Theory of Integrable Systems and Their Applications, 56 hours

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Contents

1 Integrability, 4 hours

The phenomenon of the integrability of quantum systems can be understood by means of their relation to the linear ones via separation of variables. Namely, the quantum system is integrable, if its non-linear equations of motion can be represented as the zero-curvature conditions of some integrable linear system. Physically this means, that the interaction of such systems reduces to the elastic scattering and the only result of it consists in the exchange of quantum numbers (momenta etc.) of the scattered particles. Accordingly, the S-matrix of the theory is factorized into the product of blocks, corresponding to $2 \to 2$ scattering and also $1 \to 1$ in the presence of a boundary.

1.1 Zamolodchikov's S-matrix

A set of (annihilation) operators $Z_a(\lambda)$ satisfying the Zamolodchikov algebra: has been proposed for an algebraic description of the factorizable scattering.

$$Z_a(\lambda)Z_b(\mu) = S_{ab,cd}(\lambda-\mu)Z_d(\mu)Z_c(\lambda) \tag{1.1}$$

where S is $n^2 \times n^2$ matrix. The consistency condition of this system, which follows from the associativity property of the triple product $Z_{a_1}(\lambda_1)Z_{a_2}(\lambda_2)Z_{a_3}(\lambda_3)$, that is the Yang-Baxter equation for the S-matrix:

$$\begin{aligned} &S_{a_j a_k}(\lambda_j-\lambda_k)S_{a_j a_l}(\lambda_j-\lambda_l)S_{a_k a_l}(\lambda_k-\lambda_l) = \\ &\quad S_{a_k a_l}(\lambda_k-\lambda_l)S_{a_j a_l}(\lambda_j-\lambda_l)S_{a_j a_k}(\lambda_j-\lambda_k). \end{aligned} \tag{1.2}$$

Extending this algebra by adding n conjugated (creation) operators $Z_a^\dagger(\mu)$, one gets the Zamolodchikov - Faddeev algebra:

$$Z_a(\lambda)Z_b^\dagger(\mu) = \delta_{ab}\delta(\lambda-\mu) + Z_c^\dagger(\mu)\hat{S}_{ac,bd}(\lambda-\mu)Z_d(\lambda), \tag{1.3}$$

or in matrix notations:

$$A(\lambda)\otimes A(\mu)\equiv A_1(\lambda)A_2(\mu)=S_{12}(\lambda-\mu)A_2(\mu)A_1(\lambda), \tag{1.4}$$
$$A_1^\dagger(\lambda)A_2^\dagger(\mu)=A_2^\dagger(\mu)A_1^\dagger(\lambda)S_{21}^\dagger(\mu-\lambda),$$
$$A_1(\lambda)\otimes A_1^\dagger(\mu)=I_1\delta(\lambda-\mu)+A_2^\dagger(\mu)\hat{S}_{12}(\mu-\lambda)A_2(\lambda),$$

where $A(\lambda)$ and $A^\dagger(\mu)$ are the column $(Z_1(\lambda),...,Z_n(\lambda))^t$ and the row $(Z_1^\dagger(\mu),...,Z_n^\dagger(\mu)$ correspondingly, subscripts refer to the corresponding isotopic spaces $\mathbb{C}^n\otimes\mathbb{C}^n\equiv V_1V_2$ and $S_{21}=\mathcal{P}S_{12}\mathcal{P}$, $\hat{S}_{12}=\mathcal{P}S_{12}$, $\mathcal{P}$ is the permutation operator in $\mathbb{C}^n\otimes\mathbb{C}^n$. The complete scattering matrix $S(\{\lambda_k\})$ of the M particle is factorized then into the ordered product of $M(M-1)/2$ two-particle S-matrices (1.1). For example the S-matrix of the j-th particle on the other $M-1$ particles is given by $t(\lambda_j;\{\lambda_m\})$, i.e. the particular value of the transfer matrix for $\lambda=\lambda_j$:

$$t(\lambda;\{\lambda_m\})=tr_aT(\lambda;\{\lambda_m\})\equiv tr_a\prod_k S_{ak}(\lambda-\lambda_k), \tag{1.5}$$

1.2 Transfer matrix

The trace in this expression is taken over the auxiliary space V_a, while the transfer matrix acts in the quantum space $\otimes_{k=1}^M V_k$. In the framework of the quantum inverse scattering method (QISM) [4] instead of the original non-linear problem the auxiliary linear one is considered:

$$\frac{d}{dx}T(\lambda,x)=L(\lambda,x)T(\lambda,x) \tag{1.6}$$

or

$$T(n+1,\lambda)=L_{n+1}(\lambda)T(n,\lambda)$$

in discrete case.

This is the Lax operator of the QISM. The solution of (1.6):

$$T(\lambda,x)=P\exp(\int^x L(\lambda,y)dy), \tag{1.7}$$

$$T(n,\lambda)=L_n(\lambda)L_{n-1}(\lambda)...L_1(\lambda)$$

in discrete case, defines the monodromy matrix $T(\lambda)$.

Its entries are the new variables (the quantum scattering data), which commutation relations are defined by

$$\sum_{j_1,j_2=1}^n R_{i_1i_2,j_1j_2}(\lambda-\mu)T_{j_1k_1}(\lambda)T_{j_2k_2}(\mu)= \tag{1.8}$$
$$\sum_{j_1,j_2=1}^n T_{i_2j_2}(\mu)T_{i_1j_1}(\lambda)R_{j_1j_2,k_1k_2}(\lambda-\mu).$$

We see that integrable systems are specified by the R-matrix, which acts on $\mathbb{C}^n \otimes \mathbb{C}^n$ and satisfies the Yang-Baxter equation

$$\sum_{j_1,j_2,j_3=1}^{n} R_{i_1 i_2,j_1 j_2}(\lambda) R_{j_1 i_3,k_1 j_3}(\lambda+\mu) R_{j_2 j_3,k_2 k_3}(\mu) = \tag{1.9}$$
$$\sum_{j_1,j_2,j_3=1}^{n} R_{i_2 i_3,j_2 j_3}(\mu) R_{i_1 j_3,j_1 k_3}(\lambda+\mu) R_{j_1 j_2,k_1 k_2}(\lambda).$$

In general the R-matrix depends on the spectral parameter λ and other parameters. Although there is no complete mathematical theory of the Yang-Baxter equation, a variety of solutions are known as well as different fields of their application. They are classified by the Lie algebra, its irreducible representations, and the spectral parameter dependence: rational, trigonometric and elliptic ones. Given a solution $R(\lambda)$ one can define the quadratic algebra $\mathcal{T}_R$ of $n \times n$ matrix elements T_{ij}, which is generated by eq. (1.8). The associative algebra $\mathcal{T}_R$ realizes the representation space of a quantum integrable system. The commutative integrals of motion are $t(\lambda) = \mathrm{tr} T(\lambda)$, which follows from (1.8), taking the trace of $T_1 T_2 = R_{12}^{-1} T_2 T_1 R_{12}$. The algebra $\mathcal{T}_R$ possesses the co-multiplication property: if $T_1(\lambda)$ and $T_2(\lambda)$ are two representations of $\mathcal{T}_R$ in the quantum spaces V_1 and V_2, then the matrix

$$T_{ik}(\lambda) = T_{1,ij}(\lambda) T_{2,jk}(\lambda) \tag{1.10}$$

is a representation of $\mathcal{T}_R$ in the tensor product $V_1 \otimes V_2$. This property allows to represent $T(\lambda)$ as a product of elementary representations, the so called Lax operators $L_i(\lambda)$. It follows from $SL(2)$ symmetry of the R-matrix that an arbitrary constant $d \times d$ matrix K provides the simplest representation of the algebra $\mathcal{T}_R$.

1.3 Quantum determinant

This algebra has a central element, the quantum determinant of $T(\lambda)$:

$$\begin{aligned}\Delta(\lambda) \equiv \det{}_q T(\lambda) = D(\lambda+\eta/2)A(\lambda-\eta/2) - B(\lambda-\eta/2)C(\lambda+\eta/2 = \\ A(\lambda-\eta/2)D(\lambda+\eta/2) - C(\lambda-\eta/2)C(\lambda+\eta/2) = \\ A(\lambda+\eta/2)D(\lambda-\eta/2) - B(\lambda+\eta/2)C(\lambda-\eta/2) = \\ D(\lambda+\eta/2)A(\lambda-\eta/2) - C(\lambda+\eta/2)B(\lambda-\eta/2) =\end{aligned} \tag{1.11}$$

which has the following remarkable properties

$$\det{}_q T_1(\lambda) T_2(\lambda) = \det{}_q T_1(\lambda) \det{}_q T_2(\lambda) \tag{1.12}$$

and

$$\det{}_q K = \det K. \tag{1.13}$$

The next representation is given by Lax operator, mentioned above, which takes the especially simple form for the XXX spin chain:

$$L(\lambda) = \lambda + \eta \sum_{\alpha=1}^{3} S_\alpha \sigma_\alpha = \begin{pmatrix} \lambda + \eta S^0 & \eta S^- \\ -\eta S^+ & \lambda - \eta S^0 \end{pmatrix} \tag{1.14}$$

where operators S_α belonging to some irreducible representation of $sl(2)$ have commutation relation:

$$[S_i^0, S_j^\pm] = \pm\delta_{ij} S_i^\pm, \qquad [S_i^+, S_j^-] = -2\delta_{ij} S_i^0.$$

Note that the R-matrix itself can be choosen as a Lax operator, if the auxiliary space is two-dimensional. We have

$$\det_q L(\lambda) = \lambda^2 - \eta^2(\mathcal{C} + 1/4), \quad \mathcal{C} = (S^0)^2 - \frac{1}{2}(S^+S^- + S^-S^+). \tag{1.15}$$

Since the $L(\lambda)$-operator, being the elementary representation of $\mathcal{T}_R$, satisfies the Yang-Baxter relation and the R-matrix depends only on the difference of the spectral parameters, the shift $L(\lambda) \to L(\lambda - \omega)$ defines an automorphism in $\mathcal{T}_R$:

$$R(\lambda - \omega_1 + S_1^\alpha \sigma_\alpha)\ (\lambda - \omega_2 + S_2^\beta \sigma_\beta) = (\lambda - \omega_2 + S_2^\beta \sigma_\beta)(\lambda - \omega_1 + S_1^\alpha \sigma_\alpha) R \tag{1.16}$$

Separating the terms, linear in λ in this equation one deduces that the R-matrix is $SL(2)$-invariant

$$[R; S_1^\alpha + S_2^\alpha] = 0 \tag{1.17}$$

and depends only on difference $\omega_{12} = \omega_1 - \omega_2$. The $SL(2)$-invariance implies that the R-matrix has to have the form

$$R = \sum \rho_j(\omega_{12}) P_j, \tag{1.18}$$

where P_j are the projectors corresponding to the decomposition of the tensor product of two initial representations into the sum of irreducible representations labelled by spin j.

Furthermore the part of eq.(1.16), which contains no λ gives for $\rho_j(\omega_{12})$ the recurrence relation:

$$\rho_{j+1}(\omega_{12}) = \frac{\omega_{12} + \eta(j+1)}{\omega_{12} - \eta(j+1)} \rho_j(\omega_{12}), \tag{1.19}$$

which determines R up to a scalar factor.

1.4 Properties of the R-matrix

Particular solutions of the Y-B equation have properties, which are important for different applications, but which are not necessarily valid for a given solution:
regularity

$$R(0) = \rho(0)^{1/2} \mathcal{P}_{12}$$

P-symmetry

$$\mathcal{P} R_{12}(\lambda) \mathcal{P} \equiv R_{21}(\lambda) = R_{12}(\lambda)$$

T-symmetry

$$R_{12}^{t_1 t_2}(\lambda) = R_{12}(\lambda)$$

unitarity

$$R_{12}(\lambda) R_{21}(-\lambda) = \rho(\lambda) I$$

crossing symmetry

$$R_{12}(\lambda) = V_{(1)} R_{12}^{t_2}(-\lambda - \eta) V_{(1)}^{-1}$$

quasiclassical property

$$R(\lambda, \eta) = I + \eta r(\lambda) + \mathcal{O}(\eta^2),$$

Here the superscript t denotes matrix transposition, $r(\lambda)$ is the classical R-matrix, $\rho(\lambda)$ is an even scalar function, η is the crossing parameter and V determines the crossing matrix $M \equiv V^t V = M^t$. The quasiclassical property gives rise to the direct connection of the quantum model to the corresponding classical one. Many R-matrices have only the combined PT-symmetry: $R_{12}^t(\lambda) = R_{21}(\lambda)$. The regularity is used to extract from $t(\lambda)$ the local integrals of motion.

Thus the general solution of (1.16) is given by

$$T(\lambda, \vec{\omega}) = K L_N(\lambda - \omega_N) ... L_2(\lambda - \omega_2) L_1(\lambda - \omega_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{1.20}$$

because any permutation of the multipliers gives the equivalent result in the algebra $\mathcal{T}_R$. Notice, that the L-operator is noting else than a R-matrix, acting in auxiliary and quantum spaces $\mathbb{C}^2 \otimes V_i$: $L_i(\lambda) \equiv R_{ai}(\lambda)$.

The corresponding quantum determinant is

$$\Delta(\lambda) = \det_q T(\lambda) = \det K \prod_{i=1}^{N} ((\lambda - \omega_i)^2 - \eta^2(\mathcal{C}_i + 1/4)) \tag{1.21}$$

Now, it follows from

$$R_{12}(\lambda - \mu) T^{(1)}(\lambda, \vec{\omega}) T^{(2)}(\mu, \vec{\omega}) = T^{(2)}(\mu, \vec{\omega}) T^{(1)}(\lambda, \vec{\omega}) R_{12}(\lambda - \mu) \tag{1.22}$$

that

$$[t(\lambda, \vec{\omega}); t(\lambda, \vec{\omega})] = 0, \tag{1.23}$$

where $t(\lambda, \vec{\omega}) = \mathrm{tr} T(\lambda, \vec{\omega})$, The trace is taken over the auxiliary space.

Among the integrals of motion (1.23) we look for local ones, i.e. quantities $H^{(k)}$ $k = 1, 2, 3, ..,$ which can be expressed as the sum of local operators,

$$H^{(k)} = \sum_{i=1}^{N} H^{(k)}_{i, i-1, ..., i-k+1} \tag{1.24}$$

The periodicity, $N + 1 \equiv 1$, is supposed. The local densities $H^{(k)}_{i, i-1, ..., i-k+1}$ should involve only k adjancent spins $S_i, S_{i+1}, ..., S_{i-k+1}$. An important case when such local integrals exist is that of the homogeneous spin chain, corresponding to equal spins $\Delta_i = \Delta$ and zero shifts $\vec{\omega} = 0$. It has the important property of translational invariance. The corresponding R-matrix is regular. The similarity transformation

$$U S_i^{\alpha} U^{-1} = S_{i+1}^{\alpha}, \qquad U S_N U^{-1} = \mathcal{K}_1 S_1 \mathcal{K}_1^{-1}, \tag{1.25}$$

where $\mathcal{K}$ permutes the boundary matrix K and with the Lax operator L_1: $KL_1(\lambda) = \mathcal{K}^{-1}L(\lambda)K\mathcal{K}$. This transformation generalizes the ordinary translation for the periodic chain ($K = 1$) to the twisted periodic boundary condition, specified by the matrix K and $U^N \neq 1$ in contrast to the case $K = 1$, when operator U takes the especially simple form: $U = \mathcal{P}_{12}\mathcal{P}_{23}...\mathcal{P}_{N-1N}$. The unitarity of U allows to represent it in exponential form

$$U = e^{iP}, \tag{1.26}$$

where operator P has the physical meaning of the total momentum of the chain. The hamiltonian of the model then acquires the form

$$H = \frac{d}{d\lambda}t(\lambda)|_{\lambda=0} = \sum_{i=1}^{N}\frac{d}{d\lambda}\mathcal{P}_{i,i+1}R_{i,i+1}(\lambda)|_{\lambda=0}. \tag{1.27}$$

2 Bäclund transformation, 8 hours

2.1 Toda chain

In this subsection we define the Bäcklund transformation for the periodic Toda lattice. The model is described by the Lax or monodromy matrix:

$$L(u) = \ell_n(u)\dots\ell_2(u)\ell_1(u), \qquad \ell_j(u) = \ell_j(u; X_j, x_j) = \begin{pmatrix} u + X_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix}. \tag{2.28}$$

The Bäcklund transformation B_λ depends on a complex parameter λ and defines the mapping from the variables (X, x) to (Y, y):

$$X_j = e^{x_j - y_j} + e^{y_{j+1} - x_j} - \lambda \tag{2.29}$$

$$Y_j = e^{x_j - y_j} + e^{y_j - x_{j-1}} - \lambda \tag{2.30}$$

One can rewrite (2.29):

$$e^{y_{j+1}} = e^{x_j}(X_j + \lambda) - e^{2x_j - y_j},$$

and iteratively express e^{y_j} through e^{y_1} as fractional-linear function. Solving it for $j = N$ one come to quadratic equation for e^{y_1} and deduces, that the transformation B_λ is a two-values algebraic function in terms of X_j and e^{x_j}.

Exercise *Check this statement.*

We do not present the manifest form of solution here, because the simple implicit formulas (2.29-2.30) are sufficient for our purposes.

Note that equations (2.29-2.30) are local: they involve only the variables with the indices differing by 0 and 1. Note that even for real λ's resolving the equations (2.29-2.30) can produce complex values of Y_j and y_j

Then the canonicity of Bäclund transformation can be stated: the variables (Y_j, y_j) are canonical. Indeed the equations (2.29-2.30) can be rewritten in the form:

$$X_j = \frac{\partial F_\lambda}{\partial x_j}, \qquad Y_j = \frac{\partial F_\lambda}{\partial y_j}, \tag{2.31}$$

with generating function:

$$F_\lambda(y;x)=\sum_{i=1}^{N}(e^{x_j-y_j}-e^{y_{i+1}-x_j}-\lambda(x_j-y_j)). \tag{2.32}$$

Exercise *Check that (2.31) reproduces the Bäclund transformation (2.29-2.30)*

Another remarkable property is the invariance of conserved quantities (Hamiltonians):

$$H_j(X,x)=H_j(Y,y), \qquad j=1,\dots,N. \tag{2.33}$$

It follows from the existence an invertible matrix M entering in Darboux transformation:

$$M(u,\lambda)L(u,Y,y)=L(u,X,x)M(u,\lambda). \tag{2.34}$$

One can choose Darboux matrix as $M(u,\lambda)=M_1(u,\lambda)$, where the local matrices $M_j(u,\lambda)$ correspond to gauge transformation:

$$M_{j+1}(u,\lambda)\ell_j(u,Y_j,y_j)=\ell_j(u,X_j,x_j)M_j(u,\lambda). \tag{2.35}$$

Indeed, multiplying the Lax operator (2.28) in terms of $(Y,y\ )$ by $M_{N+1}(u,\lambda)=M_1(u,\lambda)$ from the left and using (2.35) one arrives at (2.34) after N steps.

Using definitions (2.28) and (2.29-2.30) one can easily check that the local Darboux matrix:

$$M_j(u,\lambda)=\begin{pmatrix} u-\lambda+e^{y_j-x_{j-1}} & -e^{y_j} \\ e^{-x_{j-1}} & -1 \end{pmatrix}, \tag{2.36}$$

satisfies to (2.35).

Exercise *Check this statement.* (Hint *exclude λ using (2.29-2.30)*)

These two properties: canonicity and invariance of Hamiltonians constitute the definition of what is called an integrable map. It is possible to prove the *commutativity* of Bäclund transformation:

$$B_{\lambda_1}\circ B_{\lambda_2}=B_{\lambda_2}\circ B_{\lambda_1}. \tag{2.37}$$

In order to define the *spectrality* of Bäclund transformation it is convenient to introduce the quantity μ, canonically conjugated to λ:

$$\mu=\frac{\partial F_\lambda}{\partial\lambda}=\sum_{j=1}^{N}(x_j-y_j). \tag{2.38}$$

The spectrality of BT then means that the pair (e^μ,λ) lies on the spectral curve of the Lax matrix. Since one has $\det L(u)=1$, it means that both e^μ and $e^{-\mu}$ are eigenvalues of $L(\lambda)$:

$$W(e^{\pm\mu},\lambda)\equiv\det(e^{\pm\mu}-L(\lambda))=0.$$

(it does not matter if one takes $L(\lambda;X,x)$ or $L(\lambda;Y,y)$ since they are isospectral).

In order to prove this relation one should to show that, say e^{μ} is an eigenvalue of the matrix $L(\lambda; Y, y)$. The corresponding eigenvector we denote ω_1:

$$L(\lambda; Y, y)\omega_1 = e^{\mu}\omega_1. \tag{2.39}$$

It can be constructed explicitly: as it follows from (2.36) $\det(M_j(u, \lambda)) = \lambda - u$ and the matrix $M_j(\lambda, \lambda)$ degenerates into a projector for $u = \lambda$:

$$M_j(\lambda, \lambda) = \begin{pmatrix} e^{y_j} \\ 1 \end{pmatrix} \times (e^{-x_{j-1}}, -1) \tag{2.40}$$

and up to an overall factor has the unique null-vector

$$\omega_j = \begin{pmatrix} e^{x_{j-1}} \\ 1 \end{pmatrix}, \qquad M_j(\lambda, \lambda)\omega_j = 0. \tag{2.41}$$

Multiplying the identity (2.34) with $M = M_1$ by ω_1 one concludes:

$$M_1(\lambda, \lambda)L(\lambda; Y, y)\omega_1 = 0. \tag{2.42}$$

Then the uniqueness of the null-vector ω_1 of M_1, implies that ω_1 is an eigenvector of the monodromy matrix $L(\lambda; Y, y)$. Multiplying the identity (2.35) by ω_j one obtains:

$$M_{j+1}(\lambda, \lambda)\ell_j(\lambda, Y, y)\omega_j = 0, \qquad \Rightarrow \qquad \ell_j(\lambda, Y, y)\omega_j \sim \omega_{j+1}.$$

One then calculates:

$$\ell_j(\lambda, Y_j, y_j)\omega_j = e^{x_{j-1}-y_j}\omega_{j+1},$$

where in expression for $\ell_j(\lambda, Y_j, y_j)$ the matrix element $\lambda + Y_j$ is replaced by $e^{x_j - y_j} + e^{y_j - x_{j-1}}$ by (2.30).

Then, acting by $L(\lambda, Y, y)$ on ω_1 one deduces:

$$L(\lambda, Y, y)\omega_1 = e^{\sum_{j=1}^{N}(x_{j-1}-y_j)}\omega_{N+1} = e^{\mu}\omega_1.$$

One can also transform $\ell_j(\lambda, Y_j, y_j)$ using triangular matrix N:

$$\hat{\ell}_j \equiv N_{j+1}^{-1}\ell_j(\lambda, Y_j, y_j)N_j = \begin{pmatrix} e^{y_j - x_{j-1}} & 0 \\ e^{-y_j} & e^{x_{j-1}-y_j} \end{pmatrix}, \qquad N_j = \begin{pmatrix} 1 & e^{x_{j-1}} \\ 0 & 1 \end{pmatrix}, \tag{2.43}$$

then for the transfer-matrix one has:

$$t(\lambda) = \mathrm{tr}(\ell_N(\lambda) \dots \ell_1(\lambda)) = \mathrm{tr}(\hat{\ell}_N(\lambda) \dots \hat{\ell}_1(\lambda)) = e^{\mu} + e^{-\mu}, \tag{2.44}$$

here the transition from ℓ to $\hat{\ell}$ is allowed because all internal matrices N_j cancel each to other and edge matrices canceled due to periodicity $N_{N+1} = N_1$ and the cyclic property of trace. The last relation follows as the product of triangular matrices is also triangular matrix and it becomes evident that $L(\lambda)$ has two eigenvalues $e^{\pm\mu}$.

2.2 Duality

The system under consideration along with description in terms 2×2 Lax matrix $L(u)$ admits description in terms of $N \times N$ Lax matrix $\mathcal{L}(v)$. These are dual with respect to interchanging the spectral parameters u and v:

$$(-1)^{n-1} \det(u - \mathcal{L}(v)) = \det(v - L(u)). \tag{2.45}$$

To define the dual Lax matrix $\mathcal{L}(v)$ one can take an eigenvector $\theta_1(u)$ of $L(u)$ corresponding to the eigenvalue v:

$$L(u)\theta_1 = v\theta_1, \tag{2.46}$$

and define by induction:

$$\theta_{j+1} = \ell_j(u)\theta_j, \qquad\qquad j = 1, \ldots, N. \tag{2.47}$$

It follows from (2.46) that θ_j is *quasiperiodic*: $\theta_{n+1} = v\theta_1$. The function $\theta_j(u)$ is called Baker-Akhiezer function. Denoting the components of the vector θ_j as φ_j and ψ_j one can rewrite (2.47) as:

$$\begin{pmatrix} \varphi_{j+1} \\ \psi_{j+1} \end{pmatrix} = \begin{pmatrix} u + X_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix} \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}. \tag{2.48}$$

This matrix equation is equivalent to following sets of equations:

$$u\varphi_j = \varphi_{j+1} - X_j\varphi_j + e^{x_j}\psi_j, \qquad j = 1, \ldots, N-1, \qquad u\varphi_N = v\varphi_1 - X_N\varphi_N + e^{x_N}\psi_N, \tag{2.49}$$

$$\psi_{j+1} = e^{-x_j}\varphi_j, \qquad\qquad j = 1, \ldots, N-1, \qquad\qquad v\psi_1 = e^{-x_N}\varphi_N, \tag{2.50}$$

Eliminating ψ_j one obtains a second-order finite-difference equation for φ_j:

$$\begin{gathered} u\varphi_1 = \varphi_2 - X_1\varphi_1 + \frac{1}{v}e^{x_1 - x_N}\varphi_N, \\ u\varphi_j = \varphi_{j+1} - X_j\varphi_j + e^{x_j - x_{j-1}}\varphi_{j-1}, \qquad j = 1, \ldots, N-1, \\ u\varphi_N = v\varphi_1 - X_N\varphi_N + e^{x_N - x_{N-1}}\varphi_{N-1}, \end{gathered} \tag{2.51}$$

which can be rewritten in matrix form as:

$$\mathcal{L}(v)\Phi = u\Phi, \quad \Phi = \begin{pmatrix} \varphi_1 \\ \ldots \\ \varphi_N \end{pmatrix}, \qquad \mathcal{L}(v) = \begin{pmatrix} -X_1 & 1 & \ldots & 0 & \frac{1}{v}e^{x_1 - x_N} \\ e^{x_2 - x_1} & -X_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -X_{N-1} & 1 \\ v & 0 & \ldots & e^{x_N - x_{N-1}} & -X_N \end{pmatrix}, \tag{2.52}$$

where $\mathcal{L}(v)$ is the dual Lax matrix. As for 2×2 matrix $L(u)$ in this case $\mathcal{L}(v)$ also must exist a Darboux matrix M intertwining $\mathcal{L}(v; X, x)$ and $\mathcal{L}(v; Y, y)$. The explicit expression for M, as for $\mathcal{L}(v)$, can be found from the Baker-Akhiezer function. Let vectors θ_j and $\tilde{\theta}_j$ refer, respectively, to $\mathcal{L}(v; X, x)$ and $\mathcal{L}(v; Y, y)$ and are related each to other as: $\theta_j = M\tilde{\theta}_j$:

$$\begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix} = \begin{pmatrix} u - \lambda + e^{y_j - x_{j-1}} & -e^{y_j} \\ e^{-x_{j-1}} & -1 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_j \\ \tilde{\psi}_j \end{pmatrix}. \tag{2.53}$$

Now taking the first line

$$\varphi_j = (u - \lambda + e^{y_j - x_{j-1}})\tilde{\varphi}_j - e^{y_j}\tilde{\psi}_j,$$

and substituting

$$u\tilde{\varphi}_j = \tilde{\varphi}_{j+1} - Y_j\tilde{\varphi}_j + e^{y_j - y_{j-1}}\tilde{\varphi}_{j-1}, \qquad \tilde{\psi}_j = e^{-y_{j-1}}\tilde{\varphi}_{j-1},$$

from $\tilde{\theta}_{j+1} = \ell_j(u; Y_j, y_j)\tilde{\theta}_j$, as well as $Y_j = e^{x_j - y_j} + e^{y_j - x_{j-1}} - \lambda$ from (2.30), one obtains, after making the necessary correction for $j = N$ the following result:

$$\varphi_j = \tilde{\varphi}_{j+1} - e^{x_j - y_j}\tilde{\varphi}_j, \qquad j = 1, \dots, N-1 \tag{2.54}$$

$$\varphi_N = v\tilde{\varphi}_1 - e^{x_N - y_N}\tilde{\varphi}_N.$$

or, in matrix form, $\Theta = \mathcal{M}\Theta$, with

$$\mathcal{M}(v) = = \begin{pmatrix} -e^{x_1 - y_1} & 1 & \dots & 0 & 0 \\ 0 & -e^{x_2 - y_2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -e^{x_{N-1} - y_{N-1}} & 1 \\ v & 0 & \dots & 0 & -e^{x_N - y_N} \end{pmatrix}. \tag{2.55}$$

By construction, one has:

$$\mathcal{M}(v)\mathcal{L}(v; Y, y) = \mathcal{L}(v; X, x)\mathcal{M}(v). \tag{2.56}$$

Alternatively, one could introduce the inverse to (2.35) $\tilde{M}_j = (\lambda - u)M_j^{-1}$:

$$\tilde{M}_j(u, \lambda) = \begin{pmatrix} 1 & -e^{y_j} \\ e^{-x_{j-1}} & \lambda - u - e^{y_j - x_{j-1}} \end{pmatrix}, \tag{2.57}$$

such that

$$\tilde{M}_{j+1}(u, \lambda)\ell_j(u, X_j, x_j) = \ell_j(u, Y_j, y_j)\tilde{M}_j(u, \lambda).$$

Repeating the same calculation, starting from $\tilde{\theta}_j = \tilde{M}_j\theta_j$ one will obtain $\tilde{\Theta} = \tilde{\mathcal{M}}\Theta$, with

$$\mathcal{M}(v) = = \begin{pmatrix} 1 & 0 & \dots & 0 & -\frac{1}{v}e^{y_1 - x_N} \\ -e^{y_2 - x_1} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -e^{y_N - x_{N-1}} & 1 \end{pmatrix}, \tag{2.58}$$

satisfying

$$\tilde{\mathcal{M}}(v)\mathcal{L}(v; X, x) = \mathcal{L}(v; Y, y)\tilde{\mathcal{M}}(v). \tag{2.59}$$

One has $\tilde{\mathcal{M}} \neq \mathcal{M}^{-1}$, however formulas (2.56) and (2.59) are compatible because of the remarkable factorization property of $\mathcal{L}(v)$:

$$\mathcal{L}(v; X, x) - \lambda I = \mathcal{M}(v)\tilde{\mathcal{M}}(v), \qquad \mathcal{L}(v; Y, y) - \lambda I = \tilde{\mathcal{M}}(v)\mathcal{M}(v). \tag{2.60}$$

2.3 Quantum case

As usual the quantization assumes substitution $X_j = -i\partial_j$. Then starting from the quantum Lax matrix:

$$\ell_j = \begin{pmatrix} u - \partial_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix}, \tag{2.61}$$

one observes that it obeys the quadratic relation:

$$R_{12}(u_1 - u_2)\ell^{(1)}(u_1)\ell^{(2)}(u_2) = \ell^{(2)}(u_2)\ell^{(1)}(u_1)R_{12}(u_1 - u_2), \tag{2.62}$$

where $R_{12}(u) = uI + i\mathcal{P}_{12}$. As at classical level it possesses co-multiplication property and implies:

$$R_{12}(u_1 - u_2)L^{(1)}(u_1)L^{(2)}(u_2) = L^{(2)}(u_2)L^{(1)}(u_1)R_{12}(u_1 - u_2),$$

as well as the commutativity of transfer-matrices: $[t(u_1), t(u_2)] = 0$. The original idea of Baxter which enabled him to solve the XYZ spin chain was to construct an one-parametric family of operators Q_λ commuting with the integrals of motion (Hamiltonians of the model):

$$[Q_\lambda, t(u)] = 0, \tag{2.63}$$

and hence sharing with $t(u)$ the common set of eigenvectors. Moreover, Q_λ must satisfy the *Baxter equation*:

$$Q_\lambda t(\lambda) = \Delta_+(\lambda)Q_{\lambda+i} + \Delta_-(\lambda)Q_{\lambda-i} \tag{2.64}$$

where $\Delta_\pm(\lambda)$ are some scalar functions determined by the parameters of model. Applying the Baxter equation to a common eigenvector of Q_λ and $t(\lambda)$ one can replace these operators in by their eigenvalues. The resulting second order finite-difference equation for the eigenvalues of Q_λ considered in an appropriate functional class allows then to determine the spectrum of $t(\lambda)$. Baxter succeeded to built a Q-operator for the XYZ spin chain as a trace of a monodromy matrix:

$$Q_\lambda = \mathrm{tr}_V\, \mathbb{L}_N(\lambda) \dots \mathbb{L}_1(\lambda), \tag{2.65}$$

constructed with a specially chosen auxiliary space V.

Such properties of Bäcklund transformation as the invariance of Hamiltonians (2.33) and spectrality (2.44) are the classical counterparts of such properties of Q-operator as commutativity (2.63) and the Baxter equation (2.64). The first one is obvious. The shift operators $\lambda \to \lambda \pm i$ are expressed as $\exp(\pm i\partial_\lambda) = \exp(\pm\mu)$ where μ is the canonical momentum conjugate to λ. So one can rewrite (2.65) in the form $t(\lambda) = \Delta_+(\lambda)e^{-\mu} + \Delta_-(\lambda)e^{\mu}$ which gives (2.44) in the classical limit (for the Toda lattice $\Delta_\pm = 1$).

2.4 Q-operator for Toda lattice

The Q-operator is represented as the trace of the monodromy matrix (2.65) taking for the auxiliary space V the space $C[s]$ of polynomials in variable s. The corresponding representation of the yangian $\mathcal{Y}[gl_2]$ then is realized as the Lax operator of the quantum DST model:

$$M(u, \lambda) = \begin{pmatrix} u - \lambda - i\partial_s & -s \\ -i\partial_s & -1 \end{pmatrix}. \tag{2.66}$$

To prove the commutativity (2.63) it is sufficient to establish the identity:

$$M(u,\lambda)\ell(u)\mathbb{L}_\lambda = \mathbb{L}\lambda\ell(u)M(u,\lambda) \tag{2.67}$$

which can be considered as a Yang-Baxter equation (2.62) with the following layout of spaces: $V_1 = \mathbb{C}^2$, $V_2 = \mathbb{C}[s]$, $V_3 = \mathcal{L}_2(\mathbb{R}^N)$. One can use this equation to determine $\mathbb{L}_\lambda$ rewriting it as the system of equations for the kernel $\mathbb{L}_\lambda(t,y|s,x)$ of $\mathbb{L}_\lambda$:

$$\begin{pmatrix} u-\lambda-it\partial_t & -t \\ -i\partial_t & -1 \end{pmatrix}\begin{pmatrix} u-i\partial_y & -e^y \\ e^{-y} & 0 \end{pmatrix}\mathbb{L}_\lambda(t,y|s,x) = \tag{2.68}$$

$$= \begin{pmatrix} u+i\partial_x & -e^x \\ e^{-x} & 0 \end{pmatrix}\begin{pmatrix} u-\lambda+i+i\partial_s & -s \\ i\partial_s & -1 \end{pmatrix}\mathbb{L}_\lambda(t,y|s,x).$$

The solution is unique, up to a scalar factor:

$$\mathbb{L}_\lambda(t,y|s,x) \sim \delta(s-e^y)\exp(ite^{-x} - ie^{x-y} + i\lambda(x-y)). \tag{2.69}$$

The kernel of Q_λ then given by:

$$Q_\lambda(y|x) = \int ds_N \ldots \int ds_1 \prod_{j=1}^N \mathbb{L}_\lambda(s_{j+1},y_j|s_j,x_j). \tag{2.70}$$

The integration over s_j here reduces, due to the delta-function to the substitution $s_j = e^{y_j}$.

Finally, one has:

$$Q_\lambda(y|x) = \prod_{j=1}^N \exp(ie^{y_{j+1}-x_j} - ie^{x_j-y_j} + i\lambda(x_j-y_j)). \tag{2.71}$$

$Q_\lambda(y|x) = \exp(-iF_\lambda(y|x))$ where $F_\lambda(y|x)$ is the generating function (2.32) of the classical BT, that is the semiclassical formula $Q_\lambda(y|x) = \exp(-\frac{i}{\hbar}F_\lambda(y|x))$ is exact in this case. It is an accidental peculiarity of Toda lattice which usually does not hold for other models.

Now one has to prove the Baxter equation (2.64) for Q_λ. Note that the kernel (2.71) factorizes as:

$$Q_\lambda(y|x) = \prod_{j=1}^N w_j(\lambda), \qquad w_j(\lambda) = \exp(ie^{y_{j+1}-x_j} - ie^{x_j-y_j} + i\lambda(x_j-y_j)). \tag{2.72}$$

Applying then $t(\lambda)$ to $Q_\lambda(y|x)$ one observes that each $\ell_j(\lambda; -i\partial_{y_j}, y_j)$ acts locally only on $w_j(\lambda)$ and obtain:

$$t(\lambda)Q_\lambda(y|x) = \operatorname{tr}(\ell_N(\lambda)w_N(\lambda))\ldots(\ell_1(\lambda)w_1(\lambda)) = Q_\lambda(y|x)\operatorname{tr}\tilde{\ell}_N\ldots\tilde{\ell}_1, \tag{2.73}$$

where

$$\tilde{\ell}_j = \ell_j(\lambda)\ln w_j(\lambda) = \begin{pmatrix} e^{y_j-x_{j-1}} + e^{x_j-y_j} & -e^{y_j} \\ e^{-y_j} & 0 \end{pmatrix}. \tag{2.74}$$

Then one can use the triangular gauge transformation $\hat{\ell}_j = N_{j+1}^{-1}\tilde{\ell}_j N_j$ with N_j the same as in classical case (2.43) with the same matrix and the resulting matrix blj given by the same matrix $\hat{\ell}_j$ (2.43). Noticing that

$$\frac{w_j(\lambda+i)}{w_j(\lambda)} = e^{y_j - x_{j-1}}, \qquad \frac{w_j(\lambda-i)}{w_j(\lambda)} = e^{x_{j-1}-y_j}, \tag{2.75}$$

one obtain desirable result:

$$t(\lambda)Q_\lambda = Q_{\lambda+i} + Q_{\lambda-i}. \tag{2.76}$$

2.5 KdV hierarchy

The Kortewegde Vries (KdV) equation describes a mathematical model of waves on shallow water surfaces. It is notable as the prototypical example of an exactly solvable model, that is, a non-linear partial differential equation, solutions to which can be specified exactly. The solutions include prototypical examples of solitons. KdV can be solved by means of the inverse scattering transform. The KdV equation is a nonlinear, dispersive partial differential equation for a function u of two real variables, space x and time t:

$$4\partial_t u - \partial_x^3 u - 6u\partial_x u = 0, \tag{2.77}$$

with ∂_x and ∂_t denoting partial derivatives with respect to x and t.

Let us consider solutions in which a fixed wave form (given by $f(X)$) maintains its shape as it travels to the right at phase speed c. Such a solution is $u(x,t) = f(x+ct-a) = f(X)$. Substituting it into the KdV equation gives the ordinary differential equation

$$-c\frac{df}{dX} + \frac{d^3 f}{dX^3} + 6f\frac{df}{dX} = 0,$$

or, after integration with respect to X,

$$-cf + \frac{d^2 f}{dX^2} + 3f^2 = A$$

where A is a constant of integration. Considering the independent variable X above as a virtual time variable, this means f satisfies Newton's equation of motion in a cubic potential. If the potential function $V(X)$ has local maximum at $X = 0$, there is a solution in which $f(X)$ starts at this point at 'virtual time' $-\infty$, eventually slides down to the local minimum, then back up the other side, reaching an equal height, then reverses direction, ending up at the local maximum again at infinite time. In other words, $f(X)$ approaches 0 as $X \to \pm\infty$. This is the characteristic shape of the solitary wave solution. The solution is

$$u(x,t) = \frac{1}{2c}\mathrm{ch}^{-2}\left[\frac{\sqrt{c}}{2}(x+ct-a)\right]$$

The KdV equation has infinitely many integrals, which can be given explicitly as

$$\int_{-\infty}^{+\infty} P_{2n-1}(u, \partial_x u, \partial_x^2 u, \ldots)dx,$$

where the polynomials P_n are defined recursively by

$$P_1 = u,$$

$$P_n = -\frac{dP_{n-1}}{dx} + \sum_{i=1}^{n-2} P_i\, P_{n-1-i} \quad \text{for} \qquad n \geq 2.$$

The first few integrals of motion are: the momentum $\int u dx$, the energy $\int u^2 dx$, $\int [\frac{1}{3}u^3 - (\partial_x u)^2] dx$.

The KdV equation

$$4\dot{u} = 6uu' + u''',$$

is rewritten as the Lax equation:

$$L_t = [L, A] \equiv LA - AL$$

with a SturmLiouville operator L:

$$L = \partial_x^2 + u, \tag{2.78}$$

$$A = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u'$$

and this accounts for the infinite number of first integrals of the KdV equation.

The Kortewegde Vries equation (2.77) is the EulerLagrange equation of motion which follows from the Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\partial_x\psi\, \partial_t\psi + (\partial_x\psi)^3 - \frac{1}{2}\left(\partial_x^2\psi\right)^2 \tag{2.79}$$

with $u = \frac{\partial\psi}{\partial x} = \partial_x\psi$.

The KdV equation give rise an infinite sequence of partial differential equations (KdV hierarchy) which starts with the Kortewegde Vries equation itself.

Introduce an algebra of *Pseudo-Differential Operators* (PDO), formal series:

$$R = \sum_{i=-\infty}^{n} X_i\partial^i = \sum_{i=0}^{n} X_i\partial^i + \sum_{i<0} X_i\partial^i \equiv R_+ + R_-, \tag{2.80}$$

where the inverse derivative is defined via generalized Leibnitz rule:

$$\partial^k f = \sum_{i=0}^{\infty} \left(\begin{array}{c} k \\ i \end{array} \right) f^{(i)}\partial^{k-i}, \tag{2.81}$$

here $\left(\begin{array}{c} k \\ i \end{array} \right) = \frac{k(k-1)\ldots(k-i+1)}{i!}$. If $k \geq 0$, this series terminates because $\left(\begin{array}{c} k \\ i \end{array} \right) = 0$ if $i > k$. For instance $\partial f = f\partial + f'$, $\partial^2 f = f\partial^2 + 2f'\partial + f''$. For $k < 0$ the series is actually infinite and one has, for instance, $\partial^{-1} f = \sum_{i=0}^{\infty}(-1)^i f^{(i)}\partial^{-1-i}$, where $\left(\begin{array}{c} -1 \\ i \end{array} \right) = (-1)^i$ is used.

Using PDO one can write down an infinite series of operators, commuting with given differential operator L. Indeed, suppose we are given by operator (2.78), it obviously commutes with operator

$$L^{\frac{1}{2}} = L_+^{\frac{1}{2}} + L_-^{\frac{1}{2}} = \partial_x + \frac{u}{2}\partial^{-1} - \frac{u'}{4}\partial^{-2} + o(\partial^{-3}).$$

So one concludes, that for arbitrary integer m, commutator $[(L^{\frac{m}{2}})_+, L]$ does not contains derivatives:

$$[(L^{\frac{m}{2}})_+, L] = [L^{\frac{m}{2}} - (L^{\frac{m}{2}})_-, L] = -[(L^{\frac{m}{2}})_-, L], \tag{2.82}$$

the operator $(L^{\frac{m}{2}})_-$ has order -1, operator L has order 2 and commutator decreases order by one. So that means every odd positive integer $m = 2k + 1$ gives rise an equivalent hamiltonian dynamic according to equation:

$$\frac{\partial L}{\partial \tau_k} = [H_{2k+1}, L] = [L_+^{\frac{2k+1}{2}}, L], \tag{2.83}$$

note that even integers $m = 2k$ correspond to trivial dynamics with zero Hamiltonian (the commutator (2.82) vanishes). The time $t = \tau_1$ corresponds to KdV equation itself and $H_3 = A$ in (2.78).

Exercise *Suppose operator L satisfies (2.83) for some operator H_k. Prove that spectrum of L is independent on τ_k, i.e. $L(\tau_k) = U^{-1}(\tau_k)L_0U(\tau_k)$.*
The set of functionals:

$$I_k = \int L_+^{\frac{2k+1}{2}}\, dx \tag{2.84}$$

forms the infinite variety integrals of motion.

The similar equation

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \tag{2.85}$$

where L is general operator of form $L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \dots$ describes more general Kadomcev-Petviashvili hierarchy.

Exercise *Show that general operator of form $\partial^n + a\partial^{n-1} + \dots$ can be transformed to form $\partial^n + c\partial^{n-2} + \dots$.*
One can make *reduction* to return to soliton-hierarchies imposing additional constraint

$$L^m \in D_k,$$

(some integer power of L is a differential operator). At $m = 2$ one comes to KdV hierarchy, at $m = 3$ one comes to Boussinesq hierarchy. Operator L has form:

$$L = \partial^2 + u\partial + v,$$

We compute

$$L^{1/3} = \partial + \frac{1}{3}u\partial^{-1} + o(\partial^{-2}), \qquad H_2 = L_+^{2/3} = \partial^2 + \frac{2}{3}u,$$

and

$$[H_2, L] = (2v' - u'')\partial + v'' - \frac{2}{3}u''' - \frac{2}{3}uu'.$$

It is equivalent to set

$$\dot{u} = 2v' - u'', \qquad \dot{v} = v'' - \frac{2}{3}u''' - \frac{2}{3}uu'.$$

One can eliminate v between the two equations and the result is the Boussinesq equation:

$$\ddot{u} = -\frac{1}{3}u''' - \frac{4}{3}(uu')'.$$

3 Separation of Variables, 4 hours

This two lections are devoted to Hamilton-Jacobi Method and Baxter Q-operator in the problem of Separation of variables.

3.1 Hamilton-Jacobi equation and Separation of variables

The principle of least (stationary) action determines the trajectory of classical system in configuration space by requirement of the minimal value of action integral:

$$S = \int_{t_1}^{t_2} L dt. \tag{3.86}$$

For a wide class of problems, the Lagrangian depends only on the generalized coordinates and their first time derivatives: $L = L(q_i, \dot{q}_i)$, so one has for variation of an action:

$$\delta S = \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt. \tag{3.87}$$

Assuming the initial and final state of the system fixed $\delta q(t_1) = \delta q(t_2) = 0$ one deduces the extremum condition of trajectory, Euler-Lagrange equations or equations of motion of the system:

$$\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}. \tag{3.88}$$

On the trajectories of the actual motion that satisfy the Euler-Lagrange, the integrand in (3.87) vanishes and one the first term in (3.87) determines dependence the action on generalized coordinates of final state $q^i(t_2)$ at fixed initial ones $\delta q^i(t_1) = 0$:

$$\delta S = \sum_i p_i \delta q^i, \tag{3.89}$$

which means that partial derivatives

$$\frac{\partial S}{\partial q^i} = p_i. \tag{3.90}$$

By the very definition of the action, its total time derivative along the trajectory is

$$\frac{dS}{dt} = L, \tag{3.91}$$

from the other hand, from eq. (3.90) one deduces:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q^i}\dot{q}^i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}^i, \tag{3.92}$$

or

$$\frac{\partial S}{\partial t} = L - \sum_i p_i \dot{q}^i \equiv -H. \tag{3.93}$$

Formulas (3.90) and (3.93) together can be written as:

$$dS = \sum_i p_i dq^i - H dt. \tag{3.94}$$

The Hamilton-Jacobi equation is a first-order, non-linear partial differential equation:

$$H + \frac{\partial S}{\partial t} = 0, \tag{3.95}$$

where $H = H\left(q_1, \cdots, q_s; \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_s}; t\right)$ is the Hamiltonian of system, $S = S(q_1, q_2 \cdots q_s, t)$ is the classical action, q_i are the s generalized coordinates ($i = 1, 2 \ldots s$) which define the configuration of the system, and t is time.

Hamilton - Jacobi equation is directly related to the classical mechanics, however, it is well suited for communication between classical and quantum mechanics and can be to get directly from the Schrödinger equation in the approximation of a rapidly oscillating wave function (higher frequencies and wave numbers).

In classical mechanics, there is usually a special canonical transformation of the classical Hamiltonian, which leads to this non-linear differential equation of the first order, the solution of which describes the behavior of dynamic systems.

Along with the s Euler-Lagrange equations and $2s$ Hamilton equations, the single Hamilton-Jacobi equation is also the basis of a general method for integrating the equations of motion.

The independent variables of the Hamilton - Jacobi are s coordinates and time, so the total integral of the equation contains $s + 1$ arbitrary constants:

$$S = f(t, q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_s) + A, \tag{3.96}$$

where α_i and A are arbitrary constants and A enter additively because S enters in (3.95) only through its derivatives.

Let us now consider the canonic transformation of variables (p, q) with generating function $f(t, q, \alpha)$, depending on coordinates old q^i and new momenta α_i. New coordinates are denoted as β_i. So, according to (3.94), the variational principles in both systems differ at most by total differential:

$$\sum_i p_i dq^i - H dt = \sum_i \alpha_i d\beta^i - H' dt + dF, \tag{3.97}$$

or

$$df = d(f - \sum_i \alpha_i \beta^i) = \sum_i p_i dq^i + \sum_i \beta^i d\alpha_i - (H' - H)dt, \tag{3.98}$$

here f is just our generating function, depending on q^i and α_i, so one has:

$$p_i = \frac{\partial f}{\partial q^i}, \qquad \beta^i = \frac{\partial f}{\partial \alpha_i}, \qquad H' = H + \frac{\partial f}{\partial t}. \tag{3.99}$$

However, the generating function (3.96) is a solution of Hamilton-Jacobi equation, so H' vanishes identically and in new variables canonical equations takes the form:

$$\dot{\alpha}_i = 0, \qquad \dot{\beta}^i = 0, \qquad \Rightarrow \alpha_i = const, \qquad \beta^i = const, \tag{3.100}$$

then s equations $\frac{\partial f}{\partial \alpha_i = \beta^i}$ allow to express coordinates q^i as functions of time and $2s$ constants α and β.

Separation of variables Let a coordinate (say q_1) and the corresponding momentum $\frac{\partial S}{\partial q_1}$ enter the equation in the form:

$$\frac{\partial S}{\partial t} + H\left(\varphi_1\left(q_1, \frac{\partial S}{\partial q_1}\right), q_2, \ldots, q_s, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_s}\right) = 0. \tag{3.101}$$

So one can look for solution of form:

$$S = S'(q_i, t) + S_1(q_1, t), \tag{3.102}$$

so the total integral of Hamilton-Jacobi equation will take the form:

$$\Phi(t, \varphi_1(q_1, \frac{\partial S_1}{\partial q_1}), q_i, \frac{\partial S'}{\partial q_i}) = 0. \tag{3.103}$$

Suppose we are given by solution (3.102) then (3.103) is satisfying identically at varying q_i. Because, only function $\varphi_1(q_1, \frac{\partial S_1}{\partial q_1})$ can change upon varying q_1, (3.103) implies

$$\varphi_1(q_1, \frac{\partial S_1}{\partial q_1}) = \alpha_1 = const, \tag{3.104}$$

and

$$\Phi(t, \alpha_1, q_i, \frac{\partial S'}{\partial q_i}) = 0. \tag{3.105}$$

The first is an ordinary differential equation, from which $S_1(q_1)$ can be determined, while the latter corresponds to problem with $s - 1$ degrees of freedom.

As an example the Hamiltonian with spherical symmetry

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) + U(r, \theta, \varphi), \qquad U(r, \theta, \varphi) = a(r) + \frac{b(\theta)}{r^2}, \tag{3.106}$$

can be considered. The variable φ is cyclic, so corresponding momentum conserved:

$$p_\varphi = const$$

and substituting

$$S = p_\varphi \varphi + S_1(r) + S_2(\theta), \tag{3.107}$$

in Hamilton-Jacobi equation one obtains:

$$\left(\frac{dS_2}{d\theta}\right)^2 + 2mb(\theta) + \frac{p_\varphi^2}{\sin^2\theta} = \beta,$$

$$\left(\frac{dS_1}{dr}\right)^2 + 2ma(r) + \frac{\beta}{r^2} = E.$$

Integrating one has:

$$S = -Et + p_\varphi \varphi + \int d\theta \sqrt{\beta - 2mb(\theta) - \frac{p_\varphi^2}{\sin^2\theta}} + \int dr \sqrt{2m[E - a(r)] - \frac{\beta}{r^2}}. \tag{3.108}$$

Taking derivatives with respect to arbitrary constants p_φ, β and E and putting result equal to new constants one will find the solution of equations of motion.

3.2 Separation of variables in classical $SL(3)$ Heisenberg chain

Let we are given by a completely integrable Hamiltonian system with D degrees of freedom. According to Liouville-Arnold definition of complete integrability it means that the system possesses exactly D independent Hamiltonians H_j commuting with respect to the Poisson bracket:

$$\{H_j, H_k\} = 0, \qquad j, k = 1, \ldots, D. \tag{3.109}$$

The separation of variables is understood as construction of D pairs of canonical variables x_j, p_j $(j = 1, \ldots, D)$

$$\{x_j, x_k\} = \{p_j, p_k\} = 0, \qquad \{p_j, x_k\} = \delta_{jk} \tag{3.110}$$

and D functions Φ_j such that

$$\Phi_j(x_j, p_j, H_1, H_2, \ldots, H_D) = 0, \qquad j = 1, 2, \ldots, D \tag{3.111}$$

where H_j are the Hamiltonians (3.109) in involution.

The $SL(N)$-symmetric magnetic chain is described in terms of the variables $S^{(m)}_{\alpha\beta}$, $(\alpha, \beta = 1, \ldots, N;\ m = 1, \ldots, M;\ \sum_{\alpha=1}^N S^{(m)}_{\alpha\alpha} = 0)$ which subject to the Poisson brackets

$$\{S^{(m)}_{\alpha_1\beta_1}, S^{(n)}_{\alpha_2\beta_2}\} = \delta_{mn}(S^{(m)}_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1} - S^{(m)}_{\alpha_2\beta_1}\delta_{\alpha_1\beta_2}). \tag{3.112}$$

The center of this algebra is generated by the eigenvalues $l^{(m)}_\alpha$ of the matrices $S^{(m)}$

$$\det(u + S^{(m)}) = \prod_{\alpha=1}^N (u + l^{(m)}_\alpha), \qquad \sum_{\alpha=1}^N l^{(m)}_\alpha = 0. \tag{3.113}$$

The Poisson bracket (3.112) is hence nondegenerate on the manifold (3.113) having dimension $D = MN(N-1)/2$ for the case of generic orbit (all eigenvalues of $S^{(m)}$ supposed to be distinct). We always assume that the orbit is generic. Let Z be an invertible $N \times N$ number matrix having N distinct eigenvalues and let δ_m $(m = 1, \ldots, M)$ be some fixed numbers. u is a complex parameter (spectral parameter). Define the product (monodromy matrix)

$$T(u) = Z(u - \delta_M + S^{(M)}) \ldots (u - \delta_2 + S^{(2)})(u - \delta_1 + S^{(1)}). \tag{3.114}$$

Proposition 1 Matrix elements of $T(u)$ have the Poisson brackets

$$\{T_{\alpha_1\beta_1}(u), T_{\alpha_2\beta_2}(v)\} = \frac{1}{u-v}(T_{\alpha_2\beta_1}(u)T_{\alpha_1\beta_2}(v) - T_{\alpha_1\beta_2}(u)T_{\alpha_2\beta_1}(v)) \tag{3.115}$$

The proof follows from the fact that the factors $(u-\delta_m+S^{(m)})$ have the same Poisson brackets (3.115) which reproduce themselves for the product $T(u)$ (Lie-Poisson group structure). Due to notation ${}^1T = T\otimes \mathrm{id}$, ${}^1T = \mathrm{id}\otimes T$ one can write this formula in a compact form:

$$\{{}^1T(u), {}^2T(v)\} = \frac{1}{u-v}[\mathcal{P}, {}^1T(u)\otimes {}^2T(v)], \tag{3.116}$$

where $\mathcal{P}$ is permutation operator on $\mathbb{C}^N\otimes\mathbb{C}^N$.

The spectral invariants $t_\nu(u)$ of the matrix $T(u)$ are defined as the elementary symmetric polynomials of its eigenvalues

$$t_\nu(u) \equiv \mathrm{tr}\bigwedge^\nu T(u), \qquad\qquad \nu = 1,\dots,N.$$

For instance, $t_1(u) = \mathrm{tr}\,T(u)$, $t_2(u) = \frac{1}{2}(\mathrm{tr}^2\,T(u) - \mathrm{tr}\,T^2(u)), \dots t_N(u) = \det T(u) \equiv d(u)$. The central functions $l_\alpha^{(m)}$ are contained in the determinant $d(u) = \det T(u)$.

Proposition 2 The non-leading coefficients at powers of u of the polynomials $t_\nu(u)$, $\nu = 1,\dots,(N-1)$, form a commutative family of $MN(N-1)/2$ independent Hamiltonians.

Proof. The polynomial $t_\nu(u)$ has power νM in u and contributes νM Hamiltonians (its leading coefficient is a number), the total number of Hamiltonians is $M(1+2+\dots+(N-1)) = MN(N-1)/2$. The commutativity of $t_\nu(u)$

$$\{t_\mu(u), t_\nu(v)\} = 0, \qquad\qquad \forall u, v$$

is then a consequence of the fundamental Poisson bracket (3.115).

By virtue of this proposition and since the number of Hamiltonians constructed $D = MN(N-1)/2$ equals exactly half dimension of the phase space the system is completely integrable.

Conjecture 1 There exist functions $\mathcal{A}$ and $\mathcal{B}$ on $GL(N)$ such that: $\mathcal{A}(T)$ is an algebraic function and $\mathcal{B}(T)$, respectively, is a polynomial of degree $D = MN(N-1)/2$ of the matrix elements $T_{\alpha\beta}$ and the variables x_j, P_j $(j = 1,\dots,D)$ defined from the equations

$$\mathcal{B}(T(x_j)) = 0, \qquad\qquad P_j = \mathcal{A}(T(x_j)) \tag{3.117}$$

have the Poisson brackets

$$\{x_j, x_k\} = \{P_j, P_k\} = 0, \qquad\qquad \{P_j, x_k\} = P_j\delta_{jk} \tag{3.118}$$

and are bound to the Hamiltonians $t_\nu(u)$ by the relations

$$\det(P_j - T(x_j)) = 0 \tag{3.119}$$

This relation means that P_j is an eigenvalue of the matrix $T(u)$ when $u = x_j$. Putting $P_j = \exp p_j$ one sees that (3.119) fits the form (3.110) since the spectral invariants of $T(u)$ contain only the integrals of motion.

We are going to prove this Conjecture for the cases $N = 2$ and $N = 3$.

$SL(2)$ **case**

The construction of the separation variables for $N = 2$ is to fix notation and to prepare the discussion of more difficult $N = 3$ case. The system under consideration has M degrees of freedom. The spectral invariants of $T(u)$ are now: $t(u) = t_1(u) = \operatorname{tr} T(u)$, $d(u) = t_2(u) = \det T(u)$ the trace $t(u)$ containing M integrals of motion. Let us define:

$$\mathcal{A}(T) = T_{11}, \qquad \mathcal{B}(T) = T_{12} \tag{3.120}$$

and x_j , P_j then are given by the formulas (3.117). The polynomial $B(u) = \mathcal{B}(T(u))$ has M zeroes because its leading coefficient Z_{12} has to be nonzero. It can always be gain by a similarity transform $QT(u)Q^{-1}$ which affects neither basic Poisson brackets (3.115), nor Hamiltonians $t(u)$, since the matrix Z has nondegenerate spectrum. The matrix $T(u)$ becomes triangular at $u = x_j$, so the quantity P_j is an eigenvalue of $T(x_j)$ and satisfies the secular equation (3.119) which in the two-dimensional case has the form

$$P_j^2 - t(x_j)P_j + d(x_j) = 0, \qquad j = 1, \ldots, M. \tag{3.121}$$

This equation defines a hyperelliptic algebraic curve relating P_j and x_j. The proof **Conjecture 1** consists of calculation the Poisson brackets of Ps and xs.

Theorem 1

The Poisson brackets for P_j and x_j are given by (3.118).

Proof.

Let $A(u) = \mathcal{A}(T(u))$ and $B(u) = \mathcal{B}(T(u))$. Choosing particular values of indices in (3.115) one obtains the identities

$$\{A(u), A(v)\} = 0 \tag{3.122}$$

$$\{B(u), B(v)\} = 0 \tag{3.123}$$

$$\{A(u), B(v)\} = \frac{A(u)B(v) - B(u)A(v)}{u - v}. \tag{3.124}$$

The commutativity of B's leads to the commutativity of x_j (zeroes of $B(u)$). The Poisson brackets containing P_j can be calculated using implicit definition of x_j. It follows from $B(x_j) = 0$ that

$$0 = \{F, B(x_j)\} = \{F, B(u)\}_{u=x_j} + B'(x_j)\{F, x_j\}$$

or

$$\{F, x_j\} = -\{F, B(u)\}_{u=x_j}/B'(x_j)$$

for any function F. Similarly

$$\{P_j, F\} = \{A(x_j), F\} = \{A(u), F\}_{u=x_j} + A'(x_j)\{x_j, F\}$$

Now it is seen that $\{P_j, x_k\} = P_j\delta_{jk}$:

$$\{P_j, x_k\} = \{A(u), x_k\}_{u=x_j} + A'(x_j)\{x_j, x_k\} =$$

$$= -\frac{\{A(u), B(v)\}_{u=x_j\ v=x_k}}{B'(x_k)} = \frac{1}{x_j - x_k}\frac{B(x_j)A(x_k) - A(x_j)B(x_k)}{B'(x_k)},$$

the second term in first row is already shown to vanish (3.123). The last expression vanishes for $x_j \neq x_k$ due to $B(x_j) = B(x_k) = 0$ and is evaluated via L'Hopital rule for $x_j = x_k$. The commutativity of P's can be shown similarly starting from (3.122).

$SL(3)$ **case**

The polynomial $T(u)$ at $N = 3$ takes values in 3×3 matrices. The number of degrees of freedom is $D = 3M$. There are three spectral invariants of $T(u)$:

$$t_1(u) = \operatorname{tr} T(u) = \lambda_1 + \lambda_2 + \lambda_3$$

$$t_2(u) = \frac{1}{2}(\operatorname{tr}^2 T(u) - \operatorname{tr} T^2(u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$d(u) = \det T(u) = \lambda_1\lambda_2\lambda_3.$$

They are the coefficients of the characteristic polynomial for $T(u)$

$$det(\lambda - T(u)) = \lambda^3 - t_1(u)\lambda^2 + t_2(u)\lambda - d(u).$$

This secular equation defines a nonhyperelliptic algebraic curve. Introduce the matrix $\mathcal{U}(T)$ for any $T \in GL(3)$:

$$\mathcal{U}(T) = T \wedge T = (T^{-1})^t \det T =$$

$$= \begin{pmatrix} T_{22}T_{33} - T_{23}T_{32} & T_{23}T_{31} - T_{21}T_{33} & T_{21}T_{32} - T_{22}T_{31} \\ T_{13}T_{32} - T_{12}T_{33} & T_{11}T_{33} - T_{13}T_{31} & T_{12}T_{31} - T_{11}T_{32} \\ T_{12}T_{23} - T_{13}T_{22} & T_{13}T_{21} - T_{11}T_{23} & T_{11}T_{22} - T_{12}T_{21} \end{pmatrix}.$$

The matrix elements $\mathcal{U}_{\alpha\beta}$ are algebraic adjuncts of $T_{\alpha\beta}$.

Let $U(u) = \mathcal{U}(T(u))$. The Poisson brackets for T and U are calculated easily from (3.115):

$$\{{}^1T(u), {}^2U(v)\} = -\frac{1}{u - v}[\mathcal{P}^{t_2}, {}^1T(u)^2U(v)] \tag{3.125}$$

or

$$\{T_{\alpha_1\beta_1}(u), U_{\alpha_2\beta 2}(v)\} = \frac{1}{u - v}\sum_{\gamma=1}^{3}(-\delta_{\alpha_1\alpha_2}T_{\gamma\beta_1}(u)U_{\gamma\beta_2}(v) + T_{\alpha_1\gamma}(u)U_{\alpha_2\gamma}(v)\delta_{\beta_1\beta_2})$$

and

$$\{{}^1U(u), {}^2U(v)\} = \frac{1}{u - v}[\mathcal{P}, {}^1U(u)^2U(v)] \tag{3.126}$$

or

$$\{U_{\alpha_1\beta_1}(u), U_{\alpha_2\beta_2}(v)\} = \frac{1}{u - v}(U_{\alpha_2\beta_1}(u)U_{\alpha_1\beta_2}(v) - U_{\alpha_1\beta_2}(u)U_{\alpha_2\beta_1}(v))$$

(the superscript t_2 in (3.125) denotes the transposition with respect to the second space in $\mathbb{C}^3 \times \mathbb{C}^3$). The experience of the Inverse Spectral Transform Method and $SL(2)$ case suggests that in $SL(3)$ case the separated coordinates x_j, $j = 1, \ldots, 3M$ should be defined

as zeroes of some polynomial B(u) of degree $3M$ and the corresponding momenta p_j should be bound to x_j by the secular equation:

$$P_j^3 - t_1(x_j)P_j^2 + t_2(x_j)P_j - d(x_j) = 0, \qquad P_j = \exp p_j$$

It means that P_j is an eigenvalue of the matrix $T(x_j)$. So there exist such a similarity transformation:

$$T(x_j) \to \tilde{T}(x_j) = K_j T(x_j) K_j^{-1},$$

for each j that the matrix $\tilde{T}(x_j)$ is block-triangular

$$\tilde{T}_{12}(x_j) = \tilde{T}_{13}(x_j) = 0 \tag{3.127}$$

and P_j is the eigenvalue of $T(x_j)$ splitted from the upper block

$$P_j = \tilde{T}_{11}(x_j) \tag{3.128}$$

The problem is now reduced to determining the polynomial $B(u)$ and the matrices K_j. Take the simplest possible triangular, one-parametric matrix $K(k)$

$$K(k) = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix

$$\tilde{T}(u,k) \equiv K(k)T(u)K^{-1}(k)$$

depends on two parameters: u and k. Considering the condition (3.127) as the set of two algebraic equations

$$\tilde{T}_{12}(x,k) = T_{12}(x) + kT_{22}(x) - kT_{11}(x) - k^2 T_{21}(x) = 0 \qquad \tilde{T}_{13}(x,k) = T_{13}(x) + kT_{23}(x) = 0$$

for two variables x and k and eliminating k one obtains the polynomial equation for x

$$T_{23}(T_{12}T_{23} - T_{13}T_{22}) - T_{13}(T_{13}T_{21} - T_{11}T_{23}) = 0$$

or

$$T_{23}(x)U_{31}(x) - T_{13}(x)U_{32}(x) = 0 \tag{3.129}$$

The matrix Z is assumed to have simple spectrum, so the leading coefficient $\mathcal{B}(Z)$ of the polynomial $\mathcal{B}(T(u))$ can always be made nonzero by a similarity transformation $QT(u)Q^{-1}$, the equation (3.129) being thus of degree $3M$. From $\tilde{T}_{13} = 0$ one has $k = -T_{13}(x)/T_{23}(x)$. Substituting it into the definition (3.128) of P one obtains

$$P = T_{11}(x) + kT_{21}(x) = -U_{32}(x)T_{23}(x) \tag{3.130}$$

In this way one obtains $3M$ pairs of variables x_j, P_j. To prove the **Conjecture 1** one has to show that they have good Poisson brackets.

Theorem 2

The Poisson brackets between x_j and P_k are given by (3.118).

Proof.

Setting

$$\mathcal{A}(T) \equiv -\frac{\mathcal{U}_{32}(T)}{T_{23}} \qquad \mathcal{B}(T) = T_{23}\mathcal{U}_{31}(T) - T_{13}\mathcal{U}_{32}(T) \tag{3.131}$$

and denoting $A(u) = \mathcal{A}(T(u))$, $B(u) = \mathcal{B}(T(u))$ one can easily calculate the following Poisson brackets using (3.115), (3.125), (3.126):

$$\{A(u), A(v)\} = \{B(u), B(v)\} = 0 \tag{3.132}$$

$$\{A(u), B(v)\} = \frac{1}{u-v}\left(A(u)B(v) - B(u)A(v)\frac{T_{23}^2(v)}{T_{23}^2(u)}\right) \tag{3.133}$$

from which Poisson brackets for x_j and P_j are derived immediately in the same manner as in the $SL(2)$ case.

4 Theory of instanton. 8 hours

As the nonperturbative effect in QFT or QM is understood an effect which can not be seen in perturbation theory.

4.1 Nonperturbative effects

The one type of this is due to instantons, i.e. to nontrivial solutions to the classical equations of motion. If g is the coupling constant, these effects behave as

$$f(g) = e^{-A/g}. \tag{4.134}$$

This quantity is small at small g's, but on the other hand it is completely invisible in perturbation theory, since it displays an essential singularity at $g = 0$. (The formal Taylor series expansion around the zero vanishes).

$$f(g) = \sum_{n=0}^{\infty} \frac{g^n}{n!} f^{(n)}(0) = 0.$$

These (instanton) effects are responsible of one of the most important quantum-mechanical effect: tunneling through a potential barrier. This effect qualitatively changes the quantum structure of vacuum. In a potential with a perturbative ground state degeneracy (say $V(x) = (x^2 - a^2)^2$) tunneling effects lift the degeneracy. There a single ground state, and the difference of energies of the ground state and the first excited state is an instanton effect of the form (4.134),

$$E_1(g) - E_0(g) \sim e^{-A/g}$$

In a potential with a metastable vacuum (say $V(x) = -x^3 + ax^2 + bx$) the perturbative vacuum obtained by small quantum fluctuations around this metastable vacuum will eventually decay. It means that the ground state energy has a small imaginary part,

$$E_0(g) = \mathrm{Re}E_0(g) + i\mathrm{Im}E_0(g), \qquad \mathrm{Im}E_0(g) \sim e^{-A/g}.$$

which dependence on g is typical instanton effect. Some of these effects appear in quantum field theories, they are an important source of information about the dynamics of these theories. However, there are many important phenomena in QFT, related to strong coupling, like confinement and chiral symmetry breaking in QCD, which can not be explained in terms of instantons.

Another type of nonperturbative method in QFT is based on resumming an infinite subset of diagrams in perturbation theory. It is nonperturbative because, typically, the effects that one discovers in this way cannot be seen at any finite order of perturbation theory. As an illustration one can consider the following series:

$$f_0(g) = g - g\log g + \frac{g}{2}(\log g)^2 - \frac{g}{6}(\log g)^3 + \dots \tag{4.135}$$

So order by order in perturbation theory, one has the property

$$\lim_{g\to 0} f_0(g) = 0.$$

QM as a one-dimensional field theory

Recall that the ground state energy of a quantum mechanical system in a potential $W(q)$ can be extracted from the small temperature behavior of the thermal partition function,

$$Z(\beta) = \mathrm{tr}\, e^{-\beta H(\beta)}, \tag{4.136}$$

as

$$E = -\lim_{\beta\to\infty} \frac{1}{\beta}\log Z(\beta). \tag{4.137}$$

In terms of the path integral

$$Z(\beta) = \int \mathcal{D}[q(t)] e^{-S(q)}, \tag{4.138}$$

where $S(q)$ is the action of the Euclidean theory,

$$S(q) = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt[\frac{1}{2}(\dot{q}(t))^2 + W(q(t))], \tag{4.139}$$

and the path integral is over periodic trajectories $q(-\beta/2) = q(\beta/2)$. Note that the Euclidean action can be regarded as an action in Lagrangian mechanics,

$$S(q) = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt[\frac{1}{2}(\dot{q}(t))^2 - V(q(t))], \tag{4.140}$$

where the potential is $V(q) = -W(q)$.

4.2 Nonperturbative aspects of gauge theories

The generators of the Lie algebra T a are chosen to be antiHermitian, and satisfy the commutation relations

$$[T^a, T^b] = f^{abc}T^c.$$

(For $SU(2)$, for example, one takes $T^a = -\frac{i}{2}\sigma^a$ and the structure constants are $f^{abc} = \varepsilon^{abc}$.)

The Cartan inner product is defined by

$$(T^a, T^b) = \delta^{ab}, \qquad (T^a, T^b) = -2\,\mathrm{tr}(T^a T^b).$$

The Euclidean action for pure YangMills has form

$$S_E = \frac{1}{4g^2}\int d^4x (F_{\mu\nu}, F^{\mu\nu}). \tag{4.141}$$

The Lagrangian of QCD is written as

$$\mathcal{L} = \frac{1}{g^2}[\frac{1}{4}(F_{\mu\nu}, F^{\mu\nu}) + \sum_{f=1}^{N_f} \bar{\phi}_f (iD_\mu \gamma^m u - m_f)\phi_f], \tag{4.142}$$

where the covariant derivative is: $D_\mu = \partial_\mu + iA_\mu$. It is more convenient to use rescaled fields, in such a way that the coupling constant appears only in the vertices of the theory: $A_\mu = gA_\mu$, $\phi = g\phi$. The Yang-Mills type theories are renormalizable at the quantum level (g is dimensionless), and they exhibit a running coupling constant and asymptotic freedom. Denoting

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi}$$

the renormalized coupling constant in the MS scheme, at the subtraction point μ one obtains the β-function in form:

$$\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \dots.$$

The one-loop coefficient of β-function

$$\beta_0 = \beta_{0g} + \beta_{0f} = -\frac{1}{4\pi}(\frac{11N_c}{3} - \frac{2N_f}{3})$$

is scheme-independent, where N_c is the number of colors and N_f the number of massless quark flavours. β_{0g} and β_{0f} denote respectively the gluon and fermion contribution to the one-loop β-function. If the number of flavours is smaller than the number of colors, the first coefficient of the beta function is negative and the theory is asymptotically free. Then the quantity

$$\Lambda^2 = \mu^2 e^{\frac{1}{\beta_0 \alpha_s(\mu)}}$$

is in fact independent of μ, at leading order, and therefore defines a RG-invariant scale or the so-called dynamically generated scale of QCD. The phenomenon that a theory with a dimensionless coupling constant g generates a dimensionful scale is called dimensional transmutation.

4.3 Topological charge and θ vacua

The YangMills theory, besides the standard YM action contains another term that can be added to the action. This term is called the topological charge and it is given by

$$Q = \int d^4x q(x), \qquad q(x) = \frac{1}{32\pi^2}(F,F) = \frac{1}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}, \tag{4.143}$$

This term is compatible with the gauge invariance and renormalizability, so it is natural to add it to the action and to take as the Euclidean YM Lagrangian

$$\mathcal{L} = \frac{1}{4g^2}(F^{\mu\nu}, F_{\mu\nu}) - i\theta q(x), \tag{4.144}$$

where θ is a new parameter in the QCD action. As we will see below that (12.227) is quantized for any classical, continuous field configuration with a finite action. The different observables of QCD should be sensitive to the changes of θ parameter. One such observable is the ground state energy density $E(\theta)$, computed at large, finite volume V as

$$\exp(-VE(\theta)) = \int [\mathcal{D}A] e^{-\int d^4x \mathcal{L}_\theta}. \tag{4.145}$$

This function $E(\theta)$ has two properties: the path integral with the insertion of $e^{i\theta Q}$, $\theta \neq 0$ should be smaller than the path integral without the insertion, at $\theta = 0$, because when $\theta \neq 0$ one integrates an oscillating function with a positive measure. One concludes that $E(0) \leq E(\theta)$, $\theta \neq= 0$ and the ground state energy has an absolute minimum at $\theta = 0$. As we will see, smooth field configurations with a finite action have quantized values of Q. So we expect $E(\theta)$ to be periodic, with period 2π:

$$E(\theta + 2\pi) = E(\theta).$$

Note that, in the limit of infinite volume, smooth configurations of finite action give just a zero-measure set in the path integral, and we could think that the value of $E(\theta)$ is dominated by field configurations with non-integer Q. The function $E(\theta)$ can be expanded around $\theta = 0$:

$$E(\theta) - E(0) = \frac{1}{2}\chi_t^V \theta^2 s(\theta), \qquad s(\theta) = 1 + \sum_{n=1}^{\infty} b_{2n}\theta^{2n}.$$

As $q(x)$ is odd under parity reversal, only even powers of $q(x)$ have nonzero vacuum expectation values (because the vacuum is symmetric under parity transformation), and only even powers of θ appear in the expansion of $E(\theta)$. The coefficient χ_t^V is an important quantity, which measures the leading dependence of $E(\theta)$ on the θ angle around $\theta = 0$. It is called the topological susceptibility and it has the form:

$$\chi_t^V = \left(\frac{d^2E}{d\theta^2}\right)_{\theta=0} = \frac{\langle Q\rangle}{V} = \int_V d^4x \langle q(x)q(0)\rangle. \tag{4.146}$$

The last equality follows from the relation:

$$\langle Q\rangle = \int_V d^4x \int_V d^4\langle 0|q(x)q(y)|0\rangle = \int_V d^4x \int_V d^4\langle 0|q(x-y)q(0)|0\rangle = V\chi_t^V,$$

where translation invariance of the vacuum has been used. Since $\theta = 0$ is a minimum of $E(\theta)$, one has $\chi_t^V \geq 0$. The infinite-volume limit of the quantity χ_t^V will be denoted by $\chi_t = \lim_{V\to\infty} \chi_t^V$. Although observables in YM theory should be sensitive to the θ parameter, this dependence is very subtle, because (12.227) is a total divergence,

$$q(x) = \partial_\mu K^\mu, \qquad K_\mu = \frac{1}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}(A_\nu, \partial_\lambda A_\rho + \frac{2}{3}A_\lambda A_\rho). \tag{4.147}$$

The threeform here is the so-called ChernSimons term. It means, that

$$q(p) = \int d^4x e^{-ipxq(x)},$$

vanishes at zero momentum, since it is of the form $p^\mu K_\mu(p)$. The topological susceptibility is given by

$$\chi_t = \lim_{k\to 0} U(k),$$

where $U(k) = \int d^4x e^{ikx}\langle q(x)q(0)\rangle$. We can write

$$U(k) = \int \frac{d^4p'}{(2\pi)^4}\langle q(-k)q(p')\rangle$$

As $q(0) = 0$, this quantity vanishes order by order in perturbation theory. But this does not mean that it vanishes tout court. It is possible that after adding an infinite number of diagrams (or a subset of them), and then taking the limit $k \to 0$, one will obtain a nonzero result. For example, after adding up an infinite number of diagrams (the so-called planar diagrams) in the $1/N$ expansion, one finds a nonzero value for the topological susceptibility.

Using Stokes theorem, one obtains for the topological charge

$$Q = \int d\Sigma_\mu K^\mu. \tag{4.148}$$

Let us choose as surface of integration two spatial planes at $t = \pm\infty$, so that

$$Q = \int d^3\vec{x} K^0(t \to \infty, \vec{x}) - \int d^3\vec{x} K^0(t \to -\infty, \vec{x}) \equiv K_+ - K_-. \tag{4.149}$$

These operators are Hermitian, and related each to other by time reversal, so their spectra should coincide. Denoting their eigenstates $|n_\pm\rangle$: $K_\pm|n_\pm\rangle = n|n_\pm\rangle$ one can expand the physical vacuum as

$$|\theta\rangle = \sum_n c_n(\theta)|n_+\rangle = \sum_n c_n(\theta)|n_-\rangle. \tag{4.150}$$

This follows from time reversal invariance of the vacuum: the time reversal operator leave the vacuum is unchanged, it maps the first sum to the second one and vise versa. Note that $|\theta\rangle$ is just the vacuum for the YangMills field theory which includes a theta term. There is following identity,

$$i\frac{\partial}{\partial\theta}\langle\theta|\mathcal{O}|\theta\rangle = i\frac{\partial}{\partial\theta}\langle 0|\mathcal{O}e^{-\int d^4x\mathcal{L}(\theta)}|0\rangle = \int d^4x\langle 0|q(x)\mathcal{O}e^{-\int d^4x\mathcal{L}(\theta)}|0\rangle = \int d^4x\langle\theta|q(x)\mathcal{O}|\theta\rangle \tag{4.151}$$

so the operator $i\partial_\theta$ is equivalent to the insertion of Q. Due to (4.149) one finds

$$i\frac{\partial}{\partial\theta}\langle\theta|\mathcal{O}|\theta\rangle = \langle\theta|K_+\mathcal{O}|\rangle - \langle\theta|K_-\mathcal{O}|\rangle.$$

Here a time-ordering prescription that K_+ should be inserted to the left and K_- to the right is used. If one plugs in the expansion (4.150):

$$i\frac{\partial}{\partial\theta}\sum_{n,k} c_n^*(\theta)c_k(\theta) = \sum_{n,k}(n-k)c_n^*(\theta)c_k(\theta),$$

which leads to $c_n = Ce^{in\theta}$, where C is an overall constant. In terms of the eigenstates of $K_\pm$, one finds that $|\theta\rangle = \sum_n e^{in\theta}|n\rangle$ after setting the overall constant C to 1 for simplicity. So far we do not have information about the structure of the spectrum of $K_\pm$, because all of the n are identical, so that the structure above collapses to something trivial. However, the existence of YM instantons implies that all $n \in \mathbb{Z}$ exist.

4.4 Instantons in YangMills theory

By their definition, instantons in YangMills theory are field configurations which solve the equations of motion and have finite action. These configurations are important in a semi-classical analysis, since they might serve a starting point for a perturbation expansion. The finite action condition gives constraints on the large distance behavior of the fields. In order to see how they must behave as $r \to \infty$, one can note that schematically the Euclidean action can be written as

$$S_E \sim \int dr r^3 F^2$$

In order this to be finite, the integrand has to go at least like $1/r^2$. For example, one could has $F \sim \frac{1}{r^3}$ as $r \to \infty$. This leads to the following behavior:

$$A(r) \sim \frac{1}{r^2}, \qquad r \to \infty.$$

However, A is well defined only up to a gauge transformation, so one can has the more general behavior

$$A_\mu \to g\partial_\mu g^{-1} + \mathcal{O}(r^{-2}), \qquad r \to \infty. \tag{4.152}$$

This means that the gauge potential becomes pure gauge at infinity. In order to the limiting behavior to be well-defined as $r \to \infty$, one can define the function g on the boundary at infinity $\mathbb{S}^3 \subset \mathbb{R}^4$. This is achieved, for instance, if g depends only on the angular variables of $\mathbb{R}^4$. Hence, any solution like the above defines a map from $\mathbb{S}^3$ to the gauge group, i.e. g: $\mathbb{S}^3 \to G$. Under gauge transformation, g is changed. What is a gauge-invariant concept is the homotopy type of mappings from $\mathbb{S}^3$ to G? As in the theory of solitons, these homotopy classes are characterized by $\pi_3(G)$. A toy example is given by instantons in Euclidean twodimensional space with $U(1)$ gauge group. The role of homotopy group here plays $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. Homotopy classes are characterized by an integer n. A map is defined by the covering $g(n)(\theta) = e^{in\theta}$.

Let us consider $G = SU(2)$. Arbitrary element of $SU(2)$ can be written as

$$g = a + i\vec{b}\cdot\vec{\sigma}, \qquad\qquad a^2 + \vec{b}^2 = 1,$$

so $SU(2)$ is homeomorphic to $\mathbb{S}^3$. Consider maps of the form

$$g: \ \mathbb{S}^3 \to \mathbb{S}^3. \tag{4.153}$$

Corresponding homotopy group is $\pi_3(\mathbb{S}^3) = \mathbb{Z}$, due to Hurewicz theorem, which holds in this case thanks to the fact that $\pi_3(\mathbb{S}^3) = 0$. The theorem relates homotopy groups to homology groups, which are typically much easier to calculate. In this case it says that $\pi_3(\mathbb{S}^3) = H_2(\mathbb{S}^3)$, $\pi_3(\mathbb{S}^3) = H_3(\mathbb{S}^3) = \mathbb{Z}$. It follows from (4.153) that the homotopy classes related to the gauge group $SU(2)$ are labeled by an integer number n, so called winding number. An explicit expression for a map $g: \ \mathbb{S}^3 \to SU(2)$ with winding number n has form

$$g^{(n)}(x) = \left(\frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{r}\right)^n. \tag{4.154}$$

At $n = 0$, this map is trivial, while for $n = 1$ it is the identity. Note that this map can be expressed in terms of angular variables.

So, at least for $SU(2)$, every field configuration of finite action is characterized by its winding number n. It can be seen that the winding number of a gauge field is the value of the topological charge (12.227). Starting from the expression (4.147) and integrating over the boundary at infinity, which is a three-sphere $\mathbb{S}^3$, one obtains for a gauge field satisfying (4.152), that the field strength $F_{\mu\nu}$ vanishes at infinity. Hence, on $\mathbb{S}^3$ one has

$$\varepsilon_{\mu\nu\lambda\rho}\partial^\lambda A^\rho = -\varepsilon_{\mu\nu\lambda\rho}A^\lambda A^\rho,$$

and

$$Q = -\frac{1}{48\pi^2}\int d\Sigma^\mu \varepsilon_{\mu\nu\lambda\rho}(A^\nu, A^\lambda A^\rho). \tag{4.155}$$

By using the boundary behavior of the gauge potentials, one can also rewrite this quantity as

$$Q = -\frac{1}{48\pi^2}\int d\theta_1 d\theta_2 d\theta_3 \varepsilon^{ijk}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g).$$

This quantity is a homotopy invariant and expresses the winding number associated to the homotopy class of g.

Exercise

The integral expression (4.155) can be used to verify that $g^{(1)}$, as given in (4.154), indeed has $n = 1$. The inverse map is given by

$$g^{-1} = \frac{x_4 - i\vec{x}\cdot\vec{\sigma}}{r}.$$

Then one finds,

$$Q = -\frac{1}{24\pi^2}\int(-\frac{12x_\mu}{|x|^4})d\Sigma^\mu = \frac{1}{2\pi^2}\int(\frac{x_\mu}{|x|^4})x^\mu|x|^2 d\Omega_3 = \frac{1}{2\pi^2}\int d\Omega_3 = 1.$$

We have seen that, if there are field configurations of finite action, they will be classified by an integer winding number. Now we have to construct explicitly configurations with finite action which solve the equations of motion, and lead to different vacua of the YangMills theory. In each of the topological sectors, there is a configuration which minimizes the action, and therefore solves the equation of motion. So we will see that there is an infinite set of classical vacua enumerated by an integer n. Starting from the identity

$$\int d^4x(F \pm \tilde{F})^2 \geq 0.$$

one finds

$$\frac{1}{4g^2}\int d^4x(F,F) \geq \mp\frac{1}{4g^2}\int d^4x(F,\tilde{F}),$$

or

$$S \pm \frac{8\pi^2 n}{g^2} \geq 0 \qquad \Rightarrow S \geq \frac{8\pi^2|n|}{g^2}.$$

To saturate the inequality, we notice that S is always positive. Therefore if $n > 0$ is positive one has $F = \tilde{F}$,

$$S = \frac{8\pi^2 n}{g^2},$$

i.e. the gauge field is self-dual (SD) and we have a gauge theory instanton. If $n < 0$ is negative one has $F = -\tilde{F}$,

$$S = -\frac{8\pi^2 n}{g^2},$$

and the corresponding gauge field is anti-self-dual (ASD), which describes a gauge theory anti-instanton. At these conditions the corresponding gauge field minimizes the action (solves the EOM) for a fixed topological class given by n. Unlike the standard equations of motion of Yang-Mills theory, these are first order equations. As we will see, this can be related to Bogomol'nyi-Prasad-Sommerfield conditions in supersymmetry. The explicit asymptotic expression can be down for the instanton configuration with gauge group $SU(2)$ and $n = 1$ (the one-instanton solution). To do this, one sets $A_\mu = -(\partial_\mu g)g^{-1}$, where $g = g^{(1)}$, and $g^{(n)}$ is the the map (4.154). As $\partial_4 g = -\frac{x_4}{r^2}g + \frac{1}{r}$, $\partial_i g = -\frac{xi}{r^2}g + i\frac{\sigma_i}{r}$, $i = 1,2,3$. Then one finds $A_4 = i\frac{\vec{x}\cdot\vec{\sigma}}{r^2}$, $A_i = -\frac{i}{r^2}(x_4\sigma_i + \epsilon_{ijk}x_j\sigma_k)$, where the relation $\sigma_i(\vec{x}\cdot\vec{\sigma} = x_i + i\epsilon_{ijk}x_j\sigma_k$ was used. One can write $A_\mu = -\frac{i}{2}\sigma_a A^a_\mu$ and introduce the 't Hooft matrices $\eta^a_{\mu\nu}$ by $\eta^a_{ij} = \epsilon_{aij}$, $\eta^a_{i4} = \delta_{ai}$, $\eta^a_{4i} = -\delta_{ai}$, with $i,j = 1,2,3$ to rewrite $A^a_\mu = 2\eta^a_{\mu\nu}\frac{x^\nu}{r^2}$. This asymptotic form suggests the ansatz for the exact form $A^a_\mu = 2\eta^a_\mu\frac{x^\nu}{r^2}f(r^2)$ where $f(r^2) \to 1$ at $r \to \infty$. The regularity at the origin requires: $f(r^2) \sim r^2$ at $r \to 0$. So one computes

$$S \sim \int_0^\infty dr[\frac{r}{2}(f')^2 + \frac{2}{r}f^2(1-f)^2].$$

The second order equation of motion for f has form $-\frac{d}{dr}(r\frac{df}{dr}) + \frac{4}{r}f(1-f)(1-2f) = 0$. It has three constant solutions: $f = 0$ is the trivial gauge connection, $f = 1$ is a pure gauge transformation with winding number 1, and $f = 1/2$ called half gauge transformation or meron. Also we have a space-dependent solution

$$f(r) = \frac{r^2}{r^2 + \rho^2}. \tag{4.156}$$

It corresponds the one-instanton solution of $SU(2)$ Yang-Mills theory. The resulting configuration interpolates between the trivial vacuum $f = 0$ at the origin and the homotopically non-trivial gauge transformation with $n = 1$as $r \to \infty$, and at large r it is indeed of the form. Here ρ is an integration constant which can be treated as the size of the instanton. In this case, the relevant symmetry is the scale invariance of the classical YangMills action. However, here we have already fixed some integration constants: the above solution is centered at the origin, but one can also write a more general solution

$$A^a_\mu = 2\eta^a_{\mu\nu}\frac{(x-x_0)^\nu}{(x-x_0)^2+\rho^2},$$

where x_0 is the position of the center of the instanton. Due to translation invariance, this gives four extra collective coordinates. One obtains this expression by solving the original equation of motion of the YangMills action, which are second order, but one can solve instead the first order equation $F = \tilde{F}$. Substituting this ansatz we find the following first order equation for f,

$$f(1-f) - r^2\frac{df}{dr^2} = 0,$$

which leads to the constant solutions $f = 0, 1$ and to the one-instanton solution (4.156). Note that the meron solution $f = 1/2$ does not solve the first order equation, but it leads to an infinite action.

4.5 Instantons in supersymmetric quantum mechanics

Now we pass to a very interesting variant of the quantum mechanical models that we have been looking at: supersymmetric quantum mechanics. On top of the usual bosonic operators $\hat{q}$, $\hat{p}$, the model includes Grassmann variables $\hat{\psi}_{1,2}$ which obey anticommutation relations,

$$\{\hat{\psi}_\alpha, \hat{\psi}_\beta\} = \delta_{\alpha\beta}.$$

It is more convenient to consider the creation and annihilation operators $\hat{\psi}_\pm = \frac{1}{\sqrt{2}}(\hat{\psi}_1 \pm \hat{\psi}_2)$, which satisfy $\{\hat{\psi}_+, \hat{\psi}_-\} = 1$, $\hat{\psi}^2_\pm = 0$. This algebra has matrix form

$$\hat{\psi}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\psi}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Wave-functions correspond to vector-valued objects,

$$\Psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}.$$

Operators have following representation:

$$\hat{\psi}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\psi}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and $[\hat{\psi}_1, \hat{\psi}_2] = \frac{i}{2}\sigma_3$. The Hamiltonian of the system is

$$\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{q}) - \frac{i}{2}Y(\hat{q})[\hat{\psi}_1, \hat{\psi}_2]. \tag{4.157}$$

On the space of wavefunctions it takes the form

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + V(q) + \frac{1}{2}Y(q)\sigma_3.$$

σ_3 commutes with the Hamiltonian, so we can diagonalize it simultaneously. One can study the spectrum by considering wavefunctions of the form $\begin{pmatrix} \phi_1(x) \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \phi_2(x) \end{pmatrix}$.

This quantum-mechanical system is supersymmetric if the functions $V(q)$ and $Y(q)$ in $\hat{H}$ satisfy to conditions: $V(q) = \frac{1}{2}W^2(q)$, $Y(q) = W'(q)$. $W(q)$ is called the superpotential. One can see it in two equivalent ways: in the Hamiltonian picture, one just notes that there are two conserved fermionic charges:

$$\hat{Q}_+ = (p - iW)\hat{\psi}_+, \qquad \hat{Q}_- = (p + iW)\hat{\psi}_-,$$

which satisfy $H = \frac{1}{2}\{\hat{Q}_+, \hat{Q}_-\}$. In matrix notation, they have the form

$$\hat{Q}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i(\partial_q + W(q)) \\ 0 & 0 \end{pmatrix}, \qquad \hat{Q}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -i(\partial_q - W(q)) & 0 \end{pmatrix}.$$

From the other hand in the Lagrangian picture, one just have to show the existence of two fermionic symmetries in the Lagrangian. This is possible to do by using standard superspace techniques. The Lagrangian in components

$$L = \frac{1}{2}\dot{q}^2 + \frac{i}{2}(\psi_-\dot{\psi}_+ - \dot{\psi}_-\psi+) + \frac{1}{2}D^2 + Df'(q) + [\psi_-, \psi_+]2f''(q)$$

is invariant under

$$i\delta q = \epsilon_-\psi - -\epsilon_+\psi+, \qquad \delta\psi_\pm = \mp i\epsilon_\mp D + \epsilon_\mp\dot{q}, \qquad \delta D = \epsilon_-\dot{\psi} - +\epsilon_+\dot{\psi} + .$$

Here D is an auxiliary field one can integrate it out to obtain $D = -f'(q)$. Upon setting $f'(q) = W(q)$ one recovers the Lagrangian above. Theory with Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(q) + \frac{1}{2}W'(q)\sigma_3,$$

the fermionic sectors with σ_3 eigenvalues ± 1 have two different potentials, $V_\pm(q) = \frac{1}{2}W^2(q) \pm \frac{1}{2}W'(q)$.

5 Yang-Baxter Equation, 4 hours

The best illustration of different spectral dependence of the model is the Heisenberg chain. It describes the interaction between nearest-neighbor spins. The Hamiltonian of isotropic chain of spins one-half has form:

$$H^{(r)} = \sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z) \tag{5.158}$$

here $\vec{\sigma}_n$ are Pauli matrices, subjected for closed chains by periodicity condition $n+N=n$. It corresponds to the Heisenberg model with the R-matrix linearly depending on spectral parameter:

$$R_{XXX}(u) = uI\otimes I + \eta\vec{\sigma}\otimes\vec{\sigma} = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}, \tag{5.159}$$

here $a(u)=u+\eta$, $b(u)=u$, $c(u)=\eta$ and parameter η is a characteristic of a chain and can be reabsorbed into rescaling of spectral parameter u.

The rational (polynomial) dependence appears for R-matrices, describing higher spins. It is so called XXX Heisenberg chain. The Yang-Baxter equation (1.9), the underlying integrability of model then is equivalent to the algebra $su(2)$ of Pauli matrices

$$[\sigma^i/2, \sigma^j/2] = \varepsilon^{ijk}\sigma^k/2.$$

5.1 Quantum deformation

Spin chain, placed in a uniform magnetic field corresponds to an integrable generalization of the model considered having cylindrical symmetry and describing by Hamiltonian:

$$H^{(t)} = \sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z), \tag{5.160}$$

here the last term obviously destroys rotational symmetry of the model from $SO(3)$ to $SO(2)$ at $\Delta\neq 1$. The interaction with uniform magnetic field can be included here by term

$$\sum_{n=1}^{N} h_n\sigma_n^z,$$

however it does not affect the spectral dependence of the R-matrix, which we are interested in. The corresponding R-matrix $R_{XXZ}(u)$ of XXZ Heisenberg chain has the same form (5.159) with

$$a(u)[u+1]_q, \qquad b(u)=[u]_q, \qquad c(u)=[\eta]_q, \tag{5.161}$$

here $[x]_q \equiv (q^x - q^{-x})/(q-q^{-1})$ is q-number. Here $q=e^{i\gamma}$ is an arbitrary complex number and anisotropy parameter $\Delta=\cos\gamma$ is related to q as follows:

$$2\Delta = [2]_q.$$

It is not hard to see that $[x]_q \to x$ at $q\to 1$. One can see that the integrals of motion demonstrate the trigonometric dependence on the spectral parameter.

The most general interaction between nearest-neighbor spins one-half is described by XYZ Heisenberg model:

$$H^{(e)} = \sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x + \Delta\sigma_n^y\sigma_{n+1}^y + \Gamma\sigma_n^z\sigma_{n+1}^z). \tag{5.162}$$

This Hamiltonian does not have rotational symmetry and depends on two additional parameters Γ and Δ. The corresponding R-matrix of XYZ Heisenberg model provides the solution to YBE (1.9) with elliptic dependence on spectral parameter

$$R_{XYZ}(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix}, \tag{5.163}$$

here

$$a(u) = sn(u+\eta), \qquad b(u) = sn(u), \qquad c(u) = sn(\eta), \qquad d(u) = ksn(u+\eta)sn(u)sn(\eta),$$

where $sn(x)$ is elliptic sinus of x and k is modulus of elliptic function.

5.2 Types of YBE solutions

The general statement about solutions of classical YBE (which can be obtained as a quasi-classical limit of (1.9):

$$[r_{13}(u), r_{23}(v)] + [r_{12}(u-v), r_{13}(u) + r_{23}(v)] = 0, \tag{5.164}$$

is following: Let $\{X^a\}$ be a basis of the Lie algebra g, which is defined by the commutation relations

$$[X^a, X^b] = C_c^{ab} X^c,$$

and let g_{ab} be the inverse of the CartanKilling metric related to this basis. Drinfeld proved that under suitable non-degeneracy conditions on $r(u)$ the only solutions to cYBE (5.164) are given by:

- rational solution: $r(u) = g_{ab}\frac{X^a \otimes X^(b)}{u}$,
- trigonometric solution: $r(u) = \sum_{n=-\infty}^{\infty}(\mathbb{A}^n \otimes \mathbb{I})g_{ab}\frac{X^a \otimes X^(b)}{u-n\omega}$,
- elliptic solution: $r(u) = \sum_{n,m=-\infty}^{\infty}(\mathbb{A}^n\mathbb{B}^m \otimes \mathbb{I})g_{ab}\frac{X^a \otimes X^(b)}{u-n\omega_1-m\omega_2}, \qquad [\mathbb{A}, \mathbb{B}] = 0.$

Here $\mathbb{A}$ and $\mathbb{B}$ denote two finite order commuting automorphisms of g not having a common fixed vector. It follows that the elliptic solution can be defined only in the case $g = s\ell(n)$, because $s\ell(n)$ is the only simple Lie algebra possessing such two automorphisms.

5.3 Open chain

The periodicity condition imposed for closed chains, provides quantization allowed excitations of the chain from the condition that the length of the chain must fit an integer number of wavelengths. In this case, the higher integrals of motion correspond to the harmonics of the main excitation.

In the case of an open chain should be imposed boundary conditions that ensure the emergence of a standing wave excitations due to reflection at the edge sites of the chain.

The analysis of the integrable systems is modified upon imposing boundary conditions different from the periodic ones. This case is related to the factorizable scattering of particles with internal degrees of freedom on a half-line. The algebraic description involves additionally a boundary operator B into the ZF-algebra:

$$Z_a(\lambda)B = K_{ab}(\lambda)Z_b(-\lambda)B \tag{5.165}$$

Then the two particles factorizability gives rise the reflection equation (compare (1.1):

$$S_{12}(\lambda-\mu)K_1(\lambda)S_{21}(\lambda+\mu)K_2(\mu) = K_2(\mu)S_{12}(\lambda+\mu)K_1(\lambda)S_{21}(\lambda-\mu) \tag{5.166}$$

in addition to the Yang-Baxter equation (1.2). The reflection matrix has the same properties as the R-matrix, regularity: $K(0) = I$; unitarity: $K(\lambda)K(-\lambda) = I$, T-symmetry: $K^t(\lambda) = K(\lambda)$; the crossing symmetry is more elaborated and it involves the S-matrix as well.

Then the boundary operator B can be constructed by

$$B = \exp(\int \phi(\lambda)d\lambda) \tag{5.167}$$

from the combination,

$$\phi(\lambda) = Z_a(-\lambda)K_{ab}(\lambda)Z_b(-\lambda), \tag{5.168}$$

which is a "local" field $\phi(\lambda)$: $[\phi(\lambda);\phi(\mu)] = 0$. Due to the $SL(2)$-symmetry of the S-matrix, the corresponding K-matrix can be transformed $K \to K' = GKG^{-1}$ with arbitrary G and the general solution of the reflection equation (5.166) for the rational case, which we are interested in, is

$$K(\lambda) = \xi I + \lambda E, \qquad\qquad E^2 = I. \tag{5.169}$$

The reflection equation has an important covariance property: if $T(\lambda)$ and $K(\lambda)$ satisfy the relations (1.9) and (5.166) then $K'(\lambda) = T(\lambda)K(\lambda)T(-\lambda)^{-1}$ is also a solution of (5.166), provided the entries of $K(\lambda)$ and $T(\lambda)$ commute, $[K_{ab}(\lambda), T_{cd}(\lambda)] = 0$. The proof follows easily by the substitution of $K'(\lambda)$ into (5.166) and by using the fundamental Y.B. relation in the different form

$$T^{-1}_{(2)}(-\mu)R_{12}(\lambda+\mu)T_{(1)}(\lambda) = T^{-1}_{(1)}(\lambda)R_{12}(\lambda+\mu)T_{(2)}(-\mu).$$

If the matrix $T(\lambda)$ is constructed as an ordered product of N independent Lax operators, then $K'(\lambda)$ can be interpreted as the monodromy matrix of N site lattice model with a boundary interaction described by the operator valued entries of the matrix $K(\lambda)$. It is called Sklyanin's monodromy matrix. The corresponding transfer matrix is defined as the trace

$$t(\lambda) = \mathrm{tr}\bar{K}(\lambda)T(\lambda)K(\lambda)T^{-1}(-\lambda),$$

where the matrix $\bar{K}(\lambda)$ is any solution of (5.166), corresponding to the other boundary, is commutative

$$[t(\lambda), t(\mu)] = 0.$$

In the context of the Heisenberg chain equation (5.166) takes the form:

$$R_{12}(\lambda-\mu)K^-_{(1)}(\lambda)R^{t_1t_2}_{12}(\lambda+\mu)K^-_{(2)}(\mu) = \tag{5.170}$$
$$K^-_{(2)}(\mu)R_{12}(\lambda+\mu)K^-_{(1)}(\lambda)R^{t_1t_2}_{12}(\lambda-\mu),$$
$$R_{12}(-\lambda+\mu)(K^+_{(1)})^{t_1}(\lambda)M^{-1}_{(1)}R^{t_1t_2}_{12}(-\lambda-\mu-2\eta)M_{(1)}(K^+_{(2)})^{t_2}(\mu) = \tag{5.171}$$
$$(K^+_{(2)})^{t_2}(\mu)M_{(1)}R_{12}(-\lambda-\mu-2\eta)M^{-1}_{(1)}(K^+_{(1)})^{t_1}(\lambda)R^{t_1t_2}_{12}(-\lambda+\mu)$$

where M is crossing matrix, defined above. In practice, if $K^-(\lambda)$ is a solution of (61) then $K^+(\lambda) = (K^-(-\lambda-\eta))^t M$ is a solution of (62). The eq. (5.166) has an important covariance property: if $T(\lambda,\vec{\omega})$ and $K_\pm(\lambda)$ satisfies the relations (**??**) and (61), (62) then Sklyanin's monodromy matrix:

$$U(\lambda,\vec{\omega}) = T(\lambda,\vec{\omega})K^-(\lambda)\tilde{T}(\lambda,\vec{\omega}), \tag{5.172}$$

where $\tilde{T}(\lambda,\vec{\omega}) = R_{Na}(\lambda-\omega_N)...R_{2a}(\lambda-\omega_2)R_{1a}(\lambda-\omega_1)$, (cr. with (5.167)), satisfies the relation

$$R_{12}(\lambda-\mu)U_{(1)}(\lambda,\vec{\omega})R^{t_1t_2}_{12}(\lambda+\mu)U_{(2)}(\mu,\vec{\omega}) = U_{(2)}(\mu,\vec{\omega})R_{12}(\lambda+\mu)U_{(1)}(\lambda,\vec{\omega})R^{t_1t_2}_{12}(\lambda-\mu). \tag{5.173}$$

Indeed, we note that unitarity and crossing symmetry together imply the relation

$$M_{(1)}R^{t_2}_{12}(-\lambda-\eta)M^{-1}_{(1)}R^{t_2}_{12}(\lambda-\eta) = \rho(\lambda). \tag{5.174}$$

Furthermore, we see that unitarity implies $T(\lambda,\vec{\omega})\tilde{T}(-\lambda,\vec{\omega}) = \prod\rho(\lambda-\omega_i)$. Therefore, up to a scalar factor, $\tilde{T}(-\lambda,\vec{\omega})$ is the inverse of $T(\lambda,\vec{\omega})$.

The commutativity of the transfer matrix $t(\lambda,\vec{\omega})$ implies integrability of the open quantum spin chain with the hamiltonian:

$$H = \sum_{i=1}^{N-1} H_{ii+1} + 1/2(K^{(1)}_-)^t + \frac{\mathrm{tr}_0 K^{(0)}_+(0)H_{N0}}{\mathrm{tr}K_+(0)} \tag{5.175}$$

whose two-site terms are given by

$$H_{ii+1} = \frac{d}{d\lambda}\mathcal{P}_{ii+1}R_{ii+1}(\lambda)|_{\lambda=0} \tag{5.176}$$

in the standard fashion.

6 Lattice models, 2 hours

Ferromagnetism arises when a collection of atomic spins align such that their associated magnetic moments all point in the same direction, yielding a net magnetic moment which is macroscopic in size. The simplest theoretical description of ferromagnetism is called the Ising model. This model was invented by Wilhelm Lenz in 1920: it is named after Ernst Ising, a student of Lenz who chose the model as the subject of his doctoral dissertation in 1925.

Consider N atoms in the presence of a z-directed magnetic field of strength H. Suppose that all atoms are identical spin-1/2 systems. It follows that either $s_i = +1$ (spin up) or $s_i = -1$ (spin down), where s_i is (twice) the z-component of the ith atomic spin. The total energy of the system is written:

$$E = -J \sum_{<ij>} s_i \, s_j - \mu \, h \sum_{i=1,N} s_i. \tag{6.177}$$

Here, $< ij >$ refers to a sum over nearest neighbour pairs of atoms. Furthermore, J is called the exchange energy, whereas μ is the atomic magnetic moment. Equation (6.177) is the essence of the Ising model. The physics of the Ising model is as follows. The first term on the right-hand side of Eq. (6.177) shows that the overall energy is lowered when neighbouring atomic spins are aligned. This effect is mostly due to the Pauli exclusion principle. Electrons cannot occupy the same quantum state, so two electrons on neighbouring atoms which have parallel spins (i.e., occupy the same orbital state) cannot come close together in space. No such restriction applies if the electrons have anti-parallel spins. Different spatial separations imply different electrostatic interaction energies, and the exchange energy, J, measures this difference. Note that since the exchange energy is electrostatic in origin, it can be quite large: i.e., $J \sim 1$ eV. This is far larger than the energy associated with the direct magnetic interaction between neighbouring atomic spins, which is only about 10^{-4} eV. However, the exchange effect is very short-range; hence, the restriction to nearest neighbour interaction is quite realistic.

Our first attempt to analyze the Ising model will employ a simplification known as the mean field approximation. The energy of the i-th atom is written

$$e_i = -\frac{J}{2} \sum_{k=1,z} s_k \, s_i - \mu \, h \, s_i, \tag{6.178}$$

where the sum is over the z nearest neighbours of atom i. The factor 1/2 is needed to ensure that when we sum to obtain the total energy,

$$E = \sum_{i=1,N} e_i, \tag{6.179}$$

we do not count each pair of neighbouring atoms twice. We can write

$$e_i = -\mu \, H_{\rm eff} \, s_i, \tag{6.180}$$

where

$$h_{\rm eff} = h + \frac{J}{2\,\mu} \sum_{k=1,z} s_k.$$

Here, $h_{\rm eff}$ is the effective magnetic field, which is made up of two components: the external field, h, and the internal field generated by neighbouring atoms. Consider a single atom in a magnetic field h_m. Suppose that the atom is in thermal equilibrium with a heat bath of temperature T. According to the well-known Boltzmann distribution, the mean spin of the atom is

$$\bar{s} = \frac{{\rm e}^{+\beta\,\mu\,h_m} - {\rm e}^{-\beta\,\mu\,h_m}}{{\rm e}^{+\beta\,\mu\,h_m} + {\rm e}^{-\beta\,\mu\,h_m}}, \tag{6.181}$$

where $\beta = 1/kT$, and k is the Boltzmann constant. The above expression follows because the energy of the "spin up" state ($s = +1$) is $-\mu\, h_m$, whereas the energy of the "spin down" state ($s = -1$) is $+\mu\, h_m$. Hence,

$$\bar{s} = \tanh(\beta\, \mu\, h_m). \tag{6.182}$$

Let us assume that all atoms have identical spins: i.e., $s_i = \bar{s}$. This assumption is known as the "mean field approximation". We can write

$$h_{\rm eff} = h + \frac{z\, J\, \bar{s}}{2\, \mu}. \tag{6.183}$$

Finally, we can combine Eqs. (6.182) and (6.183) (identifying h_m and $h_{\rm eff}$) to obtain

$$\bar{s} = \tanh\left\{\beta\, \mu\, h + \beta\, z\, J\, \bar{s}/2\right\}. \tag{6.184}$$

Note that the heat bath in which a given atom is immersed is simply the rest of the atoms. Hence, T is the temperature of the atomic array. It is helpful to define the critical temperature,

$$T_c = \frac{z\, J}{2\, k}, \tag{6.185}$$

and the critical magnetic field,

$$h_c = \frac{k\, T_c}{\mu} = \frac{z\, J}{2\, \mu}. \tag{6.186}$$

Equation (6.182) reduces to

$$\bar{s} = \tanh\left\{\frac{T_c}{T}\left(\frac{h}{h_c} + \bar{s}\right)\right\}. \tag{6.187}$$

The above equation cannot be solved analytically. However, it is fairly easily to solve numerically using the following iteration scheme:

$$\bar{s}_{i+1} = \tanh\left\{\frac{T_c}{T}\left(\frac{h}{h_c} + \bar{s}_i\right)\right\}. \tag{6.188}$$

The above formula is iterated until $\bar{s}_{i+1} \rightarrow \bar{s}_i$. It is helpful to define the net magnetization,

$$M = \mu \sum_{i=1,N} s_i = \mu\, N\, \bar{s}, \tag{6.189}$$

the net energy,

$$E = \sum_{i=1,N} e_i = -N\, k\, T_c \left(\frac{h}{h_c} + \bar{s}\right) \bar{s}, \tag{6.190}$$

and the heat capacity,

$$C = \frac{dE}{dT}. \tag{6.191}$$

7 Heisenberg magnet, Bethe Ansatz method, 8 hours

The Heisenberg model was proposed in 1928 to describe ferromagnetism phenomenon. Classical Heisenberg model can be formulated as follows: take a d-dimensional lattice, and a set of spins of the unit length

$$\vec{s}_i \in \mathbb{R}^3, \qquad |\vec{s}_i| = 1,$$

each one placed on a lattice node. The model is defined through the following Hamiltonian:

$$\mathcal{H} = -\sum_{i,j} \mathcal{J}_{ij} \vec{s}_i \cdot \vec{s}_j$$

with

$$\mathcal{J}_{ij} = \begin{cases} J & \text{if } i,j \text{ are neighbors} \\ 0 & \text{else.} \end{cases}$$

a coupling between spins.

For quantum mechanical reasons, the dominant coupling between two dipoles may cause nearest-neighbors to have lowest energy when they are aligned. Under this assumption the Hamiltonian can be written in the form

$$\hat{H} = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} - h \sum_{j=1}^{N} \sigma_j$$

where J is the coupling constant for a 1-dimensional model consisting of N dipoles, represented by classical vectors (or "spins") s_j, subject to the periodic boundary condition $\sigma_{N+n} = \sigma_n$. The Heisenberg model is a more realistic model in that it treats the spins quantum-mechanically, by replacing the spin by a quantum operator (Pauli spin-1/2 matrices at spin 1/2), and the coupling constants J_x, J_y, and J_z. As such in 3-dimensions, the Hamiltonian is given by

$$\hat{H} = -\frac{1}{2} \sum_{j=1}^{N} (J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z - h\sigma_j^z), \tag{7.192}$$

where the h on the right-hand side indicates the external magnetic field, with periodic boundary conditions, and at spin s=1/2, spin matrices given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Hamiltonian then acts upon the tensor product $(\mathbb{C}^2)^{\otimes N}$, of dimension 2^N. The objective is to determine the spectrum of the Hamiltonian, from which the partition function can be calculated, from which the thermodynamics of the system can be studied. The most widely known is the Heisenberg XXZ model, which gives rise the new branch of mathematics *quantum algebras* occurs in the case $J = J_x = J_y \neq J_z = \Delta$. The spin 1/2 Heisenberg model in one dimension may be solved exactly using the Bethe ansatz [8], while other approaches do so without Bethe ansatz [7]. The physics of the Heisenberg model strongly depends on the sign of the coupling constant J and the of dimension of the space-time. For

positive J the ground state of the model is always ferromagnetic. At negative J the ground state is antiferromagnetic in two and three space-time dimensions, it is from this ground state that the Hubbard model is given. In one dimension the nature of correlations in the antiferromagnetic Heisenberg model depends on the spin of the magnetic dipoles. If the spin is given by positive integer then only short-range order is present. A system of positive half-integer spins exhibits quasi-long range order.

7.1 Coordinate Bethe Ansatz

The simplest model of N interacting atoms (1D Heisenberg model) has been solved by Hans Bethe in *Bethe, ZS. f. Phys. (1931).* The energy levels of each atom are given by two spin states of one valence electron. When interaction is switched off (lattice spacing is large) all 2^N energy levels become degenerate. Taking into account the Coulomb interaction leads to a splitting of the energy levels and the exchange of spin states of pairs of electrons localized in neighboring atoms.

It leads to the Hamiltonian of ferromagnetic 1D XXX Heisenberg model (7.192) with $J_x = J_y = J_z = J$:

$$H = -J \sum_{n=1}^{N} \mathcal{P}_{n,n+1} = -\frac{J}{2} \sum_{n=1}^{N} \vec{\sigma}_n \vec{\sigma}_{n+1},$$

where $\mathcal{P}_{12} = \frac{1}{2}(1 + \vec{\sigma}_1 \vec{\sigma}_2)$ is the permutation operator.

In order to avoid the need to consider the boundary effects one considers a closed chain, imposing periodic boundary conditions $n + N = n$. The open chain can be considered as well, upon placing special reflection matrices at the edge sites.

Upon taking into account the average of the magnetic interaction between the electrons, a so-called Heisenberg-Ising Hamiltonian H_Δ appears:

$$H_\Delta = -J \sum_{n=1}^{N} S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta(S_n^z S_{n+1}^z - \frac{1}{4}), \tag{7.193}$$

anisotropy parameter Δ is set to 1 only in the isotropic case, when only the exchange forces. Here the constant contribution to Hamiltonian H_Δ is added to the ground ferromagnetic state $|S^z = \frac{N}{2}\rangle \equiv |0\rangle$ has zero energy. Hamiltonian of XXZ model H_Δ commutes wit z-projection of spin: $[H_\Delta, S^z] = 0$. In isotropic XXX case one has $[H, \vec{S}] = 0$.

The more general XYZ Hamiltonian (7.192) has been considered by Bonner, Sutherland and Baxter. It depends on two anisotropy parameters Δ and Γ:

$$J_x : J_y : J_z = (1 - \Gamma) : (1 + \Gamma) : \Delta.$$

Diagonalization method proposed by Bethe, applicable to both Hamiltonian $H = H_{XXX}$ and to the H_Δ. Diagonalization of more general Hamiltonian H_{XYZ} requires using of so called Algebraic Bethe ansatz method. Hence the conservation of the z-projection of total spin is quite important in this context. The eigenstate of the Hamiltonian H_Δ $|M\rangle$ corresponding to eigenvalue $S^z = \frac{N}{2} - M$ (here M is the number of reversed spins) is given by superposition of basis states:

$$|n_1 n_2 \dots n_M\rangle = S_{n_1}^- S_{n_2}^- \dots S_{n_M}^- |0\rangle.$$

Non-diagonal matrix elements of Hamiltonian:

$$\langle\{n'\}|\sum_n S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+|\{n\}\rangle,$$

are responsible for transitions $|n_1 n_2 \dots n_M\rangle \to |n'_1 n'_2 \dots n'_M\rangle$, where the configuration $\{n'\}$ differs from $\{n\}$ by transposition of single pair of anti-parallel spins:

$$n'_1 = n_1, \dots, n'_\alpha = n_\alpha \pm 1, \dots, n'_M = n_M. \tag{7.194}$$

The periodicity condition for amplitudes has the form:

$$a(n_1 n_2 \dots n_M) = a(n_2 \dots n_M n_1 + N),$$

where sets $\{n\}$ are ordered: $1 \le n_1 < n_2 \dots < n_M \le N$. The diagonal matrix elements of Hamiltonian:

$$2\Delta\langle\{n\}|\sum_m (S_m^z S_{m+1}^z - \frac{1}{4})|\{n\}\rangle,$$

then counts the number of admissible transitions in units Δ. The equation for amplitudes is:

$$\sum_{\{n'\}}[a(\{n'\}) - \Delta a(\{n\})] = 2Ea(\{n\}), \tag{7.195}$$

where $\{n'\}$ are admissible transitions defined by (7.194).

7.2 Separation of variables

The state with all spins up, as mentioned above, has zero energy, consider then the state with single spin down $M = 1$:

$$a(n+1) + a(n-1) - 2\Delta a(n) = 2Ea(n),$$

with solution:

$$a(n) = e^{ikn}, \qquad E = \cos k - \Delta,$$

where wave number k is quantized due to periodicity condition:

$$e^{ikN} = 1, \qquad kN = 2\pi\lambda, \qquad \lambda = 0, 1, 2, \dots N-1.$$

In the case of two spins down $M = 2$, the ansatz is:

$$a(n_1, n_2) = e^{i\psi_{12} + ik_1 n_1 + ik_2 n_2} + e^{-i\psi_{12} + ik_1 n_2 + ik_1 n_2}.$$

This result is extended on $M > 2$ due to the separation of variables:
1) amplitudes $a(\{n\})$ extended for coinciding values $n_1 \le n_2 \le \dots \le n_M$
2) in eq. (7.195) summation goes over all transitions (7.194).
Then eq. (7.195) becomes a finite-difference equation with constant coefficients:

$$\sum_{\alpha=1}^{M}(a(n_1, \dots, n_\alpha + 1, \dots, n_M) + a(n_1, \dots, n_\alpha - 1, \dots, n_M) -$$

$$-2\Delta a(n_1,\ldots,n_\alpha,\ldots,n_M) = 2Ea(n_1,n_2,\ldots,n_M). \tag{7.196}$$

3) only two contributions possible: $n'_\alpha = n_\alpha + 1 = n_{\alpha+1}$ and $n'_\beta = n_\beta - 1 = n_{\beta-1}$. Considering down spins at n_α and $n_{\alpha+1}$ one obtains that the forbidden terms cancel each to other or (7.195) and (7.196) are equivalent (compatible) at

$$a(n_1,\ldots,n_\alpha+1,n_\alpha+1,\ldots,n_M)+a(n_1,\ldots,n_\alpha,n_\alpha,\ldots,n_M)-2\Delta a(n_1,\ldots,n_\alpha,\ldots,n_M)=0.$$

The equation (7.196) admits separation of variables in any coordinates and has solutions of form:

$$e^{ik_1n_1+\ldots+ik_Mn_M},$$

parameterized by set of momenta $\{k\}$. Corresponding energy is given by:

$$E=\sum_{\alpha=1}^{M}(\cos k_\alpha - \Delta).$$

The solution (Bethe ansatz) is given by sum over $M!$ permutations of momenta $\{k\}$:

$$a(n_1,\ldots,n_M) = \sum_{P\in\pi_M} A(P)\exp(\sum_{\alpha=1}^{M} ik_{P_\alpha}n_\alpha). \tag{7.197}$$

The coefficients $A(P)$ are determined from compatibility condition:

$$\sum_P (e^{ik_{P_\alpha}+k_{P_{\alpha+1}}} - 2\Delta e^{ik_{P_{\alpha+1}}} + 1)e^{ik_{P_1}n_1+\ldots+i(k_{P_\alpha}+k_{P_{\alpha+1}})n_\alpha+\ldots} = 0.$$

Defining antisymmetric phases $\psi_{\alpha\beta} = -\psi_{\beta\alpha}$:

$$e^{i\psi_{\alpha\beta}} = -\frac{e^{ik_\alpha+ik_\beta} - 2\Delta e^{ik_\alpha} + 1}{e^{ik_\alpha+ik_\beta} - 2\Delta e^{ik_\beta} + 1},$$

one obtains:

$$A(P\mathcal{P}_{\alpha\alpha+1}) = A(P)e^{-i\psi_{P_\alpha,P_{\alpha+1}}},$$

then one can write unique solution of this relation and (7.197) as:

$$A(P) = \exp(\frac{i}{2}\sum_{\alpha<\beta}\psi_{P_\alpha,P_\beta}).$$

7.3 Periodicity condition

The periodic boundary conditions impose on amplitudes following restriction:

$$A(P) = A(PC)e^{ik_{P_1}N}, \tag{7.198}$$

where C is the cyclic permutation $C\alpha = \alpha + 1$, so one has:

$$A(PC) = \exp(\frac{i}{2}\sum_{1\le\alpha<\beta<M}\psi_{P_{\alpha+1}P_{\beta+1}} + \frac{i}{2}\sum_{1\le\alpha<M}\psi_{P_{\alpha+1}P_1}) =$$

$$= \exp(\frac{i}{2} \sum_{1\leq\alpha<\beta<M} \psi_{P_\alpha P_\beta} + i \sum_{1\leq\alpha<M} \psi_{P_\alpha P_1}),$$

then the Bethe equations (7.198) take the form:

$$e^{ik_{P_1}N + i\sum_{1<\alpha\leq M}\psi_{P_\alpha P_1}} = 1, \qquad \forall P,$$

so, there exists the set of integer numbers $\{\lambda_\alpha\}$ such that:

$$Nk_\alpha = 2\pi\lambda_\alpha + \sum_\beta \psi_{\alpha\beta}. \tag{7.199}$$

Eqs. (7.199) are complicated set of algebraic equations of high power with respect to admissible values of variables e^{ik_α}.

7.4 Algebraic Bethe ansatz

The Algebraic Bethe ansatz is an alternative approach to solution of Heisenberg model, developed by L. D. Faddeev and collaborators, which based on Inverse Scattering Method and allow to solve also XYZ model.

For the rational case (XXX Heisenberg model) the R-matrix has the form

$$R_{12}(u) = u + \eta\mathcal{P}_{12},$$

It can be written as:

$$R_{12}(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}.$$

here

$$a(u) = \phi(u+\eta), \qquad b(u) = \phi(u), \qquad c(u) = \phi(\eta),$$

and $\phi(u) = u$ for the rational case and $\phi(u) = \sin(u)$ for the trigonometric case. The R-matrix satisfies the unitarity condition

$$R_{12}(u)R_{21}(-u) = a(u)a(-u).$$

The solution of the integrable model suppose the construction of complete set of common eigenvectors of Hamiltonian and integrals of motion, which in this context are given by the generating function - transfer matrix:

$$t(u) = \mathrm{tr}T(u) = \mathrm{tr}\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = A(u) + D(u). \tag{7.200}$$

where the monodromy matrix $T(u)$ is given by product ordered along the lattice (1.7) or (3.114). Then the matrix elements $B(u)$ and $C(u)$ can be treated as annihilation and creation operators correspondingly. The reference state $\Omega = \{\uparrow \ldots \uparrow\rangle\} = \prod_{n=1}^N \omega_n$ with all spins down (pseudovacuum state) is an eigenvector of the Hamiltonian. Indeed, one has:

$$L_n(u)\omega_n = \begin{pmatrix} \phi(u+\eta/2) & * \\ 0 & \phi(u-\eta/2) \end{pmatrix}\omega_n,$$

so taking product, one obtains:

$$T(u)\Omega = \begin{pmatrix} \phi^N(u+\eta/2) & * \\ 0 & \phi^N(u-\eta/2) \end{pmatrix} \Omega,$$

or $C(u)\Omega = 0$, $A\Omega = \phi^N(u+\eta/2)\Omega$, $D\Omega = \phi^N(u-\eta/2)\Omega$ and Ω is an eigenvector of $t(u)$ and H.

Other eigenvectors have the form:

$$\Psi_M(u_1, u_2, \ldots, u_M) = B(u_1)\ldots B(u_M)\Omega. \tag{7.201}$$

The $RTT = TTR$ relation (1.8) then gives commutation relation for operators $A = T_{11}$, $B = T_{12}$, $C = T_{21}$ and $D = T_{22}$:

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)),$$

which equivalent to

$$[B(u), B(v)] = 0, \tag{7.202}$$

$$A(u)B(v) = \frac{u-v+\eta}{u-v}B(v)A(u) - \frac{\eta}{u-v}B(u)A(v), \tag{7.203}$$

etc. Using these relations, one can commute $A(u)$ and $D(u)$ in (7.201) to the right and act on Ω:

$$A(u)\Psi(u_1, \ldots, u_M) = (u+\eta/2)^N \prod_{k=1}^{M} \frac{u-u_k+\eta}{u-u_k}\Psi(u_1, \ldots, u_M) +$$

$$+ \sum_{k=1}^{M} \Lambda_k(u, u_1, \ldots, u_k)B_1(u_1)\ldots\hat{B}_k(u_k)\ldots B_M(u_M)B(u)\Omega,$$

where the hat on B_k means that this factor is omitted. Here the first (wanted) term is obtained when one takes M times the first term in (7.202) and the remaining terms are unwanted. The unwanted term with coefficient $\Lambda_k(u, u_1, \ldots, u_k)$ is obtained when the first time the second term in (7.202) is taken and then $M-1$ times the first one is taken, so one has:

$$\Lambda_1(u, u_1, \ldots, u_k) = (-\frac{\eta}{u-u_1})(u_1+\eta/2)^N \prod_{k=2}^{M} \frac{u_1-u_k+\eta}{u_1-u_k}.$$

Then one notices that due to commutativity (7.202) the coefficients $\Lambda_k(u, u_1, \ldots, u_k)$ with $k > 1$ can be obtained from $\Lambda_1(u, u_1, \ldots, u_k)$ by simple substitution $u_1 \to u_k$.

In similar way the wanted and unwanted terms computed for $D(u)\Psi$ and demanding cancelation of all unwanted terms one obtains:

$$(u_j+\eta/2)^N \prod_{k\neq j}^{M} \frac{u_j-u_k+\eta}{u_j-u_k} = (u_j-\eta/2)^N \prod_{k\neq j}^{M} \frac{u_j-u_k-\eta}{u_j-u_k},$$

or in more conventional form:

$$\left(\frac{u_j+\eta/2}{u_j-\eta/2}\right)^N = \prod_{k\neq j}^{M} \frac{u_j-u_k+\eta}{u_j-u_k-\eta}. \tag{7.204}$$

Then if the set $\{u_k\}$ is solution to Bethe equations (7.204), the vector

$$\Psi_M(u_1, u_2, \ldots, u_M) = B(u_1) \ldots B(u_M)\Omega,$$

will be eigenvector of transfer matrix (and Hamiltonian) with eigenvalue

$$E(u, \{u_k\}) = (u + \eta/2)^N \prod_{k=1}^{M} \frac{u - u_k + \eta}{u - u_k} + (u - \eta/2)^N \prod_{k=1}^{M} \frac{u - u_k - \eta}{u - u_k}.$$

Exercise
Show that $S^z\Psi_M(u_1, u_2, \ldots, u_M) = (\frac{N}{2} - M)\Psi_M(u_1, u_2, \ldots, u_M)$ *and* $S^+\Psi_M(u_1, u_2, \ldots, u_M) = 0$.

The solution for XXZ Heisenberg model is obtained straightforward, while for XYZ model as reference state Ω the state, annihilated by some combination of "annihilation" and "creation" operators $B(u)$ and $C(u)$ should be chosen.

8 Ising model, square lattice, 2 hours

Making in (7.193) substitution $2J = \beta/\Delta$ and taking limit $\Delta \to \infty$ one comes to Ising Hamiltonian:

$$H_{Ising} = \beta \sum_{n=1}^{N} (S_n^z S_{n+1}^z - \frac{1}{4}). \tag{8.205}$$

The Ising model solved in one- and two-dimensional cases.

8.1 One-dimensional Ising model

The most general Hamiltonian of 1-dim Ising model is:

$$H = -\sum_{n=1}^{N-1} J_n \sigma_n \sigma_{n+1}. \tag{8.206}$$

The discrete variable σ_n takes two values: ± 1. The statistical sum of the model is given by

$$Z_N = \sum_{\sigma_1, \ldots, \sigma_N} e^{\beta \sum_{n=1}^{N-1} J_n \sigma_n \sigma_{n+1}} = \sum_{\sigma_1, \ldots, \sigma_{N-1}} e^{\beta \sum_{n=1}^{N-2} J_n \sigma_n \sigma_{n+1}} \sum_{\sigma_N = -1}^{1} e^{\beta J_{N-1} \sigma_{N-1} \sigma_N} =$$

$$= 2Z_{N-1} \cosh \beta J_{N-1} = 2^N \prod_{n=1}^{N-1} \cosh \beta J_n,$$

here we used $\cosh \beta J_{N-1}\sigma_{N-1} = \cosh \beta J_{N-1}$ as $\sigma_{N-1} = \pm 1$.

8.2 Two-dimensional Ising model on square lattice

The Hamiltonian of the Ising model in the absence of magnetic field is:

$$H(\sigma) = -\sum_{j=1}^{N} E_j \sigma(p_j)\sigma(q_j),$$

where p_j and q_j are edge points of the links e_j of some graph $G = \{e_1 e_2 \dots e_N\}$. It is possible to show that the statistical sum of the two-dimensional Ising model is reduced to the sum over cycles drawn on two-dimensional square lattice and is given by:

$$Z(\beta) = \sum_{\sigma} e^{-\beta H(\sigma)} = 2^{\#V} \prod_{j=1}^{N} \cosh(\beta E_j) \sum_{\xi} \prod_{e_i \in \xi} \tanh(\beta E_i),$$

where ξ is subgraph, cycle on graph G. The square of generating function

$$F = \sum_{\xi} \prod_{e_i \in \xi} \tanh(\beta E_i),$$

is given by determinant of $2N \times 2N$ Kac-Ward matrix:

$$A_{\mathbf{e}_j \mathbf{e}_k} = \begin{cases} 1 & \text{if} \quad \mathbf{e}_j = \mathbf{e}_k \\ -d_j e^{\frac{i}{2}\widehat{(\mathbf{e}_j, \mathbf{e}_k)}} & \text{if} \quad f(\mathbf{e}_j) = b(\mathbf{e}_k), \quad f(\mathbf{e}_k) \neq b(\mathbf{e}_j) \\ 0 & \text{otherwise} \end{cases}$$

where $b(\mathbf{e})$ and $f(\mathbf{e})$ is beginning and end of directed link, and $\widehat{(\mathbf{e}_j, \mathbf{e}_k)}$ is the angle between links $\mathbf{e}_j$ and $\mathbf{e}_k$. Here $\mathbf{e_j}$ is the link of graph and one variable $d(\mathbf{e})$ corresponds to opposite directed links $\mathbf{e}_j$ and $\mathbf{e}_{j+N}$:

$$\mathbf{e}_j \to d_j, \quad 1 \leq j \leq N, \qquad \mathbf{e}_j \to d_{j-N}, \quad N < j \leq 2N.$$

The sign factor chosen in such way implies that path with repeated constraints cancel to each other. This result has been proved for square lattice by Onsager and later it was proved using finite combinatorial methods for general rectangular lattice.

9 Graphen, 4 hours

9.1 Hexagonal Lattice, Brillouin zone, Massless Dirac Fermions

Graphene is a mono-layer graphitic sheet of carbon atoms forming a honeycomb hexagonal lattice, which is given by a superposition of two triangular sublattices **A** and **B**. The atoms belonging to the sublattice **B** denoted by hollow rings on figure and solid rings correspond to sublattice **A**. The elementary cell is rhombic (depicted at the central part of figure) and contains two atoms of carbon. The distance d between atoms relates to the large diagonal of the cell as $1 : 3$.

$H^{0\frac{1}{2}}$ transforms covariantly with weights $0, \frac{1}{2}$.
Consider first the homogeneous closed chain, which consists out of three fermions.

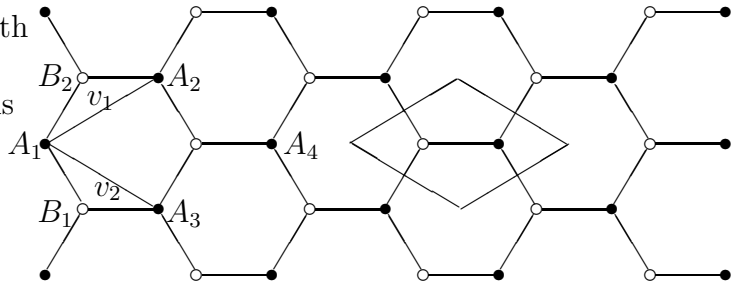

Fig.2

Three of four electrons of carbon atom hybridize to form three σ-orbitals directed to bounds with three nearest atoms, while the fourth electron forms a π-orbital, orthogonal to the lattice plane. This latter is responsible for the conductivity properties of graphene.Three radius-vectors directed from an atom in one sublattice towards three nearest neighbor atoms, belonging to other sublattice are given by:

$$A \to B: \qquad \mathbf{u}_1 = (-d, 0), \qquad \mathbf{u}_2 = (\frac{1}{2}d, \frac{\sqrt{3}}{2}d), \qquad \mathbf{u}_3 = (\frac{1}{2}d, -\frac{\sqrt{3}}{2}d), \qquad |\mathbf{u}_i| = d. \tag{9.207}$$

Triple $\{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3\}$ makes inverse mapping: $B \to A$. The motion of conducting electron conditioned by hopping with amplitude t onto the next atoms is described by a Hamiltonian in the approximation of tightly bound electrons:

$$H = -t\sum_{\alpha\in A}\sum_{i=1}^{3} a^\dagger(\mathbf{r}_\alpha)b(\mathbf{r}_\alpha + \mathbf{u}_i) - t\sum_{\alpha\in B}\sum_{i=1}^{3} b^\dagger(\mathbf{r}_\alpha)a(\mathbf{r}_\alpha - \mathbf{u}_i) =$$

$$= -t\sum_{\alpha\in A}\sum_{i=1}^{3}(a^\dagger(\mathbf{r}_\alpha)b(\mathbf{r}_\alpha + \mathbf{u}_i) + b^\dagger(\mathbf{r}_\alpha + \mathbf{u}_i)a(\mathbf{r}_\alpha)), \tag{9.208}$$

where operators $a^\dagger(\mathbf{r}_\alpha)$ and $b^\dagger(\mathbf{r}_\alpha)$ ($a(\mathbf{r}_\alpha)$ and $b(\mathbf{r}_\alpha)$) create (annihilate) electrons related to sublattice A and B correspondingly:

$$\{a^\dagger(\mathbf{r}_\alpha); a(\mathbf{r}_{\alpha'})\} = \delta_{\alpha\alpha'} = \{b^\dagger(\mathbf{r}_\alpha); b(\mathbf{r}_{\alpha'})\}.$$

To find dispersion relation one just uses disposition symmetry of carbon atoms in the lattice looking for the energy eigenvector in form:

$$|\psi\rangle = C_A\sum_{\alpha\in A} e^{i\mathbf{k}\mathbf{r}_\alpha}a^\dagger(\mathbf{r}_\alpha)|0\rangle + C_B\sum_{\alpha\in B} e^{i\mathbf{k}\mathbf{r}_\alpha}b^\dagger(\mathbf{r}_\alpha)|0\rangle. \tag{9.209}$$

Then acting on this state by Hamiltonian (9.208) one finds:

$$H|\psi\rangle = -t\sum_{i=1}^{3} C_B e^{i\mathbf{k}\mathbf{u}_i}\sum_{\alpha\in A} e^{i\mathbf{k}\mathbf{r}_\alpha}a^\dagger(\mathbf{r}_\alpha)|0\rangle - t\sum_{i=1}^{3} C_A e^{-i\mathbf{k}\mathbf{u}_i}\sum_{\alpha\in B} e^{i\mathbf{k}\mathbf{r}_\alpha}b^\dagger(\mathbf{r}_\alpha)|0\rangle = E|\psi\rangle,$$

i.e. one comes to two-dimensional eigenproblem:

$$\begin{pmatrix} 0 & -tX \\ -tX^* & 0 \end{pmatrix}\begin{pmatrix} C_A \\ C_B \end{pmatrix} = E\begin{pmatrix} C_A \\ C_B \end{pmatrix}, \qquad X = \sum_{i=1}^{3} e^{i\mathbf{k}\mathbf{u}_i}, \tag{9.210}$$

from which eigenvalues are easily deduced:

$$E = \pm t\sqrt{XX^*} = \pm t|X| = \pm t\sqrt{1 + 4\cos(\frac{\sqrt{3}}{2}k_y d)[\cos(\frac{3}{2}k_x d) + \cos(\frac{\sqrt{3}}{2}k_y d)]}. \qquad (9.211)$$

So the one-particle energy spectrum of the graphene sheet in tightly bound electrons approximation consists of two surfaces with $E_k > 0$ and $E_k < 0$ which touch each to other in six Fermi points corresponding to zero energy condition (which is equivalent to equations: $\sin(\frac{3}{2}k_x d) = 0$, $\cos(\frac{3}{2}k_x d) + 2\cos(\frac{\sqrt{3}}{2}k_y d) = 0$:

$$(k_x, k_y) = (0, \pm\frac{4\pi}{3\sqrt{3}d}), \qquad (k_x, k_y) = (\frac{2\pi}{3d}, \pm\frac{2\pi}{3\sqrt{3}d}), \qquad (k_x, k_y) = (-\frac{2\pi}{3d}, \pm\frac{2\pi}{3\sqrt{3}d}). \quad (9.212)$$

These points form a regular hexagon in the momentum space (Brillouin zone), only two of which are nonequivalent, because the opposite sides of hexagon are identified, i.e. next to neighboring corners are equivalent.

Besides interaction with three nearest-neighbor atoms belonging to another sublattice the tight-binding Hamiltonian for electrons in graphene can include also interaction with six next-nearest-neighbor atoms belonging to the same sublattice:

$$H' = -t'\sum_{\alpha\in A}\sum_{j=1}^{6}(a^\dagger(\mathbf{r}_\alpha)a(\mathbf{r}_\alpha + \mathbf{v}_j) + b^\dagger(\mathbf{r}_\alpha)b(\mathbf{r}_\alpha + \mathbf{v}_j)), \qquad (9.213)$$

where $\mathbf{v}_1 = \frac{d}{2}(3, \sqrt{3})$, $\mathbf{v}_2 = d(0, \sqrt{3})$, $\mathbf{v}_3 = \frac{d}{2}(-3, \sqrt{3})$, $\mathbf{v}_4 = \frac{d}{2}(-3, -\sqrt{3})$, $\mathbf{v}_5 = d(0, -\sqrt{3})$ and $\mathbf{v}_6 = \frac{d}{2}(3, -\sqrt{3})$, i.e. $|\mathbf{v}_j| = d\sqrt{3}$. Upon taking into account this addition the eigenproblem (9.210) changes to

$$\begin{pmatrix} -t'Y & -tX \\ -tX^* & -t'Y \end{pmatrix}\begin{pmatrix} C_A \\ C_B \end{pmatrix} = E\begin{pmatrix} C_A \\ C_B \end{pmatrix}, \qquad Y = \sum_{j=1}^{6} e^{i\mathbf{k}\mathbf{v}_j}, \qquad (9.214)$$

and just shifts energy value E (9.211) to $E + t'Y$ with $Y = 4\cos(\frac{\sqrt{3}}{2}k_y d)[\cos(\frac{3}{2}k_x d) + \cos(\frac{\sqrt{3}}{2}k_y d)] - 2$.

Note that theoretical estimation for t' is $0,02t < t' < 0,2t$, the experiment gives $t \approx 2,8$ eV and $t' \approx 0,1$ eV. It means that the long-wave (low-energy electron excitations) approach is a good approximation. It can be reached by passing $d \to 0$ in vicinity of nonequivalent Fermi points $\mathbf{K}_\pm = (0, \pm\frac{4\pi}{3\sqrt{3}d})$:

$$H_\pm = \lim_{d\to 0}\frac{1}{d}\begin{pmatrix} 0 & -tX \\ -tX^* & 0 \end{pmatrix}\Big|_{\mathbf{k}=\mathbf{K}_\pm+\mathbf{q}} = \frac{3}{2}t\begin{pmatrix} 0 & iq_x \pm q_y \\ -iq_x \pm q_y & 0 \end{pmatrix} = \hbar v(-q_x\sigma_2 \pm q_y\sigma_1), \qquad (9.215)$$

where $v = \frac{3}{2}t\hbar^{-1}$ is the Fermi velocity. Combining these contributions and carrying out Fourier transformation $\mathbf{q} \to -i\partial$ one comes to:

$$H = -i\hbar v(\gamma^x\partial_x + \gamma^y\partial_y), \qquad \gamma^x = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad \gamma^y = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \qquad (9.216)$$

Hamiltonian (9.216) acts on bispinor $\psi = (\psi_+^A, \psi_+^B, \psi_-^A, \psi_-^B)^T$, where superscripts A and Bcorrespond to two sublattices, while subscripts + and − correspond to two nonequivalent Fermi points. In this way the electronic processes in graphene are effectively described in low-energy approximation in continuous limit by "relativistic" (2+1)-dimensional Dirac equation where the role of speed of light c plays Fermi velocity $v \approx c/300$. The next-to-nearest interaction (terms proportional to t') contribute as a constant term shifting energy value and the second derivatives $H' \approx \frac{t'}{d}(-3 - \frac{9}{16}d^2(\partial_x^2 + 4\partial_y^2))$.

The rotations around z-axis are generated by operator $\Sigma = \frac{1}{2i}\alpha^x\alpha^y = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$. The graphene lattice is invariant under 2π-rotations:

$$\psi(\mathbf{r})|_{\varphi=2\pi} = \exp(2\pi i\Sigma)\psi(\mathbf{r})|_{\varphi=0} = -\psi(\mathbf{r})|_{\varphi=0}, \tag{9.217}$$

while rotation on angle π (inversion transformation: $x \to -x$, $y \to -y$) should be accompanied by $A \leftrightarrow B$ and $+ \leftrightarrow -$ transformations,in other words the parity transformation is generated by matrix $i\gamma_0 = i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$: $i\gamma_0(\psi_+^A, \psi_+^B, \psi_-^A, \psi_-^B)^T = i(\psi_-^B, \psi_-^A, \psi_+^B, \psi_+^A)^T$

9.2 Fullerene, nanotubes

Fullerenes are molecules composed of carbon atoms and were discovered in 1985. They are a new allotropic molecules of spherical form of carbon, containing more than 20 carbon atoms, which are connected with each other by three chemical bonds. Scientific and practical interest in studying fullerenes appeared in 1990, after the invention of the method enabling their production in bulk.

It is appeared that fullerene molecules can be considered unique instruments of scientific knowledge and can be used in resolving old scientific problems from an absolutely new position. Nature has united many opposite properties and notions in the fullerene C_{60} molecule.

Fullerene C_{60} is considered as a missed link between organic and inorganic matter. It is simultaneously a molecule and a particle. The diameter of a C_{60} molecule is about 1 nanometer, which is equal to the dispersion limit between the colloidal and molecular state of substances. Inside fullerene one sees only a void with a diameter of 0.4 nm and, above that, penetrated with electromagnetic fields. By this reason fullerene molecules are called vacuum bubbles, for which the well-known thesis that Nature abhors a vacuum is false.

Some molecular structures of proteins resembling fullerene in form were known, before the discovery of fullerenes. Some viruses and other vital biological structures contain such structures. The geometrical sizes of the fullerene C_{60} coincides with spherical clusters of secondary structures of the DNA molecule.

At the beginning of the last century, academician Vernadsky noted that living matter is characterized by high symmetry: many biologically important molecules have a fifth-order symmetry axis and are characterized by the golden section concept. C_{60} molecule has 6 fifth-order axes. It is the only unique known example molecule in nature with such high symmetry.

Carbon nanotubes are carbon allotropes with a cylindrical nanostructure. Up to now constructed nanotubes with ratio length-to-diameter of up to 132,000,000:1. It is signifi-

cantly larger than for any other material. These carbon molecules have unusual properties, which can be useful for nanotechnology, electronics, optics, materials science, technology and other fields.

Thanks to their extraordinary thermal conductivity and mechanical and electrical properties, carbon nanotubes find applications as additives to various structural materials. Nanotubes allow to form a tiny portion of the material in some baseball bats, golf clubs, or car parts. Nanotubes belong to the fullerene structural family. Their name comes from their long, hollow structure with the walls formed by one-atom-thick sheets of carbon (graphene).

The sheets are rolled at specific and discrete ("chiral") angles. The combination of the rolling angle and radius decides the nanotube properties: whether the individual nanotube shell is a metal or semiconductor.

Nanotubes are classified as single-walled and multi-walled.

10 Fusion Method, 2 hours

Algebraic Bethe Ansatz was already presented above. In this section we will describe the fusion method developed by efforts of Kulish, Sklyanin and Reshetikhin.

10.1 Higher spin solutions to YBE

Multiplying YBE

$$R_{12}(u-v)R_{13}(u-\alpha_3)R_{23}(v-\alpha_4) = R_{23}(v-\alpha_4)R_{13}(u-\alpha_3)R_{12}(u-v), \qquad (10.218)$$

by $R_{14}(u-a_4)R_{24}(v-a_4)$ and using (10.218) twice, one deduces, that

$$(T^{a_3a_4}_{b_3b_4})^{b_1}_{c_1}(u) \equiv R^{b_1a_3}_{d_1b_3}(u-\alpha_3)R^{d_1a_4}_{c_1b_4}(v-\alpha_4),$$

satisfies to $RTT = TTR$ equation (1.8). So having the fundamental solution to YBE (1.9) one obtains a composite solution $T \sim RR$ with arbitrary shifts α_1 and α_2. In simplest case of $s\ell_2$-algebra in corresponds to fusion of two spins one half. According to spin addition rule, it contains the contribution of spin one:

$$\frac{1}{2}[R^{b_1a_3}_{d_1b_3}(u+1-\alpha_3)R^{d_1a_4}_{c_1b_4}(v-\alpha_3) + R^{b_1a_4}_{d_1b_4}(u+1-\alpha_3)R^{d_1a_3}_{c_1b_3}(v-\alpha_3)],$$

which is obtained upon symmetrization of indices (a_3a_4), (b_3b_4) and $\alpha_3 = 1+\alpha_4$. The latter corresponds to multiplication on projector $R_{34}(\alpha_3-\alpha_4) = R_{34}(1) = I + P_{34} = 2P^+$ on symmetric subspace in $V_3 \otimes V_4$. Multiplication from the left is equivalent to multiplication from the right due to (1.9). Due to insertion of projection operator the remaining part, corresponding to spin zero is decoupled resulting solution $T^{(+)}$ becomes linear with respect to spectral parameter (second factor containing u enters as an overall scalar factor, which can be omitted).

Similarly, the contribution of spin zero is obtained upon antisymmetrization of indices (a_3a_4), (b_3b_4) and $\alpha_3 = \alpha_4 - 1$, which is equivalent to insertion of $P^- = \frac{1}{2}(I + P_{34})$.

This process can be continued to obtain an arbitrary half-integer spin, resulting to the Lax operator (1.14) linear by spectral parameter.

Then the $RLL = LLR$ relation (1.8) can be treated as a defining relation for the more general object, *Universal* $\mathcal{R}$-operator, acting in tensor product of two arbitrary representation spaces. In simplest case of $s\ell_2$-symmetry it means that the RLL-relation defined on space $V_{\ell_1} \otimes V_{\ell_2} \otimes V_{\frac{1}{2}}$, Lax operators act on product $V_{\frac{1}{2}} \otimes V_{\ell_i}$, while $\mathcal{R}$-operator acts on $V_{\ell_1} \otimes V_{\ell_2}$. It satisfies to symmetry relation (1.17) and is given by expansion over projection operators (1.18) where coefficients are determined from the recursion relation (1.19).

10.2 Universal R-operator

Moreover, it is possible to prove the more general tri-linear relation (10.218) for the $\mathcal{R}$-operator, defining on $V_{\ell_1} \otimes V_{\ell_2} \otimes V_{\ell_3}$. The latter has to take place as a consistency condition of RLL-relation:

$$\mathcal{R}_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}_{12}(u-v). \tag{10.219}$$

Indeed, let us consider product of three Lax-operators: $L_1(u)L_2(v)L_3(w)$. It can be transposed into $L_3(w)L_2(v)L_1(u)$ by two different ways:

$$\begin{aligned}
&\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u-w)\mathcal{R}_{23}(v-w)L_1(u)L_2(v)L_3(w) = \\
&\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u-w)L_1(u)[\mathcal{R}_{23}(v-w)L_2(v)L_3(w)] = \\
&= \mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u-w)L_1(u)[L_3(w)L_2(v)\mathcal{R}_{23}(v-w)] = \\
&= \mathcal{R}_{12}(u-v)[\mathcal{R}_{13}(u-w)L_1(u)L_3(w)]L_2(v)\mathcal{R}_{23}(v-w) = \\
&= \mathcal{R}_{12}(u-v)[L_3(w)L_1(u)\mathcal{R}_{13}(u-w)]L_2(v)\mathcal{R}_{23}(v-w) = \\
&= L_3(w)[\mathcal{R}_{12}(u-v)L_1(u)L_2(v)]\mathcal{R}_{13}(u-w)\mathcal{R}_{23}(v-w) = \\
&= L_3(w)L_2(v)L_1(u)\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u-w)\mathcal{R}_{23}(v-w),
\end{aligned}$$

here we used RLL-relation for objects in square brackets and commutativity of the objects acting in different spaces like $\mathcal{R}_{12}$ and L_3. From the other hand the same result can be obtained using $\mathcal{R}_{23}(v-w)\mathcal{R}_{13}(u-w)\mathcal{R}_{12}(u-v)$.

Exercise
Prove that $\mathcal{R}_{23}(v-w)\mathcal{R}_{13}(u-w)\mathcal{R}_{12}(u-v)$ *transposes* $L_1(u)L_2(v)L_3(w)$ *to* $L_3(w)L_2(v)L_1(u)$.

11 Application to Quantum Chromodynamics, 4 hours

It is generally believed that the integrable models are the key to strong coupling problem. Up to date the significant progress of Quantum Field Theory is recorded only for theories with non-trivial interaction in the region of weak coupling where the interaction only slightly distorts the picture of free (non-interacting) theory and the perturbation theory works.

This is why now so popular different duality transformations, which involve not only field variables but also coupling constants and map the strong coupling regime of one theory to the weak coupling regime of other.

Another way to overcome the strong coupling problem is to relate corresponding theory with exactly solvable system in some limit.

11.1 Regge limit

The one of the most important theory with strong coupling is Quantum Chromodynamics, at low energies the coupling constant of the strong interaction α_S is exceed the unity and the representation of hadronic processes in form of Feynman diagram is meaningless. It decreases with increasing energy and at high energies becomes small enough. The hard hadronic processes is convenient to represent as scattering of two particles into $n-2$, so the amplitude $A_{2\to n-2}$ has n external legs. In Regge limit

$$(p_1+p_2)^2 = s \gg -t \sim M^2 \tag{11.220}$$

here s is an energy in the center of mass, M is a characteristic hadronic mass scale and t is momentum transfer $A_{2\to n-2}$ is represented by hamiltonian of $s\ell_2$ Heisenberg spin chain with non-compact spins.

In Regge theory the high-energy asymptotics of the hadron-hadron scattering amplitudes is determined by singularities of partial waves in the complex angular momentum plane. It was observed that the regularity of quantum mechanics, which relates high-energy scattering amplitudes to the singularities of the partial waves in the complex angular momentum plane is valid for quantum field theory as well. Namely, in Regge kinematics the hadron-hadronic scattering amplitude $A(s,t)$

$$A(s,t) = is \int\limits_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dJ}{2\pi i} \left(\frac{s}{M^2}\right)^J f(J,t) \tag{11.221}$$

is governed by singularities of the partial waves $f(J,t)$, by Regge poles and Regge cuts. According to bootstrap conjecture all particle-like excitations correspond to some Regge singularity and are related to each other via unitarity of the S-matrix and sum rules.

However the programme of building the axiomatic quantum field theory from assumptions of only unitarity and analyticity of the S-matrix failed, because Regge theory itself does not allow to calculate the positions of these singularities. So now, QCD as a theory of strong interaction is called to describe the Regge behavior of the scattering amplitudes. It was shown that in leading logarithmic approximation (LLA), which is the natural approximation in the Regge limit of QCD, $A(s,t)$ can be expressed as a sum of Feynman diagrams describing the multiple exchange of reggeized gluons in the t-channel.

The perturbative expressions for corresponding Feynman diagrams including large logarithmic factors $\alpha^n log^m s$ (m=n,n-1,...) have to be resummed to all orders in α_s, because bare gluons and quarks are not a good approximation in the Regge limit. The leading contribution ($m=n$) comes from ladder diagrams, corresponding to exchange of n-reggeons in t-channel. Being built from an infinite number of perturbative gluons, the reggeons carry the quantum numbers of the quark or gluon and become a new collective excitation in the Regge limit. The leading logarithmic approximation results in an asymptotics of amplitudes, which violates the Froissart bound. However unitarity is restored by taking into account sub-leading contributions. In the generalized leading logarithmic approximation some minimal set of non-leading terms is included to restore unitarity. In the frame of effective theory the interaction of the reggeons is determined by LLA.

The key point of this approach is assumption that number of colors N_c is large (or multicolor limit $N_c \to infty$). It implies that all Feynman diagrams are planar (can be drown on plane without intersections of the inner lines), all non-planar diagrams are suppressed by powers of $1/N_c$. It means that if n scattered particles are positioned along the circle, they will interact only with nearest neighborhoods.

The Regge limit condition implies that momenta of scattered particles are large in comparison with hadronic mass scale and the four-dimensional space-time is separated on product of two planes: *longitudinal* containing 4-momenta p_1 and p_2 (it entirely determined by kinematics of the process) and *transversal*, containing whole information about dynamics of the process. In turn the expression for amplitude in transversal plane is separated on holomorphic and anti-homorphic parts, which are given by two copies of 1-dimensional Heisenberg Hamiltonians of $s\ell_2$-magnet with representations of complex spins $\ell = s + i\nu$ places at the sites of chain. Here s is positive half-integer and ν is anomalous scaling dimension of corresponding scattered particle.

Such complex spins correspond to infinite-dimensional non-compact representation space, unlike to positive half-integer spins, corresponding to finite-dimensional representations, considering in statistical mechanics.

11.2 Integral kernel

The $\mathcal{R}$-operator for such values of spins can be defined by means of defining equation (10.219). It can be represented as an integral operator with kernel:

$$\mathcal{R}(x_1, x_2|x_1', x_2') = c(u)\frac{(x_1 - x_2)^{u-\ell_1-\ell_2+1}(x_1' - x_2')^{u+\ell_1+\ell_2-1}}{(x_1 - x_2')^{u+1+\ell_1-\ell_2}(x_1' - x_2)^{u+1-\ell_1+\ell_2}}, \tag{11.222}$$

where $c(u)$ is an arbitrary function, ℓ_1, ℓ_2 are arbitrary spins of representations V_1, V_2 and on test function $\psi(x_1, x_2)$ $\mathcal{R}$ acts as follows:

$$(\mathcal{R}\psi)(x_1, x_2) = \int \mathcal{R}(x_1, x_2|x_1', x_2')\psi(x_1', x_2')dx_1'dx_2'.$$

The kernel of holomorphic part of scattering amplitude is then given by Hamiltonian:

$$H_n = \frac{\partial}{\partial u}\log \mathrm{tr}_0(\mathcal{R}_{01}(u)\mathcal{R}_{02}(u)\dots\mathcal{R}_{0n}(u)).$$

12 Calogero-Moser Model, 6 hours

Quantum Calogero-Moser-Sutherland model describes the set of N identical particles on a circle interacting pairwise with inverse square potential. Calogero-Moser-Sutherland model attracted some attention due to conformal character of interaction potential. The new aspects of their algebraic structure and quantum integrability was later clarified by [3]. The Calogero Model is a rare example of integrable many-body problem. The studying of the Hamiltonian H_c was started by Calogero [2], who computed the spectrum, eigenfunctions and scattering states in the confined and free cases. The Perelomov observed the complete

quantum integrability of the model, he stated that exist N commuting, algebraically independent operators and H_c is one among them. The complete integrability of the classical Hamiltonian of Calogero-Moser-Sutherland was proved by Moser.

Unfortunately, in a series of integrable theories, Calogero model stands alone. A powerful Inverse Scattering Method is applicable to it with considerable restrictions. The key object in ISM is R-matrix depending on spectral parameter. The R-matrix can associate to this model, but it is dynamic: its matrix elements are not c-numbers and depend on the coordinates. Moreover, the dependence on the spectral parameter can be obtained only for elliptical extension. It makes us look for other ways to describe this model.

The Lax method being applied to the integrable model allows to trace its integrability to zero-curvature condition of $L - A$ pair. Namely the equations of motion of the model express the consistency condition of some (usually more wide) linear (free) system.

The Calogero model in external harmonic field

$$H_C = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + \frac{\omega^2 q_i^2}{2} \right) + \sum_{i<j} \frac{g^2}{(q_i - q_j)^2} \tag{12.223}$$

can be obtained by $SU(N)$ reduction from the matrix model

$$H = \frac{1}{2} \operatorname{tr} P^2 + \frac{\omega^2}{2} \operatorname{tr} Q^2 \tag{12.224}$$

where Q and P are hermitean matrices:

$$\{P_{ij}, Q_{i',j'}\} = \delta_{ij'}\delta_{ji'}, \quad \{P_a, Q_b\} = \delta_{ab}, \tag{12.225}$$

which is equivalent to the homogeneous $(N^2 - 1)$-dimensional oscillator.

Consider the Calogero matrix Hamiltonian without oscillator term $\omega = 0$. Then (12.224) tell us $H_0 = \frac{1}{2} tr(P^2)$. The canonical relations (12.224) are invariant under (canonical) similarity transformation:

$$P \to P' = U^{-1} P U, \qquad Q \to Q' = U^{-1} Q U,$$

with numerical unitary matrix $U = \exp i\varepsilon$. The Noether current corresponding to this transformation is $tr(P\delta Q)$ for infinitesimal ε one has $\delta Q = i[Q; \varepsilon]$ and $tr(P\delta Q) = tr(i\varepsilon[P; Q])$. So one deduces, that the Noether charges for this transformation are:

$$J_{ij} = i[P; Q]_{ij} = const. \tag{12.226}$$

Now, using that symmetry one can turn the matrix Q to diagonal form:

$$Q = diag(q_1, q_2, \ldots, q_N). \tag{12.227}$$

Then one can define matrix P from (12.226):

$$J_{ij} = i \sum_j (P_{ik} q_k \delta_{kj} - q_i \delta_{ik} P_{kj}) = i P_{ij} (q_j - q_i), \qquad \Rightarrow \qquad P_{ij} = \frac{i J_{ij}}{q_i - q_j}. \tag{12.228}$$

Thus the result of reduction of matrix model under consideration is entirely determined by the numerical matrix J with zeros at diagonal. The simplest case of zero (rank) matrix $J = 0$ corresponds to diagonal matrix P (diagonal elements can not be derived from (12.228) due to vanishing the diagonal elements of J) and to the case of N free particles.

The next in complexity case is rank two matrix:

$$J_{ij} = \delta_{ij} - w_i w_j, \qquad w_i = 1, \quad i = 1, \dots N, \tag{12.229}$$

the rank one matrix is excluded due to vanishing diagonal elements. Let ψ_i are components of column ψ eigenvector corresponding to eigenvalue λ:

$$\psi_i - \sum_{j=1}^{N} \psi_j = \lambda \psi_i, \qquad \Rightarrow \qquad (1-\lambda)\psi_i = (1-\lambda)\psi_j,$$

one sees, that eigenvalue $\lambda_1 = 1$ has multiplicity $N-1$, while eigenvalue $\lambda_2 = 1-N$ has multiplicity one, i.e. the matrix (12.230) indeed has rank two.

So finally, matrix P is:

$$P_{ij} = p_i \delta_{ij} + i\frac{1-\delta_{ij}}{x_i - x_j} = \begin{cases} p_i, & i = j \\ \frac{i}{q_i - q_j}, & i \neq j \end{cases} \tag{12.230}$$

Substituting (12.228) for this case into (12.224) one will come to Hamiltonian of Calogero model:

$$H = trP^2 = \sum_{i,j=1}^{N} P_{ij}P_{ji} = \sum_{i=1}^{N} P_{ii}^2 + 2\sum_{i<j} P_{ij}P_{ji} = \sum_{i=1}^{N} p_i^2 + 2\sum_{i<j} \frac{1}{(x_i - x_j)^2}.$$

Then the $N-1$ integrals of motion are given by $I_j = trP^j$, $j \neq 2$. In particular,

$$trP^3 = \sum_{i=1}^{N} p_i^3 + 3\sum_{i,j=1}^{N} \frac{1-\delta_{ij}}{x_i - x_j} p_i \frac{1}{x_i - x_j} - i\sum_{i,j,k=1}^{N} \frac{(1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})}{(x_i - x_j)(x_j - x_k)(x_k - x_k)} \tag{12.231}$$

12.1 Elliptic Calogero model

The elliptic Calogero-Moser model describes the system of N one-dimensional particles interacting via two-particle potential

$$V(q_{\alpha\beta}) = \wp(q_{\alpha\beta}), \qquad q_{\alpha\beta} = q_\alpha - q_\beta, \tag{12.232}$$

where $\wp$ is the Weierstrass function with the periods ω_1 and ω_2. Hamiltonian of the model is

$$\mathfrak{H} = \sum_{\alpha=1}^{N} p_\alpha^2 + \sum_{\alpha\neq\beta} V(_{\alpha\beta}), \tag{12.233}$$

where q_α and p_α are coordinates and momenta of particles.

As we already seen above for the commutativity of the spectral invariants $\mathrm{tr}(L(u))^n$ of the Lax operator $L(u)$ it is necessary and sufficient that the Poisson bracket between two Lax operators could be represented in the commutator form: $\{L^{\alpha_1}_{\beta_1}(u), L^{\alpha_2}_{\beta_2}(v)\} =$

$$=\sum_{\gamma,\gamma_2}\big(r^{\alpha_1\alpha_2}_{\gamma_1\beta_2}(u,v)L^{\gamma_1}_{\beta_1}(u)-L^{\alpha_1}_{\gamma_1}(u)r^{\gamma_1\alpha_2}_{\beta_1\beta_2}(u,v)-r^{\alpha_2\alpha_1}_{\gamma_2\beta_1}(v,u)L^{\gamma_2}_{\beta_2}(u)-L^{\alpha_2}_{\gamma_2}(v)r^{\gamma_2\alpha_1}_{\beta_2\beta_1}(v,u)\big) \quad (12.234)$$

or in compact notations:

$$\{L^{(1)}(u), L^{(2)}(v)\} = [r^{(12)}(u,v), L^{(1)}] - [r^{(21)}(v,u), L^{(2)}(v)],$$

where $L^{(1)} = L \otimes \mathbb{I}$, $L^{(2)} = \mathbb{I} \otimes L$, $r^{(21)} = Pr^{(12)}P$ and P is permutation operator.

The Lax operator is given by the $N \times N$ matrix

$$L^\alpha_\beta(u) = p_\alpha \delta^a_\beta + iQ(u, q_{\alpha\beta})(1 - \delta^\alpha_\beta), \quad (12.235)$$

where $Q(u, q_{\alpha\beta})$ is expressed in terms of Weierstrass σ functions

$$Q(u,q) = \frac{\sigma(u+q)}{\sigma(u)\sigma(q)}, \quad (12.236)$$

and

$$\sigma x = x \prod_{m,n\neq 0} (1 - \frac{x}{\omega_{mn}}) \exp[\frac{x}{\omega_{mn}} + \frac{1}{2}(\frac{x}{\omega_{mn}})^2], \quad (12.237)$$

where $\omega_{mn} = 2m\omega_1 + 2n\omega_2$, ω_1, ω_2 is a pair of periods.

The ζ and $\wp$ functions are derivatives of $\sigma(x)$

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}, \qquad \wp(x) = -\zeta'(x).$$

They have properties:

$$\sigma(x + 2\omega_j) = -\sigma(x) \exp[2(x + \omega_j)\zeta(\omega_j)],$$

$$\zeta(x + 2\omega_j) = \zeta(x) + 2\zeta(\omega_j), \qquad \wp(x + 2\omega_j) = \wp(x),$$

$$\sigma(-x) = -\sigma(x), \qquad \zeta(-x) = -\zeta(x), \qquad \wp(-x) = \wp(x).$$

In the vicinity of $x = 0$ the Weierstrass functions have the expansions

$$\sigma(x) = x + O(x^5), \qquad \zeta(x) = x^{-1} + O(z^3), \qquad \wp(x) = x^{-2} + O(x^2).$$

The spectral invariants $\mathrm{tr}(L(u))^n$ give as usual integrals of motion. $n = 1, 2$ correspond to total momentum of system $\mathfrak{P} = \sum_{\alpha=1}^N p_\alpha$ and Hamiltonian (12.233):

$$\mathfrak{P} = \mathrm{tr}L(u), \qquad \mathrm{tr}(L(u))^2 = \mathfrak{H} - V(u). \quad (12.238)$$

The RLL-relation (12.234) holds for following matrix r:

$$r^{(12)}(u,v) = a\sum_{\alpha=1}^N E^\alpha_\alpha \otimes E^\alpha_\alpha + \sum_{\alpha\neq\beta} c_{\alpha\beta} E^\alpha_\beta \otimes E^\beta_\alpha + \sum_{\alpha\neq\beta} d_{\alpha\beta}(E^\alpha_\beta \otimes E^\beta_\alpha + E^\beta_\beta \otimes E^\alpha_\alpha), \quad (12.239)$$

where

$$a = r^{\alpha\alpha}_{\alpha\alpha} = -\zeta(u-v)-\zeta(v), \quad c_{\alpha\beta} = r^{\alpha\beta}_{\beta\alpha} = -Q(u-v, q_{\alpha\beta}), \quad d_{\alpha\beta} = r^{\alpha\alpha}_{\alpha\beta} = r^{\beta\alpha}_{\beta\beta} = -\frac{1}{2}Q(v, q_{\alpha\beta}).$$

12.2 Dynamical R-matrix

It is known that for pure numerical r-matrices, the consistency condition for classical rLL-relation (12.234) is just Jacoby identity for r-matrices:

$$[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] - [r^{(13)}, r^{(32)}] = 0.$$

However it is not enough in the case under consideration. The r-matrix (12.239) is not numerical, it is dynamical, i.e. depends on field variables (coordinates $q_{\alpha\beta}$).

The consistency condition for (12.234) looks like:

$$\{\{L^{(1)}, L^{(2)}\}, L^{(3)}\} + \{\{L^{(2)}, L^{(3)}\}, L^{(1)}\} + \{\{L^{(3)}, L^{(1)}\}, L^{(2)}\} = 0. \tag{12.240}$$

It implies the following relation:

$$[R^{(123)}, L^{(1)}] + [R^{(231)}, L^{(2)}] + [R^{(312)}, L^{(3)}] = 0,$$

where

$$R^{(123)} = [r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] - [r^{(13)}, r^{(32)}] - \{r^{(13)}, L^{(2)}\} + \{r^{(12)}, L^{(3)}\}.$$

Trying anzatz $R^{(123)} = [X^{(123)}, L^{(2)}] - [Y^{(123)}, L^{(3)}]$ one obtains $Y^{(123)} = X^{(312)}$ and one comes to following generalized Yang=Baxter equation for classical r-matrix:

$$[r^{(12)}, r^{(13)}]+[r^{(12)}, r^{(23)}]-[r^{(13)}, r^{(32)}]-\{r^{(13)}, L^{(2)}\}+\{r^{(12)}, L^{(3)}\}-[X^{(123)}, L^{(2)}]+[X^{(312)}, L^{(3)}] = 0. \tag{12.241}$$

Then is possible to show, that the solution to (12.241) is achieved at:

$$X^{(123)}(u, v, w) = -i \sum_{\alpha \neq \beta} Q(w, q_{\alpha\beta})[-\frac{5}{8} E^{\alpha}_{\alpha} \otimes E^{\alpha}_{\alpha} + \frac{1}{8} E^{\beta}_{\beta} \otimes E^{\beta}_{\beta} + \frac{1}{4} E^{\alpha}_{\alpha} \otimes E^{\beta}_{\beta} + \frac{1}{4} E^{\beta}_{\beta} \otimes E^{\alpha}_{\alpha}] \otimes E^{\alpha}_{\beta}.$$

So the Poisson algebra of Lax operator L and dynamical r-matrix in elliptic Calogero model is not closed: r satisfies a generalized version of the classical Yang-Baxter equation (12.241). It involves a new object, the X matrix and possible give rise to infinite sequence of r-matrices, depending on coordinate q_{α}.

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