Lectures on Integrable Models in Field/String Theory

Lecture 1

Introduction

Hamilton's principle. Noether's theorem. Gauge symmetry. Noether's second theorem. Hamiltonian description. Faddeev-Jackiw formalism.

Lecture 2

Symplectic geometry

The Hamiltonian dynamics. Definitions and notations. Useful formulas and identities. Hamiltonian vector fields. Darboux theorem. Symplectic structure on TQ, T^*Q and on the space of solutions. Moment map. Co-cycles of Lie algebras and central extensions.

Lecture 3

The $SL(2,\mathbb{R})$ group

The $sl(2,\mathbb{R})$ algebra. The Killing form. The exponential map: $sl(2,\mathbb{R}) \to SL(2,\mathbb{R})$. The adjoint representation. Coordinates on $SL(2,\mathbb{R})$. Functions, vector fields, 1-forms and the metric on the $SL(2,\mathbb{R})$ group manifold.

Lecture 4

Particle dynamics on symmetric spaces

The Liouville model. The dynamics of a particle in $SU(2)$ (classical and quantum theories). Dynamics of a relativistic particle in $SL(2,\mathbb{R})$. Particle dynamics in AdS space. The dynamics of a massless particle.

Lecture 5

Gauging and Hamiltonian reduction

Gauging of Noether symmetries. Singular Lagrangian. First order formalism. Reductions of differential forms. Examples: Mechanical model of QED. Gauging of the particle dynamics on $SU(2)$ and $SL(2,\mathbb{R})$ group manifolds.

Lecture 6

The method of co-adjoint orbits

Co-adjoint representation of Lie groups. Co-adjoint orbits. Symplectic forms and Hamiltonian vector fields on co-adjoint orbits. Geometric quantization. Choice of polarization. Irreducible representations.

Lecture 7

Geometric quantization and coherent states

Symmetries and coherent states. Examples: Weyl group, $SL(2,\mathbb{R})$ and $SU(2)$ coherent states. Symbol calculus. Moyal quantization. Coherent state formalism and geometric quantization.

Lecture 8

The Lagrangian formulation of $SL(2,\mathbb{R})$ WZW theory

σ-models in 2-dimensions. The $SL(2,\mathbb{R})$ target space. 2-forms on $SL(2,\mathbb{R})$ group manifold and the $SL(2,\mathbb{R})$ WZW Lagrangian. The general solution and global symmetries. The $SU(2)$ WZW Lagrangian and the WZ term. Symmetries and integration of dynamical equations.

Lecture 9

The Symplectic structure of 2d free-field theory

Free field theory on a cylinder and a strip. Canonical form. Chiral fields and the chiral symplectic form. The Poisson brackets algebra of chiral fields. 'Vertex functions' and their algebra. The energy momentum tensor and the conformal symmetry.

Lecture 10

The Hamiltonian formulation of WZW theory

Canonical structure of WZW theory. The chiral symplectic form. The Poisson brackets algebra of chiral WZ fields. Kac-Moody algebra. The Sugawara energy momentum tensor. $SU(2)$ and $SL(2,\mathbb{R})$ WZW models.

Lecture 11

Gauging of WZW theory

Vector and axial gauging of $SL(2,\mathbb{R})$ WZW theory. $U(1)$ gauging and $SL(2,\mathbb{R})/U(1)$ black hole model. $\mathbb{R}^1$ gauging. Nilpotent gauging and Liouville theory. Hamiltonian reduction and free-field parametrization.

Lecture 12

Canonical quantization of 2d CFT

Canonical quantization of free-field theory. 2d conformal symmetry and Virasoro algebra. Vertex operators and their algebra. Canonical map to Liouville theory. Construction of Liouville vertex operators and calculation of the reflection amplitude.

Lecture 13

Geometric quantization of infinite dimensional symmetries

The co-adjoint orbits of Virasoro group. Symplectic structure and Poisson brackets. Transformation to free-field variables. Coherent states of infinite dimensional translation group and 2d conformal group. Transition amplitudes between the coherent states. Kac-Moody group.

Lecture 14

String dynamics in Minkowsky space

Dynamical equations and gauge fixing. Integration of equations of motion. Light-cone gauge quantization. Covariant quantization and critical dimension. Polyakov method and non-critical strings. Static gauge quantization.

Lecture 15

The AdS/CFT correspondence

String dynamics in AdS and $AdS \times S$ spaces. Lax pair representation of the dynamical equations. Integration by the Pohlmeyer method. Light-cone quantization of $AdS_5 \times S^5$ string dynamics. Static gauge quantization of AdS strings. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Main ideas of the correspondence. Integrable structures of the dual theories.

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By George Jorjadze
Free University of Tbilisi and
Razmadze Mathematical Institute of Javakhishvili State University