\partial_\alpha\partial_\beta h^{\rho)\mu\nu\gamma} - \partial_\mu\partial_\nu \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}] J^{(2)}_{\lambda\rho} \\
+ \quad & \frac{3}{2}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho \varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho} J^{(2)}_{\beta\gamma}. \qquad (3.56)
\end{aligned}$$

Using relation (3.55) we can bring improvement for discrepancy in numbers a front of $J^{(4)}$ terms in (3.15). The second relation (3.56) is suitable for cancellation of the fifth line of (3.15) connected with $J^{(2)}$ current. Finally we obtain instead of (3.15) the following relation:

$$\begin{aligned}
\delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) &= \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho} J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi \\
&+ \frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu \varepsilon^{(\alpha\beta\gamma} h^{\nu\lambda\rho)\mu}\right] \tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\
&+ \frac{1}{6}\left[\partial_\alpha \varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu \varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right] J^{(4)}_{\lambda\rho\beta\gamma} \\
&+ \frac{1}{6}\left[\partial_\alpha\partial_\beta\partial_\gamma \varepsilon^{(\mu\nu\lambda} h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right] J^{(4)}_{\mu\nu\lambda\rho} \\
&+ \frac{1}{2}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho \varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho} J^{(2)}_{\beta\gamma}\,, \qquad (3.57)
\end{aligned}$$

where modified $\tilde{J}^{(6)}$ is

$$\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} = J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} + \frac{1}{9}\partial_{(\alpha}\partial_\beta J^{(4)}_{\gamma\nu\lambda\rho)} + \frac{1}{3}\partial_{(\nu}\partial_\lambda\partial_\rho\partial_\alpha J^{(2)}_{\beta\gamma)}. \qquad (3.58)$$

The last term in expression (3.57) also could be canceled due to relation

$$\begin{aligned}
& \left[\partial_\alpha \varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu \varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right] \partial_\lambda\partial_\rho J^{(2)}_{\beta\gamma} \\
+ \quad & \left[\partial_\alpha\partial_\beta\partial_\gamma \varepsilon^{(\mu\nu\lambda} h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right] \partial_\mu\partial_\nu J^{(2)}_{\lambda\rho} = \\
& 6\partial_\mu\partial_\nu\partial_\lambda\partial_\rho \varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho} J^{(2)}_{\beta\gamma}. \qquad (3.59)
\end{aligned}$$

But in this case we see that this we can do only after adding to initial $J^{(4)}$ current a symmetrized double gradient of $J^{(2)}$:

$$\tilde{J}^{(4)}_{\nu\lambda\rho\gamma} = J^{(4)}_{\nu\lambda\rho\gamma} + \frac{1}{2}\partial_{(\nu}\partial_\lambda J^{(2)}_{\rho\gamma)}, \qquad (3.60)$$

which is possible but needs special consideration out of our restrictions on spin 4 gauge field. So we prefer to keep initial spin 4 current unchanged and cancel this term by traceless Stueckelberg like transformation of the spin two gauge field from linear coupling with $J^{(2)}$ current:

$$\delta_1 h^{\beta\gamma}_{(2)} \sim \partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}. \tag{3.61}$$

Note that this transformation is always traceless due to tracelessness of gauge parameter in Fronsdal formulation. So we need to work out only first four lines of (3.55)

INTEGRATION AND INTERACTION

Now we start to integrate expression:

$$\begin{aligned} & \frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\ + \quad & \frac{1}{6}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma} \\ + \quad & \frac{1}{6}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho}. \end{aligned} \tag{3.62}$$

To extract interactions and linear on gauge field transformations we can use the following important formulas:

$$\partial_\mu\varepsilon^{\alpha\beta\gamma} = \delta_0 h^{\alpha\beta\gamma}_\mu - \partial^{(\alpha}\varepsilon^{\beta\gamma)}_\mu, \tag{3.63}$$

$$\partial_\mu\partial_\nu\varepsilon^{\alpha\beta\gamma} = \frac{1}{2}\partial_{(\nu}\delta_0 h^{\alpha\beta\gamma}_{\mu)} - \frac{1}{2}\partial^{(\alpha}\delta_0 h^{\beta\gamma)}_{\mu\nu} + \partial^{(\alpha}\partial^\beta\varepsilon^{\gamma)}_{\mu\nu}, \tag{3.64}$$

$$\partial_\mu\partial_\nu\partial_\lambda\varepsilon^{\alpha\beta\gamma} = \frac{1}{3}\partial_{(\nu}\partial_\lambda\delta_0 h^{\alpha\beta\gamma}_{\mu)} - \frac{1}{6}\partial^{(\alpha}\partial_{(\lambda}\delta_0 h^{\beta\gamma)}_{\mu\nu)} + \frac{1}{3}\partial^{(\alpha}\partial^\beta\delta_0 h^{\gamma)}_{\mu\nu\lambda} - \partial^\alpha\partial^\beta\partial^\gamma\varepsilon_{\mu\nu\lambda}. \tag{3.65}$$

Using (3.63) we can immediately integrate first line of (3.62) and obtain interaction Lagrangian:

$$L^1_2 = \frac{1}{10}h^{\alpha\beta\gamma}_\mu h^{\nu\lambda\rho\mu}\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}. \tag{3.66}$$

The remaining part of first line of (3.62) can be removed by gauge transformation of the spin 6 field

$$\delta_1 h^{\mu\nu\lambda\alpha\beta\gamma}_{(6)} = \varepsilon^{\rho(\alpha\beta}\partial_\rho h^{\gamma\mu\nu\lambda)} + \partial^{(\alpha}\varepsilon^{\beta\gamma}_\rho h^{\mu\nu\lambda)\rho}, \tag{3.67}$$

in cubic part (3.19). Then we turn to the second and third line of (3.62). Applying (3.62) and (3.64) to the first terms of second and third lines of (3.62) we see immediately that for integration of two terms $\delta_0 h\partial\partial h + \partial\partial\delta_0 hh = \delta_0(h\partial\partial h)$ coming from different brackets we need the same coefficient a front of second and third line of (3.62) and corresponding identity (3.55) helped us to improve this discrepancy in (3.15). Finally using (3.64) for other terms we arrive to the following interaction terms coming from these two lines:

$$L_2^2 = -\frac{2}{3}h_{\mu}^{\alpha\beta\gamma}\partial_\alpha\partial_\beta h^{\mu\nu\lambda\rho}J^{(4)}_{\nu\lambda\rho\gamma} + \frac{1}{2}\partial_\nu h_{\mu}^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\lambda\rho\beta\gamma} - \frac{1}{4}\partial^\alpha h_{\mu\nu}^{\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\lambda\rho\beta\gamma}$$
$$-\partial^\beta h_{\mu\nu}^{\alpha\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\lambda\rho\beta\gamma} + \frac{1}{3}\partial^\beta h_{\mu\nu\lambda}^{\gamma}\partial^\alpha h^{\mu\nu\lambda\rho}J^{(4)}_{\rho\alpha\beta\gamma}. \tag{3.68}$$

From remainder we can extract $\delta_1 h^{\mu\nu\lambda\rho}$

$$\delta_1 h^{\mu\nu\lambda\rho} \sim$$
$$\varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho} + \partial^{(\mu}\varepsilon_{\gamma}^{|\alpha\beta|}\partial_\alpha\partial_\beta h^{\nu\lambda\rho)\gamma} + \partial^{(\mu}\partial^{\nu}\varepsilon_{\beta\gamma}^{|\alpha|}\partial_\alpha h^{\lambda\rho)\beta\gamma} + \partial^{(\mu}\partial^{\nu}\partial^{\lambda}\varepsilon_{\alpha\beta\gamma}h^{\rho)\alpha\beta\gamma}. \tag{3.69}$$

After all this manipulation we still have four remaining terms of two types:

First two remaining terms contain divergences of spin 4 current:

$$\partial_\lambda\partial^{(\nu}\varepsilon^{\beta\gamma\mu)}h_{\mu\nu}^{\rho\lambda}\partial^\alpha J^{(4)}_{\rho\alpha\beta\gamma} - \frac{2}{3}\partial^\beta\partial^{(\gamma}\varepsilon^{\mu\nu\lambda)}h_{\mu\nu\lambda}^{\rho}\partial^\alpha J^{(4)}_{\rho\alpha\beta\gamma} =$$
$$-\frac{1}{6}\partial^{(\mu}\big(\partial_\beta\partial_\gamma\varepsilon_{\alpha}^{\nu\lambda}h^{\rho)\alpha\beta\gamma}\big)J^{(4)}_{\mu\nu\lambda\rho} + \frac{1}{18}\partial^{(\mu}\big(\partial^\nu\partial^\lambda\varepsilon_{\alpha\beta\gamma}h^{\rho)\alpha\beta\gamma}\big)J^{(4)}_{\mu\nu\lambda\rho}. \tag{3.70}$$

We can cancel them introducing first order deformation of $\delta_0 h^{\mu\nu\lambda\rho}$

$$\bar{\delta}_0 h^{\mu\nu\lambda\rho} \sim \partial^{(\mu}\big(\partial_\beta\partial_\gamma\varepsilon_{\alpha}^{\nu\lambda}h^{\rho)\alpha\beta\gamma}\big) - \frac{1}{3}\partial^{(\mu}\big(\partial^\nu\partial^\lambda\varepsilon_{\alpha\beta\gamma}h^{\rho)\alpha\beta\gamma}\big). \tag{3.71}$$

Second two remainders contain contractions between derivatives of gauge parameter and gauge fields:

$$-\frac{4}{3}\partial^\alpha\varepsilon_{\mu}^{\beta\gamma}\partial_\alpha\partial_\beta h^{\nu\lambda\rho\mu}J^{(4)}_{\nu\lambda\rho\gamma} - 2\partial^\alpha\partial^\beta\varepsilon_{\mu\nu}^{\gamma}\partial_\alpha h^{\lambda\rho\mu\nu}J^{(4)}_{\lambda\rho\beta\gamma}. \tag{3.72}$$

Using the following (up to total derivatives) identity:

$$\partial_\mu A\partial^\mu BC = \frac{1}{2}(AB\Box C - \Box ABC - A\Box BC), \tag{3.73}$$

and taking into account that for our traceless and transversal gauge field on-shell condition means just $\Box h^{\mu\nu\lambda\rho} = 0$ and our gauge parameter is also harmonic, we can transform this terms for *on-shell spin 4 gauge field* to the:

$$
\begin{aligned}
&-\frac{2}{3}\varepsilon^{\beta\gamma}_{\mu}\partial_{\beta}h^{\nu\lambda\rho\mu}\Box J^{(4)}_{\nu\lambda\rho\gamma} - \partial^{\beta}\varepsilon^{\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu}\Box J^{(4)}_{\lambda\rho\beta\gamma} = \\
&\frac{1}{12}\{ - 8\varepsilon^{\beta\gamma}_{\mu}\partial_{\beta}h^{\nu\lambda\rho\mu} - 24\partial^{\beta}\varepsilon^{\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu} - 12\partial^{\mu}\varepsilon^{\beta\gamma}_{\nu}h^{\lambda\rho\mu\nu}\}\Box J^{(4)}_{\nu\lambda\rho\gamma} \\
&+\frac{1}{4}\delta_0\{h^{\beta\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu}\}\Box J^{(4)}_{\lambda\rho\beta\gamma},
\end{aligned} \tag{3.74}
$$

where we used again (3.63) to integrate term in last line. Investigating expression in brackets of the second line of (3.74) we see that it is exactly trace of our spin 6 gauge field transformation introduced in (3.67)

$$
\delta_1 h^{\mu\nu\lambda\rho\alpha}_{\alpha}\Box J^{(4)}_{\mu\nu\lambda\rho} = \Big\{8\varepsilon^{\alpha\beta\rho}\partial_{\alpha}h^{\mu\nu\lambda}_{\beta} + 12\partial^{\beta}\varepsilon^{\lambda\rho}_{\alpha}h^{\mu\nu\alpha}_{\beta} + 24\partial^{\mu}\varepsilon^{\nu\beta}_{\alpha}h^{\lambda\rho\alpha}_{\beta}\Big\}\Box J^{(4)}_{\mu\nu\lambda\rho}. \tag{3.75}
$$

So we completely get rid of all reminders and obtained additional cubic interaction for trace of spin 6 field:

$$
\frac{1}{12}h^{\lambda\rho\beta\gamma\alpha}_{\alpha}\Box J^{(4)}_{\lambda\rho\beta\gamma}, \tag{3.76}
$$

and one more interacting term for second order on gauge field spin four interaction:

$$
L^3_2 = -\frac{1}{4}h^{\beta\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu}\Box J^{(4)}_{\lambda\rho\beta\gamma}. \tag{3.77}
$$

Summing all $L^i_2, i = 1, 2, 3$ we arrive to the quartic interaction (3.25)

3.6 GENERALIZED CURVATURE AND CHRISTOFFEL SYMBOLS FOR SPIN $2 \leq S \leq 4$

To finalize our consideration the natural task should be consideration of the commutator of linear on gauge field gauge transformation (3.69) obtained during construction of our quartic interaction. To do that we should first consider some exact relations for generalized Christoffel symbols and Curvatures introduced first time in [65]. The alphabet of $s = 4$ generalized curvature and Christoffel symbols are considered in details in [66]. Here we present exact formulas for $s = 2, 3, 4$ and derive nice reduction relations connected with antisymmetrization of possible pairs of indices. So introducing gauge fields for spin $s \leq 4$

$$h^{(4)}(x)_{\mu\nu\lambda\rho}, h^{(3)}(x)_{\mu\nu\lambda}, h^{(2)}(x)_{\mu\nu}. \tag{3.78}$$

we can define the hierarchy of the Generalized Christoffel symbols and Curvature for spin 4:

$$\Gamma^{(1)}_{\alpha;\mu\nu\lambda\rho}(h^{(4)}) = \partial_\alpha h^{(4)}_{\mu\nu\lambda\rho} - \partial_{(\mu} h^{(4)}_{\nu\lambda\rho)\alpha}, \tag{3.79}$$

$$\Gamma^{(2)}_{\alpha\beta;\mu\nu\lambda\rho}(h^{(4)}) = \partial_\alpha\partial_\beta h^{(4)}_{\mu\nu\lambda\rho} - \frac{1}{2}\partial_{<\alpha}\partial_{(\mu} h^{(4)}_{\nu\lambda\rho)\beta>} + \partial_{(\mu}\partial_\nu h^{(4)}_{\lambda\rho)\alpha\beta}, \tag{3.80}$$

$$\begin{aligned}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) &= \partial_\alpha\partial_\beta\partial_\gamma h^{(4)}_{\mu\nu\lambda\rho} - \frac{1}{3}\partial_{<\alpha}\partial_\beta\partial_{(\mu} h^{(4)}_{\nu\lambda\rho)\gamma>} + \frac{1}{3}\partial_{<\alpha}\partial_{(\mu}\partial_\nu h^{(4)}_{\lambda\rho)\beta\gamma>} \\ &- \partial_{(\mu}\partial_\nu\partial_\lambda h^{(4)}_{\rho)\alpha\beta\gamma},\end{aligned} \tag{3.81}$$

$$\begin{aligned}R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) = \Gamma^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) &= \partial_\eta\partial_\alpha\partial_\beta\partial_\gamma h^{(4)}_{\mu\nu\lambda\rho} - \frac{1}{4}\partial_{<\eta}\partial_\alpha\partial_\beta\partial_{(\mu} h^{(4)}_{\nu\lambda\rho)\gamma>} \\ &+ \frac{1}{6}\partial_{<\eta}\partial_\alpha\partial_{(\mu}\partial_\nu h^{(4)}_{\lambda\rho)\beta\gamma>} - \frac{1}{4}\partial_{<\eta}\partial_{(\mu}\partial_\nu\partial_\lambda h^{(4)}_{\rho)\alpha\beta\gamma>} + \partial_\mu\partial_\nu\partial_\lambda\partial_\rho h^{(4)}_{\eta\alpha\beta\gamma},\end{aligned} \tag{3.82}$$

with the following rule for zero order gauge transformation equipped by third rank symmetric tensor parameter:

$$\delta^{(\varepsilon)}_0 \Gamma^{(1)}_{\alpha;\mu\nu\lambda\rho}(h^{(4)}) = -2\partial_{(\mu}\partial_\nu \varepsilon^{(3)}_{\lambda\rho)\alpha}, \tag{3.83}$$

$$\delta^{(\varepsilon)}_0 \Gamma^{(2)}_{\alpha\beta;\mu\nu\lambda\rho}(h^{(4)}) = 3\partial_{(\mu}\partial_\nu\partial_\lambda \varepsilon^{(3)}_{\rho)\alpha\beta}, \tag{3.84}$$

$$\delta^{(\varepsilon)}_0 \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) = -4\partial_\mu\partial_\nu\partial_\lambda\partial_\rho \varepsilon^{(3)}_{\alpha\beta\gamma}, \tag{3.85}$$

$$\delta^{(\varepsilon)}_0 R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) = 0. \tag{3.86}$$

The generalized curvature is invariant as it should be.

Corresponding hierarchy of the Connections and Curvature can be defined spin three:

$$\Gamma^{(1)}_{\alpha;\mu\nu\lambda}(h^{(3)}) = \partial_\alpha h^{(3)}_{\mu\nu\lambda} - \partial_{(\mu}h^{(3)}_{\nu\lambda)\alpha}, \tag{3.87}$$

$$\Gamma^{(2)}_{\alpha\beta;\mu\nu\lambda}(h^{(3)}) = \partial_\alpha\partial_\beta h^{(3)}_{\mu\nu\lambda} - \frac{1}{2}\partial_{<\alpha}\partial_{(\mu}h^{(3)}_{\nu\lambda)\beta>} + \partial_{(\mu}\partial_\nu h^{(3)}_{\lambda)\alpha\beta}, \tag{3.88}$$

$$R^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda}(h^{(3)}) = \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda}(h^{(3)}) = \partial_\alpha\partial_\beta\partial_\gamma h^{(3)}_{\mu\nu\lambda} - \frac{1}{3}\partial_{<\alpha}\partial_\beta\partial_{(\mu}h^{(3)}_{\nu\lambda)\gamma>} + \frac{1}{3}\partial_{<\alpha}\partial_{(\mu}\partial_\nu h^{(3)}_{\lambda)\beta\gamma>} - \partial_\mu\partial_\nu\partial_\lambda h^{(3)}_{\alpha\beta\gamma}. \tag{3.89}$$

With corresponding gauge transformation with traceless symmetric second rank tensor parameter:

$$\delta^{(\varepsilon)}_0\Gamma^{(1)}_{\alpha;\mu\nu\lambda}(h^{(3)}) = -2\partial_{(\mu}\partial_\nu\varepsilon^{(2)}_{\lambda)\alpha}, \tag{3.90}$$

$$\delta^{(\varepsilon)}_0\Gamma^{(2)}_{\alpha\beta;\mu\nu\lambda}(h^{(3)}) = 3\partial_\mu\partial_\nu\partial_\lambda\varepsilon^{(2)}_{\alpha\beta}, \tag{3.91}$$

$$\delta^{(\varepsilon)}_0 R^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda}(h^{(3)}) = 0. \tag{3.92}$$

And the same type relations for spin two:

$$\Gamma^{(1)}_{\alpha;\mu\nu}(h^{(2)}) = \partial_\alpha h^{(2)}_{\mu\nu} - \partial_{(\mu}h^{(2)}_{\nu)\alpha}, \tag{3.93}$$

$$R^{(2)}_{\alpha\beta;\mu\nu}(h^{(2)}) = \Gamma^{(2)}_{\alpha\beta;\mu\nu}(h^{(2)}) = \partial_\alpha\partial_\beta h^{(2)}_{\mu\nu} - \frac{1}{2}\partial_{<\alpha}\partial_{(\mu}h^{(3)}_{\nu)\beta>} + \partial_\mu\partial_\nu h^{(2)}_{\alpha\beta}, \tag{3.94}$$

$$\delta^{(\varepsilon)}_0\Gamma^{(1)}_{\alpha;\mu\nu}(h^{(2)}) = -2\partial_\mu\partial_\nu\varepsilon^{(1)}_\alpha, \tag{3.95}$$

$$\delta^{(\varepsilon)}_0 R^{(2)}_{\alpha\beta;\mu\nu}(h^{(2)}) = 0. \tag{3.96}$$

General formulation for the general spin can be found in original paper [65] (see also [66], [67] for generalization to the AdS space). Here in this Appendix we presented detailed formulas for spin $2 \leq s \leq 4$ to prove reduction relation between different objects to use that for calculation f the commutator of two δ_1 variations.

To derive these relations we should choose one representative from each set of symmetrized indices in antisymmetrize one pair of indices from two sets of symmetrized indices in (3.81) and compare with (3.88) then we arrive to:

$$\Gamma^{(3)}_{\alpha\beta[\gamma;\rho]\mu\nu\lambda}(h^{(4)}) = \frac{4}{3}\Gamma^{(2)}_{\alpha\beta;\mu\nu\lambda}(H^{(3)}_{[\gamma\rho]}), \tag{3.97}$$

$$\Gamma^{(2)}_{\alpha[\beta;\rho]\mu\nu\lambda}(h^{(4)}) = \frac{3}{2}\Gamma^{(1)}_{\alpha;\mu\nu\lambda}(H^{(3)}_{[\beta\rho]}), \tag{3.98}$$

$$\Gamma^{(1)}_{[\alpha;\rho]\mu\nu\lambda}(h^{(4)}) = 2H^{(3)}_{[\alpha\rho];\mu\nu\lambda}, \tag{3.99}$$

where

$$H^{(3)}_{[\gamma\rho];\mu\nu\lambda} = \partial_{[\gamma}h^{(4)}_{\rho]m\nu\lambda} \tag{3.100}$$

is first skew derivative of our spin four gauge field. So we see that antisymmetrized derivative pair behaves inert in symmetric construction of the Generalized Christoffel symbols and leads to the reduction relation (3.97)-(3.99).

The same type of antisymmetrization reduction relations exist for spin two-three case:

$$\Gamma^{(2)}_{\alpha[\beta;\lambda]\mu\nu}(h^{(3)}) = \frac{3}{2}\Gamma^{(1)}_{\alpha;\mu\nu\lambda}(H^{(2)}_{[\beta\rho]}), \tag{3.101}$$

$$\Gamma^{(1)}_{[\alpha;\lambda]\mu\nu}(h^{(3)}) = 2H^{(2)}_{[\alpha\lambda];\mu\nu}, \tag{3.102}$$

$$H^{(2)}_{[\alpha\lambda];\mu\nu} = \partial_{[\alpha}h^{(3)}_{\lambda]\mu\nu}. \tag{3.103}$$

Then combining (3.97) and (3.102) we can derive second reduction formula for spin four third Christoffel symbol:

$$\Gamma^{(3)}_{\alpha\left[\beta[\gamma;\rho]\lambda\right]\mu\nu}(h^{(4)}) = 2\Gamma^{(1)}_{\alpha;\mu\nu}(H^{(2)}_{[\gamma\rho][\beta\lambda]}), \tag{3.104}$$

$$H^{(2)}_{[\gamma\rho][\beta\lambda];\mu\nu} = \partial_{\left[\beta\right.}\partial_{[\gamma}h^{(4)}_{\left.\rho]\lambda\right]\mu\nu}. \tag{3.105}$$

Gauge transformations for reduced Christoffels look like:

$$\delta^{\varepsilon}_{0}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho}(H^{(3)}_{[\eta\alpha]}) = 3\partial_{\nu}\partial_{\lambda}\partial_{\rho}E^{(2)}_{[\eta\alpha];\beta\gamma}(\varepsilon^{(3)}), \tag{3.106}$$

$$E^{(2)}_{[\eta\alpha];\beta\gamma}(\varepsilon^{(3)}) = \partial_{[\eta}\varepsilon^{(3)}_{\alpha]\beta\gamma}, \tag{3.107}$$

$$\delta^{\varepsilon}_{0}\Gamma^{(1)}_{\gamma;\lambda\rho}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) = -2\partial_{\lambda}\partial_{\rho}E^{(1)}_{[\eta\alpha][\sigma\beta]\gamma}(\varepsilon^{(3)}), \tag{3.108}$$

$$E^{(1)}_{[\eta\alpha][\sigma\beta];\gamma}(\varepsilon) = \partial_{\left[\sigma\right.}\partial_{[\eta}\varepsilon^{(3)}_{\left.\alpha]\beta\right]\gamma}, \tag{3.109}$$

with corresponding gauge transformation for antisymmetrized derivatives of gauge field:

$$\delta_0^\varepsilon H^{(3)}_{[\eta\alpha];\mu\nu\lambda} = \partial_{(\mu} E^{(2)}_{[\eta\alpha];\nu\lambda)}, \tag{3.110}$$

$$\delta_0^\varepsilon H^{(2)}_{[\eta\alpha][\sigma\beta];\nu\lambda} = \partial_{(\nu} E^{(1)}_{[\eta\alpha][\sigma\beta];\lambda)}. \tag{3.111}$$

Reduction relations work also for gauge invariant curvatures

$$R^{(4)}_{\eta\alpha\beta[\gamma;\rho]\mu\nu\lambda}(h^{(4)}) = \frac{5}{4} R^{(3)}_{\eta\alpha\beta;\mu\nu\lambda}(H^{(3)}_{[\gamma\rho]}), \tag{3.112}$$

$$R^{(3)}_{\eta\alpha[\beta;\lambda]\mu\nu}(h^{(3)}) = \frac{4}{3} R^{(2)}_{\eta\alpha;\mu\nu}(H^{(2)}_{[\beta\lambda]}). \tag{3.113}$$

Then we end this Appendix with some Bianchi Identity like formulas which we widely use in our calculation in this paper:

$$\partial_\eta \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) = \frac{1}{4} \partial_{(\mu} \Gamma^{(3)}_{|\alpha\beta\gamma|;\nu\lambda\rho)\eta}(h^{(4)}) + R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}), \tag{3.114}$$

$$R^{(4)}_{(\eta\alpha\beta\gamma;\mu)\nu\lambda\rho}(h^{(4)}) = 0, \quad R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h^{(4)}) = R^{(4)}_{\mu\nu\lambda\rho;\eta\alpha\beta\gamma}(h^{(4)}), \tag{3.115}$$

$$\partial_{[\delta} R^{(4)}_{\eta]\alpha\beta\gamma;\mu\nu\lambda\rho} = \frac{1}{5} \partial_{(\mu} R^{(4)}_{\nu\lambda\rho)[\delta;\eta]\alpha\beta\gamma}. \tag{3.116}$$

3.7 EXPLICITLY COMPUTING COMMUTATOR USING WOLFRAM MATHEMATICA

In this appendix we will introduce whole process of modelling Higher spin objects in Wolfram Mathematica and will explicitly compute the commutator using Mathematica code. We will introduce other objects as well such as generalized Christopher's symbols and more. Here we mainly used *xTras* package form *xAct* bundle.

SETUP

We start our project by simply defining all the required object which will be used along the way, such as manifold, tangent bundle, metric, higher spin fields and gauge parameters.

```
<< xAct`xTras`;
DefManifold[M, dim, IndexRange[a, m]];
AddIndices[TangentM, {n, o, p, q, r, s, t}]
DefMetric[-1, metric[-a, -b], PD, PrintAs -> "η", FlatMetric -> True,
    SymbolOfCovD -> {",", "∂"}]
SetOptions[SymmetryOf, ConstantMetric -> True];
DefTensor[H[-a, -b, -c, -d], M, Symmetric[{-a, -b, -c, -d}], PrintAs -> "h"]
DefTensor[ϵ[a, b, c], M , Symmetric[{a, b, c}], PrintAs -> "ϵ"];
DefTensor[ω[a, b, c], M , Symmetric[{a, b, c}], PrintAs -> "ω"]
DefTensor[A[a], M , PrintAs -> "a"]
DefTensor[B[a], M , PrintAs -> "b"]
```

Listing 3.1: Setup, Manifold, Metric, Gauge field and Parameter definitions

In the above (3.1) code snipped we defined the manifold M without curvature, also we have defined gauge field H as a symmetric tensor of rank four. We have also defined ϵ, ω gauge parameters as symmetric tensors of the third rank and some auxiliary symmetric tensors and vectors which will be used along the way.

Next we define the generating function for the first order variation.

```
δ1HG[H_, ϵ_] := Module[{ANS},
 ANS[d_, e_, f_, l_] := Module[{a, b, c, A, B, C, D, res},
   A = ϵ[a, b, c]*PD[-a]@PD[-b]@PD[-c]@H[d, e, f, l];
   B = PD[d]@ ϵ[a, b, c]*PD[-a]@PD[-b]@H[e, f, l, -c] //
     Symmetrize[#, {d, e, f, l}] &;
   C = PD[d]@PD[e]@ϵ[a, b, c]*PD[-a]@H[f, l, -b, -c] //
     Symmetrize[#, {d, e, f, l}] &;
   D = PD[d]@PD[e]@PD[f]@ϵ[a, b, c]*H[l, -a, -b, -c] //
     Symmetrize[#, {d, e, f, l}] &;
```

```
    res = A + 4*B + 6* C + 4 *D // CollectTensors;
    ANS[d, e, f, l] = res;
    res
    ];
  ANS
  ]
```

Listing 3.2: Module for generating first order variation tensor

The beauty of this implementation is that it is a function of functions and due to encapsulation it can be reused with different fields and parameters. It requires arguments such as H and ϵ which are arbitrary variable tensors of rank 4 and 3 respectively (in our case of course they are gauge field and gauge parameter) and returns another function which can be treated as a tensor. On the last line of above figure is the explicit form of first order variation same as (3.27).

Before moving forward it is worth to give more explanation of the syntax which is used in the above code snippet as well as an *important trick* which significantly reduces the computation time.

- The **@** operator executes the function standing on the left hand side and passes the right hand side of it as an argument.
- The // operator executes the function in the right hand side and passes the left hand side of it as an argument, just like the opposite of the @ operator
- Expressions similar to **Symmetrize** [**#, {d, e, f, l}**] **&;** are wrapped functions, they always contain # element somewhere in the argument list and are always ending with &. The wrapper function will plug anything given into the slot of # and execute the expression. In this particular case it will symmetrize everything with respect to {d, e, f, l} indices.
- The last and probably the **most important** thing is caching, which is implemented here to significantly reduce the computation time. The thing is that if something is defined in

Mathematica using := then it is basically a delayed expression and it will execute every time it is used unlike the expressions defined using =. On line 11 we cache the result in the same object and next time if we need to get the exact same object the whole code will not be executed, instead the previously computed result will be returned.

The function can be used in a following way:

```
δ1H = δ1HG[H, ϵ];
δ1H[i, j, k, l]*A[-i]*A[-j]*A[-k]*A[-l] // CollectTensors
```

Listing 3.3: Usage of the module

On the first line we initialized our tensor to use H as a gauge field and ϵ as a corresponding gauge parameter. The result is a tensor of the first order variation of H by ϵ. If we think about the physical meaning of this object, it is a tensor, but programmatically in Mathematica it is yet another module which takes four arguments. This four arguments are the tensor index slots which gives us a possibility to reuse the tensor with different indices without creating the whole object from scratch. We contracted the result with A vectors which acts as a symmetrization operation and makes the expressions more human readable and compact. As a result we get:

$$(4h_{ajkl}\partial_i\partial_c\partial_b\epsilon^{jkl} + 6\partial_b\partial_a\epsilon^{jkl}\partial_l h_{cijk} + 4\partial_a\epsilon^{jkl}\partial_l\partial_k h_{bcij} + \epsilon^{jkl}\partial_l\partial_k\partial_j h_{abci})a^a a^b a^c a^i \tag{3.117}$$

Now we have the left hand side of the (3.4), we move forward into modelling the first element of right hand side using the same approach.

```

δHG[H_, ϵ_] := Module[{ANS},
  ANS[d_, e_, f_, l_] := Module[{a, b, c, A, B, C, D, res},
    A = PD[a]@PD[b]@PD[c]@H[d, e, f, l] ;
    B = -PD[a]@PD[b]@PD[d]@H[e, f, l, c] //
        Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    C =  PD[a]@PD[d]@PD[e]@H[f, l, c, b] //
        Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    D =  -PD[d]@PD[e]@PD[f]@H[l, c, b, a] //
```

```
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    res = ϵ[-a, -b, -c] *(A + 4*B + 6*C + 4*D) // CollectTensors;
    ANS[d, e, f, l] = res;
    res
    ];
  ANS
  ]
```

Listing 3.4: Generating module for $\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$

Although the second term of the rhs is not important we will model it as well to be able to programmatically check the correctness of our expressions.

```

alphaG[H_, ϵ_] := Module[{ANS},
 ANS[j_, k_, l_] := Module[{e, f, i, A, B, C, D, E, F, res},
   A = ϵ[e, f, i]*PD[j]@PD[k]@H[l, -e, -f, -i] //
     Symmetrize[#, {j, k, l}] &;
   B = -PD[j]@ ϵ[e, f, i]*PD[k]@H[l, -e, -f, -i] //
     Symmetrize[#, {j, k, l}] &;
   C = PD[j]@PD[k]@ϵ[e, f, i]*H[l, -e, -f, -i] //
     Symmetrize[#, {j, k, l}] &;
   D = 3/2 * PD[j]@ϵ[e, f, i]*PD[-e]@H[k, l, -f, -i] //
     Symmetrize[#, {j, k, l}] &;
   E = -3/2 * ϵ[e, f, i]*PD[-e]@PD[j]@H[k, l, -f, -i] //
     Symmetrize[#, {j, k, l}] &;
   F = ϵ[e, f, i]*PD[-e]@PD[-f]@H[j, k, l, -i] //
     Symmetrize[#, {j, k, l}] &;
   res = A + B + C + D + E + F // CollectTensors;
   ANS[j, k, l] = res;
   res
   ];
 ANS
 ]
```

```
alpha = alphaG[H, ϵ]
```

Listing 3.5: Generating module for the $\Lambda_{\nu\lambda\rho}(\varepsilon, h)$

Now we have all three terms of the (3.4) equation and can check it to make sure there is no mistake in the modelled object. To do so, we can simply plug all the objects into the (3.4) equation and see that equation actually takes place.

```
    (δ1H[i, j, k, l] - δHϵ[i, j, k, l] - 4*PD[i]@alpha[j, k, l] )
    *A[-i]*A[-j]*A[-k]*A[-l] // CollectTensors
```

Listing 3.6: Check of the equation $\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} - \varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) - \partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon, h) = 0$

We focus on computing the commutator using the first term only, because the $\partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon, h)$ term is a full derivative and does not play a role. To do so we need to compute the variation of $\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$ by the second gauge parameter ω. It can be done easily by plugging the $\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$ into the generating function of variation with respect to ω parameter.

```
δHϵω=δHG[δHϵ,ω];
δHϵω[i, j, k, l]
```

Listing 3.7: Computing $\delta_1^{(\omega)}\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$

This last object is very compute intensive and requires a lot of time to compute.

We almost have the commutator, the last thing is to compute the variations in a reverse order, first by ω and then by ϵ and subtract from each other. To do so there is no need to compute the whole variation from scratch, it is sufficient to implement a function which can swap gauge variables and by using it we can compute the other term very quickly.

```
swap[exp_, ϵ_, ω_, σ_] := Module[{r1, r2, r3},
  r1 = MakeRule[{ω[i, j, k], σ[i, j, k]},
    PatternIndices -> All, MetricOn -> All, UseSymmetries -> True];
  r2 = MakeRule[{ϵ[i, j, k], ω[i, j, k]},
    PatternIndices -> All, MetricOn -> All, UseSymmetries -> True];
  r3 = MakeRule[{σ[i, j, k], ϵ[i, j, k]},
    PatternIndices -> All, MetricOn -> All, UseSymmetries -> True];
  exp /. r1 /. r2 /. r3
```

```
]
```

Listing 3.8: Swapping module

The **swap** function takes expression as the first argument and in that expression swaps the variables provided in the second and third argument using temporary variable from the last argument. It is important that the last 3 arguments have the same type to have correct results. In our specific case we will provide the expression from the second line of the listing (3.7) and will change $\omega <-> \epsilon$.

```
swapped = δHεω[i, j, k, l] // swap[#, ε, ω, σ] &
```

Listing 3.9: Usage of swapping module

And finally we compute the commutator

```
comm = A[-i]*A[-j]*A[-k]* A[-l]*
(swapped -δHεω[i, j, k, l]) //CollectTensors;
```

Listing 3.10: Computation of the commutator

The *comm* term in the above figure is the commutator of gauge transformation. Now when we have the commutator successfully modelled in Mathematica the second task will be to simplify it and classify all terms. This will be done by modelling other objects such as Christofel symbols and more using the same methodology, observing the expressions and guessing the ansatz then subtracting the expression from the commutator and repeating the cycle until all terms are classified.

To finalize the appendix we will introduce the implementation of the third and forth order Christofel symbols as well:

```
Gamma3G[H_] := Module[{ANS},
 ANS[a_, b_, c_, d_, e_, f_, l_] := Module[{A, B, C, D, res},
   A = PD[a]@PD[b]@PD[c]@H[d, e, f, l] ;
   B = -PD[a]@PD[b]@PD[d]@H[e, f, l, c] //
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
   C =  PD[a]@PD[d]@PD[e]@H[f, l, c, b] //
```

```
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    D =  -PD[d]@PD[e]@PD[f]@H[l, c, b, a] //
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    res = (A + 4*B + 6*C + 4*D) // CollectTensors;
    ANS[a, b, c, d, e, f, l] = res;
    res
    ];
  ANS
  ]
```

Listing 3.11: Third Christoffel symbol $\Gamma_{abc,defl}$

```
Gamma3G[H_] := Module[{ANS},
  ANS[a_, b_, c_, d_, e_, f_, l_] := Module[{A, B, C, D, res},
    A = PD[a]@PD[b]@PD[c]@H[d, e, f, l] ;
    B = -PD[a]@PD[b]@PD[d]@H[e, f, l, c] //
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    C =  PD[a]@PD[d]@PD[e]@H[f, l, c, b] //
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    D =  -PD[d]@PD[e]@PD[f]@H[l, c, b, a] //
      Symmetrize[#, {a, b, c}] & // Symmetrize[#, {d, e, f, l}] &;
    res = (A + 4*B + 6*C + 4*D) // CollectTensors;
    ANS[a, b, c, d, e, f, l] = res;
    res
    ];
  ANS
  ]
```

Listing 3.12: Fourth Christoffel symbol $\Gamma_{abcd,ijkl}$

They take as an argument symmetric tensor, which is gauge field in our case and return a tensor like object of rank 7 and rank 8.

3.8 CONCLUSION

In this chapter, we considered the possibility to construct local quartic interaction in the special case of two higher spin gauge fields and two scalars. Restricting ourselves to the traceless and transverse physical gauge for spin four HS fields we obtained a solution of the Noether's equation adding cubic interaction with additional spin six field in the form of gauge field-current (3.19). As an additional bonus of this construction, we derived linear on-field gauge transformation for spin four filed (3.27) getting a possibility to investigate the corresponding commutator. Performing complicated calculations and using the formalism of generalized Christoffel symbols [65] we classified the right-hand side of commutator and understood that general algebra of $\delta_1^{(\varepsilon)}$ transformation closes on not only spin four gauge transformation. The r.h.s. of commutator includes the sector of transformations of spin four field in respect to gauge transformations with parameters corresponding to spin four and higher up to spin six gauge fields. All other terms include only transformations with parameters coming from transformations of additional fields with mixed symmetry of indices including one or two antisymmetric pairs (3.17). Investigation and analyses of this mixed symmetry sector of r.h.s of commutator can be done in future in separate work.

We also demonstrated the approach on modeling various objects in Wolfram Mathematica, specifically we demonstrated the process of computation of commutator and along the way modelled first order variation, Christoffel symbols and more. The approach is generic enough to be applied on other problems as well in the field of Higher spin gauge theories.

Summary

In this thesis, we have solved various non-trivial problems in the field of Higher Spin theory, such as qubic or special quartic interactions of interacting Lagrangians in flat and AdS spaces. The thesis is based on the following published articles [1–5], where we presented all results included in thesis and aimed for further development of theories in the field of higher spins.

- We were able to successfully finalize the radial pullback procedure for the main term of cubic self-interaction by solving all necessary recurrence relations. The solutions of these equations lead to the possibility of obtaining the full set of AdS_{d+1} dimensional interacting terms with all curvature corrections, including trace and divergence terms from any interaction term in $d + 2$ dimensional flat space.

- Using Wolfram Mathematica programming language, we developed general methods for solving recurrence relations which we faced during the radial pullback procedure. These methods are also generic enough to be used for solving different recurrence relations in other problems.

- We have considered local quartic interaction between higher-spin gauge field and scalar field. Using the full power of Neother's procedure, we successfully constructed interacting Lagrangian for this special case in a physical gauge.

- We successfully constructed and analyzed the commutator of the linearized higher spin gauge field transformation and successfully classified the right-hand side of the commutator. As an important result, we discovered that on the right-hand side of the commutator, there are terms with mixed symmetry.

- Using Wolfram Mathematica programming language, we have developed generic functions and methods which introduced the possibility of working with symmetric tensors with higher rank using the polynomial notation, which significantly makes computation much easier and human-readable.

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