QUANTUM FIELD THEORY IN CURVED SPACETIME

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Contents

These lectures provide an introduction to quantum field theory on curved backgrounds. The course is divided into three parts. In the first one we give a short introduction to classical and quantum fields in Minkowski spacetime, and to General Relativity as a classical theory of gravitation. In the second part, classical and quantum fields in curved spacetime are considered. The general procedures for the regularization and renormalization are described. In the third part, the applications of general scheme are given to several examples. In particular, we consider quantum fields in de Sitter and anti-de Sitter spacetimes.

Notations

Spacetime vectors: $a^\mu = (a^0, a^i)$.

Greek indices correspond to spacetime components and Latin indices are for spatial components.

Partial derivative: $\frac{\partial}{\partial x^\mu} = \partial_\mu =, \mu$

Metric tensor: $g_{\mu\nu}$. In flat spacetime in Galilean coordinates: $g_{\mu\nu}$ =diag$(1, -1, \ldots, -1)$.

Covariant components: $a_\mu = g_{\mu\nu} a^\nu = \sum_\nu g_{\mu\nu} a^\nu$

Covariant derivative: ∇_μ, $;\mu$

Riemann tensor: $R^{\mu}_{\cdot\nu\rho\sigma} = \partial_\sigma \Gamma^\mu_{\nu\rho} - \cdots$, $R_{\mu\nu} = R^{\rho}_{\cdot\mu\rho\nu}$.

Units: $\hbar = c = 1$

Chapter 1

Introduction

Quantum Field Theory as a theory of elementary particles

Quantum Field Theory is a physical theory of elementary particles and their interactions. It has emerged as the most successful physical framework describing the subatomic world. The quantum field theoretical predictions for the interactions between electrons and photons have proved to be correct to within one part in 10^8. Furthermore, it can adequately explain the interactions of three of the four known fundamental forces in the universe. Quantum field is a fundamental physical concept within the framework of which the properties of elementary particles and their interacions are formulated and described.

Quantum field theory with an external background is an adequate model for studying quantum processes in the cases when a part of the quantized field is strong enough to be treated as a given and a classical one. Numerous problems in QED and QCD with superstrong electromagnetic fields, which must be treated nonperturbatively, are at present investigated in this framework, with applications to astrophysics and condensed matter physics (e.g. graphene physics).

The success of quantum field theory as a theory of subatomic forces is today embodied in what is called the Standard Model. In fact, at present, there is no known experimental deviation from the Standard Model (excluding gravity). Standard Model is based on the gauge group $SU(3) \times SU(2) \times U(1)$. As a result of theoretical and experimental successes, the Standard Model was rapidly recognized to be a first-order approximation to the ultimate theory of particle interactions. The spectrum of the Standard Model for the left-handed fermions is schematically listed here, consisting of the neutrino ν, the electron e, the "up" and "down" quarks, which come in three "colors," labelled by the index i. This pattern is then repeated for the other two generations:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \begin{pmatrix} u^i \\ d^i \end{pmatrix}; \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \begin{pmatrix} c^i \\ s^i \end{pmatrix}; \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} \begin{pmatrix} t^i \\ b^i \end{pmatrix}.$$

In the Standard Model, the forces between the leptons and quarks are mediated by the photon for electromagnetic interactions, by the massive vector mesons for the weak interactions and the massless gluons for the strong interactions:

$$\gamma, W^{\pm}, Z, A^a_\mu.$$

In addition, the Standard Model contains a scalar particle, Higgs boson, which is responsible for the particle masses. On 4 July 2012, the ATLAS and CMS experiments at Large Hadron Collider (CERN) announced they had observed a new particle in the mass region around 126 GeV. This particle is consistent with the Higgs boson predicted. If further confirmed, this will be the first elementary scalar particle discovered in nature.

Gravity

Although the gravitational interaction was the first of the four forces to be investigated classically, it was the most difficult one to be quantized. The quantization of the gravitational field remains among the most fundamental problems of theoretical physics. In elementary particle physics, usually the gravitational interaction is ignored. This is argued by smallness of the gravitational coupling constant comparing to those for other interactions. However, there are a number of examples showing that despite the smallness of corrections, they lead to qualitatively new effects. In particular, the quantum gravitational effects may provide a solution for quantum field-theoretical divergences. Using some general physical arguments, one could calculate the mass and spin of the gravitational interaction. Since gravity was a long-range force, it should be massless. Since gravity was always attractive, this meant that its spin must be even. (Spin-one theories, such as electromagnetism, can be both attractive and repulsive.) Since a spin-0 theory was not compatible with the known bending of starlight around the sun, we were left with a spin-two theory. A spin-two theory could also be coupled equally to all matter fields, which was consistent with the equivalence principle. These heuristic arguments indicated that Einstein's theory of general relativity should be the classical approximation to a quantum theory of gravity.

The problem, however, was that quantum gravity had a dimensionfull coupling constant and hence was nonrenormalizable. This coupling constant, in fact, was Newton's gravitational constant, the first important universal physical constant to be isolated in physics. Another fundamental problem with quantum gravity was that the strength of the interaction was exceedingly weak, and hence very difficult to measure. Once gravity was quantized, the energy scale at which the gravitational interaction became dominant was set by Newton's constant G_N. To see this, let r be the distance at which the gravitational potential energy of a particle of mass M equals its rest energy, so that $G_N M^2/r = Mc^2$. Let r also be the Compton wavelength of this particle, so that $r \approx \hbar/Mc$. Eliminating M and solving for r, we find that r equals the Planck length, 10^{-33} cm, or 10^{19} GeV:

$$\begin{aligned} l_P &= (\hbar G_N/c^3)^{1/2} = 1.61605 \times 10^{-33}\text{cm}, \\ M_P &= (\hbar c/G_N)^{1/2} = 1.22105 \times 10^{19}\text{GeV}/c^2. \end{aligned}$$

This is beyond the range of our instruments for the foreseeable future.

Yet another problem arose when one tried to push the theory of gravity to its limits. Phenomenologically, Einstein's general relativity has proved to be an exceptionally reliable tool over cosmological distances. However, when one investigated the singularity at the center of a black hole or the instant of the Big Bang, then the gravitational fields became singular, and the theory broke down. One expected quantum corrections to dominate in those important regions of space-time. However, without a quantum theory of gravity, it was impossible to make any theoretical calculation in those interesting regions of space and time.

Completely satisfactory quantum theory of gravity remains elusive. Perhaps the most hopeful current approaches are supergravity and superstring theories, in which the graviton is regarded as only one member of a multiplet of gauge particles including both fermions and bosons.

In the absence of a viable theory of quantum gravity, can one say anything at all about the influence of the gravitational field on quantum phenomena? In the early days of quantum theory, many calculations were undertaken in which the electromagnetic field was considered as a classical background field, interacting with quantized matter. Such a semiclassical approximation yields some results that are in complete accordance with the full theory of QED. One may therefore hope that a similar regime exists for quantum aspects of gravity, in which the gravitational field is retained as a classical background, while the matter fields are quantized in the usual way. Adopting General Relativity as a description of gravity, one is led to the subject of *quantum field theory in a curved background.*

If the gravitational field is treated as a small perturbation, and attempts are made to quantize it along the lines of QED, then the square of the Planck length appears in the role of coupling constant. Unlike QED, however, the Planck length has dimensions. Effects can become large when the length and time scales of quantum processes of interest fall below the Planck value. When this happens, the higher orders of perturbation expansion breaks down. The Planck values therefore mark the frontier at which a full theory of quantum gravity, preferably non-perturbative, must be invoked. Nevertheless, one might hope that when the distances and times involved are much larger than the Planck values, the quantum effects of the gravitational field will be negligible.

In the absence of horizons and singularities, the formal construction of quantum field theories in curved backgrounds (fields as operator valued generalized functions, algebra of observables, commutators) closely follows the quantum field theory in flat spacetime. The difficulties arise in constructing of second quantized theories. Two mutually related problems arise at this stage. The first one is the construction of the Hilbert space of states for a quantum field and the second one is the obtainment of finite results for physical observables.

The construction of the space of states for a quantum field is reduced to the definition of the vacuum state and the interpretation of the field in terms of particles. The different sets of mode functions used in the quantization procedure lead to different definitions of the notion of particle. In flat spacetime the corpuscular interpretation of a free field is based on the invariance with respect to the Poincaré group. In an arbitrary curved background such a principle is absent. The second problem, the extraction of finite results from diverging expressions for physical observables, is more complicated compared to the corresponding procedure in flat spacetime. New types of divergences appear which are absent in usual quantum field theory. As a result, the corresponding renormalization procedure is more involved.

Outline

In these lectures we will describe the basics of quantum field theory in background of curved spacetimes and the points mentioned above will be clarified. At present there are several books covering various aspects of this topic [1]-[9]. A number of review papers (see, for example, [10]-[15]) and lecture notes [16]-[21] are also available. The lectures are organized as follows. In Chapter 2 we consider classical fields in Minkowski spacetime. The action principle, field equations and the Noether's theorem will be discussed. Examples of scalar, Dirac spinor and gauge fields are considered. A short review of the quantization procedure in Minkwoski spacetime is presented in Chapter 3. The canonical quantization procedure is described for real and complex scalar fields, for Dirac field and for gauge fields.

The consideration of curved backgrounds we start with a short review of General Relativity and cosmology in Chapter 4. Then, in Chapter 5, we pass to classical fields propagating in curved backgrounds. For description of the influence of gravity on matter the tetrad formalism is introduced. An application to Dirac spinor field is given. Properties of the fields under conformal transformations of the metric tensor are discussed. The quantization of fields in curved backgrounds is considered in Chapter 6. The Bogoliubov transformations and the uniqueness of the vacuum state are discussed. The response of the Unruh-DeWitt particle detector is studied for a scalar field. Examples of inertial and uniformly accelerated detectors in Minkowski spacetime are considered. In quantum field theory on curved spacetimes, among the most important objects are two-point functions for a quantum field. In Chapter 7 we consider the De Witt-Schwinger expansion for the Feynman Green function for a scalar field. This is an important step in the renormalization procedure and allows to clarify the structure of divergences appearing in the expectation values of physical observables in the coincidence limit. The renormalization of the effective action for scalar, spinor and electromagnetic fields is discussed in Chapter 8. We consider conformal anomalies and the expectation value of the energy-momentum tensor in conformally related problems. Various examples are studied with explicit expressions for the vacuum energy-momentum tensor. Then the renormalization of the energy-momentum tensor in general backgrounds is discussed. In Chapters 6,7,8 we mainly follow Ref. [1].

Chapter 9 starts the applications of general procedure to special problems. First we consider flat spacetimes different from the Minkowski one. They include spacetimes with compact dimensions and problems in presence of boundaries on which the field operator obeys some prescribed boundary conditions. As important physical characteristics of the vacuum state we evaluate the expectation values of the field squared, of the energy-momentum tensor and of the current density. Another example of flat spacetime different from the Minkwoski one, is the Rindler spacetime, employed for description of quantum fields by uniformly accelerated observers. A quantum scalar field in Rindler spacetime and the Unruh effect are considered in Chapter 10. Then, we pass to examples with curved backgrounds. In order to have exactly solvable problem, we consider maximally symmetric spacetimes, namely, de Sitter and anti-de Sitter spacetime. The most frequently used coordinate systems and the corresponding mode functions for a scalar field with general curvature coupling parameter in de Sitter and anti-de Sitter spacetimes with arbitrary number of spatial dimensions are described in Chapter 11 and Chapter 12, respectively. As an example of application of the mode functions, the two-point function is evaluated. And finally, in Chapter 13 we describe the general procedure for the evaluation of the two-point functions in maximally symmetric spaces, based on the direct solution of the corresponding equation. As examples, the scalar two-point functions in de Sitter and anti-de Sitter spacetimes are considered. We also consider the renormalization procedure for the energy-momentum tensor of a scalar field in de Sitter spacetime.

Chapter 2

Fields in flat spacetime

2.1 Lorentz group

The basic fields of physics transform as irreducible representations of the Lorentz and Poincare groups. The complete set of finite-dimensional representations of the orthogonal group comes in two classes, the tensors and spinors. We define the Lorentz group as the set of all 4×4 real matrices that leave the following invariant:

$$s^2 = (x^0)^2 - (x^i)^2 = g_{\mu\nu} x^\mu x^\nu . \tag{2.1}$$

A Lorentz transformation can be parametrized by:

$$x'^\mu = \Lambda^{\mu}_{\cdot\nu} x^\nu , \tag{2.2}$$

with

$$g_{\rho\sigma} \Lambda^{\rho}_{\cdot\mu} \Lambda^{\sigma}_{\cdot\nu} = g_{\mu\nu} \Rightarrow g = \Lambda^T g \Lambda . \tag{2.3}$$

We say that $g_{\mu\nu}$ is the metric of the Lorentz group. We call the Lorentz group $O(3,1)$. Taking the determinant of (2.3) we gets $\det\Lambda = \pm 1$. The transformations with $\det\Lambda = +1$ ($\det\Lambda = -1$) are called proper (inproper) Lorentz transformations. For an infinitesimal Lorentz transformation one has

$$\Lambda^{\mu}_{\cdot\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\cdot\nu} . \tag{2.4}$$

Now from (2.3) we see that $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$. From here it follows that there are six independent infinitesimal Lorentz transformations. Three of them correspond to spatial rotations and three - to boosts. The transformations with $\Lambda^{0}_{\cdot 0} \geqslant 1$ form a subgroup called orthochronous Lorentz transformations. The infinitesimal transformation (2.4) is orthochronous and proper Lorentz transformation. The examples of transformations that do not belong to this subgroup are parity and time reversal transformations with $\Lambda^{\mu}_{\cdot\nu} = P^{\mu}_{\cdot\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $\Lambda^{\mu}_{\cdot\nu} = T^{\mu}_{\cdot\nu} = \mathrm{diag}(-1,1,1,1)$, respectively. These transformations are discrete ones and cannot be reached by compounding infinitesimal Lorentz transformations. Generally, a theory is said to be Lorentz invariant if is invariant under the orthochronous and proper subgroup only.

Spacetime splits up into distinct regions that cannot be connected by a Lorentz transformation. If x and y are two position vectors, then these regions can be labeled by the value of the invariant distance s^2:

$$\begin{aligned} (x-y)^2 &> 0: \text{ time-like} \\ (x-y)^2 &= 0: \text{ light-like} \\ (x-y)^2 &< 0: \text{ space-like} \end{aligned} \tag{2.5}$$

We introduce the operator $L^{\mu\nu}$ in order to define the action of the Lorentz group on fields:

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu = i(x^\mu \partial^\nu - x^\nu \partial^\mu), \tag{2.6}$$

where $p_\mu = i\partial_\mu$. This generates the algebra of the Lorentz group:

$$[L^{\mu\nu}, L^{\rho\sigma}] = ig^{\nu\rho}L^{\mu\sigma} - ig^{\mu\rho}L^{\nu\sigma} - ig^{\nu\sigma}L^{\mu\rho} + ig^{\mu\sigma}L^{\nu\rho}. \tag{2.7}$$

2.2 Poincaré group

We can generalize the Lorentz group by adding translations:

$$x'^\mu = \Lambda^\mu_{\cdot\nu} x^\nu + a^\mu. \tag{2.8}$$

The Lorentz group with translations now becomes the Poincaré group. Because the Poincaré group includes four translations in addition to three rotations and three boosts, it is a 10-parameter group. In addition to the usual generator of the Lorentz group, we must add the translation generator $P_\mu = -i\partial_\mu$.

The Poincaré algebra is given by the usual Lorentz algebra, plus some new relations:

$$[L_{\mu\nu}, P_\rho] = ig_{\nu\rho}P_\mu - ig_{\mu\rho}P_\nu, \quad [P_\mu, P_\nu] = 0. \tag{2.9}$$

These relations mean that two translations commute, and that translations transform as a vector under the Lorentz group.

2.3 Group of conformal transformations

The group of conformal transformation is defined as the subgroup of general coordinate transformations that leave the metric tensor $g_{\mu\nu}(x)$ invariant up to a rescaling. If the transformation $x^\mu \to x'^\mu$ is an element of the group then for the metric transformation one has

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{2.10}$$

Under this transformation the angle between two vectors is preserved. The conformal group contains the Poincaré group as a subgroup. For the elements of this subgroup $\Omega(x) = 1$.

In order to examine the generators of the conformal group let us consider infinitesimal transformations $x'^\mu = x^\mu + \xi^\mu$. By taking into account that under the general coordinate transformation $x^\mu \to x'^\mu$ the metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x),$$

we get

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - g_{\mu\beta}\partial_\nu \xi^\beta - g_{\alpha\nu}\partial_\mu \xi^\alpha.$$

For a conformal transformation one has $g'_{\mu\nu}(x') - g_{\mu\nu}(x) = \left[\Omega^2(x) - 1\right] g_{\mu\nu}$ and this relation is sepcified to

$$\left[\Omega^2(x) - 1\right] \delta^\beta_\mu = -g^{\beta\nu} g_{\mu\alpha}\partial_\nu \xi^\alpha - \partial_\mu \xi^\beta.$$

Taking $\beta = \mu$ and summing over μ we find

$$\Omega^2(x) - 1 = -\frac{2}{D+1}\partial_\mu \xi^\mu.$$

Substituting this into the previous relation one gets the final result

$$g^{\beta\nu}g_{\mu\alpha}\partial_\nu\xi^\alpha + \partial_\mu\xi^\beta = \frac{2}{d}\partial_\alpha\xi^\alpha\delta^\beta_\mu, \tag{2.11}$$

where d is the number of spacetime dimensions.

For a conformal transformation around the flat metric, $g_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, 1)$, the general result (2.11) takes the form

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \frac{2}{d}\partial_\alpha\xi^\alpha g_{\mu\nu}. \tag{2.12}$$

By acting on both sides of this relation with ∂^μ we get

$$\Box\xi_\nu + (1 - 2/d)\,\partial_\nu\partial_\alpha\xi^\alpha = 0,$$

where $\Box = \partial^\mu\partial_\mu$. Now we act with $\Box$ on the left- and right-hand sides of (2.12):

$$\partial_\mu\Box\xi_\nu + \partial_\nu\Box\xi_\mu = \frac{2}{d}g_{\mu\nu}\Box\partial_\alpha\xi^\alpha.$$

Combining the last two relations we obtain the equation

$$[g_{\mu\nu}\Box + (d-2)\,\partial_\mu\partial_\nu]\,\partial_\alpha\xi^\alpha = 0. \tag{2.13}$$

This shows that the case of $d = 2$ dimensional spacetime is special.

For $d > 2$, equation (2.13) is cubic in derivatives and nondegenerate. From here it follows that the functions $\xi^\alpha(x)$ can be at most quadratic. One has the following possibilities

$\xi^\mu = a^\mu$,	translations,
$\xi^\mu = \Lambda^{\mu}_{\cdot\nu}x^\nu$,	rotations,
$\xi^\mu = \lambda x^\mu$,	scale transformations,
$\xi^\mu = b^\mu x^2 - 2x^\mu b_\alpha x^\alpha$,	special conformal transformations.

For the finite scale and special conformal transformations one has $D : x'^\mu = \lambda x^\mu$ and

$$K_\mu : x'^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2b_\alpha x^\alpha + x^2 b^2}.$$

In combination with the Poincaré group they form the conformal group. The corresponding generators are

$$P_\mu,\ L_{\mu\nu},\quad K_\mu = -i\left(x^2\partial_\mu - 2x_\mu x^\alpha\partial_\alpha\right),\quad D = -ix^\alpha\partial_\alpha.$$

In addition to the commutation relations for the Poincaré group we have

$$\begin{aligned}[][P_\mu, K_\nu] &= 2iL_{\mu\nu} - 2ig_{\mu\nu}D, \quad [L_{\mu\nu}, K_\rho] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu),\\ [P_\mu, D] &= iP_\mu, \quad [D, K_\nu] = iK_\nu, \quad [L_{\mu\nu}, D] = 0.\end{aligned}$$

The total number of the paramaters for the conformal group is equal to $d(d+1)/2 + d + 1 = (d+1)(d+2)/2$. In $d = 4$ dimensional spacetime the conformal group is 15 parametric group.

The case of $d = 2$ spacetime dimensions requires a special consideration. In this case the conformal group is infinite dimensional. The corresponding transformations are reduced to the analytic coordinate transformations.

2.4 Fields

Fields are functions (single component or multi-component) of spacteime coordinates x^μ given in every reference frame. If $\omega = (\Lambda^{\mu}_{\cdot\nu}, a^\mu)$ is a set of parameters describing translations and rotations, then under the transformation $x \to x'$ the field $u(x)$ transforms as

$$u(x) \to u'(x') = \Omega(\omega)u(x). \tag{2.14}$$

For a given Lorentz transformation we have a matrix $\Omega(\omega)$. To the unit element of the Lorentz group corresponds the unit matrix and $\Omega_{\Lambda_1\Lambda_2} = \Omega_{\Lambda_1}\Omega_{\Lambda_2}$. Hence, the set of matrices $\Omega(\omega)$ realize a representation of the group. The order of the matrices determines the rang of the representation, which coincides with the number of the components of $u(x)$. The fields and their transformations can be obtained by studying the finite dimensional irreducible representations of the Lorentz group. Finite dimensional representations of the Lorentz group are decomposed into two classes: single valued and double valued, i.e., the map $\Lambda \to \Omega_\Lambda$ is single valued and double valued. The first class of the representations corresponds to tensorial representations. The field functions which transform by tensorial representations are called as *tensors*. They can be directly observable (electromagnetic field). For the second class of representations the map is double valued: $\Lambda \to \pm\Omega_\Lambda$. These representations are called as spinorial and the corresponding fields are called spinor fields. Under the Lorentz transformation $x'^\mu = \Lambda^{\mu}_{\cdot\nu} x^\nu$ the tensor field transforms as

$$T'^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}_{\cdot\nu_1}\cdots\Lambda^{\mu_n}_{\cdot\nu_n} T^{\nu_1\cdots\nu_n}.$$

The transformation law for spinors is more complicated. Under the translations, $x'^\mu = x^\mu + a^\mu$, one has $u'(x') = u(x)$ for both tensors and spinors.

Consider the transformation corresponding to the spatial inversion:

$$x \to x' = Px, \quad x'^0 = x^0, \quad x'^i = -x^i.$$

For tensor fields $P^2 = 1$. 0-rang tensor with the property $Pu(x) = \pm u(x)$ is called as a scalar or pseudoscalar for upper and lower signs respectively. 1-rang tensor with the transformation $Pa^0(x) = a^0(x)$, $Pa^i(x) = -a^i(x)$ is called as a vector. 1-rang tensor with the transformation $Pa^0(x) = -a^0(x)$, $Pa^i(x) = a^i(x)$ is called as a pseudovector (or axial vector).

2.5 Action Principle

The dynamics of a field $\phi(x)$ is determined by the Lagrangian density $\mathcal{L}(\phi(x), \partial_\mu\phi(x))$, which is a function of both the field $\phi(x)$ as well as its space-time derivatives. The action is given by a four dimensional integral over a Lagrangian density:

$$S = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)), \tag{2.15}$$

integrated between initial and final times t_1 and t_2. The action functional obeys several conditions. First of all, it is invariant under the transformations of the Poincaré group. As a consequence of this the field equations are tensorial relations. Next, we require that the action should be real. This condition is needed in order to have a conservation of probability in quantum field theory. Another requirement is that the field equations are differential equations of the order not higher than two. The classical systems with higher order differential equations of motion usually allow noncausal solutions. From this requirement it follows that the Lagrangian density should contain the fields and their first derivatives with respect to spacetime coordinates. The higher derivatives may be contained in the form of the total derivative only. Depending on the field under consideration, the action may have additional symmetries. Examples are the gauge symmetries.

We can derive the classical equations of motion by minimizing the action:

$$\begin{aligned}\delta S &= 0 = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\partial_\mu\phi\right] \\ &= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right)\right]. \end{aligned} \tag{2.16}$$

The last term is a total derivative and vanishes for any $\delta\phi$ that decays at spatial infinity and obeys $\delta\phi|_{t=t_1,t_2} = 0$. The last term vanishes at the endpoint of the integration; so we arrive at the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{2.17}$$

These equations are easily generalized for the systems with higher derivatives:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu \phi)} - \cdots = 0.$$

In classical physics the extrema of the action are employed only. Two actions with same extrema yield to the same physics. In quantum physics the whole action functional is relevant for the dynamics of the system.

The simplest example is given by a scalar field $\phi(x)$ with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{2.18}$$

with

$$\partial^\mu\phi = \eta^{\mu\nu}\partial_\nu\phi = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x^l}\right). \tag{2.19}$$

In analogy with the case of a point particle, the first and second terms in the right-hand side are called as kinetic and potential terms. The corresponding field equation has the form

$$\partial_\mu\partial^\mu\phi + V'(\phi) = 0.$$

For a free scalar field one has $V(\phi) = m^2\phi^2/2$ with the mass m. The corresponding field equation is the Klein-Gordon equation.

2.6 Symmetries of the action and conservation laws

Another important role of the action is that its symmetries determine the conserved quantities. One can consider three types of symmetries:

1. Space-time symmetries include Poincaré group. These symmetries are noncompact, that is, the range of their parameters does not contain the endpoints. For example, the velocity of a massive particle can range from 0 to c, but cannot reach c.

2. Internal symmetries are ones that mix particles among each other, for example, symmetries like $SU(N)$ that mix N quarks among themselves. These internal symmetries rotate fields and particles in an abstract, "isotopic space," in contrast to real space-time. These groups are compact, that is, the range of their parameters is finite and contains their endpoints. For example, the rotation group is parametrized by angles that range between 0 and n or 2π. These internal symmetries can be either global (i.e., independent of space-time) or local, as in gauge theory, where the internal symmetry group varies at each point in space and time.

3. Supersymmetry nontrivially combines both space-time and internal symmetries.

2.6.1 Space-time symmetries

We can use the symmetries of the action to derive conservation principles. We start with spacetime symmetries.

Translations

First we consider the current associated with a translation:

$$x^\mu \to x^\mu + a^\mu. \tag{2.20}$$

Under this displacement, a field $\phi(x)$ transforms as $\phi(x) \to \phi(x+a)$. For small a^μ, the change in the field is given by:

$$\delta\phi = \phi(x+a) - \phi(x) = a^\mu \partial_\mu \phi(x), \quad \delta\partial_\mu\phi = a^\nu \partial_\nu \partial_\mu \phi. \tag{2.21}$$

The variation of Lagrangian is given by:

$$\delta\mathcal{L} = a^\mu \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\partial_\mu\phi. \tag{2.22}$$

Substituting the variation of the fields and using the equations of motion, we find:

$$\delta\mathcal{L} = a^\mu \partial_\mu \mathcal{L} = a^\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right). \tag{2.23}$$

From here it follows that

$$\partial_\mu T^\mu_\nu = 0, \tag{2.24}$$

where we have defined the *canonical energy-momentum tensor*:

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \tag{2.25}$$

The conserved charges corresponding to the energy-momentum tensor are the energy and momentum:

$$P^\mu = (E, P^i), \quad P^\mu \equiv \int d^3x\, T^\mu_0, \quad \frac{dP^\mu}{dt} = 0. \tag{2.26}$$

The conservation of energy-momentum is therefore a consequence of the invariance of the action under translations, which in turn corresponds to invariance under the time and space displacements.

There is, however, a certain ambiguity in the definition of the canonical energy-momentum tensor. The energy-momentum tensor is not a measurable quantity, but the integrated charges correspond to the physical energy and momentum, and hence are measurable. We can add to the energy-momentum tensor a term $\partial_\lambda E^{\lambda\mu\nu}$, where $E^{\lambda\mu\nu} = -E^{\mu\lambda\nu}$. Because of this antisymmetry, this tensor satisfies trivially: $\partial_\lambda \partial_\mu E^{\lambda\mu\nu} = 0$. So we can make the replacement:

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\lambda E^{\lambda\mu\nu}. \tag{2.27}$$

This new energy-momentum tensor is conserved, like the previous one. We can choose this tensor such that the new energy-momentum tensor is symmetric.

The addition of this extra tensor to the energy-momentum tensor does not affect the energy and the momentum, which are measurable quantities. If we take the integrated charge, we find that the contribution from $E^{\lambda\mu\nu}$ vanishes:

$$P^\mu \to P^\mu + \int d^3x\, \partial_\lambda E^{\lambda 0 \mu} = P^\mu + \int_S dS_i\, E^{i0\mu} = P^\mu,$$

as long as $E^{i0\mu}$ vanishes sufficiently rapidly at infinity.

For the case of a scalar field one has the Lagrangian density (2.18) one has $\partial\mathcal{L}/\partial(\partial_\mu\phi) = \eta^{\mu\nu}\partial_\nu\phi$ and hence

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}.$$

In this case the canonical energy-momentum tensor is automatically symmetric. For other fields (for example, for the electromagnetic field) this is not the case.

Rotations

Let us now construct the current associated with Lorentz transformations. We define how a four-vector x^μ changes under a Lorentz transformation:

$$\delta x^\mu = \epsilon^\mu_\nu x^\nu,\ \delta\phi(x) = \epsilon^\mu_\nu x^\nu\partial_\mu\phi(x), \tag{2.28}$$

where ϵ^μ_ν is an infinitesimal, antisymmetric constant matrix (i.e. $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$). By taking into account the field equation, we have

$$\delta\mathcal{L} = \epsilon^\mu_\nu x^\nu\partial_\mu\mathcal{L} = \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\delta\phi\right) = \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\epsilon^\mu_\nu x^\nu\partial_\mu\phi\right). \tag{2.29}$$

From here it follows that

$$\begin{aligned} 0 &= \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\epsilon^\mu_\nu x^\nu\partial_\mu\phi\right) - \epsilon^\mu_\nu x^\nu\partial_\mu\mathcal{L} \\ &= \epsilon_{\mu\nu}\left\{\partial_\rho\left[x^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\partial^\mu\phi - \eta^{\mu\rho}\mathcal{L}\right)\right] + \eta^{\mu\nu}\mathcal{L}\right\} = \epsilon_{\mu\nu}\partial_\rho\left(x^\nu T^{\rho\mu}\right). \end{aligned}$$

This gives us the conserved current:

$$\partial_\rho\mathcal{M}^{\rho,\mu\nu} = 0, \quad \mathcal{M}^{\rho,\mu\nu} = T^{\rho\nu}x^\mu - T^{\rho\mu}x^\nu, \tag{2.30}$$

and the conserved charge:

$$M^{\mu\nu} = \int d^3x\,\mathcal{M}^{0,\mu\nu}, \quad \frac{d}{dt}M^{\mu\nu} = 0. \tag{2.31}$$

The derivation above is presented for a scalar field. For general case if we define $\mathcal{M}^{\rho,\mu\nu}$ in accordance of (2.30), then

$$\partial_\rho\mathcal{M}^{\rho,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}.$$

Hence, the conservation of angular momentum requires a symmetric energy-momentum tensor.

Another reason for requiring a symmetric energy-momentum tensor is that in general relativity, the gravitational field tensor, which is symmetric, couples to the energy-momentum tensor. By the equivalence principle, the gravitational field couples equally to all forms of matter via its energy-momentum content. Hence, when we discuss general relativity, we will need a symmetric energy-momentum tensor.

2.6.2 Internal symmetries

In general, an action may be invariant under an internal transformation of the fields. Consider the symmetry under the transformation when the fields ϕ^α vary according to some small parameter $\delta\epsilon^\alpha$. The variation of the action

$$S = \int d^4x\,\mathcal{L}(\phi^\alpha(x), \partial_\mu\phi^\alpha(x)),$$

under the field variation $\delta\phi^\alpha$ is given by

$$\begin{aligned}\delta S &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi^\alpha}\delta\phi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\partial_\mu\phi^\alpha\right] \\ &= \int d^4x\, \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha\right] \equiv \int d^4x\, \partial_\mu j^\mu_\alpha \delta\epsilon^\alpha. \end{aligned} \tag{2.32}$$

Here we have used the equations of motion and defined the current j^μ_α:

$$j^\mu_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\beta)}\frac{\delta\phi^\beta}{\delta\epsilon^\alpha}. \tag{2.33}$$

If the action is invariant under this transformation, then the current is conserved:

$$\partial_\mu j^\mu_\alpha = 0. \tag{2.34}$$

From this conserved current, we can establish a conserved charge:

$$Q_\alpha = \int d^3x\, j^0_\alpha. \tag{2.35}$$

In summary, the symmetry of the action implies the conservation of a current, which in turn implies a conservation principle:

Symmetry → Current conservation → Conserved charge.

2.7 Noether's theorem

In the general form the Noether's theorem state that to any finite-parametric (depending on s constant parameters) continuous transformation of the fields and coordinates for which the variation of the action vanishes (on the solutions of the field equations), correspond s dynamic invariants, i.e. conserved quantities which are functions of fields and their derivatives.

We consider infinitesimal transformation of coordinates and fields

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \quad \phi^\beta(x) \to \phi'^\beta(x') = \phi^\beta(x) + \delta\phi^\beta(x).$$

The variations δx^μ and $\delta\phi_\mu(x)$ are expressed in terms of infinitesimal linear independent parameters of the transformation $\delta\omega^n$:

$$\delta x^\mu = X^\mu_n \delta\omega^n, \quad \delta\phi^\beta(x) = \Phi^\beta_n \delta\omega^n. \tag{2.36}$$

Note that $\delta\phi_{\mu,\nu} \neq (\delta\phi_\mu)_{,\nu}$, i.e., the operations δ and ∂_x are not commutative. We introduce the variation of the functional form (Lie variation)

$$\bar{\delta}\phi^\beta(x) = \phi'^\beta(x) - \phi^\beta(x),$$

which can be written in the form

$$\begin{aligned}\bar{\delta}\phi^\beta(x) &= \phi'^\beta(x'^\mu - \delta x^\mu) - \phi^\beta(x) = \phi'^\beta(x'^\mu) - \phi^\beta(x) - \partial_\mu\phi^\beta\delta x^\mu \\ &= \delta\phi^\beta(x) - \partial_\mu\phi^\beta\delta x^\mu = \left(\Phi^\beta_n - \partial_\mu\phi^\beta X^\mu_n\right)\delta\omega^n. \end{aligned} \tag{2.37}$$

For the variation of the action one has

$$\delta S = \int d^4x' \mathcal{L}'(x') - \int d^4x \mathcal{L}(x),$$

where

$$\mathcal{L}'(x') = \mathcal{L}'(\phi'^{\beta}(x'), \phi'^{\beta}_{,\nu}(x')) = \mathcal{L}(x) + \delta\mathcal{L}(x),$$

and

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\phi^{\beta}}\delta\phi^{\beta} + \frac{\partial\mathcal{L}}{\partial(\phi^{\beta}_{,\nu})}\delta\phi^{\beta}_{,\nu} = \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^{\mu}}\delta x^{\mu}.$$

Here $\bar{\delta}\mathcal{L}(x)$ is the variation due to the variation of the form of the functions ϕ^{β} and $\phi^{\beta}_{,\nu}$:

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\phi^{\beta}}\bar{\delta}\phi^{\beta} + \frac{\partial\mathcal{L}}{\partial(\phi^{\beta}_{,\nu})}\bar{\delta}\phi^{\beta}_{,\nu},$$

and the second term describes the variation due to the coordinate variation. Hence,

$$\delta S = \int d^4x \left(\bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^{\mu}}\delta x^{\mu}\right) + \int d^4x'\mathcal{L}(x) - \int d^4x\mathcal{L}(x).$$

For the volume one has

$$d^4x' = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}d^4x \approx \left(1 + \frac{\partial\delta x^{\mu}}{\partial x^{\mu}}\right)d^4x.$$

Hence,

$$\int d^4x'\mathcal{L}(x) - \int d^4x\mathcal{L}(x) = \int d^4x\mathcal{L}(x)\frac{\partial\delta x^{\mu}}{\partial x^{\mu}},$$

and we can write

$$\delta S = \int d^4x \left[\bar{\delta}\mathcal{L}(x) + \frac{d}{dx^{\mu}}(\mathcal{L}\delta x^{\mu})\right].$$

By taking into account the field equation

$$\frac{\partial\mathcal{L}}{\partial\phi^{\beta}} = \partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^{\beta}_{,\nu}}\right),$$

we find

$$\bar{\delta}\mathcal{L}(x) = \partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^{\beta}_{,\nu}}\right)\bar{\delta}\phi^{\beta} + \frac{\partial\mathcal{L}}{\partial(\phi^{\beta}_{,\nu})}\partial_{\nu}\bar{\delta}\phi^{\beta} = \partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^{\beta}_{,\nu}}\bar{\delta}\phi^{\beta}\right),$$

and

$$\delta S = \int d^4x\,\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^{\beta}_{,\mu}}\bar{\delta}\phi^{\beta} + \mathcal{L}\delta x^{\mu}\right).$$

By taking into account (2.36) and (2.37), we find

$$\delta S = -\int d^4x\,\delta\omega^n\partial_{\mu}\theta^{\mu}_{n},$$

where

$$\theta^{\mu}_{n} = \left(\frac{\partial\mathcal{L}}{\partial\phi^{\beta}_{,\mu}}\partial_{\nu}\phi^{\beta} - \mathcal{L}\delta^{\mu}_{\nu}\right)X^{\nu}_{n} - \frac{\partial\mathcal{L}}{\partial\phi^{\beta}_{,\mu}}\Phi^{\beta}_{n}. \tag{2.38}$$

From $\delta S = 0$ it follows that

$$\int d^4x\,\partial_{\mu}\theta^{\mu}_{n} = 0.$$

By making use of the Gauss theorem we find that

$$C_n = \int d^3x\,\theta^{0}_{n} = \text{const},\ n = 1, 2, \ldots, s, \tag{2.39}$$

does not depend on time.

For the special case of spacetime translations one has $x'^\mu = x^\mu + \delta x^\mu$. Taking $\delta\omega^n = \delta x^n$, $n = 0, \ldots, 3$, we see that $X^\nu_n = \delta^\nu_n$, $\Phi^\beta_n = 0$. In this case θ^μ_n coincides with the canonical energy-momentum tensor:

$$\theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \phi^\beta_{,\mu}} \partial_\nu \phi^\beta - \mathcal{L}\delta^\mu_\nu = T^\mu_\nu.$$

2.7.1 Angular momentum tensor and spin tensor

For infinitesimal 4-rotations

$$x'^\mu = x^\mu + \epsilon^\mu_\nu x^\nu, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}.$$

As transformation parameters we can take six linearly independent components $\epsilon^{\mu\nu} = \delta\omega^{\mu\nu}$, $\mu < \nu$. The indices (μ, ν) determine the plane where the rotation takes place. One has

$$\begin{aligned} \delta x^\mu &= X^\mu_n \delta\omega^n = \sum_{\nu<\rho} X^\mu_{\nu\rho}\delta\omega^{\nu\rho} = \delta^\mu_\rho \epsilon^\rho_\nu x^\nu = \eta_{\nu\sigma}\delta^\mu_\rho \epsilon^{\rho\sigma} x^\nu = \eta_{\sigma\rho}\delta^\mu_\nu \epsilon^{\nu\rho} x^\sigma \\ &= \sum_{\nu<\rho} \eta_{\sigma\rho}\delta^\mu_\nu \epsilon^{\nu\rho} x^\sigma + \sum_{\nu>\rho} \eta_{\sigma\rho}\delta^\mu_\nu \epsilon^{\nu\rho} x^\sigma = \sum_{\nu<\rho} \epsilon^{\nu\rho}\left(\eta_{\sigma\rho}\delta^\mu_\nu x^\sigma - \eta_{\sigma\nu}\delta^\mu_\rho x^\sigma\right), \end{aligned}$$

from which it follows that

$$X^\mu_{\nu\rho} = \eta_{\sigma\rho}\delta^\mu_\nu x^\sigma - \eta_{\sigma\nu}\delta^\mu_\rho x^\sigma = \delta^\mu_\nu x_\rho - \delta^\mu_\rho x_\nu, \ \nu < \rho.$$

The total variation of the field we present in the form

$$\phi'^\beta(x') = \phi^\beta(x) + \delta\phi^\beta, \quad \delta\phi^\beta = \sum_{\nu,\rho<\sigma} A^\beta_{\nu\rho\sigma}\phi^\nu(x)\delta\omega^{\rho\sigma}.$$

For scalar fields $A^\beta_{\nu\rho\sigma} = 0$ and for vector fields $A^\beta_{\nu\rho\sigma} = \eta_{\nu\sigma}\delta^\beta_\rho - \eta_{\nu\rho}\delta^\beta_\sigma$, $\rho < \sigma$. Indeed, for a vector field

$$\begin{aligned} a'^\beta &= \frac{\partial x'^\beta}{\partial x^\rho} a^\rho = a^\beta + \epsilon^\beta_\nu a^\nu = a^\beta + \eta_{\nu\sigma}\epsilon^{\beta\sigma} a^\nu = a^\beta + \eta_{\nu\sigma}\delta^\beta_\rho \epsilon^{\rho\sigma} a^\nu \\ &= a^\beta + \sum_{\nu,\rho<\sigma}\left(\eta_{\nu\sigma}\delta^\beta_\rho - \eta_{\nu\rho}\delta^\beta_\sigma\right)\epsilon^{\rho\sigma} a^\nu. \end{aligned}$$

From

$$\delta\phi^\beta(x) = \Phi^\beta_n \delta\omega^n = \sum_{\nu,\rho<\sigma}\left(\eta_{\nu\sigma}\delta^\beta_\rho - \eta_{\nu\rho}\delta^\beta_\sigma\right)\epsilon^{\rho\sigma}\phi^\nu,$$

it follows that

$$\Phi^\beta_{\rho\sigma} = \left(\eta_{\nu\sigma}\delta^\beta_\rho - \eta_{\nu\rho}\delta^\beta_\sigma\right)\phi^\nu,$$

for vector fields. In general case,

$$\delta\phi^\beta(x) = \Phi^\beta_n \delta\omega^n = \sum_{\nu,\rho<\sigma} A^\beta_{\nu\rho\sigma}\phi^\nu(x)\delta\omega^{\rho\sigma},$$

with $n = (\rho\sigma)$, from which it follows that

$$\Phi^\beta_n = \Phi^\beta_{\rho\sigma} = A^\beta_{\nu\rho\sigma}\phi^\nu(x), \ \rho < \sigma.$$

Hence, for the angular momentum tensor we find

$$\begin{aligned} \theta^{\mu}_{\rho\sigma} &\equiv M^{\mu}_{\rho\sigma} = \left(\frac{\partial \mathcal{L}}{\partial \phi^{\beta}_{,\mu}} \partial_\nu \phi^{\beta} - \mathcal{L} \delta^{\mu}_{\nu} \right) X^{\nu}_{\rho\sigma} - \frac{\partial \mathcal{L}}{\partial \phi^{\beta}_{,\mu}} \Phi^{\beta}_{\rho\sigma} \\ &= \left(T^{\mu}_{\rho} x_\sigma - T^{\mu}_{\sigma} x_\rho \right) - \frac{\partial \mathcal{L}}{\partial \phi^{\beta}_{,\mu}} A^{\beta}_{\nu\rho\sigma} \phi^{\nu}. \end{aligned}$$

The term $T^{\mu}_{\rho} x_\sigma - T^{\mu}_{\sigma} x_\rho$ corresponds to the orbital momentum and the term

$$S^{\mu}_{\rho\sigma} = -\frac{\partial \mathcal{L}}{\partial \phi^{\beta}_{,\mu}} A^{\beta}_{\nu\rho\sigma} \phi^{\nu}.$$

describes the polarization properties of the field and corresponds to the spin of particles in quantum theory.

2.8 Scalar field

We start with the simplest case of a free real scalar field $\phi(x)$. The corresponding Lagrangian has the form

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right), \tag{2.40}$$

The field equation obtained from (2.17) is the standard Klein-Gordon equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{2.41}$$

The canonical energy-momentum tensor is given by the expression

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + \frac{1}{2} m^2 g_{\mu\nu} \phi^2.$$

Complex scalar field describes charged particles. For the corresponding Lagrangian one has

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi, \tag{2.42}$$

where the star stands for the complex conjugate and ϕ and ϕ^* are treated independently. The field equations for both the fields are the Klein-Gordon equation. If we insert the decomposition

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1 + i \phi_2 \right)$$

into the action, then we find the sum of two independent actions for real scalar fields ϕ_1 and ϕ_2. The action is invariant under the transformation

$$\phi \to e^{-i\alpha} \phi, \quad \phi^* \to e^{i\alpha} \phi^*, \tag{2.43}$$

which generates $U(1)$ symmetry. For an infinitesimal transformation one has $\delta\phi = -i\alpha\phi$, $\delta\phi^* = i\alpha\phi^*$ and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^*, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi.$$

By taking into account that $\delta\epsilon = \alpha$, for the current density, we get

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \epsilon} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta \phi^*}{\delta \epsilon} = i \phi^* \partial^\mu \phi - i \phi \partial^\mu \phi^*.$$

Hence, for a charged scalar field the conserved current is given by the expression

$$j_\mu = i \phi^* \partial_\mu \phi - i \phi \partial_\mu \phi^*. \tag{2.44}$$

The conserved charges arising from currents of this type have the interpretation of electric charge or particle number. The expression for the canonical energy-momentum tensor of a complex scalar field reads

$$T_{\mu\nu} = \partial_\mu \phi^* \partial_\nu \phi + \partial_\nu \phi^* \partial_\mu \phi - g_{\mu\nu} \partial_\rho \phi^* \partial^\rho \phi + m^2 g_{\mu\nu} \phi^* \phi.$$

2.9 Dirac spinor field

Spinor fields are transformed under spinor representations of the Poincaré group. After the quantization they describe particles with half-integer spins. The simplest case is Dirac spinor field having spin 1/2. It is described by a four-component spinor ψ. The corresponding Lagrangian has the form

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi\right) - m\bar{\psi}\psi, \tag{2.45}$$

where the Dirac matrices γ^μ satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

(Clifford algebra), and $\bar{\psi} = \psi^+\gamma^0$ is the Dirac conjugate spinor. The field equation corresponding to (2.45) is the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{2.46}$$

Acting on the both sides of the Dirac equation by the operator $(i\gamma^\nu\partial_\nu + m)$, we can see that each component of the Dirac spinor obeys the Klein-Gordon equation

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\psi = 0. \tag{2.47}$$

The Dirac equation can be written in the form

$$i\frac{\partial\psi}{\partial t} = \left(-i\alpha\cdot\nabla + \beta m\right)\psi,$$

where the hermitian matrices $\alpha = (\alpha^1, \alpha^2, \alpha^3)$, β are related to the Dirac matrices in accordance with

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i.$$

From here it follows that for the Dirac spinor field the Hamiltonian is given by

$$H = -i\alpha\cdot\nabla + \beta m.$$

Spinors transform under some representation $S(\Lambda)$ of the proper Lorentz group:

$$\psi'(x') = S(\Lambda)\psi(x). \tag{2.48}$$

In order for the Dirac equation to be Lorentz covariant, we must have the following relation:

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu_\nu\gamma^\nu. \tag{2.49}$$

Let us introduce the following matrices:

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu].$$

They are generators of the Lorentz group in the spinor representation. In terms of these matrices, we can find an explicit representation of the $S(\Lambda)$ matrix:

$$S(\Lambda) = \exp\left(-i\sigma_{\mu\nu}\omega^{\mu\nu}/4\right). \tag{2.50}$$

For spatial rotations $S(\Lambda)$ is a unitary matrix, whereas for boosts it is hermitian.

Taking the representation of the Dirac matrices with the properties $(\gamma^0)^+ = \gamma^0$, $(\gamma^i)^+ = -\gamma^i$, for the Dirac conjugate we obtain the equation

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0.$$

Under a Lorentz transformation,

$$\bar{\psi}'(x') = \bar{\psi}(x) S^{-1}(\Lambda).$$

We also introduce

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Based on these properties we can form covariant tensors:

Scalar	$\bar{\psi}\psi$
Vector	$\bar{\psi}\gamma^\mu\psi$
Tensor	$\bar{\psi}\sigma^{\mu\nu}\psi$
Pseudoscalar	$\bar{\psi}\gamma_5\psi$
Pseudovector	$\bar{\psi}\gamma_5\gamma^\mu\psi$

It is often convenient to find an explicit representation of the Dirac matrices. The most common representation of these matrices is the *Dirac representation*:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.51}$$

where σ^i are the familiar Pauli matrices.

Now let us consider the transformation of the spinor ψ with respect to the spatial reflection $\Lambda^{\mu}_{\cdot\nu} = \mathcal{P}^{\mu}_{\cdot\nu} = \mathrm{diag}(1, -1, -1, -1)$. The corresponding matrix should again obey the relation (2.49). It can be checked that the spinor transformation has the form

$$\psi'(x') = \eta_P \gamma^0 \psi(x),$$

where η_P is an arbitrary phase factor.

2.9.1 Conserved currents

Similar to the case of a charged scalar field, the Lagrangian (2.45) is invariant under the transformation

$$\psi \to e^{-i\alpha}\psi, \quad \bar{\psi} \to e^{i\alpha}\bar{\psi}, \tag{2.52}$$

corresponding to $U(1)$ symmetry. Assuming that the parameter α is small, one has $\delta\psi = -i\alpha\psi$, $\delta\bar{\psi} = i\alpha\bar{\psi}$. By taking into account that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{i}{2}\bar{\psi}\gamma^\mu, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -\frac{i}{2}\gamma^\mu\psi,$$

for the spinor current density we get

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\frac{\delta\psi}{\delta\epsilon} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\frac{\delta\bar{\psi}}{\delta\epsilon} \\ &= \bar{\psi}\gamma^\mu\psi. \end{aligned} \tag{2.53}$$

For the canonical energy-momentum tensor one has

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\nu\psi + \partial_\nu\bar{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \delta^\mu_\nu \mathcal{L}.$$

By taking into account that the Lagrangian vanishes on the solutions of the field equation, we obtain

$$T_{\mu\nu} = \frac{i}{2}\left[\bar{\psi}\gamma_\mu\partial_\nu\psi - \left(\partial_\nu\bar{\psi}\right)\gamma_\mu\psi\right].$$

In this form the energy-momentum tensor is not symmetric. The symmetrized tensor is given by

$$T_{\mu\nu} = \frac{i}{2}[\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi - (\partial_{(\mu}\bar{\psi})\gamma_{\nu)}\psi].$$

By using the Dirac equation and the equation (2.47) it is easy to see that this tensor obeys the conservation law $\partial_\mu T^\mu_\nu = 0$.

For the spin tensor we have the expression

$$S^\mu_{\rho\sigma} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}A_{\rho\sigma}\psi - \bar{\psi}\bar{A}_{\rho\sigma}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = -\frac{i}{2}\bar{\psi}\left(\gamma^\mu A_{\rho\sigma} - \bar{A}_{\rho\sigma}\gamma^\mu\right)\psi.$$

From the transformation rule (2.48) with (2.50) one has

$$\delta\psi = -(i/4)\sigma_{\rho\sigma}\delta\omega^{\rho\sigma}\psi = \sum_{\rho<\sigma} A_{\rho\sigma}\psi\delta\omega^{\rho\sigma},$$

and hence

$$A_{\rho\sigma} = -\frac{i}{2}\sigma_{\rho\sigma}, \quad \bar{A}_{\rho\sigma} = \frac{i}{2}\sigma_{\rho\sigma}.$$

This gives the following result

$$S^{\mu\rho\sigma} = \frac{1}{4}\bar{\psi}\left(\gamma^\mu\sigma^{\rho\sigma} + \sigma^{\rho\sigma}\gamma^\mu\right)\psi.$$

2.10 Gauge fields

We start with the simplest case of abelian gauge field.

2.10.1 Abelian gauge field

Field function $u(x)$ for the matter fields enters in the Lagrangian in the form of the product $u(x)u^*(x)$. As a consequence, the matter fields are determined up to phase factor. The corresponding Lagrangian is invariant under the field transformation

$$u(x) \to e^{i\alpha}u(x), \quad u^*(x) \to e^{-i\alpha}u^*(x). \tag{2.54}$$

This corresponds to the abelian group $U(1)$. Now we require the invariance in the case when the parameter α depends on x:

$$u(x) \to u'(x) = e^{i\alpha(x)}u(x), \quad u^*(x) \to u'^*(x) = e^{-i\alpha(x)}u^*(x), \tag{2.55}$$

i.e., the relative phase at two different points is arbitrary. It can be easily seen that the Lagrangian for complex fields is not invariant under this local transformation. The invariance can be achieved if we introduce an additional vector field A_μ which transforms as

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - (1/e)\partial_\mu\alpha(x), \tag{2.56}$$

and make the replacements

$$\begin{aligned} \partial_\mu u(x) &\to D_\mu u(x) = (\partial_\mu + ieA_\mu(x))\,u(x), \\ \partial_\mu u^*(x) &\to D^*_\mu u^*(x) = (\partial_\mu - ieA_\mu(x))\,u^*(x). \end{aligned} \tag{2.57}$$

The operator D_μ is called as covariant derivative. It is easily seen that the covariant derivative transforms as:

$$D'_\mu u'(x) = e^{i\alpha(x)}D_\mu u(x), \quad D'^*_\mu u'^*(x) = e^{-i\alpha(x)}D^*_\mu u'(x),$$

and, hence, the Lagrangian remains invariant.

In the action we should also add the part corresponding to free field A_μ, which should be invariant under the transformation (2.56). It is taken in the form

$$\mathcal{L}(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.58}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.59}$$

being the field tensor. The constant e is identified with the charge and the vector field A_μ is identified with the electromagnetic field. The fields compensating the changes of the matter fields gauge are called as gauge fields. In the example we have considered the gauge group is abelian and gauge field is called abelian as well. Note that gauge fields are massless. The pass from the partial derivatives to covariant ones introduces the interaction between the matter and gauge fields. The interaction introduced in this way is called a minimal interaction.

As an example we can consider a fermionic field ψ. By making the replacement (2.57) and adding the Lagrangian (2.58) to the Dirac Lagrangian we get the Lagrangian for Quantum Electrodynamics

$$\mathcal{L} = \frac{i}{2} \left(\bar{\psi} \gamma^\mu D_\mu \psi - \left(D^*_\mu \bar{\psi} \right) \gamma^\mu \psi \right) - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{2.60}$$

Note that the interaction term is given by $e \bar{\psi} \gamma^\mu A_\mu \psi = e j^\mu A_\mu$.

2.10.2 Non-abelian gauge fields

In a similar way one can consider Quantum Chromodynamics describing the interactions between the quarks. The latter are presented by three color fields q_j, $j = 1, 2, 3$ with the free Lagrangian (for simplicity wwe consider a single flavor)

$$\mathcal{L}_0 = \frac{i}{2} \left(\bar{q}_j \gamma^\mu \partial_\mu q_j - \left(\partial_\mu \bar{q}_j \right) \gamma^\mu q_j \right) - m \bar{q}_j q_j.$$

We require the invariance of the Lagrangian under the local phase transformation, corresponding to the gauge group is $SU(3)$,

$$q'(x) = e^{i\alpha_a(x) T_a} q(x), \tag{2.61}$$

where T_a, $a = 1, 2, \ldots, 8$, is the set of linear independent trace free 3×3 matrices, α_a are the group parameters and the summation over a is understood. From the condition $\det e^{i\alpha_a(x) T_a} = 1$ it follows that $\mathrm{Tr}(T_a) = 0$. From the unitarity of $e^{i\alpha_a(x) T_a}$ one gets $\alpha_a(x) T_a = \alpha^*_a(x) T^+_a$ and, hence, for hermitian matrices T_a the group parameters are real. The group $SU(3)$ is non-abelian. For the commutator of the matrices T_a one has $[T_a, T_b] = i f_{abc} T_c$ with the structure constants f_{abc}. The latter are antisymmetric with respect to all indices.

In order to have invariance with respect to the local gauge transformation (2.61), we introduce the gauge fields A^a_μ, $a = 1, 2, \ldots, 8$, with the gauge transformation

$$A'^a_\mu = A^a_\mu - f_{abc} \alpha_b A^c_\mu - \frac{1}{g} \partial_\mu \alpha_a.$$

The corresponding covariant derivative is defined as

$$D_\mu = \partial_\mu + i g T_a A^a_\mu.$$

The expression for the field tensor is more complicated

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu.$$

And finally the QCD Lagrangian is presented as

$$\mathcal{L} = \frac{i}{2}\left(\bar{q}_j \gamma^\mu D_\mu q_j - \left(D_\mu^* \bar{q}_j\right)\gamma^\mu q_j\right) - m\bar{q}_j q_j - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}.$$

The latter describes the interaction between the quarks and vector gluon fields A_μ^a.

Chapter 3

Quintizing fields in Minkowski spacetime

3.1 Methods of quantization

We presented a short review of classical filed theory. We now make the transition to the quantum theory in Minkowski bulk. There are many excellent books on quantum fields in Minkowski spacetime. We mainly follow the books [22]-[27].

Different types of quantization schemes have been proposed, each with their own merits and drawbacks:

1. The most direct method is the canonical quantization program. Canonical quantization closely mimics the development of quantum mechanics. Time is singled out as a special coordinate and manifest Lorentz invariance is lost. The advantage of canonical quantization is that it quantizes only physical modes and unitarity of the system is manifest.

2. The Gupta-Bleuler or covariant quantization method will also be mentioned in this chapter. Contrary to canonical quantization, it maintains full Lorentz symmetry, which is a great advantage. The disadvantage of this approach is that ghosts or unphysical states of negative norm are allowed to propagate in the theory, and are eliminated only when constraints to the state vectors are applied.

3. The path integral method is perhaps the most elegant and powerful of all quantization programs. One advantage is that one can easily go back and forth between many of the other quantization programs to see the relationships between them. The path integral approach is based on simple, intuitive principles that go to the very heart of the assumptions of quantum theory. The disadvantage of the path integral approach is that functional integration is a mathematically delicate operation that may not even exist in Minkowski space.

We start with the canonical quantization scheme in Minkowski spacetime.

3.2 Klein-Gordon scalar field

For a free scalar field the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right). \tag{3.1}$$

The canonical quantization program begins with fields ϕ and their conjugate momentum fields π, which satisfy equal time commutation relations among themselves. Then the time evolution of these quantized fields is governed by a Hamiltonian. Thus, we closely mimic the dynamics found in ordinary quantum mechanics. We begin by singling out time as a special coordinate and then defining the canonical conjugate field to ϕ:

$$\pi(\mathbf{x},t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi(\mathbf{x},t))} = \partial_t \phi(\mathbf{x},t). \tag{3.2}$$

The Hamiltonian is obtained in the standard way:

$$\mathcal{H} = \pi \partial_t \phi(\mathbf{x},t) - \mathcal{L} = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right].$$

Then the transition from classical mechanics to quantum field theory begins when we postulate the commutation relations between the field and its conjugate momentum:

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}). \tag{3.3}$$

All other commutators are set equal to zero.

Let us consider the transformation of the scalar field operator under the Lorentz transformation Λ. In quantum theory we associate a unitary operator $U(\Lambda)$ to each proper orthochronous Lorentz transformation with the property $U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda)$. For an infinitesimal Lorentz transformation (2.4) one can write $U(1+\omega) = I + (i/2)\omega_{\mu\nu}L^{\mu\nu}$ with hermitian operators $L^{\mu\nu} = -L^{\nu\mu}$ being the generators of the Lorentz group. Now, by using the relation $U(\Lambda^{-1}\Lambda'\Lambda) = U^{-1}(\Lambda)U(\Lambda')U(\Lambda)$, it can be seen that $U^{-1}(\Lambda)L^{\mu\nu}U(\Lambda) = \Lambda^{\mu}_{\cdot\rho}\Lambda^{\nu}_{\cdot\sigma}L^{\rho\sigma}$. In this type of relations for each tensor index its own factor $\Lambda^{\mu}_{\cdot\rho}$ appears. In particular, for the momentum 4-vector P^μ one has $U^{-1}(\Lambda)P^\mu U(\Lambda) = \Lambda^{\mu}_{\cdot\rho}P^\rho$. For a quantum scalar field $\phi(x)$ the corresponding relation reads

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \tag{3.4}$$

For the derivative of the scalar field one has the relation $U^{-1}(\Lambda)\partial^\mu\phi(x)U(\Lambda) = \Lambda^{\mu}_{\cdot\nu}\partial'^{\nu}\phi(\Lambda^{-1}x)$, where the derivative ∂'^ν is taken with respect to the coordinates $x' = \Lambda^{-1}x$. In particular, from here it follows that the Klein-Gordon equation is Lorentz invariant. Transformation rules for dynamical variables constructed from the scalar field operator will contain a factor $\Lambda^{\mu}_{\cdot\nu}$ for each tensorial index. For example, for the current density and the energy-momentum tensor one has

$$\begin{aligned} U^{-1}(\Lambda)j^\mu(x)U(\Lambda) &= \Lambda^{\mu}_{\cdot\nu}j'^{\nu}(\Lambda^{-1}x), \\ U^{-1}(\Lambda)T^{\mu\nu}(x)U(\Lambda) &= \Lambda^{\mu}_{\cdot\rho}\Lambda^{\nu}_{\cdot\sigma}T'^{\rho\sigma}(\Lambda^{-1}x). \end{aligned} \tag{3.5}$$

Our strategy will be to find a specific Fourier representation of the commutation relation (3.3) in terms of plane waves. When these plane-wave solutions are quantized in terms of harmonic oscillators, we will be able to construct the multiparticle Hilbert space. We want a decomposition of the scalar field where the energy k^0 is positive, and where the Klein-Gordon equation is explicitly obeyed. Therefore, we choose:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4x\, \delta(k^2-m^2)\theta(k^0)\left[A(k)e^{-ik\cdot x} + A^+(k)e^{ik\cdot x}\right], \tag{3.6}$$

where θ is a step function, $A(k)$ are operator-valued Fourier coefficients, and $k\cdot x = k_\mu x^\mu = k^0 t - \mathbf{k}\cdot\mathbf{x}$. We can simplify this expression by integrating out dk^0:

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}}\left[a_{\mathbf{k}}e^{-ik\cdot x} + a^+_{\mathbf{k}}e^{ik\cdot x}\right] \\ &= \int d^3k\, \left[a_{\mathbf{k}}\phi_{\mathbf{k}}(x) + a^+_{\mathbf{k}}\phi^*_{\mathbf{k}}(x)\right], \\ \pi(x) &= \int d^3k\, i\omega_k\left[-a_{\mathbf{k}}\phi_{\mathbf{k}}(x) + a^+_{\mathbf{k}}\phi^*_{\mathbf{k}}(x)\right], \end{aligned} \tag{3.7}$$

where $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$,

$$\phi_{\mathbf{k}}(x) = \frac{e^{-ik\cdot x}}{\sqrt{2(2\pi)^3\omega_k}}, \quad A(k) = \sqrt{2\omega_k} a_{\mathbf{k}}. \tag{3.8}$$

From (3.3) it follows that

$$\left[a_{\mathbf{k}}, a^+_{\mathbf{k}'}\right] = \delta(\mathbf{k} - \mathbf{k}'), \tag{3.9}$$

and all other commutators are zero.

Exercize: Show that from (3.9) the commutation relation (3.3) follows.

Note that the functions $\phi_k(x)$ are solutions of the classical Klein-Gordon equation normalized by the condition

$$(\phi_{\mathbf{k}}(x), \phi_{\mathbf{k}'}(x)) = \delta^3(\mathbf{k} - \mathbf{k}'),$$

where the scalar product is defined in accordance with

$$(\phi_1, \phi_2) = -i \int d^3x\, (\phi_1 \partial_t \phi_2^* - (\partial_t \phi_1)\phi_2^*) \equiv -i \int d^3x\, \phi_1 \overleftrightarrow{\partial}_t \phi_2^*.$$

Exercize: Show that if the field obeys Klein-Gordon equation then the scalar product does not depend on time.

Now we can calculate the Hamiltonian in terms of the Fourier modes:

$$\begin{aligned} H &= \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \\ &= \frac{1}{2}\int d^3k\, \omega_k \left(a_{\mathbf{k}} a^+_{\mathbf{k}} + a^+_{\mathbf{k}} a_{\mathbf{k}}\right) = \int d^3k\, \omega_k \left(a^+_{\mathbf{k}} a_{\mathbf{k}} + \delta^3(\mathbf{k} - \mathbf{k})/2\right). \end{aligned}$$

Note that $\delta^3(\mathbf{k} - \mathbf{k}) = (2\pi)^{-3} \int d^3x\, e^{i(\mathbf{k}-\mathbf{k})\cdot\mathbf{r}} = V/(2\pi)^3$, where V is the volume of the whole space. Now we see that the vacuum energy density is given by

$$\varepsilon_{\mathrm{vac}} = \frac{H_{\mathrm{vac}}}{V} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega_k.$$

Similarly, we can evaluate the momentum:

$$\begin{aligned} \mathbf{P} &= -\int d^3x\, \pi \nabla\phi = \frac{1}{2}\int d^3k\, \mathbf{k} \left(a_{\mathbf{k}} a^+_{\mathbf{k}} + a^+_{\mathbf{k}} a_{\mathbf{k}}\right) \\ &= \int d^3k\, \mathbf{k} \left(a^+_{\mathbf{k}} a_{\mathbf{k}} + \delta^3(\mathbf{k} - \mathbf{k})/2\right). \end{aligned}$$

The energy and momentum can be obtained by using the energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}\eta_{\mu\nu}m^2\phi^2.$$

For them one has

$$H = \int d^3x\, T_{00}, \quad P_i = \int d^3x\, T_{0i}.$$

Now we construct the eigenstates of the Hamiltonian to find the spectrum of states. We define the "vacuum" state as follows:

$$a_{\mathbf{k}} |0\rangle = 0.$$

By convention, we call a_k an "annihilation" operator. We define a one-particle state via the "creation" operator a^+_k:

$$a^+_{\mathbf{k}} |0\rangle = |\mathbf{k}\rangle.$$

One can write down the N -particle Fock space:

$$|\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_N\rangle = a^+_{\mathbf{k}_1} a^+_{\mathbf{k}_2} \cdots a^+_{\mathbf{k}_N} |0\rangle .$$

For a state consisting of n_k identical particles with momentum k:

$$|n_{\mathbf{k}}\rangle = \frac{(a^+_{\mathbf{k}})^{n_{\mathbf{k}}}}{\sqrt{n_{\mathbf{k}}!}} |0\rangle , \quad |n_{\mathbf{k}_1} n_{\mathbf{k}_2} \cdots n_{\mathbf{k}_m}\rangle = \prod_{i=1}^{m} \frac{(a^+_{\mathbf{k}_i})^{n_{\mathbf{k}_i}}}{\sqrt{n_{\mathbf{k}_i}!}} |0\rangle$$

Particle number operator is defined as

$$N_{\mathbf{k}} = a^+_{\mathbf{k}} a_{\mathbf{k}}, \quad N_{\mathbf{k}} |n_{\mathbf{k}}\rangle = n_{\mathbf{k}} |n_{\mathbf{k}}\rangle .$$

The operators H and $\mathbf{P}$ commute with $N_{\mathbf{k}}$.

In the vacuum state one has

$$\langle 0| \mathbf{P} |0\rangle = 0, \quad \varepsilon_{\text{vac}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k = \frac{1}{4\pi^2} \int_0^\infty dk\, k^2 \sqrt{k^2 + m^2}.$$

The vacuum or zero point energy is the sum of the ground state energies for elementary oscillators. In non-gravitational physics we can rescale the energy or renormalize, even by an infinite amount. This is done by defining a normal ordering operation , denoted by : :, in which one demands that wherever a product of creation and annihilation operators appears, it is understood that all annihilation operators stand to the right of the creation operators.

Note that in the vacuum state the field has no definite value. It fluctuates near $\phi = 0$. These quantum fluctuations are called vacuum or zero-point fluctuations. Among the physical effects of the vacuum fluctuations are the Lamb shift of the atomic energy levels and the Casimir effect.

In the quantization procedure, as a complete set of mode functions, we have taken plane waves. Of course, we could take another complete set, for example, spherical waves. The relation between quantization schemes based on different sets of the mode functions will be discussed below when we will consider the quantum fields in curved backgrounds.

Summarizing, the general scheme for canonical quantization is reduced to the following steps:

1. Take a complete set of solutions to the Klein-Gordon equation $\{\phi_\alpha, \phi^*_\alpha\}$, where α is a collective index which specifies the quantum numbers. Mode functions are orthonormalized in accordance with

$$(\phi_\alpha, \phi_{\alpha'}) = \delta_{\alpha\alpha'}.$$

2. Expand field operator:

$$\phi = \sum_\alpha \left(a_\alpha \phi_\alpha + a^+_\alpha \phi^*_\alpha \right), \tag{3.10}$$

with the annihilation and creation operators a_α, a^+_α, and the commutation relations

$$[a_\alpha, a^+_{\alpha'}] = \delta_{\alpha\alpha'}, \quad [a_\alpha, a_{\alpha'}] = 0, \quad [a^+_\alpha, a^+_{\alpha'}] = 0, \tag{3.11}$$

$\delta_{\alpha\alpha'}$ is understood as Kronecker delta for discrete quantum numbers and as Dirac delta function for continuous ones.

3. Construct the Fock space:

$$a_\alpha |0\rangle = 0, \quad |n_{\alpha_1} n_{\alpha_2} \cdots n_{\alpha_m}\rangle = \prod_{i=1}^{m} \frac{(a^+_{\alpha_i})^{n_{\alpha i}}}{\sqrt{n_{\alpha_i}!}} |0\rangle . \tag{3.12}$$

Vacuum expectation value of the energy-momentum tensor

Energy-momentum tensor is a bilinear form in field. We substitute the expansion (3.10) of the field operator into this form and use the relations (3.11), (3.12). In this way it can be seen that

$$\langle 0|\, \partial_\mu \phi \partial_\nu \phi \,|0\rangle = \sum_\alpha \partial_\mu \phi_\alpha \partial_\nu \phi^*_\alpha .$$

Hence, we find

$$\langle 0|\, T_{\mu\nu}\{\phi, \phi\} \,|0\rangle = \sum_\alpha T_{\mu\nu}\{\phi_\alpha, \phi^*_\alpha\}.$$

3.3 Quantization of a complex scalar field

For a complex scalar field the Lagrangian density is given by the expression

$$\mathcal{L} = \partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi. \tag{3.13}$$

We decompose the field into two real fields:

$$\phi = (\phi_1 + i\phi_2)/\sqrt{2},$$

and quantize the real fields in accordance withe scheme dsecribeb above. Let $a_{i\alpha}$, $a^+_{i\alpha}$, $i = 1, 2$, be the annihilation and creation operators for the field ϕ_i:

$$[a_{i\alpha}, a^+_{j\alpha'}] = \delta_{\alpha\alpha'} \delta_{ij},$$

and the other commutators vanish. Now we define the operators

$$\begin{aligned} a_\alpha &= (a_{1\alpha} + i a_{2\alpha})/\sqrt{2}, \quad a^+_\alpha = (a^+_{1\alpha} - i a^+_{2\alpha})/\sqrt{2}, \\ b_\alpha &= (a_{1\alpha} - i a_{2\alpha})/\sqrt{2}, \quad b^+_\alpha = (a^+_{1\alpha} + i a^+_{2\alpha})/\sqrt{2}, \end{aligned}$$

For these operators, the new commutation relations read:

$$[a_\alpha, a^+_{\alpha'}] = [b_\alpha, b^+_{\alpha'}] = \delta_{\alpha\alpha'}.$$

The expansion for the field operator takes the form

$$\phi = \sum_\alpha \left(a_\alpha \phi_\alpha + b^+_\alpha \phi^*_\alpha \right). \tag{3.14}$$

The operators a_α, a^+_α are interpreted as annihilation and creation operators for particles and the operators b_α, b^+_α are interpreted as annihilation and creation operators for antiparticles. Assuming that the spectrum for α is discrete, the Hamiltonian takes the form

$$H = \sum_\alpha \omega_\alpha (a^+_\alpha a_\alpha + b^+_\alpha b_\alpha \ + 1)$$

The Fock spaces are constructed in a similar way. For the charge one has

$$Q = \int d^3x \, j_0 = \sum_\alpha (a^+_\alpha a_\alpha - b^+_\alpha b_\alpha \ ) = N_a - N_b,$$

where

$$j_\mu = i\phi^+ \partial_\mu \phi - i \left(\partial_\mu \phi^+ \right) \phi$$

is the current density operator. Note that the vacuum charge vanishes. For the operator of the energy-momentum tensor we get

$$T_{\mu\nu} = \partial_\mu \phi^+ \partial_\nu \phi + \partial_\nu \phi^+ \partial_\mu \phi - g_{\mu\nu} \partial_\rho \phi^+ \partial^\rho \phi + m^2 g_{\mu\nu} \phi^+ \phi.$$

In particular, if the quantization procedure is based on plane waves then the corresponding expansion has the form

$$\phi(x) = \int d^3k \, \left[a_{\mathbf{k}} \phi_{\mathbf{k}}(x) + b^+_{\mathbf{k}} \phi^*_{\mathbf{k}}(x) \right], \tag{3.15}$$

with the mode functions (3.8) and the commutation relations $[a_{\mathbf{k}}, a^+_{\mathbf{k}'}] = [b_{\mathbf{k}}, b^+_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}')$ with the other commutators being zero.

3.4 Dirac spinor field

The operator of spinor field obeys the Dirac equation (2.46). The quantization is done in a way similar to that for a charged scalar field. Let $\psi^{(\pm)}_\alpha$ is a complete set of positive (upper sign) and negative (lower sign) energy solutions to the Dirac equation. The corresponding time dependence is given by $\psi^{(\pm)}_\alpha \sim e^{\mp i E_\alpha t}$, where where $E_\alpha > 0$ is the energy. We assume that these solutions are normalized in accordance with

$$\int d^3x \, \psi^{(\lambda)+}_\alpha \psi^{(\lambda')}_{\alpha'} = \delta_{\lambda\lambda'} \delta_{\alpha\alpha'}. \tag{3.16}$$

For example, in the case of plane waves describing particles with definite momentum one has $\psi^{(\pm)}_\alpha \sim e^{i\mathbf{p}\cdot\mathbf{r} \mp i E_\alpha t}$ and the collective index α includes the momentum $\mathbf{p}$ and the spin projection σ with the eigenvalues $\pm 1/2$. We expand the field operator in terms of the complete set of spinors:

$$\begin{aligned} \psi &= \sum_\alpha \left(a_\alpha \psi^{(+)}_\alpha + b^+_\alpha \psi^{(-)}_\alpha \right), \\ \bar{\psi} &= \sum_\alpha \left(a^+_\alpha \bar{\psi}^{(+)}_\alpha + b_\alpha \bar{\psi}^{(-)}_\alpha \right). \end{aligned}$$

In order to satisfy the Pauli principle anticommutation relations shoud be imposed instead of commutation relations for bosonic fields. These relations are as follows:

$$\{a_\alpha, a^+_{\alpha'}\} = \delta_{\alpha\alpha'}, \quad \{b_\alpha, b^+_{\alpha'}\} = \delta_{\alpha\alpha'},$$

with figure braces standing for anticommutator: $\{a, b\} = ab + ba$. The energy is given by the expression

$$H = \int d^3x \, T^0_0 = \frac{i}{2} \int d^3x \, [\psi^+ \partial_t \psi - (\partial_t \psi^+) \psi].$$

Note that the energy density is not a positive defined quantity. Substituting the expansions for spinor operators and using the normalization condition for the mode spinors, it can be seen that the field hamiltonian is presented as

$$H = \sum_\alpha E_\alpha \left(a^+_\alpha a_\alpha - b_\alpha b^+_\alpha \right).$$

From here it also follows that we should impose anticommutation relations. Assuming that the spectrum for α is discrete and using the corresponding anticommutaion relation for operators b_α we see that

$$H = \sum_\alpha E_\alpha \left(a^+_\alpha a_\alpha + b^+_\alpha b_\alpha - 1 \right) = \sum_\alpha E_\alpha \left(N_\alpha + \bar{N}_\alpha - 1 \right),$$

with the particle and antiparticle number operators $N_\alpha = a^+_\alpha a_\alpha$ and $\bar{N}_\alpha = b^+_\alpha b_\alpha$. For the momentum we have

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha \left(N_\alpha + \bar{N}_\alpha\right).$$

Note that in the fermionic case the vacuum energy is negative. For the charge operator we get

$$\begin{aligned} Q &= \int d^3x\, \bar{\psi}\gamma^0\psi = \sum_\alpha \left(a^+_\alpha a_\alpha + b_\alpha b^+_\alpha\right) \\ &= \sum_\alpha \left(a^+_\alpha a_\alpha - b^+_\alpha b_\alpha + 1\right). \end{aligned}$$

The construction of the Fock space is similar to that for a scalar field.

As a special example for the complete set of modes $\psi^{(\pm)}_\alpha$ we can consider the plane waves. The corresponding positive-energy solution is presented as

$$\psi^{(+)} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-ik\cdot x},$$

with two-component spinors φ and χ and $k = (k^0, \mathbf{k})$, $k^0 > 0$. Substituting into the Dirac equation we get

$$\begin{aligned} \left(k^0 - m\right)\varphi - (\sigma\cdot\mathbf{k})\,\chi &= 0, \\ \left(k^0 + m\right)\chi - (\sigma\cdot\mathbf{k})\,\varphi &= 0. \end{aligned} \tag{3.17}$$

Excluding χ from this system, from the equation for φ it follows that $k^0 = \omega_k = \sqrt{\mathbf{k}^2 + m^2}$. As two independent solutions for φ we can take $\varphi = C\varphi^{(\rho)}$, $\rho = 1, 2$, with $\varphi^{(1)} = (1, 0)^T$ and $\varphi^{(2)} = (0, 1)^T$, where T stands for the transponation. The mode functios are specified by the quantum numbers $\alpha = (\rho, \mathbf{k})$. The quantum number ρ is related to the projection of the spin: $\rho = 1$ ($\rho = 2$) corresponds to the spin projection $s = 1/2$ ($s = -1/2$). The constant C is determined from the normalization condition (3.16), where now in the right-hand side $\delta_{\alpha\alpha'} = \delta_{\rho\rho'}\delta(\mathbf{k} - \mathbf{k}')$. The final expression for the normalizaed positive-energy plane wave modes reads

$$\psi^{(+)}_{\rho\mathbf{k}}(x) = \left(\frac{1 + m/\omega_k}{2\,(2\pi)^3}\right)^{1/2} \begin{pmatrix} \varphi^{(\rho)} \\ \frac{\sigma\cdot\mathbf{k}}{\omega_k + m}\varphi^{(\rho)} \end{pmatrix} e^{-ik\cdot x}. \tag{3.18}$$

The negative-energy modes are found in a similar way. The corresponding equations for the two-component spinors are obtained from (3.17) by the replacements $k^0 \to -k^0$ and $\mathbf{k} \to -\mathbf{k}$. Now we exclude the upper component and after the normalization one gets

$$\psi^{(-)}_{\rho\mathbf{k}}(x) = \left(\frac{1 + m/\omega_k}{2\,(2\pi)^3}\right)^{1/2} \begin{pmatrix} \frac{\sigma\cdot\mathbf{k}}{\omega_k + m}\varphi^{(\rho)} \\ \varphi^{(\rho)} \end{pmatrix} e^{ik\cdot x}. \tag{3.19}$$

Now we can see that the mode functions (3.18) and (3.19) are orthogonal. The corresponding expansion for the field operator takes the form

$$\psi(x) = \sum_\rho \int d\mathbf{k}\, [a_{\rho\mathbf{k}}\psi^{(+)}_{\rho\mathbf{k}}(x) + b^+_{\rho\mathbf{k}}\psi^{(-)}_{\rho\mathbf{k}}(x)], \tag{3.20}$$

and a similar expansion for $\bar{\psi}(x)$.

3.5 Electromagnetic field

The Lagrangian for the electromagnetic field is in the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(E^2 - B^2\right),$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the field tensor. The components of the field tensor are related to the electric and magnetic field strengths by

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk}B^k.$$

Because of gauge invariance, there are complications when we quantize the theory. A naive quantization of the Maxwell theory fails: The propagator does not exist. To see this, let us write down the action in the following form

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\left[\eta^{\mu\alpha}\left(\partial_\mu A_\nu\right)\partial_\alpha A^\nu - \left(\partial_\nu A^\alpha\right)\partial_\alpha A^\nu\right] \\
&= -\frac{1}{2}\partial_\mu\left(\eta^{\mu\alpha}A_\nu\partial_\alpha A^\nu - A^\alpha\partial_\alpha A^\mu\right) + \frac{1}{2}A^\mu\left[\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta - \partial_\mu\partial_\nu\right]A^\nu.
\end{aligned}$$

Hence, up to the total divergence, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}A^\mu\left[\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta - \partial_\mu\partial_\nu\right]A^\nu.$$

3.5.1 Canonical quantization in the Coulomb gauge

To begin the process of canonical quantization, we will take the Coulomb gauge, $\partial_i A^i = 0$, in which only the physical states are allowed to propagate. Let us first evaluate the canonical conjugate to the various fields. Since $\partial_0 A_0$ does not occur in the Lagrangian, this means that A_0 does not appear to propagate, which is a sign that there are redundant modes in the action. The other modes, however, have canonical conjugates:

$$\pi^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} = 0, \quad \pi^i = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} = -\partial_0 A^i + \eta^{i\alpha}\partial_\alpha A_0 = E^i.$$

If we impose canonical commutation relations, we find a further complication. We might want to impose:

$$[A_i(\mathbf{x},t), \pi^j(\mathbf{y},t)] = -i\delta_{ij}\delta(\mathbf{x}-\mathbf{y}).$$

However, this cannot be correct because we can take the divergence of both sides of the equation. The divergence of the left-hand side is zero, but the right-hand side is not. As a result, we must modify the canonical commutation relations as follows:

$$[A_i(\mathbf{x},t), \pi^j(\mathbf{y},t)] = -i\widetilde{\delta}_{ij}(\mathbf{x}-\mathbf{y}),$$

where the right-hand side must be transverse. We can take

$$\widetilde{\delta}_{ij}(\mathbf{x}-\mathbf{y}) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right).$$

The next step is to decompose the Maxwell field in terms of its Fourier modes, and then show that they satisfy the commutation relations. The decomposition is given by:

$$\mathbf{A}(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega}} \sum_{\lambda=1}^{2} \varepsilon^\lambda (a_{\lambda\mathbf{k}}e^{-ik\cdot x} + a^+_{\lambda\mathbf{k}}e^{ik\cdot x}), \tag{3.21}$$

with $\omega = |\mathbf{k}|$. In order to preserve the condition that $\mathbf{A}$ is transverse, we must impose:

$$\varepsilon^\lambda \cdot \mathbf{k} = 0, \quad \varepsilon^\lambda \cdot \varepsilon^{\lambda'} = \delta^{\lambda\lambda'}.$$

In order to satisfy the canonical commutation relations among the fields, we must impose the following commutation relations among the Fourier moments:

$$[a_{\lambda\mathbf{k}}, a^+_{\lambda'\mathbf{k}'}] = \delta_{\lambda\lambda'}\delta(\mathbf{k} - \mathbf{k}').$$

An essential point is that the sign of the commutation relations gives us positive norm states. There are no negative norm states, or ghosts, in this construction in the Coulomb gauge. Let us now insert this Fourier decomposition into the expression for the energy:

$$H = \frac{1}{2}\int d^3x(\mathbf{E}^2 + \mathbf{B}^2) = \sum_{\lambda=1}^{2}\int d\mathbf{k}\,\omega(a^+_{\lambda\mathbf{k}}a_{\lambda\mathbf{k}} + \delta^3(\mathbf{k} - \mathbf{k})/2).$$

The advantage of the canonical quantization method in the Coulomb gauge is that we always work with transverse states. Thus, all states have positive norm:

$$\langle 0|\, a_{\lambda\mathbf{k}}a^+_{\lambda'\mathbf{k}'}\,|0\rangle = \delta_{\lambda\lambda'}\delta(\mathbf{k} - \mathbf{k}').$$

3.5.2 Gupta-Bleuler quantization

The canonical quantization method, although it is guaranteed to yield a unitary theory, is cumbersome because Lorentz invariance is explicitly broken. For higher spin theories, the loss of Lorentz invariance multiplies the difficulty of any calculation. There is another method of quantization, called the Gupta-Bleuler quantization method or covariant method, which keeps manifest Lorentz invariance and simplifies the calculations. There is, however, a price that must be paid, and that is the theory allows negative norm states, or ghosts, to propagate. The resulting theory is manifestly Lorentz invariant with the presence of these ghosts, but the theory is still self-consistent because we remove these ghost states by hand from the physical states of the theory.

We begin by explicitly breaking gauge invariance by adding a noninvariant term into the action:

$$\mathcal{L} = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \tag{3.22}$$

for arbitrary α. Up to total divergence, the lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}A^\mu\left[\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta - (1 - \alpha^{-1})\partial_\mu\partial_\nu\right]A^\nu.$$

Now that we have explicitly broken the gauge invariance, the corresponding operator can be inverted to find the propagator.

The corresponding field equation has the form

$$\left[\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta - (1 - \alpha^{-1})\partial_\mu\partial_\nu\right]A^\nu = 0. \tag{3.23}$$

We will take the gauge $\alpha = 1$, so that the equation of motion now reads:

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta A^\nu = 0.$$

In this gauge, we find that A_0 is a dynamical field and hence has a canonical conjugate to it. The conjugate field of A_μ is now a four-vector:

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\mu)} = \partial_0 A_\mu.$$

Then the covariant canonical commutation relations read:

$$[A_\mu(x), \pi^\nu(x')] = i\delta^\nu_\mu \delta(\mathbf{x} - \mathbf{x}').$$

As usual, we can decompose the field in terms of the Fourier components:

$$A_\mu(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega}} (\varepsilon^\lambda_\mu a_{\lambda\mathbf{k}} e^{-ik\cdot x} + \varepsilon^\lambda_\mu a^+_{\lambda\mathbf{k}} e^{ik\cdot x}).$$

Now ε^λ_μ is a four-vector. In order for the canonical commutation relations to be satisfied, we necessarily choose the following commutation relations among the operators:

$$[a_{\lambda\mathbf{k}}, a^+_{\lambda'\mathbf{k}'}] = -\eta_{\lambda\lambda'} \delta(\mathbf{k} - \mathbf{k}').$$

The presence of the metric tensor in the commutation relation signals that the norm of the states may be negative; that is, a nonphysical, negative norm ghost is present in the theory. The norm of the state $a^+_{\lambda\mathbf{k}} |0\rangle$ can now be negative. This is the price we pay for having a Lorentz covariant quantization scheme.

Since ghosts now propagate in the theory, we must be careful how we remove them. If we take the condition $\partial_\mu A^\mu |\text{physical}\rangle = 0$, we find that this condition is too stringent; it has no solutions at all. The Gupta-Bleuler formalism is based on the observation that a weaker condition is required:

$$(\partial_\mu A^\mu)^{(+)} |\text{physical}\rangle = 0,$$

where we only allow the destruction part of the constraint to act on physical states. In momentum space, this is equivalent to the condition that $k^\mu a_{\lambda\mu} |\text{physical}\rangle = 0$. This guarantees that, although ghosts are allowed to circulate in the system, they are explicitly removed from all physical states of the theory.

We can also quantize the massive vector field in much the same way. The quantization is almost identical to the one presented before, but now the counting of physical states is different. The massless field only has two helicity components whereas the massive vector field has 3 components.

3.6 Path integral approach

In the discussion above we have displayed the canonical quantization procedure in Minkowski spacetime. In this section, on the example of a neutral scalar field, we describe the main points of the quantization procedure based on path integrals. This treatment has at least two advantages. First, the role of the classical limit is apparent and, secondly, the path integral quantization provides a direct way to the study regimes where the perturbation theory fails.

The Lagrangian density for a scalar field $\phi(x)$ is given by the expression (2.18). The central object is the vacuum to vacuum transition amplitude in the presence of the source $J(x)$. This amplitude is determined by the generating functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[i \int d^4x \, \left(\mathcal{L} + i\epsilon\phi^2/2 + J\phi\right)\right], \tag{3.24}$$

where the functional integration goes over all the field configurations $\phi(x)$. The term with $\epsilon > 0$ is introduced to improve the convergence properties of the functional integral. By using the integration by parts in the action, the functional can be presented as

$$Z[J] = \int \mathcal{D}\phi \, \exp\left\{-\frac{i}{2} \int d^4x \, \left[\phi \left(\partial_\mu \partial^\mu + m^2 - i\epsilon\right) \phi - 2J\phi\right]\right\}.$$

Let us shift the integration variable as $\phi(x) \to \phi(x) + \phi_0(x)$, where $\phi_0(x)$ obeys the Klein-Gordon equation with the source:

$$\left(\partial_\mu \partial^\mu + m^2 - i\epsilon\right) \phi_0(x) = J(x).$$

With this change, the functional integral is rewritten as

$$Z[J] = \int \mathcal{D}\phi \, \exp\left\{-\frac{i}{2}\int d^4x \, \left[\phi\left(\partial_\mu \partial^\mu + m^2 - i\epsilon\right)\phi - J\phi_0\right]\right\}. \tag{3.25}$$

For the function $\phi_0(x)$ one has

$$\phi_0(x) = -\int d^4y \, G_F(x,y) J(y),$$

where $G_F(x,y)$ is the Feynman propagator which obeys the equation

$$\left(\partial_\mu \partial^\mu + m^2 - i\epsilon\right) G_F(x,y) = -\delta^4(x-y).$$

In Minkowski spacetime the function $G_F(x,y)$ depends on the relative coordinates $x - y$ only. An important advantage of the representation (3.25) is that the parts in the exponent containing the scalar field and the source are separated. This allows to write the generating functional in the decomposed form

$$Z[J] = Z[0] \exp\left[-\frac{i}{2}\int d^4x d^4y \, J(x) G_F(x,y) J(y)\right],$$

with

$$Z[0] = \int \mathcal{D}\phi \, \exp\left\{-\frac{i}{2}\int d^4x \, \left[\phi\left(\partial_\mu \partial^\mu + m^2 - i\epsilon\right)\phi\right]\right\}.$$

In the last integral the integration goes over all the field configurations and it is just a number. We are interested in normalized transition amplitudes and the factor $Z[0]$ can be normalized as $Z[0] = 1$. This corresponds to that in the absence of sources the field being prepared in the vacuum state initially remains in that state in the future (in general, this is not the case in the presence of external gravitational and electromagnetic fields, see Chapter 7 below). With this normalization one can write

$$Z[J] = \langle 0|\, 0\rangle_J = \exp\left[-\frac{i}{2}\int d^4x d^4y \, J(x) G_F(x,y) J(y)\right].$$

For a given generating functional, the n-point Green function $G_{(n)}(x_1, \ldots, x_n)$ is evaluated by the functional differentiation:

$$G_{(n)}(x_1, \ldots, x_n) = i^{-n} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]|_{J=0}.$$

This shows that the Green functions are the coefficients in the functional expansion of $Z[J]$. All the Green functions with odd number of arguments vanish. For the functions with even number of arguments we get

$$\begin{aligned} G_{(2)}(x_1, x_2) &= iG_F(x_1, x_2), \\ G_{(4)}(x_1, x_2, x_3, x_4) &= -G_F(x_1, x_2)G_F(x_3, x_4) - G_F(x_1, x_3)G_F(x_2, x_4) \\ &\quad -G_F(x_1, x_4)G_F(x_2, x_3). \end{aligned}$$

All the Green functions are expressed in terms of the two-point function (for various types of two-point functions in Minkowski spacetime see 7.2 below). Note that we can also introduce the functional $W[J] = -i \ln Z[J]$. In the functional expansion of $W[J]$ the connected Green functions appear.

Now let us consider a self-interacting scalar field with the Lagrangian density

$$\mathcal{L}_{\rm int} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - V(\phi). \tag{3.26}$$

Again, we define the generating functional $Z[J]$ as (3.24) with the replacement $\mathcal{L} \to \mathcal{L}_{\rm int}$. It can be rewritten as

$$Z[J] = \int \mathcal{D}\phi \,\exp\left[-i\int d^4x\, V(\phi)\right]\exp\left[i\int d^4x\,\left(\mathcal{L}{+}i\epsilon\phi^2/2 + J\phi\right)\right]. \tag{3.27}$$

By taking into account that

$$\phi(x)\exp\left(i\int d^4x\, J\phi\right) = \frac{1}{i}\frac{\delta}{\delta J(x)}\exp\left(i\int d^4x\, J\phi\right),$$

we can write

$$\exp\left[-i\int d^4x\, V(\phi)\right]\exp\left(i\int d^4x\, J\phi\right) = \exp\left[-i\int d^4x\, V(\frac{1}{i}\frac{\delta}{\delta J(x)})\right]\exp\left(i\int d^4x\, J\phi\right).$$

In this form the part containing V does not depend on ϕ and can be written outside the integral:

$$Z[J] = \exp\left[-i\int d^4x\, V(\frac{1}{i}\frac{\delta}{\delta J(x)})\right]\int \mathcal{D}\phi\,\exp\left[i\int d^4x\,\left(\mathcal{L}{+}i\epsilon\phi^2/2 + J\phi\right)\right]. \tag{3.28}$$

The second factor presents the generating functional for a free field and has been discussed before. The formula (3.28) can be used for the evaluation of $Z[J]$ by the perturbation theory (see, for example, [25]).

3.7 Discrete symmetries

In addition to the proper continuous transformations, the Lorentz group includes discrete transformations. In the discussion above we have already defined the parity transformation with $\Lambda^{\mu}_{\cdot\nu} = \mathcal{P}^{\mu}_{\cdot\nu} = {\rm diag}(1,-1,-1,-1)$ and the time reversal transformation with $\Lambda^{\mu}_{\cdot\nu} = \mathcal{T}^{\mu}_{\cdot\nu} = {\rm diag}(-1,1,1,1)$. Here we consider the transformation of the fields under these transformations and also under the charge conjugation which acts not on spacetime coordinates but on internal space. As it has been mentioned before, there is a unitary operator associated to each proper orthochronous Lorentz transformation. In a similar way, we expect that there should be operators P and T associated with the parity and time reversal transformations, $P = U(\mathcal{P})$ and $T = U(\mathcal{T})$, respectively.

3.7.1 Parity transformation

First we consider the parity transformation for a scalar field $\phi(x)$. The corresponding relation reads $P^{-1}\phi(x)P = \eta_P\phi(\mathcal{P}x)$, where $\eta_P = \pm 1$ and we have taken into account that $\mathcal{P}^{-1} = \mathcal{P}$. Similar relation takes place for the Hermitian conjugate operator. For $\eta_P = 1$ ($\eta_P = -1$) we say that the field is even (odd) under parity. These cases correspond to scalars and pseudoscalars respectively. The corresponding quantum number is determined by experiment. For example, the mesons $\pi^0$and $\pi^{\pm}$, considered as elementary, are pseudoscalar particles. Substituting the expansion (3.15) and by taking into account that P is a linear operator, we find the transformation rules for the creation and annihilation operators under the parity transformation:

$$\begin{aligned} P^{-1}a_{\mathbf{k}}P &= \eta_P a_{-\mathbf{k}}, \quad P^{-1}a^{+}_{\mathbf{k}}P = \eta_P a^{+}_{-\mathbf{k}}, \\ P^{-1}b_{\mathbf{k}}P &= \eta_P b_{-\mathbf{k}}, \quad P^{-1}b^{+}_{\mathbf{k}}P = \eta_P b^{+}_{-\mathbf{k}}. \end{aligned}$$

Fixing the parity of the vacuum by convention, $P\,|0\rangle = |0\rangle$, from here for the parity of one-particle state we get

$$P\,|\mathbf{k}\rangle = Pa^{+}_{\mathbf{k}}\,|0\rangle = Pa^{+}_{\mathbf{k}}P^{-1}P\,|0\rangle = \eta_P a^{+}_{-\mathbf{k}}\,|0\rangle = \eta_P\,|-\mathbf{k}\rangle\,.$$

This just restates that the momentum changes the sign under the parity transformation. As seen, the state with a definite momentum $\mathbf{k} \neq 0$ is not an eigenstate for the parity. In the rest frame of the particle $\mathbf{k} = 0$ and the corresponding state is an eigenstate, $P\,|\mathbf{k}=0\rangle = \eta_P\,|\mathbf{k}=0\rangle$. η_P is called as intrinsic parity of a scalar particle. A similar analysis can be done for the corresponding antiparticle. In particular, we see that a particle and its conjugate antiparticle have equal intrinsic parities. The transformation rules for dynamical variables bilinear in the field operator, such as the current density and the energy-momentum tensor are obtained from (3.5) taking $U(\Lambda) = P$ and $\Lambda^{\mu}_{\cdot\nu} = \mathcal{P}^{\mu}_{\cdot\nu}$.

Now let us consider the action of the parity transformation on the Dirac spinor ψ. This action is given by the relation

$$P^{-1}\psi(x)P = \eta_P\gamma^0\psi(\mathcal{P}x), \tag{3.29}$$

with η_P being an overall intrisic parity factor. On the base of this relation and by using the expansion (3.20), the transformation rules for the annihilation and creation operators are obtained. From (3.20) it follows that

$$\psi(\mathcal{P}x) = \sum_{\rho}\int d\mathbf{k}\,[a_{\rho,-\mathbf{k}}\psi^{(+)}_{\rho,-\mathbf{k}}(x) + b^{+}_{\rho,-\mathbf{k}}\psi^{(-)}_{\rho,-\mathbf{k}}(x)].$$

Combining this with (3.29), we get

$$\begin{aligned} P^{-1}a_{\rho\mathbf{k}}P\psi^{(+)}_{\rho\mathbf{k}}(x) &= \eta_P a_{\rho,-\mathbf{k}}\gamma^0\psi^{(+)}_{\rho,-\mathbf{k}}(x), \\ P^{-1}b^{+}_{\rho\mathbf{k}}P\psi^{(-)}_{\rho\mathbf{k}}(x) &= \eta_P b^{+}_{\rho,-\mathbf{k}}\gamma^0\psi^{(-)}_{\rho,-\mathbf{k}}(x). \end{aligned}$$

From (3.18) and (3.19) one has $\gamma^0\psi^{(\pm)}_{\rho,-\mathbf{k}}(x) = \pm\psi^{(\pm)}_{\rho\mathbf{k}}(x)$ and, hence, we obtaine the transformation rules

$$\begin{aligned} P^{-1}a_{\rho\mathbf{k}}P &= \eta_P a_{\rho,-\mathbf{k}}, \quad P^{-1}b^{+}_{\rho\mathbf{k}}P = -\eta_P b^{+}_{\rho,-\mathbf{k}}, \\ P^{-1}a^{+}_{\rho\mathbf{k}}P &= \eta_P a^{+}_{\rho,-\mathbf{k}}, \quad P^{-1}b_{\rho\mathbf{k}}P = -\eta_P b_{\rho,-\mathbf{k}}, \end{aligned}$$

This shows that particles and antiparticles have opposite intrinsic parities.

For a vector field $V^{\mu}(x)$ the action of the parity transformation is given by the relation

$$P^{-1}V^{\mu}(x)P = \eta_P\mathcal{P}^{\mu}_{\cdot\nu}V^{\nu}(\mathcal{P}x),$$

where $\eta_P = 1$ for polar vectors and $\eta_P = -1$ for axial vectors. Let us consider the case of the electromagnetic field. The corresponding interaction term in the Lagrangian density $ej^{\mu}A_{\mu}$ is invariant with respect to the parity transformation and, hence, for $V^{\mu}(x) = A^{\mu}(x)$ one has $\eta_P = 1$. In particular, for the vector potential $\mathbf{A}(x)$ we get

$$P^{-1}\mathbf{A}(x)P = -\mathbf{A}(\mathcal{P}x).$$

With this result and using the expansion (3.21) one can find the transformation properties for the annihilation and creation operators. For the polarization vector $\varepsilon^{\lambda} = \varepsilon^{\lambda}(\hat{\mathbf{k}})$, with $\hat{\mathbf{k}}$ being the unit vector along the vector $\mathbf{k}$, we have $\varepsilon^{\lambda}(\hat{\mathbf{k}}) = \varepsilon^{\lambda'}(-\hat{\mathbf{k}})$, where $\lambda' = 1$ for $\lambda = 2$ and $\lambda' = 2$ for $\lambda = 1$. On the base of this, the transformation rule for the annihilation operator reads $P^{-1}a_{\lambda\mathbf{k}}P = -a_{\lambda',-\mathbf{k}}$. Hence, the photon has a negative intrinsic parity. Both its momentum and helicity change signs under the parity transformation.

3.7.2 Time reversal

Now we turn to the time inversion. Similar to the parity transformation one has $T^{-1}\phi(x)T = \eta_T\phi(\mathcal{T}x)$ with $\eta_T = \pm 1$. Unlike to the parity operator, the time inversion operator is antilinear and antiunitary: $T\left(c_1\left|\psi_1\right\rangle + c_2\left|\psi_2\right\rangle\right) = c_1^* T\left|\psi_1\right\rangle + c_2^* T\left|\psi_1\right\rangle$, $\left\langle T\psi_1(t)\right| T\psi_2(t)\rangle = \left\langle\psi_1(-t)\right|\psi_2(-t)\rangle^*$. The transformation rules for the operators acting on the Fock states are obtained on the base of the expansion (3.7). This leads to the relations $T^{-1}a_{\mathbf{k}}T = \eta_T a_{-\mathbf{k}}$ and $T^{-1}b_{\mathbf{k}}^+T = \eta_T b_{-\mathbf{k}}^+$. Having the transformation rule for the field operator, for the current density one gets

$$T^{-1}j^\mu(x)T = j_\mu(\mathcal{T}x). \tag{3.30}$$

For a Dirac spinor field ψ the transformation under the time reversal reads $T^{-1}\psi(x)T = \eta_T A\psi(\mathcal{T}x)$ with a matrix A. From the condition of the invariance of the Dirac Lagrangian $L_D(x)$ with respect to the time reversal, $T^{-1}L_D(x)T = \eta_T L_D(\mathcal{T}x)$ it follows that $A\gamma^\mu A^+ = \gamma^{\mu\mathrm{T}}$, where T stands for transponation. In order to solve this relation we can consider the representation (2.51) for the Dirac matrices. With this choice, A commutes with γ^0 and γ^2, and anticommutes with γ^1 and γ^3. From this it can be seen that, up to a phase, $A = \gamma^1\gamma^3$. Hence, the transformation rule reads

$$T^{-1}\psi(x)T = \eta_T\gamma^1\gamma^3\psi(\mathcal{T}x).$$

For the transformation of bilinear combinations of a Dirac spinor we get

$$\begin{aligned} T^{-1}\bar{\psi}(x)\psi(x)T &= \bar{\psi}(\mathcal{T}x)\psi(\mathcal{T}x),\ T^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)T = \bar{\psi}(\mathcal{T}x)\gamma^{\mu+}\psi(\mathcal{T}x), \\ T^{-1}\bar{\psi}(x)\gamma_5\psi(x)T &= -\bar{\psi}(\mathcal{T}x)\gamma_5\psi(\mathcal{T}x),\ T^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)T = \bar{\psi}(\mathcal{T}x)\gamma^{\mu+}\gamma_5\psi(\mathcal{T}x). \end{aligned}$$

Having the transformation for the field operator we can find the transformation rules for the annihilation and creation operators in the expansion (3.20):

$$T^{-1}a_{\rho\mathbf{k}}T = -(-1)^\rho\eta_T a_{\rho',-\mathbf{k}},\ T^{-1}b_{\rho\mathbf{k}}^+T = -(-1)^\rho\eta_T b_{\rho',-\mathbf{k}}^+,$$

where $\rho' = 1$ for $\rho = 2$ and $\rho' = 2$ for $\rho = 1$.

Now let us consider the electromagnetic field with the vector potential $A^\mu(x)$. From the time reversal invariance of the electromagnetic interaction $ej^\mu(x)A_\mu(x)$ and by using the result (3.30) for the current density, we get $T^{-1}A^\mu(x)T = A_\mu(\mathcal{T}x)$. Substituting the plane-wave expnsion (3.21), for the creation and annihilation operators one finds the following transformation rules

$$T^{-1}a_{\lambda\mathbf{k}}T = a_{\lambda,-\mathbf{k}},\quad T^{-1}a_{\lambda\mathbf{k}}^+T = a_{\lambda,-\mathbf{k}}^+.$$

This shows that for the photon η_T factor one has $\eta_T = 1$ and photon helicity in unchanged under the time reversal transformation.

3.7.3 Charge conjugation

Here we consider another discrete transformation - charge conjugation. It reverses the sign of charge of a given particle, converting it into the corresponding antiparticle. The corresponding unitary operator we will denote by C. It has the following properties: $C^2 = 1$, $C^+ = C^{-1} = C$. The action of the charge conjugation operator on a particle of momentum p, spin s, and charge q is given by $C\left|p,s,q\right\rangle = \eta\left|p,s,-q\right\rangle$, with η being a unimodular phase factor.

Let us start our consideration for charge conjugation from a complex scalar field $\phi(x)$. For the action of the field conjugation operator one has

$$C^{-1}\phi(x)C = \eta_C\phi^+(x), \tag{3.31}$$

with $|\eta_C| = 1$. The Lagrangian density (3.13). Substituting the expansion (3.15) and identifying the corresponding coefficients on both sides, we get

$$C^{-1}a_{\mathbf{k}}C = \eta_C b_{\mathbf{k}}, \quad C^{-1}b_{\mathbf{k}}C = \eta_C^* a_{\mathbf{k}}.$$

Assuming the C-invariance of the vacuum state, $C|0\rangle = |0\rangle$, for a one-particle state one finds $Ca_{\mathbf{k}}^+|0\rangle = Ca_{\mathbf{k}}^+C^{-1}C|0\rangle = \eta_C^* b_{\mathbf{k}}^+|0\rangle$ and $Cb_{\mathbf{k}}^+|0\rangle = Cb_{\mathbf{k}}^+C^{-1}C|0\rangle = \eta_C a_{\mathbf{k}}^+|0\rangle$. This shows that the charge conjugation transforms a particle into its antiparticle, and vice versa, without changing their momenta. For the action of C on the current density operator we get $C^{-1}j_\mu(x)C = -j_\mu(x)$. The conserved charge q defined as the spatial integral of $j^0(x)$ changes the sign under the charge conjugation, $C^{-1}qC = -q$.

For a Dirac field $\psi(x)$ we write the transformation rule in the form $C^{-1}\psi(x)C = \eta_C\tilde{B}\psi^*(x)$, where $|\eta_C| = 1$ and $\tilde{B}$ is a 4×4 unitary matrix. We can write it in terms of the spinor $\bar{\psi}(x)$ as

$$C^{-1}\psi(x)C = \eta_C B\bar{\psi}^T(x),$$

with $B = \tilde{B}\gamma_0^*$, $B^+B = 1$. The matrix B is found from the condition of the invariance of the Dirac Lagrangian under the charge conjugation. For that invariance it is sufficient to require the condition $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. The latter property is valid in any representation for Dirac matrices. In the Dirac representation one finds $C^{-1}\gamma_\mu C = -\gamma_\mu$ for $\mu = 0, 2$, and $C^{-1}\gamma_\mu C = \gamma_\mu$ for $\mu = 1, 3$. These relations are solved by the choice $B = \lambda\gamma^2\gamma^0$, $|\lambda| = 1$. By taking into account that $C^{-1}\psi^+C = \eta_C^*\psi^T\gamma_0 B^+$, for the charge conjugate of the adjoint operator one obtains

$$C^{-1}\bar{\psi}C = C^{-1}\psi^+\gamma_0 C = \eta_C^*\psi^T\gamma_0 B^+\gamma_0 = -\eta_C^*\psi^T B^+.$$

For the current density and the associated charge we get the transformation properties $C^{-1}j_\mu C = C^{-1}\bar{\psi}C\gamma_\mu C^{-1}\psi C = -j_\mu$ and $C^{-1}qC = -q$. By taking into account the expansion (3.20) and taking $\lambda = i$, for the Fock operators one finds the following relations

$$C^{-1}a_{\rho\mathbf{k}}C = \eta_C b_{\rho\mathbf{k}}, \quad C^{-1}b_{\rho\mathbf{k}}C = \eta_C^* a_{\rho\mathbf{k}}.$$

From here it follows that the field conjugation converts a particle state $a_{\rho\mathbf{k}}^+|0\rangle$ into the corresponding antiparticle state $b_{\rho\mathbf{k}}^+|0\rangle$ without changing its spin or momentum. All charges change signs according to $C^{-1}qC = -q$. A fermion and its conjugate partner have opposite parities, opposite chiralities, but equal helicities.

Now let us turn to the electromagnetic field with 4-potental $A_\mu(x)$. From the transformation rule for the current density $j_\mu(x)$ and from the invariance of the electromagnetic interaction with respect to C it follows that the field $A_\mu(x)$ should transform in accordance with

$$C^{-1}A_\mu(x)C = -A_\mu(x).$$

Substituting the plane-wave expansion (3.21), the transformation rule for the annihilation operator reads $C^{-1}a_{\lambda\mathbf{k}}C = -a_{\lambda\mathbf{k}}$. From here for a one-photon state one gets $Ca_{\lambda\mathbf{k}}^+|0\rangle = Ca_{\lambda\mathbf{k}}^+C^{-1}C|0\rangle = -a_{\lambda\mathbf{k}}^+|0\rangle$. This shows that the photon is odd under charge conjugation.

The *CPT theorem* states that the product of the parity transformation, time reversal and the charge conjugation, applied in any order, is a symmetry of a quantum theory if the corresponding Lagrangian density is Hermitian, is invariant under proper Lorentz transformations and if the fields are quantized in accordance with the usual spin–statistics connection. The validity of the theorem is based on the invariance to the group of continuous Lorentz transformations, the usual spin-statistics connection and the locality of the theory. It is not affected by whether parity transformation, time reversal and the charge conjugation separately are symmetries or not. The electromagnetic and strong interactions are invariant with respect to C-, P-, T-transformations separately. This is not the case for weak interactions. These interactions break parity and charge conjugation and also the combined CP-symmetry (and, hence, in accordance with the CPT theorem, T-invariance). One of the models to explain the dominance of matter over antimatter in the present Universe is based on the CP-violation by weak interactions.

Chapter 4

General Relativity as a classical theory of Gravitation

4.1 Gravity as a geometry

The best candidate we have for a theory of gravity is Einstein's General Theory of Relativity or General Relativity (for a review of alternative theories of gravity see, e.g., [28]). We give a brief introduction to the basics of the theory.

In Special Relativity, the invariant interval between two events is defined by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

The essence of General Relativity is to transform the gravity from being a force to being a property of spacetime. In his theory, the spacetime is not necessarily flat as it is in Minkowski spacetime but may be curved.

General Relativity is based on the equivalence principle. There are several forms of the principle:

Weak equivalence principle (WEP): The laws of free motion of test particles in the local inertial frame are the same as in Special Relativity.

Einstein equivalence principle (EEP): The laws of non-gravitational physics in the local inertial frame are the same as in Special Relativity. The outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed (Local Lorentz invariance). The outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed (Local position invariance).

Strong equivalence principle (SEP): All laws, including the gravitation, in the local inertial frame are the same as in Special Relativity.

The Einstein equivalence principle is the heart of gravitational theory. If EEP is valid, then gravitation must be a "curved spacetime" phenomenon: The effects of gravity must be equivalent to the effects of living in a curved spacetime. As a consequence of this argument, the only theories of gravity that can fully embody EEP are those that satisfy the postulates of "metric theories of gravity":

1. Spacetime is endowed with a symmetric metric.

2. The trajectories of freely falling test bodies are geodesics of that metric.

3. In local freely falling reference frames, the non-gravitational laws of physics are those written in the language of Special Relativity.

General relativity is a metric theory of gravity, but there are many others, including the Brans–Dicke theory and its generalizations.

In any gravitating system, one can at any point choose a new set of coordinates such that the gravitational field disappears. This new set of coordinates is the freely falling "elevator frame," in which space appears locally to resemble ordinary Lorentzian space. Since we need to express the physical consequences of the equivalence principle mathematically, one needs a mathematical language by which we can easily transform from one frame to another, that is, tensor calculus. We will define a general coordinate transformation as an arbitrary reparametrization of the coordinate system:

$$x'^{\mu} = x'^{\mu}(x).$$

Under reparametrizations, a scalar field transforms simply as follows:

$$\phi'(x') = \phi(x).$$

Contravariant/covariant components of vectors transform like dx^{μ}/∂_{μ}:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}, \quad \partial'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu}.$$

Given these transformation laws, we can now give the abstract definition of covariant tensors, with lower indices, and contravariant tensors, with upper indices, depending on their transformation properties:

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}, \quad A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}.$$

Similarly, we can construct tensors of arbitrary rank or indices. They transform as the product of a series of first-rank tensors (vectors).

The infinitesimal invariant distance between two points separated by dx^{μ} is given by:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}.$$

If $g_{\mu\nu}$ is defined to be a second-rank covariant tensor, then this distance ds^2 is invariant. One essential point is that it is always possible to find a local coordinate system in which we can diagonalize the metric tensor, so that $g_{\mu\nu}$ becomes the usual Lorentzian metric at a point. The tensor indices are lowered and raised with the help of the metric tensor:

$$A_{\mu} = g_{\mu\nu} A^{\nu}, \quad T^{\mu\nu} = g^{\mu\alpha} T_{\alpha}{}^{\nu},$$

where the contravariant components of the metric tensor obey the relation $g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}$.

The next step is to write down derivatives of the fields that are also covariant objects. The derivative of a scalar field is a tensor under general coordinate transformations:

$$\partial'_{\mu} \phi'(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu} \phi(x).$$

However, the partial derivative of a vector is not a tensor under general coordinate transformations. Covariant derivative:

$$\begin{aligned} \nabla_{\mu} A^{\nu} &= \partial_{\mu} A^{\nu} + \Gamma^{\nu}_{\mu\lambda} A^{\lambda}, \\ \nabla_{\mu} A_{\nu} &= \partial_{\mu} A_{\nu} - \Gamma^{\lambda}_{\mu\nu} A_{\lambda}, \end{aligned}$$

where the Γs are called Christoffel symbols,

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\sigma\nu} - \partial_{\sigma} g_{\nu\rho} \right).$$

This objet is symmetric with respect to the lower indices. The covariant derivative $\nabla_{\mu} A_{\nu}$ transforms as a second rank tensor. The Christoffel symbol is not a tensor. It vanishes i a local inertial frame.

Covariant derivatives can be constructed for increasingly complicated tensors by adding appropriate Christoffel symbols. For example, in the case of a second rank tensor one has

$$\nabla_\mu T^{\nu\lambda} = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} T^{\alpha\lambda} + \Gamma^\lambda_{\mu\alpha} T^{\nu\alpha}.$$

We see close analogy between the elements of gauge theory and general relativity. This close correspondence can be symbolically represented as follows:

$$A_\mu \to \Gamma^\mu_{\nu\rho}, \quad D_\mu \to \nabla_\mu.$$

In what follows we will use the following relations involving the covariant derivative:

$$\begin{aligned} \nabla_\mu B^\mu &= \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} B^\mu), \\ \nabla_\mu A^{\mu\nu} &= \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} A^{\mu\nu}), \\ \nabla_\mu S^\mu_\nu &= \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} S^\mu_\nu) - \frac{1}{2} S^{\mu\lambda} \partial_\nu g_{\mu\lambda}, \end{aligned} \tag{4.1}$$

for antisymmetric and symmetric tensors $A^{\mu\nu}$ and S^μ_ν. Here, g is the determinant of the metric tensor, $g = \det(g_{\mu\nu})$. For the covariant d'Alambertian acting on a scalar field one has

$$\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi). \tag{4.2}$$

Note that the covariant conservation for vector and antisymmetric tensor fields yields the conserved quantities. In general, this is not the case for symmetric tensors.

In General Relativity, the metric tensor serves as a characteristic of the gravitational field. It determines all geometrical characteristics of spacetime. In particular, for the Riemann tensor one has

$$R^\mu{}_{\nu\alpha\beta} = \partial_\beta \Gamma^\mu_{\nu\alpha} - \partial_\alpha \Gamma^\mu_{\nu\beta} + \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha} - \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta}.$$

The geometrical properties of the background manifold is encoded in the Riemann tensor. In order to have a flat spacetime it is necessary and sufficient that the Riemann tensor be zero. For the tensor $R_{\mu\nu\alpha\beta} = g_{\mu\sigma} R^\sigma{}_{\nu\alpha\beta}$ one has the following properties

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha} = R_{\alpha\beta\mu\nu},$$

and the Bianchi identity

$$\nabla_\sigma R^\mu{}_{\nu\alpha\beta} + \nabla_\beta R^\mu{}_{\nu\sigma\alpha} + \nabla_\alpha R^\mu{}_{\nu\beta\sigma} = 0.$$

In addition one has the cyclic identity

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0. \tag{4.3}$$

We can construct lower rank tensors by using the Riemann tensor. The second rank symmetric tensor

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta}.$$

is called the Ricci tensor. The scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

is called Ricci or curvature scalar. From the Bianchi identity the following relation is obtained:

$$\nabla_\nu R^\nu_\mu = \partial_\mu R/2. \tag{4.4}$$

We could have derived the curvature tensor by taking a vector A_μ and then moving it around a closed circle using parallel transport. After completing the circuit, the vector is rotated by the amount

$$\Delta A_\mu = -\frac{1}{2} R^{\nu}{}_{\mu\alpha\beta} A_\nu \Delta f^{\alpha\beta},$$

where $\Delta f^{\alpha\beta}$ is the area tensor of the closed path. From here it follows that the necessary and sufficient condition that parallel transport be independent of the path is that the curvature tensor vanishes. In curved spacetime, the second order covariant derivative acted on a vector field depends on the order of differentiations:

$$\nabla_\alpha \nabla_\nu A_\mu - \nabla_\nu \nabla_\alpha A_\mu = A_\beta R^{\beta}{}_{\mu\alpha\nu}. \tag{4.5}$$

In a $(D+1)$-dimensional spacetime the Riemann tensor can be decomposed as

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= C_{\mu\nu\alpha\beta} + \frac{1}{D-1}\left(g_{\mu\alpha} R_{\nu\beta} - g_{\mu\beta} R_{\nu\alpha} + g_{\nu\beta} R_{\mu\alpha} - g_{\nu\alpha} R_{\mu\beta}\right) \\ &\quad - \frac{R}{D(D-1)}\left(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}\right). \end{aligned}$$

The tensor $C_{\mu\nu\alpha\beta}$ is called the Weyl tensor or the conformal tensor. It shares all the properties of the Riemann tensor and $C^{\alpha}_{\cdot\nu\alpha\beta} = 0$. Let us consider a conformal transformation of the metric tensor:

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{4.6}$$

for some real function $\Omega(x)$. Under this transformation the Ricci scalar and the Ricci tensor transform as

$$\begin{aligned} R &\to \bar{R} = \Omega^{-2} R + 2D \frac{\nabla_\mu \nabla^\mu \Omega}{\Omega^3} + D(D-3) \frac{\nabla_\mu \Omega \nabla^\mu \Omega}{\Omega^4}, \\ R^{\nu}_{\mu} &\to \bar{R}^{\nu}_{\mu} = \Omega^{-2} R^{\nu}_{\mu} - (D-1) g^{\nu\alpha} \frac{\nabla_\alpha \nabla_\mu \Omega^{-1}}{\Omega} + \frac{\delta^{\nu}_{\mu} g^{\alpha\beta}}{D-1} \frac{\nabla_\alpha \nabla_\beta \Omega^{D-1}}{\Omega^{D+1}}. \end{aligned} \tag{4.7}$$

An important point is that the Weyl tensor $C^{\nu}_{\cdot\mu\alpha\beta}$ is invariant under the conformal transformations of the metric tensor: $\bar{C}^{\nu}_{\cdot\mu\alpha\beta} = C^{\nu}_{\cdot\mu\alpha\beta}$. The vanishing of the Weyl tensor is a necessary and sufficient condition for the Riemannian manifold being conformally flat. The conformal transformation (4.6) of the metric tensor should not be confused with the element (2.10) of the group of conformal transformation. The latter is a coordinate transformation whereas in the transformation (4.6) the coordinate system is fixed.

4.2 Influence of the gravitational field on non-gravitational matter

The influence of the gravitational field on non-gravitational matter is directly obtained on the base of the equivalence principle. First let us consider the motion of a test particle. The particle moves in such a way that the integral along its path is stationary:

$$\delta \int_{\text{path}} ds = 0.$$

The corresponding Euler-Lagrange equation is presented in the form

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^{\mu}_{\nu\rho} u^\nu u^\rho = 0,$$

with $u^\nu = dx^\nu / ds$ being the velocity. This equation determines the *geodesics* of the metric.

Now let us consider the general case of a physical system with the action functional

$$S_m[\phi(x)] = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)), \tag{4.8}$$

in Minkowski spacetime. In order to obtain the action of this system in curved spacetime, firstly we write the action in a local inertial frame. In accordance to the EEP, in this system the action has the form (4.8). The form of the action in arbitrary coordinate system is obtained by the coordinate transformation. As a result, the action in the presence of gravitational field is obtained from the Special Relativity action by using the following rules:

1. Replace the Minowskian metric by curved metric: $\eta_{\mu\nu} \to g_{\mu\nu}$.
2. Replace partial derivatives by covariant derivatives: $\partial_\mu \to \nabla_\mu$.
3. Replace the volume element: $d^4x \to d^4x\,\sqrt{-g}$, where $g = \det(g_{\mu\nu})$. The volume element $d^4x\,\sqrt{-g}$ is a genuine invariant.

Hence, the action for the system on background of the gravitational field described by the metric tensor $g_{\mu\nu}$ is given by the expression

$$S_m[\phi(x)] = \int d^4x\,\sqrt{-g}\,\mathcal{L}(g_{\mu\nu}, \phi(x), \nabla_\mu \phi(x)). \tag{4.9}$$

This is called as a minimal interaction. In some cases additional terms are added to the Lagrangian. Special examples will be discussed below.

4.3 Action for gravitational field

For the formulation of the theory, in addition to the matter Lagrangian, one needs to have the action for the gravitational field. In General Relativity the gravitational field is described by the metric tensor. The corresponding Lagrangian must be constructed from the metric tensor and its derivatives. The action must be invariant under general coordinate transformations and, hence, the Lagrangian should be a scalar. Next, we require that the field equations for the gravitational field must contain no more than two derivatives. Consequently, the Lagrangian should contain no more than first derivative or if there are higher derivative terms they should be in the form of a total divergence. Then the latter will not contribute to the field equation. Where is no nontrivial scalar constructed from the metric and its first derivatives.

The solution to these constraints, is given by:

$$S_g = \frac{1}{2\varkappa^2} \int d^4x\,\sqrt{-g} R, \quad \varkappa^2 = 8\pi G,$$

where R is the Ricci scalar and G is the Newton gravitational constant. We can also add the cosmological term, which is proportional to $\Lambda\sqrt{-g}$ with $\Lambda = \text{const}$. This is the Einstein-Hilbert action, which is the starting point for all calculations in General Relativity. The Ricci scalar can be written in the form

$$R = G_R + \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} w^\mu), \tag{4.10}$$

where

$$G_R = g^{\mu\nu}(\Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{\mu\alpha}\Gamma^\rho_{\nu\rho}).$$

The part G_R contains only the metric tensor and its first dervatives. Note that G_R is not a scalar. The second term in (4.10) will not contribute to the field equations.

4.4 Total action and Einstein equations

The action for a system of matter fields and the gravitational field reads

$$S = S_g + S_m = \frac{1}{2\varkappa^2}\int d^4x\,\sqrt{-g}R + \int d^4x\,\sqrt{-g}\,\mathcal{L}(g_{\mu\nu}, \phi(x), \nabla_\mu\phi(x)).$$

The equation for the gravitational field is obtained from the extremum condition for the action with respect to the variation of the metric field:

$$\delta S = 0.$$

We define the metric energy-momentum tensor as

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta\sqrt{|g|}\,\mathcal{L}}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{|g|}}\left[\frac{\partial\sqrt{|g|}\,\mathcal{L}}{\partial g^{\mu\nu}} - \partial_\rho\frac{\partial\sqrt{|g|}\,\mathcal{L}}{\partial(\partial_\rho g^{\mu\nu})}\right]. \tag{4.11}$$

This tensor is symmetric by the definition. The variation of the matter part of the action is written in the form

$$\delta S_m = \frac{1}{2}\int d^4x\,\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\int d^4x\,\sqrt{|g|}T^{\mu\nu}\delta g_{\mu\nu}.$$

From the definition of the metric energy-momentum tensor, by using the equation of motion for $\phi(x)$, it can be seen that it is covariantly conserved on the solutions of the field equations:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.12}$$

For the variation of the action one has

$$\delta S = \frac{1}{2\varkappa^2}\int d^4x\,\delta(\sqrt{-g}R) + \frac{1}{2}\int d^4x\,\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}.$$

For the variations appearing in the gravitational part of the action we have the relations

$$\begin{aligned}
\delta R &= g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}, \quad \delta R_{\mu\nu} = \nabla_\nu\delta\Gamma^\rho_{\mu\rho} - \nabla_\rho\delta\Gamma^\rho_{\mu\nu}, \\
g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\rho}\right) - \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) = \nabla_\rho\left(g^{\mu\rho}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right), \\
\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.
\end{aligned}$$

Though the Christoffel symbols do not form a tensor, the variation $\delta\Gamma^\rho_{\mu\nu}$ is a tensor and we can write

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\rho\left[\sqrt{-g}\left(g^{\mu\rho}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right)\right].$$

An equivalent representation is given by

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\rho\left[\nabla_\mu(\delta g^{\rho\mu}) - g^{\rho\mu}g_{\nu\alpha}\nabla_\mu(\delta g^{\nu\alpha})\right].$$

This shows that the term $g^{\mu\nu}\delta R_{\mu\nu}$ does not contribute to the field equations. Consequently, the variation of the action is presented as

$$\delta S = \frac{1}{2\varkappa^2}\int d^4x\,\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \frac{1}{2}\int d^4x\,\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}.$$

From $\delta S = 0$ we obtain the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\varkappa^2 T_{\mu\nu}. \tag{4.13}$$

By taking into account the identity (4.4), from the Einstein equations the covariant conservation equation (4.12) is obtained.

For the covariant divergence of the symmetric tensor one has the relation (4.1). This shows that, in general, the covariant conservation equation (4.12) does not lead to conserved integral quantities. Conserved quantities are present if the background geometry has symmetries. Let us consider a coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}, \tag{4.14}$$

with small ξ^{μ}. Under this transformation, the metric tensor transforms as

$$g'^{\mu\nu}(x'^{\alpha}) = g^{\rho\sigma}(x^{\beta})\partial_{\rho}x'^{\mu}\partial_{\sigma}x'^{\nu} \approx g^{\mu\nu}(x^{\alpha}) + g^{\mu\rho}\partial_{\rho}\xi^{\nu} + g^{\nu\rho}\partial_{\rho}\xi^{\mu}.$$

The difference $\delta_{\xi}g_{\mu\nu} = g'_{\mu\nu}(x^{\alpha}) - g_{\mu\nu}(x^{\alpha})$ describes the change in the form of the metric tensor. It can be presented in the form

$$\delta_{\xi}g_{\mu\nu} = -\nabla_{\mu}\xi_{\nu} - \nabla_{\nu}\xi_{\mu}.$$

From this relation it follows that if $\delta_{\xi}g_{\mu\nu} = 0$ then the metric tensor is form-invariant under the transformation (4.14).

From $\delta_{\xi}g_{\mu\nu} = 0$ it follows that

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0. \tag{4.15}$$

The vector field $\xi_{\mu}(x)$ obeying this equation is called a Killing vector. The symmetries of spacetime are described by Killing vectors. The problem of determining all infinitesimal isometries of the spacetime is reduced to the solution of the Killing equation for a given metric tensor. In $(D+1)$-dimensional spacetime the maximal number of independent Killing vectors is equal $(D+1)(D+2)/2$. The spaces with maximal number of Killing vectors are called maximally symmetric spaces.

If the background spacetime has symmetries and hence a Killing vector ξ_{μ}, then we have

$$\xi_{\nu}\nabla_{\mu}T^{\mu\nu} = \nabla_{\mu}\left(\xi_{\nu}T^{\mu\nu}\right) - T^{\mu\nu}\nabla_{\mu}\xi_{\nu} = \nabla_{\mu}\left(\xi_{\nu}T^{\mu\nu}\right) = 0.$$

The last relation shows that the vector $J^{\mu} = \xi_{\nu}T^{\mu\nu}$ is a conserved current:

$$\nabla_{\mu}J^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}J^{\mu}) = 0,$$

with a conserved charge

$$Q = \int d^3x\,\sqrt{-g}J^0.$$

In particular, if the metric tensor does not depend on time the vector $\xi_{\nu} = (\xi_0, 0, 0, 0)$ is a Killing vector and the corresponding conserved charge coincides with the energy.

4.5 Tetrad formalism

The evaluation of the curvature tensor for a given metric usually is a rather cumbersome procedure. In some cases the corresponding calculations are simplified by using the tetrad formalism. The tetrad formalism is required in order to intriduce the interaction of the gravitational field with fermionic fields. Here we will present the basics of the formalism.

By using the equivalence principle, we introduce local inertial coordinates ξ^a_X at a given point X. In this coordinates $ds^2 = \eta_{ab}d\xi^a_X d\xi^b_X$. In a general non-inertial frame the metric tensor is written in the form

$$g_{\mu\nu} = e^a_{\mu}(x)e^b_{\nu}(x)\eta_{ab},$$

where

$$e^a_{\mu}(x) = \left(\frac{\partial\xi^a_X(x)}{\partial x^{\mu}}\right)_{x=X}.$$

Note that the locally inertial system at every point X is fixed and, hence, if we transform the local noninertial coordinates $x^\mu \to x'^\mu$ then

$$e^a_\mu \to e'^a_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu.$$

e^a_μ can be considered as a set of 4 covariant tensors numbered by the index a. This set is called vierbein or tetrad.

For a contravariant tensor $A^\mu(x)$ we may use the tetrad to give the components of the tensor in the locally inertial coordinate system ξ^a_X:

$$A^a = e^a_\mu A^\mu.$$

In a similar way we can write

$$A_a = e^\mu_a A_\mu, \quad B^a_{\cdot b} = e^a_\mu e^\nu_b B^\mu_{\cdot\nu},$$

where

$$e^\mu_a = g^{\mu\nu}\eta_{ab}e^b_\nu.$$

From the latter relation it follows that

$$e^\mu_a e^a_\nu = \delta^\mu_\nu, \quad e^\mu_a e^b_\mu = \delta^b_a.$$

We also have the relation

$$e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab}.$$

Note that A_a, $B^a_{\cdot b}$ are scalars under the coordinate transformation $x^\mu \to x'^\mu$.

Let us consider the quantity $e^a_{\mu;\nu}$, where ; stands for the usual covariant derivative when e^a_μ is considered as a covariant vector with the components specified by index μ. We can expand this quantity in terms of tetrad as $e^a_{\mu;\nu} = \gamma^a_{\cdot bc}e^b_\mu e^c_\nu$ with the coefficients $\gamma^a_{\cdot bc}$. For the latter one gets $\gamma^a_{\cdot bc} = e^\mu_b e^\nu_c e^a_{\mu;\nu}$. We also define $\gamma_{abc} = \eta_{ad}\gamma^d_{\cdot bc} = e^\mu_b e^\nu_c e_{a\mu;\nu}$. By taking into account that $e^\mu_b e_{a\mu;\nu} = -e_{a\mu}e^\mu_{b;\nu} = -e^\mu_a e_{b\mu;\nu}$, we see that $\gamma_{abc} = -\gamma_{bac}$.

In order to find the tetrad components of the Riemann tensor we use the relation (4.5) for the vector $A_\mu = e_{a\mu}$. Note that in (4.5) the derivative ∇_ν corresponds to $;\nu$. One gets

$$e_{a\mu;\nu;\alpha} - e_{a\mu;\alpha;\nu} = e^\beta_a R_{\beta\mu\alpha\nu} = R_{a\mu\alpha\nu}.$$

From here for the tetrad components of the Riemann tensor we find

$$R_{abcd} = \left(e_{a\mu;\nu;\alpha} - e_{a\mu;\alpha;\nu}\right) e^\mu_b e^\alpha_c e^\nu_d.$$

For the covariant derivatives in this expression one has $e_{a\mu;\nu} = \gamma_{aef}e^e_\mu e^f_\nu$. In the evaluation of the second covariant derivative we again use this relation and also take into account that for γ_{aef} the covariant derivative ; is reduced to the partial derivative. In this way we can show that

$$e_{a\mu;\nu;\alpha}e^\mu_b e^\alpha_c e^\nu_d = \gamma_{abd,c} + \gamma_{aed}\gamma^e_{\cdot bc} + \gamma_{abf}\gamma^f_{\cdot dc}.$$

The corresponding relation for $e_{a\mu;\alpha;\nu}e^\mu_b e^\alpha_c e^\nu_d$ is obtained by making the replacement $c \rightleftarrows d$ in the expression of the right-hand side. For the curvature tensor this gives

$$R_{abcd} = \gamma_{abd,c} - \gamma_{abc,d} + \gamma_{aed}\gamma^e_{\cdot bc} - \gamma_{aec}\gamma^e_{\cdot bd} + \gamma_{abf}\left(\gamma^f_{\cdot dc} - \gamma^f_{\cdot cd}\right).$$

For the tetrad components of the Ricci tensor we obtain

$$R_{ab} = \gamma^c_{\cdot ab,c} - \gamma^c_{\cdot ac,b} + \gamma^c_{\cdot eb}\gamma^e_{\cdot ac} - \gamma^c_{\cdot ec}\gamma^e_{\cdot ab} + \gamma^c_{\cdot af}\left(\gamma^f_{\cdot bc} - \gamma^f_{\cdot cb}\right).$$

Having this tensor we can write the Einstein equations in tetrad formalism.

4.6 Cosmological models

The cosmological backgrounds are among the most popular geometries in quantum field theory. Due to the high symmetry, a large number of problems are exactly solvable on these backgrounds. The quantum effects such as the vacuum polarization and the creation of particles by the gravitational field play an important role in the evolution of the early universe. Recent cosmology is based on the cosmological principle in accordance of which there is a reference frame in which the large scale properties of the universe are homogeneous and isotropic. Homogeneity is the property of being identical everywhere in space, while isotropy is the property of looking the same in every direction. From the cosmological principle it follows that the space is maximally symmetric. This essentially simplifies the corresponding gravitational problem.

4.6.1 Friedmann-Robertson-Walker metric

By geometrical considerations only, it can be seen that the most general metric tensor describing a universe in which the cosmological principle is obeyed is given the Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{4.16}$$

where t is the time measured by an observer with fixed comoving coordinates (r, θ, φ). It is the proper time or the synchronous time. The radial coordinate r is dimensionless. The function $a(t)$ has dimension of a length and is called the scale factor. The curvature parameter k takes the values $k = 0, -1, 1$. In the case $k = 0$ the space is flat and for $k = 1$ ($k = -1$) the space has a constant positive (negative) curvature. For $k = 1$ one has $0 \leqslant r < 1$ and the space has the topology of S^3. The corresponding geometry with $a(t) = \text{const}$ is called as the Einstein static universe. For the models with $k = 1$ the space has finite volume $2\pi^2 a^3$, but has no boundaries.

Instead of the coordinate r, we can introduce the coordinate χ defined in accordance with

$$r = f(\chi) = \begin{cases} \chi, & 0 \leqslant \chi < \infty, \quad k = 0, \\ \sin\chi, & 0 \leqslant \chi \leqslant \pi, \quad k = 1, \\ \sinh\chi, & 0 \leqslant \chi < \infty, \quad k = -1. \end{cases}$$

The FRW line element is rewritten as

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]. \tag{4.17}$$

For the area of sphere with the radius χ one has $S(\chi) = 4\pi a^2(t) f^2(\chi)$. In the models with $k = 0, -1$, it monotonically increases with increasing χ. For $k = 1$, the function $S(\chi)$ first increases, takes its maximum value $S_m = 4\pi a^2(t)$ at $\chi = \pi/2$ and then tends to zero in the limit $\chi \to \pi$. Introducing a conformal time η in accordance with $d\eta = dt/a(t)$, the line element is written in the form

$$ds^2 = C^2(\eta)\left[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{4.18}$$

conformally related to the line element of static spacetime and $C(\eta) = a(t)$.

For the proper distance d_P of the points with the radial coordinate r from the origin $r = 0$ on has

$$d_P = a(t)\chi = a(t)\int_0^r \frac{dr'}{\sqrt{1-kr'^2}}.$$

For the radial velocity with respect to the origin this gives $v(t) = H d_P$, where $H = \dot{a}/a$ is the Hubble function and the dot stands for the derivative with respect to the proper time t. This relation for the radial velocity is called the Hubble law. The values of the Hubble function evaluated at the present time $t = t_0$ is called the Hubble constant, $H_0 = H(t_0) \approx 70$ (km/s)/Mpc (1 Mpc = 10^6 pc = 3.086×10^{19} km). The scale factor $a(t)$ describes the expansion of the universe. More directly observable cosmological quantity is the redshift z. The redshift of a luminous source is defined as $z = (\lambda_0 - \lambda_e)/\lambda_e$, where λ_0 is the wavelength of radiation from the source observed at the origin (the location of the observer) at time t_0. The wavelength of radiation emitted by the source at earlier time t is λ_e. Considering the propagation of the light from the source to the origin the relation $1 + z = a(t_0)/a(t)$ is obtained for the redshift and the scale factor.

The generalization of the FRW line element to a D-dimensional space is straightforward: the angular part $d\theta^2+\sin^2\theta d\varphi^2$ should be replaced by the line element on a unit $(D-1)$-dimensional sphere $d\Omega_{D-1}^2$. If the points of the sphere are parameterized by the angular coordinates $(\theta_1, \theta_2, \cdots, \theta_{D-1})$ with $0 \leqslant \theta_i < \pi$ for $i = 1, \ldots, D-2$, and $0 \leqslant \theta_{D-1} < 2\pi$, then

$$d\Omega_{D-1}^2 = (d\theta_1)^2 + \sum_{j=2}^{D-1} \left(\prod_{i=1}^{j-1} \sin^2\theta_i \right) (d\theta_j)^2.$$

4.6.2 Friedmann equations and the expansion of the Universe

From the Einstein equations (4.13) it follows that in FRW models the energy-momentum tensor should have a perfect fluid form $T_\nu^\mu = \mathrm{diag}(\rho, -p, -p, -p)$ with the energy density ρ and pressure p. The Einstein equations are reduced to the Friedmann cosmological equations

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}\rho, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p)\,. \end{aligned} \tag{4.19}$$

From the covariant conservation equation $\nabla_\mu T_\nu^\mu = 0$ we get

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{4.20}$$

The second equation in (4.19) is obtained from the first equation in the combination with (4.20). The first equation in (4.19) can be rewritten in the form $ka^{-2} = H^2(\rho/\rho_c - 1)$, where $\rho_c = 3H^2/(8\pi G)$ is the critical density. The space is open ($k = -1$), flat ($k = 0$) or closed ($k = 1$) according to whether the density is less that, equal to, or greater than ρ_c. In order to complete the set of the cosmological equations, the equation of state $p = p(\rho)$ must be specified. For the barotropic equation of state $p = w\rho$ with $w = \mathrm{const}$ and the cosmological equations are exactly solvable in terms of the elementary functions. The most important special cases are the radiation ($w = 1/3$), dust matter ($w = 0$) and the cosmological constant ($w = -1$). In the latter case, the energy density and the pressure are expressed in terms of the cosmological constant Λ as $\rho_\Lambda = -p_\Lambda = \Lambda/(8\pi G)$. The cosmological constant is the subject of much interest on both conceptual and observational grounds. In the modern interpretation, ρ_Λ presents the energy of the vacuum, which is understood as the ground state of a quantum system. The cosmological constant problem is among the most serious problems in cosmology. It is strictly connected with the particle physics and, probably, to quantum gravity.

The observations of Type Ia supernovae indicate that at the recent epoch the expansion of the universe is accelerating. From the second equation in (4.19) it follows that in order to have that type of expansion within the framework of general relativity one needs the source for which $\rho + 3p < 0$. This unknown form of energy is called the dark energy. Dark energy is the most accepted hypothesis to explain the accelerated expansion of the universe (other models are based on modifications of general relativity as a classical theory of gravity). The existence of dark energy is suggested by other observations as well. Measurements of cosmic microwave background (CMB) spectrum and the theory of large-scale structure indicate that the density of matter (baryons and dark matter) in the universe is only $\approx 30\%$ of the critical density. The observational data on CMB temperature anisotropies indicate that the universe is close to flat. For that, the energy density of the universe must be close to the critical density and, hence, in addition to baryons and dark matter, the presence of some other source (about 70% of the total energy in the universe) is required. In the simplest explanation, the dark energy is modelled by cosmological constant Λ and the corresponding model is called Lambda-CDM model (CDM stands for cold dark matter). The corresponding mass density is estimated to be of the order of 10^{-29} g/cm^3. In Planck units this is 10^{-120}. One of the main problems in models with cosmological constant is that most quantum field theories predict a cosmological constant in the form of the vacuum energy with the value much larger than the one suggested by the cosmological observations. In alternative models for dark energy, the accelerated expansion of the universe is caused by the potential energy of a dynamical scalar field, called quintessence field. Unlike to cosmological constant, the corresponding energy density can vary in space and time and in order not to clump like matter, the corresponding mass must be very small.

In the standard cosmological model the expansion of the Universe starts from hot superdense phase governed by ultrarelativistic particles. This phase is radiation dominated with the equation of state $p = \rho/3$ and contains Big Bang singularity at some initial time. In addition, a number of other problems are present in the standard cosmological model. A part of them (for example, horizon, flatness and monopole problems) are naturally solved in the inflationary scenario. The latter assumes the presence of the phase with accelerating expansion of the space (usually quasiexponential) in the early universe. This phase precedes the radiation dominated expansion in standard cosmology. From the observational point of view, among the most important predictions of the inflationary scenario is the generation of seeds for large scale structure formation. These seeds are sourced by the quantum fluctuations of scalar fields during the inflationary phase and can be tested on the base of the observational data from CMB temperature anisotropies.

Chapter 5

Classical fields in curved spacetime

The general procedure to write down the action for a field in curved spacetime is described before. We consider the specific cases

5.1 Scalar field

We consider a free real scalar field $\phi(x)$ in $(D+1)$-dimensional spacetime with the action

$$S = \frac{1}{2}\int d^{D+1}x\sqrt{|g|}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right), \tag{5.1}$$

where R is the Ricci scalar for background spacetime and ξ is the curvature coupling parameter. Note that we have included in the Lagrangian the term $-\xi R\phi^2$ describing a nonminimal coupling of the filed to gravity. The field with $\xi = 0$ is called a minimally coupled field. The action principle leads to the field equation (Klein-Gordon equation in curved spacetime)

$$\left(\nabla_\mu\nabla^\mu + m^2 + \xi R\right)\phi = 0. \tag{5.2}$$

Note that for the covariant d'Alambertian one has the relation (4.2).

The metric energy-momentum tensor for the general coupling has the form

$$\begin{aligned} T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi + \frac{1}{2}m^2 g_{\mu\nu}\phi^2 \\ &\quad -\xi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\phi^2 + \xi\left(g_{\mu\nu}\nabla_\rho\nabla^\rho - \nabla_\mu\nabla_\nu\right)\phi^2. \end{aligned} \tag{5.3}$$

Note that, though the Lagrangian does not contain the parameter ξ for the special case of flat bulk, the metric energy-momentum tensor depends on this parameter for the flat bulk as well. By taking into account that

$$\nabla_\rho\phi\nabla^\rho\phi = \frac{1}{2}\nabla_\rho\nabla^\rho\phi^2 - \phi\nabla_\rho\nabla^\rho\phi,$$

we can write

$$\begin{aligned} T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}\phi\left(\nabla_\rho\nabla^\rho + m^2 + \xi R\right)\phi \\ &\quad + \left[\left(\xi - 1/4\right)g_{\mu\nu}\nabla_\rho\nabla^\rho - \xi\nabla_\mu\nabla_\nu - \xi R_{\mu\nu}\right]\phi^2. \end{aligned} \tag{5.4}$$

The second term in the right-hand side vanishes on the solutions of the field equation. For the trace of the metric energy-mpmentum tensor one gets

$$T_\mu^\mu = D\left(\xi - \xi_c\right)\nabla_\rho\nabla^\rho\phi^2 + m^2\phi^2 + \frac{D-1}{2}\phi\left(\nabla_\rho\nabla^\rho + m^2 + \xi R\right)\phi, \tag{5.5}$$

with the notation

$$\xi_c = \frac{D-1}{4D}. \tag{5.6}$$

On the solutions of the field equation the metric energy-momentum tensor is traceless for a conformally coupled massless scalar field.

Let us consider a conformal transformation of the metric tensor $\bar{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x)$ with a positive function $\Omega(x)$. We also consider the field transformation

$$\phi \to \bar{\phi} = \Omega^\beta \phi,$$

with a constant β. We want to specify the conditions under which the field equation for the new field has the form of the Klein-Gordon equation:

$$\left[\left(\bar{\nabla}_\mu \bar{\nabla}^\mu + m^2 + \xi \bar{R}\right)\right] \bar{\phi} = 0.$$

By using the transformation relation (4.7) for the Ricci scalar and substituting the barred quantities in terms of unbarred ones, one obtains

$$\begin{aligned} 0 \;=\; & \Omega^{\beta-2}\left(\nabla_\mu \nabla^\mu + m^2 + \xi R\right)\phi + \Omega^{\beta-2}m^2\left(\Omega^2 - 1\right)\phi + (D + 2\beta - 1)\,\Omega^{\beta-3}\nabla^\mu \Omega \nabla_\mu \phi \\ & + (2D\xi + \beta)\,\frac{\nabla_\mu \nabla^\mu \Omega}{\Omega^3}\Omega^\beta \phi + \left[\xi D(D-3) + \beta\,(D + \beta - 2)\right]\frac{\nabla_\mu \Omega \nabla^\mu \Omega}{\Omega^4}\Omega^\beta \phi. \end{aligned}$$

By taking into account (5.2), we get the following relations for the parameters:

$$D + 2\beta = 1, \quad 2D\xi + \beta = 0, \quad \xi D(D-3) + \beta\,(D + \beta - 2) = 0,$$

and $m = 0$. From these conditions we find

$$\beta = (1 - D)/2, \quad \xi = \xi_c.$$

For these values of the parameters one has

$$\left(\bar{\nabla}_\mu \bar{\nabla}^\mu + \xi \bar{R}\right)\bar{\phi} = \Omega^{\beta-2}\left(\nabla_\mu \nabla^\mu + \xi R\right)\phi.$$

The obtained value of $\xi = \xi_c$ corresponds to a conformally coupled field. Hence, for a conformally coupled massless field the field equation is form-invariant under the conformal transformations. As it has been mentioned before, for this field the metric energy-momentum tensor is traceless. This result is a special case of a general statement about that for a conformally invariant field the energy-momentum tensor is traceless. Let us show that.

Consider a field with the Lagrangian $\mathcal{L}$. Under the variation $\delta g^{\mu\nu}$ of the metric tensor, for the variation of the action one has

$$\delta S = \int d^{D+1}x \frac{\delta\sqrt{|g|}\,\mathcal{L}}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \frac{1}{2}\int d^{D+1}x\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu}.$$

If the variation of the metric is induced by the conformal transformation (4.6) then one has $\Omega \to 1 + \delta\Omega$, $\delta g^{\mu\nu} = 2\delta\Omega g^{\mu\nu}$, and

$$\delta S = \int d^{D+1}x\sqrt{|g|}T_\mu^\mu \delta\Omega.$$

If the action is invariant under conformal transformations we have δS and from the previous relation it follows that the energy-momentum tensor is traceless.

5.2 Electromagnetic field

For the electromagnetic field in $(D+1)$-dimensional spacetime the action reads

$$S = -\frac{1}{4}\int d^{D+1}x\sqrt{|g|}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^{D+1}x\sqrt{|g|}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta},$$

with the field tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The action is invariant under the conformal transformations (4.6) in the case $D = 3$ only.

The action is invariant under the conformal transformations (4.6) in the case $D = 3$ only. For the variation of the action under the variation of the vector potential we have

$$\delta S = -\int d^{D+1}x\sqrt{|g|}\left[\nabla_\mu\left(\delta A_\nu F^{\mu\nu}\right) - \delta A_\nu \nabla_\mu F^{\mu\nu}\right].$$

The term with the total divergence is removed by using the Gauss theorem and from $\delta S = 0$ we obtain the field equation

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}F^{\mu\nu}\right) = 0. \tag{5.7}$$

We can write the field equation in terms of the vector potential. By taking into account that

$$\nabla_\mu F^{\mu\nu} = \nabla_\mu\nabla^\mu A^\nu - g^{\mu\alpha}g^{\nu\beta}\nabla_\mu\nabla_\beta A_\alpha,$$

and using the relation (4.5) for the commutator of covariant derivatives, from (5.7) we get

$$\nabla_\mu\nabla^\mu A^\nu - g^{\nu\beta}\nabla_\beta\nabla_\mu A^\mu + R^\nu_\mu A^\mu = 0. \tag{5.8}$$

Imposing the gauge condition $\nabla_\mu A^\mu = 0$, the field equation is presented as

$$\nabla_\mu\nabla^\mu A^\nu + R^\nu_\mu A^\mu = 0. \tag{5.9}$$

The metric energy-momentum tensor, obtained from (4.11), is given by the expression

$$T_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F_\nu{}^\alpha.$$

For the corresponding trace we find

$$T^\mu_\mu = \frac{D-3}{4}F_{\alpha\beta}F^{\alpha\beta}.$$

In the special case $D = 3$ the energy-momentum tensor is traceless. As we have mentioned before, this is related to the conformal invariance of the electromagnetic field in $D = 3$.

5.3 Influence of the gravity on matter: Alternative approach

5.3.1 Covariant derivative in tetrad formalism and the action functional

We have discussed the influence of gravity on bosonic fields. In order to introduce the corresponding interaction with spinor field we use the approach based on the tetrad formalism. There are two invariance principles which should be taken into account in the construction of the action on the base of this formalism:

1. Action should be generally covariant and all fields should be considered as scalars, except the tetrad.

2. From the equivalence principle it follows that in the local inertial systems the special relativity is applicable. In particular, we should have the local Lorentz invariance. This means that the field equations and the action must be invariant with respect to the local Lorents transformations:
$$A^a(x) \to \Lambda^a_{\cdot b}(x) A^b(x), \quad T_{ab}(x) \to \Lambda_a^{\cdot c}(x)\Lambda_b^{\cdot d}(x) T_{cd},$$
where
$$\eta_{ac}\Lambda^a_{\cdot b}\Lambda^c_{\cdot d} = \eta_{bd}.$$

The tetrad $e^a_\mu(x)$ is a Lorentz contravariant vector and we have the transformation law
$$e^a_\mu(x) \to \Lambda^a_{\cdot b}(x) e^b_\mu(x).$$
An arbitrary field $\psi_n(x)$ transforms as
$$\psi_n(x) \to \sum_m [D(\Lambda(x))]_{nm}\psi_m(x), \tag{5.10}$$
where $D(\Lambda)$ is the matrix representation of the Lorentz group.

The Lagrangian should be a coordinate scalar and the Lorentz scalar. In order to construct the Lagrangian we should also introduce derivatives. The partial derivative transforms like a covariant vector:
$$\partial_\mu \to \partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu.$$
Hence, in order to make the action a coordinate scalar we should introduce the derivatives
$$e^\mu_a \partial_\mu.$$
However, this combination has not a simple transformation property under the Lorentz transformation which depends on the position. If the field transforms in accordance with (5.10), then the corresponding scalar derivatives transform as
$$e^\mu_a \partial_\mu \psi(x) \to \Lambda_a^{\cdot b} e^\mu_b \partial_\mu \left[D(\Lambda(x))\psi(x)\right] = \Lambda_a^{\cdot b} e^\mu_b \left\{D(\Lambda(x))\partial_\mu\psi(x) + \left[\partial_\mu D(\Lambda(x))\right]\psi(x)\right\}.$$
We need to have in the action derivatives D_a which, in addition to the coordinate scalar, are also Lorentz vector. The latter means that under the Lorentz transformation $\Lambda^a_{\cdot b}(x)$ we should have the following relation
$$D_a\psi \to \Lambda_a^{\cdot b}(x) D(\Lambda(x)) D_b \psi(x).$$
In this case, any action depending on various fields ψ and its derivatives $D_a\psi$ automatically will not depend on the choice of local inertial frames, if it is invariant under the usual Lorentz transformations with constant matrix. From the transformation law for $e^\mu_a\partial_\mu\psi(x)$ it follows that we can construct the derivative as
$$D_a = e^\mu_a \left(\partial_\mu + \Gamma_\mu\right),$$
where the matrix Γ_μ transforms as
$$\Gamma_\mu(x) \to D(\Lambda(x))\Gamma_\mu(x)D^{-1}(\Lambda(x)) - \left[\partial_\mu D(\Lambda(x))\right] D^{-1}(\Lambda(x)).$$
The last term cancels the second term in the transformation law for $e^\mu_a\partial_\mu\psi(x)$.

In order to determine the structure of the matrices $\Gamma_\mu(x)$, it is sufficient to consider an infinitesimal Lorentz transformation:
$$\Lambda^a_{\cdot b}(x) = \delta^a_b + \omega^a_{\cdot b}(x), \quad \omega_{ab} = -\omega_{ba}.$$

In this case the matrix D in the transformation law for the field has the form

$$D(1+\omega(x)) = 1 + \frac{1}{2}\omega^{ab}(x)\Sigma_{ab}, \tag{5.11}$$

where Σ_{ab} is a set of constant matrices, antisymmetric with respect to a and b: $\Sigma_{ab} = -\Sigma_{ba}$, and obeying the commutation relations

$$[\Sigma_{ab}, \Sigma_{cd}] = \eta_{bc}\Sigma_{ad} - \eta_{ac}\Sigma_{bd} + \eta_{bd}\Sigma_{ca} - \eta_{ad}\Sigma_{cb}.$$

Let us prove the last relation. As $D(\Lambda)$ is the representation of the Lorentz group we have the relation

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2).$$

We apply this rule to the product $\Lambda(1+\omega)\Lambda^{-1}$:

$$D(\Lambda)D(1+\omega)D(\Lambda^{-1}) = D(\Lambda(1+\omega)\Lambda^{-1}) = D(1+\Lambda\omega\Lambda^{-1}).$$

In the zeroth order with respect to ω we obtain $1 = 1$. In the first order:

$$\begin{aligned} & D(\Lambda)(1+\frac{1}{2}\omega^{ab}\Sigma_{ab})D(\Lambda^{-1}) = 1 + \frac{1}{2}(\Lambda\omega\Lambda^{-1})^{ab}(x)\Sigma_{ab} \\ \Rightarrow\quad & D(\Lambda)\omega^{ab}\Sigma_{ab}D(\Lambda^{-1}) = (\Lambda\omega\Lambda^{-1})^{ab}(x)\Sigma_{ab} \\ \Rightarrow\quad & D(\Lambda)\omega^{ab}\Sigma_{ab}D(\Lambda^{-1}) = \Lambda^{c}_{\cdot a}\omega^{ab}\Lambda^{d}_{\cdot b}\Sigma_{cd}. \end{aligned}$$

As a consequence we get

$$D(\Lambda)\Sigma_{ab}D(\Lambda^{-1}) = \Lambda^{c}_{\cdot a}\Lambda^{d}_{\cdot b}\Sigma_{cd}. \tag{5.12}$$

Taking $\Lambda = 1+\omega$, $\Lambda^{-1} = 1-\omega$ (ω here may be different), using (5.11) and $\Lambda^{c}_{\cdot a} = \delta^{c}_{a} + \omega^{c}_{\cdot a}$, in the first order with respect to ω, from (5.12) we find

$$\Sigma_{ab} - \frac{1}{2}\Sigma_{ab}\omega^{st}\Sigma_{st} + \frac{1}{2}\omega^{cd}\Sigma_{cd}\Sigma_{ab} = \Sigma_{ab} + \delta^{c}_{a}\omega^{d}_{\cdot b}\Sigma_{cd} + \omega^{c}_{\cdot a}\delta^{d}_{b}\Sigma_{cd}.$$

This is simplified to

$$-\frac{1}{2}\omega^{cd}\left(\Sigma_{ab}\Sigma_{cd} - \Sigma_{cd}\Sigma_{ab}\right) = \omega^{d}_{\cdot b}\Sigma_{ad} + \omega^{c}_{\cdot a}\Sigma_{cb},$$

and, consequently,

$$-\frac{1}{2}\omega^{cd}\left(\Sigma_{ab}\Sigma_{cd} - \Sigma_{cd}\Sigma_{ab}\right) = \eta_{bc}\omega^{dc}\Sigma_{ad} + \eta_{ad}\omega^{cd}\Sigma_{cb} = -\omega^{cd}\left(\eta_{bc}\Sigma_{ad} - \eta_{ad}\Sigma_{cb}\right).$$

From here it follows that

$$-\frac{1}{2}\omega^{cd}\left(\Sigma_{ab}\Sigma_{cd} - \Sigma_{cd}\Sigma_{ab}\right) = -\frac{1}{2}\omega^{cd}\left(\eta_{bc}\Sigma_{ad} - \eta_{bd}\Sigma_{ac} - \eta_{ad}\Sigma_{cb} + \eta_{ac}\Sigma_{db}\right)$$

or

$$[\Sigma_{ab}, \Sigma_{cd}] = \eta_{bc}\Sigma_{ad} - \eta_{ac}\Sigma_{bd} + \eta_{bd}\Sigma_{ca} - \eta_{ad}\Sigma_{cb}.$$

This defines the Lie algebra for $SO(1,D)$.

From the relation

$$\Gamma_{\mu}(x) \to D(\Lambda(x))\Gamma_{\mu}(x)D^{-1}(\Lambda(x)) - \left[\partial_{\mu}D(\Lambda(x))\right]D^{-1}(\Lambda(x)),$$

it follows that under the infinitesimal Lorentz transformation:

$$\begin{aligned} \Gamma_{\mu}(x) \quad \to \quad & \left(1+\frac{1}{2}\omega^{ab}(x)\Sigma_{ab}\right)\Gamma_{\mu}(x)\left(1-\frac{1}{2}\omega^{cd}(x)\Sigma_{cd}\right) \\ & - \left[\partial_{\mu}\left(1+\frac{1}{2}\omega^{ab}(x)\Sigma_{ab}\right)\right]\left(1-\frac{1}{2}\omega^{cd}(x)\Sigma_{cd}\right) \\ = \quad & \Gamma_{\mu}(x) + \frac{1}{2}\omega^{ab}(x)[\Sigma_{ab}, \Gamma_{\mu}(x)] - \frac{1}{2}\partial_{\mu}\omega^{ab}(x)\Sigma_{ab}. \end{aligned}$$

We note that for the tetrad field one has

$$e_\nu^a(x) \to e_\nu^a(x) + \omega_{\cdot b}^a e_\nu^b(x).$$

By taking into account that $e_\nu^a(x)e_b^\nu(x) = \delta_b^a$, we get

$$\begin{aligned}
\delta_b^a &= \left(e_\nu^a(x) + \omega_{\cdot c}^a e_\nu^c(x)\right)\left(e_b^\nu(x) + \Omega_b^d e_d^\nu(x)\right) \\
&= e_\nu^a(x)e_b^\nu(x) + \Omega_b^d e_\nu^a(x)e_d^\nu(x) + \omega_{\cdot c}^a e_\nu^c(x)e_b^\nu(x).
\end{aligned}$$

From here it follows that

$$\Omega_b^d e_\nu^a(x)e_d^\nu(x) + \omega_{\cdot c}^a e_\nu^c(x)e_b^\nu(x) = 0 \Rightarrow \Omega_b^a + \omega_{\cdot b}^a = 0,$$

and, hence,

$$e_b^\nu(x) \to e_b^\nu(x) - \omega_{\cdot b}^a e_a^\nu(x).$$

By using this relation we find

$$\begin{aligned}
e_b^\nu(x)\frac{\partial}{\partial x^\mu}e_{a\nu}(x) &\to \left(e_b^\nu(x) - \omega_{\cdot b}^d e_d^\nu(x)\right)\frac{\partial}{\partial x^\mu}\left(e_{a\nu}(x) + \omega_{ac}e_\nu^c(x)\right) \\
&= e_b^\nu(x)\frac{\partial}{\partial x^\mu}e_{a\nu}(x) + \frac{\partial}{\partial x^\mu}\omega_{ab} + \omega_{ac}e_b^\nu(x)\frac{\partial}{\partial x^\mu}e_\nu^c(x) \\
&\quad -\omega_{\cdot b}^d e_d^\nu(x)\frac{\partial}{\partial x^\mu}e_{a\nu}(x).
\end{aligned}$$

Then it can be seen that

$$\Gamma_\mu(x) = \frac{1}{2}\Sigma^{ab}e_a^\nu(x)e_{b\nu;\mu}.$$

As a result, the action of the gravity on the non-gravitational matter can be taken into account if in the action of special relativity all derivatives ∂_μ are replaced by "covariant" derivatives

$$D_a = e_a^\mu\left(\partial_\mu + \Gamma_\mu\right).$$

This allows to find the action or the field equations which are invariant under general coordinate transformations. Under these transformations $e_a^\mu(x)$ are considered as covariant vectors and all other fields as scalars.

Let us introduce a connection $\omega_\mu{}^a{}_b$ defined in accordance with

$$\nabla_\mu A^a = \partial_\mu A^a + \omega_\mu{}^a{}_b A^b,$$

for a vector field A^a. One has

$$\nabla_\mu A^a = \left(\nabla_\mu e_\nu^a\right)A^\nu + e_\nu^a\nabla_\mu A^\nu = \left(\nabla_\mu e_\nu^a\right)A^\nu + e_\nu^a\partial_\mu A^\nu + e_\nu^a\Gamma_{\mu\alpha}^\nu A^\alpha.$$

From here it follows that

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b - e_\alpha^a\Gamma_{\mu\nu}^\alpha.$$

From the other side, it can be seen that the following relation takes place:

$$\omega_\mu{}^a{}_b = -e_b^\nu\left(\partial_\mu e_\nu^a - e_\alpha^a\Gamma_{\mu\nu}^\alpha\right) = -e_b^\nu e_{\nu;\mu}^a. \tag{5.13}$$

Combining this with the previous relation we see that

$$\nabla_\mu e_\nu^a = 0. \tag{5.14}$$

5.3.2 Dirac spinor field

Let us consider the special case of spinors. As we have already mentioned, spinors transform under some representation $S(\Lambda)$ of the Lorentz group:

$$\psi'(x') = S(\Lambda)\psi(x),\ S(\Lambda) = \exp[-(i/4)\sigma_{ab}\omega^{ab}],$$

where

$$\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b],$$

where γ_a are flat spacetime gamma matrices. For an infinitesimal transformation we have

$$S(\Lambda) = 1 - (i/4)\sigma_{ab}\omega^{ab}.$$

From here it follows that for spinors

$$\Sigma_{ab} = -\frac{i}{2}\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b],$$

and hence,

$$\Gamma_\mu(x) = \frac{1}{8}[\gamma^a, \gamma^b]e_a^\nu e_{b\nu;\mu}.$$

Here, $;\mu$ stands for the covariant derivative of the vector field $e_{b\nu}$ with respect to the index ν. By taking into account that

$$e_a^\nu e_{b\nu;\mu} = (e_a^\nu e_{b\nu})_{;\mu} - e^\nu_{a;\mu} e_{b\nu} = -e_b^\nu e_{a\nu;\mu},$$

we can also write

$$\Gamma_\mu(x) = \frac{1}{4}\gamma^a\gamma^b e_a^\nu e_{b\nu;\mu}.$$

The object $\Gamma_\mu(x)$ is called spin connection. By taking into account the relation

$$\gamma^{a+} = \gamma^{(0)}\gamma^a\gamma^{(0)}, \tag{5.15}$$

with $\gamma^{(0)} = \gamma^a|_{a=0}$, it can be seen that

$$\gamma^{(0)}\Gamma_\mu^+(x)\gamma^{(0)} = -\Gamma_\mu(x),$$

or

$$\Gamma_\mu^+(x)\gamma^{(0)} = -\gamma^{(0)}\Gamma_\mu(x). \tag{5.16}$$

By using the relation (5.13), the expression for the spin connection is written in the form

$$\Gamma_\mu(x) = \frac{1}{8}\omega_\mu{}^{ab}[\gamma_a, \gamma_b].$$

Now the flat spacetime Dirac equation $(i\gamma^a\partial_a - m)\psi = 0$ is generalized to

$$(i\gamma^a e_a^\mu (\partial_\mu + \Gamma_\mu) - m)\psi = 0. \tag{5.17}$$

Introduce curved spacetime Dirac matrices

$$\gamma^\mu = \gamma^a e_a^\mu,$$

and the covariant derivative for spinor fields

$$\nabla_\mu = \partial_\mu + \Gamma_\mu,$$

the Dirac equation in curved spacetime is written as

$$(i\gamma^{\mu}\nabla_{\mu}-m)\psi=0. \tag{5.18}$$

It may be shown that

$$\nabla_{\mu}\gamma^{\nu}=0.$$

Now by taking into account (5.14), this implies $\nabla_{\mu}\gamma^{a}=0$. In addition, by using (5.15), we see that

$$\gamma^{\mu+}=\gamma^{(0)}\gamma^{\mu}\gamma^{(0)}. \tag{5.19}$$

Dirac matrices obey the anticommutation relations

$$\{\gamma^{\mu},\gamma^{\nu}\}=\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}.$$

In a $(D+1)$-dimensional spacetime the Dirac matrices are $N\times N$ matrices with $N=2^{[(D+1)/2]}$, where the square brackets mean the integer part of the enclosed expression. Consequently, ψ is a N-component field.

In odd dimensional spacetimes (D is an even number) the mass term breaks C-invariance in $D=4n$, P-invariance in $D=4n,4n+2$, and T-invariance in $D=4n+2$ (with n being an integer, for a general discussion see Ref. [29]). In odd dimensions the flat spacetime γ^{D} matrix can be represented by other gamma matrices in the following way,

$$\gamma^{D}=\gamma^{D}_{\pm}=\begin{cases}\pm\gamma, & D=4n,\\ \pm i\gamma, & D=4n+2,\end{cases} \tag{5.20}$$

where $\gamma=\gamma^{0}\gamma^{1}\cdots\gamma^{D-1}$. Hence, the Clifford algebra in odd dimensions has two inequivalent representations corresponding to the upper and lower signs in Eq. (5.20).

For the commutator of covariant derivatives acting on the Dirac spinor we have the relation

$$[\nabla_{\mu},\nabla_{\nu}]\psi=-\frac{1}{8}R_{\mu\nu}{}^{ab}[\gamma_{a},\gamma_{b}]\psi, \tag{5.21}$$

where

$$R_{\mu\nu}{}^{a}{}_{b}=\partial_{\nu}\omega_{\mu}{}^{a}{}_{b}-\partial_{\mu}\omega_{\nu}{}^{a}{}_{b}+\omega_{\nu}{}^{a}{}_{c}\omega_{\mu}{}^{c}{}_{b}-\omega_{\mu}{}^{a}{}_{c}\omega_{\nu}{}^{c}{}_{b}.$$

The latter is related to the Riemann tensor by $R_{\mu\nu}{}^{a}{}_{b}=e^{a}_{\lambda}e^{\sigma}_{b}R^{\lambda}{}_{\sigma\mu\nu}$. Hence, we can also write

$$[\nabla_{\mu},\nabla_{\nu}]\psi=-\frac{1}{8}R_{\lambda\sigma\mu\nu}[\gamma^{\lambda},\gamma^{\sigma}]\psi. \tag{5.22}$$

By using this relation we can show that from here it follows that

$$\left(\gamma^{\mu}\nabla_{\mu}\right)^{2}\psi=\left(\nabla_{\mu}\nabla^{\mu}+R/4\right)\psi. \tag{5.23}$$

Indeed, by taking into account the anticommutation relations for the Dirac matrices one gets

$$\left(\gamma^{\mu}\nabla_{\mu}\right)^{2}\psi=\gamma^{\mu}\gamma^{\nu}\nabla_{\mu}\nabla_{\nu}\psi=\nabla_{\mu}\nabla^{\mu}\psi+\frac{1}{8}R_{\lambda\sigma\mu\nu}\gamma^{\nu}\gamma^{\mu}\gamma^{\lambda}\gamma^{\sigma}\psi$$

By using the cyclic identity (4.3) it can be seen that

$$R_{\lambda\sigma\mu\nu}\gamma^{\nu}\gamma^{\mu}\gamma^{\lambda}\gamma^{\sigma}=-R_{\mu\nu\lambda\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}=2R.$$

This leads to the relation (5.23).

Let us act on the Dirac equation (5.18) by the operator $-i\gamma^{\nu}\nabla_{\nu}-m$. This gives

$$\left(-i\gamma^{\nu}\nabla_{\nu}-m\right)\left(i\gamma^{\mu}\nabla_{\mu}-m\right)\psi=\left(\left(\gamma^{\mu}\nabla_{\mu}\right)^{2}+m^{2}\right)\psi=0.$$

By taking into account (5.23), we get the second order equation

$$\left(\nabla_\mu \nabla^\mu + R/4 + m^2\right)\psi = 0. \tag{5.24}$$

The Dirac adjoint spinor in curved spacetime is defined as

$$\bar{\psi} = \psi^+ \gamma^{(0)}, \tag{5.25}$$

with the flat spacetime matrix $\gamma^{(0)}$. Now, taking the hermitian conjugate of (5.18) and using the relations (5.16) and (5.19), we obtain the equation for the Dirac adjoint:

$$i\left(\bar{\nabla}_\mu \bar{\psi}\right)\gamma^\mu + m\bar{\psi} = 0, \tag{5.26}$$

where the corresponding covariant derivative is defined as

$$\bar{\nabla}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi}\Gamma_\mu. \tag{5.27}$$

The action for a Dirac spinor field is given by the expression

$$S[\psi] = \int d^{D+1}x\sqrt{|g|}\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu \nabla_\mu \psi - \left(\nabla_\mu \bar{\psi}\right)\gamma^\mu \psi\right) - m\bar{\psi}\psi\right]. \tag{5.28}$$

For the current density one has $j^\mu = \bar{\psi}\gamma^\mu\psi$. The metric energy-momentum tensor takes the form

$$T_{\mu\nu} = \frac{i}{2}\left[\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi - \left(\nabla_{(\mu}\bar{\psi}\right)\gamma_{\nu)}\psi\right], \tag{5.29}$$

where the braces in the index expression mean the symmetrization over the indices enclosed. Now, by using the Dirac equation it can be seen that $\nabla_\mu j^\mu = 0$ and $\nabla_\nu T^{\mu\nu} = 0$.

Let us consider a conformal transformation

$$\widetilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}.$$

Under this transformation $(D+1)$-bein is transformed as

$$\widetilde{e}^a_\mu(x) = \Omega(x) e^a_\mu(x), \quad \widetilde{e}^\mu_a(x) = \Omega^{-1}(x) e^\mu_a(x).$$

From here we obtain the transformation rule for the gamma matrices

$$\widetilde{\gamma}^\mu = \Omega^{-1}(x)\gamma^\mu.$$

For the connection (5.13) the transformation law is given by

$$\widetilde{\omega}_\mu{}^a{}_b = \omega_\mu{}^a{}_b + \Omega^{-1}\partial_\nu\Omega(e^a_\mu e^\nu_b - e^{a\nu} e_{b\mu}).$$

Assuming that the spinor is transformed as

$$\widetilde{\psi} = \Omega^\beta(x)\psi,$$

one finds

$$i\widetilde{\gamma}^\mu \widetilde{\nabla}_\mu \widetilde{\psi} = \Omega^{\beta-1}\left[i\gamma^\mu \nabla_\mu \psi + i(\beta + D/2)\Omega^{-1}\gamma^\mu \partial_\mu \Omega \psi\right].$$

From here it follows that if β is chosen as $\beta = -D/2$,

$$\widetilde{\psi} = \Omega^{-D/2}(x)\psi,$$

the action (5.28) is conformally invariant for a massless spinor field.

Chapter 6

Quantization of fields in curved backgrounds

6.1 Canonical quantization

There are four basic ingredients in the construction of a quantum field theory. These are

1. The Lagrangian, or equivalently, the equation of motion of the classical theory.
2. A quantization procedure, such as canonical quantization or the path integral approach.
3. The characterization of the quantum states.
4. The physical interpretation of the states and of the observables.

In flat spacetime, Lorentz invariance plays an important role in each of these steps. For example, it is a guide which generally allows us to identify a unique vacuum state for the theory. However, in curved spacetime the Lorentz symmetry is absent. This is not a crucial problem in the first two steps listed above. The formulation of a classical field theory and its formal quantization may be carried through in an arbitrary spacetime. The real differences between flat space and curved space arise in the latter two steps. In general, there does not exist a unique vacuum state in a curved spacetime. As a result, the concept of particles becomes ambiguous, and the problem of the physical interpretation becomes much more difficult.

Formally, field quantization procedure in curved spacetime is similar to that for the Minkowski spacetime case. The best way to discuss these issues in more detail is in the context of a particular model theory. Let us consider a real, massive scalar field for which the action functional and the field equation are given by (5.1) and (5.2). The scalar product for two solutions of the field equation is generalized to

$$(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma^\mu \sqrt{|g_\Sigma|} \left[\phi_1(x)\partial_\mu \phi_2^*(x) - (\partial_\mu \phi_1(x))\, \phi_2^*(x)\right], \tag{6.1}$$

where Σ is a spacelike hypersurface, $d\Sigma^\mu = n^\mu d\Sigma$ with n^μ being the future-directed unit vector normal to Σ. One can show that the scalar product is independent of Σ. Indeed, from the field equation it follows that

$$\nabla_\mu \left(\phi_2^* \nabla^\mu \phi_1 - \phi_1 \nabla^\mu \phi_2^*\right) = 0.$$

Integrating this equation over the region between two spatial hypersurfaces Σ_1 and Σ_2 and using the Stoke's theorem we get

$$\int_\Sigma d^D x \sqrt{|h|} n_\mu \left(\phi_2^* \nabla^\mu \phi_1 - \phi_1 \nabla^\mu \phi_2^*\right) |_{\Sigma=\Sigma_1}^{\Sigma=\Sigma_2} = 0,$$

where n_μ is the future directed normal to both the hypersurfaces and h is the determinant of the induced metric $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. Hence, we see that

$$\int_{\Sigma_1} d^D x \, \sqrt{|h|} n^\mu \left(\phi_2^* \nabla_\mu \phi_1 - \phi_1 \nabla_\mu \phi_2^*\right) = \int_{\Sigma_2} d^D x \, \sqrt{|h|} n^\mu \left(\phi_2^* \nabla_\mu \phi_1 - \phi_1 \nabla_\mu \phi_2^*\right).$$

For the geometries with $g_{0k} = 0$, $k = 1, \ldots, D$, taking the hypersurface $t = \text{const}$ one has $n^\mu = (n^0, 0, \ldots, 0)$, $g_{00}(n^0)^2 = 1$. In this special case the scalar product takes the form

$$(\phi_1, \phi_2) = -i \int_\Sigma d^D x \, \sqrt{|g|} g^{00} \left(\phi_1 \partial_0 \phi_2^* - \phi_2^* \partial_0 \phi_1\right). \tag{6.2}$$

In the canonical quantization scheme, the first step is to construct a complete set of mode functions $\{\phi_i(x), \phi_i^*(x)\}$ for the classical field equation (5.2) obeying the orthonormalization conditions:

$$(\phi_i, \phi_j) = \delta_{ij}, \quad (\phi_i^*, \phi_j^*) = -\delta_{ij}, \quad (\phi_i, \phi_j^*) = 0.$$

The index i stands for the set of quantum numbers labeling the modes. The symbol δ_{ij} is understood as the Kronecker delta for discrete quantum numbers and as the Dirac delta function for continuous ones. We expand the field operator in the series

$$\phi = \sum_i [a_i \phi_i(x) + a_i^+ \phi_i^*(x)], \tag{6.3}$$

with the operator coefficients

$$a_i = (\phi, \phi_i), \quad a_i^+ = -(\phi, \phi_i^*). \tag{6.4}$$

The quantization proceeds in close analogy to the Minkowskian case and is implemented by adopting the commutation relations

$$[a_i, a_j^+] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^+, a_j^+] = 0.$$

The further construction of Fock space of states is the same as described for the Minkowski space. However, in curved spacetime there is an ambiguity in the choice of a complete set of modes for the expansion of the field operator. In Minkowski space, the natural set of modes are associated with the Cartesian coordinates x^μ with the line element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. These coordinates are associated with the Poincaré group, the action of which leaves the Minkowski line element unchanged. The corresponding vacuum state is invariant under the action of the Poincaré group.

In curved spacetime, in general, the choice of the modes ϕ_i is not unique. As a consequence, there is no unique notion of the vacuum state and the notion of "particle" becomes ambiguous. One possible resolution of this difficulty is to choose some quantities other than particle content to label quantum states. Possible choices might include local expectation values, such as $\langle\phi\rangle$, $\langle\phi^2\rangle$, etc. In the particular case of an asymptotically flat spacetime, the particle concept can be used in asymptotic regions. Even this characterization is not unique. This non-uniqueness is an essential feature of the theory with physical consequences, namely the phenomenon of particle creation.

The generalization of the quantization procedures for the electromagnetic field and for a Dirac fermionic field is done in a similar way. Here we consider the normalization conditions for the corresponding mode functions. For the electromagnetic field the vector potential obeys the equation (5.8). By using this equation, we can see that for two solutions, A_1^μ and A_2^μ, the following relation takes place

$$\nabla_\mu [A_{2\nu}^* \nabla^\mu A_1^\nu - (\nabla^\mu A_2^{\nu*}) A_{1\nu} + (\nabla_\nu A_2^{\nu*}) A_1^\mu - A_2^{\mu*} \nabla_\nu A_1^\nu] = 0.$$

In a way similar to that for a scalar field, from here it follows that the integral

$$\int_\Sigma d^D x \, \sqrt{|h|} n_\mu [A_{2\nu}^* \nabla^\mu A_1^\nu - (\nabla^\mu A_2^{\nu*}) A_{1\nu} + (\nabla_\nu A_2^{\nu*}) A_1^\mu - A_2^{\mu*} \nabla_\nu A_1^\nu]$$

does not depend on the choice of the spatial hypersurface Σ and can be used for the definition of the invariant scalar product for the vector field. The integral is further simplified in the gauge $\nabla_\mu A^\mu = 0$:

$$\int_\Sigma d^D x \sqrt{|h|} n_\mu [A^*_{2\nu} \nabla^\mu A^\nu_1 - (\nabla^\mu A^{\nu *}_2) A_{1\nu}].$$

Let $\{A^\mu_{(j)}, A^{\mu *}_{(j)}\}$ is a complete set of mode functions obeying the classical field equation for the vector field and specified by a set of quantum numbers j. Then the corresponding orthonormalization condition in the gauge $\nabla_\mu A^\mu_{(j)} = 0$ is given by

$$\int_\Sigma d^D x \sqrt{|h|} n^\mu [A^*_{(j')\nu} \nabla_\mu A^\nu_{(j)} - (\nabla_\mu A^{\nu *}_{(j')}) A_{(j)}] = 4i\pi \delta_{jj'}. \tag{6.5}$$

In the special case $g_{0k} = 0$, $k = 1, \ldots, D$, identifying Σ with the hypersurface $t = \mathrm{const}$, this condition is simplified to

$$\int d^D x \sqrt{|g|} g^{00} [A^*_{(j')\nu} \nabla_0 A^\nu_{(j)} - (\nabla_0 A^{\nu *}_{(j')}) A_{(j)}] = 4i\pi \delta_{jj'},$$

with $d^D x = dx^1 \cdots dx^D$.

Now let us turn to the normalization condition for fermionic fields. By using the equations (5.18) and (5.26), we can show that for two solutions ψ_1 and ψ_2 of the Dirac equation one has

$$\nabla_\mu \left(\bar{\psi}_2 \gamma^\mu \psi_1 \right) = 0.$$

As a consequence, the integral

$$\int_\Sigma d^D x \sqrt{|h|} n_\mu \bar{\psi}_2 \gamma^\mu \psi_1$$

is independent of the choice for the spatial hypersurface Σ. On the base of this, for a complete set of the fermionic modes $\psi^{(\pm)}_j$ the orthonormalization condition is in the form

$$\int_\Sigma d^D x \sqrt{|h|} n_\mu \bar{\psi}^{(\lambda')}_{j'} \gamma^\mu \psi^{(\lambda)}_j = \delta_{\lambda\lambda'} \delta_{jj'}. \tag{6.6}$$

For background geometries with $g_{0k} = 0$, $k = 1, \ldots, D$, as the hypersurface Σ we can take the hypersurface $t = \mathrm{const}$. By taking into account that in the special case under consideration $\gamma^0 = \gamma^{(0)}/\sqrt{g_{00}}$, the condition takes the form

$$\int d^D x \sqrt{|g|/g_{00}} \psi^{(\lambda')+}_{j'} \psi^{(\lambda)}_j = \delta_{\lambda\lambda'} \delta_{jj'}.$$

Here we have taken into account that $\bar{\psi} = \psi^+ \gamma^{(0)}$.

6.2 Bogoliubov transformations

In order to see the relation between two quantization schemes based on two different sets of mode functions, in addition to the modes $\{\phi_i(x), \phi^*_i(x)\}$, consider a second complete orthonormal set of mode functions $\left\{\bar{\phi}_j(x), \bar{\phi}^*_j(x)\right\}$. The corresponding expansion for the field operator reads

$$\phi = \sum_j [\bar{a}_j \bar{\phi}_j(x) + \bar{a}^+_j \bar{\phi}^*_j(x)].$$

This decomposition defines a new vacuum state $|\bar{0}\rangle$:

$$\bar{a}_j |\bar{0}\rangle = 0$$

and a new Fock space constructed by acting on the vacuum state by the creation operators $\bar{a}_j^+$.

Both sets of modes are complete and we can write the expansion

$$\bar{\phi}_j = \sum_i \left(\alpha_{ji}\phi_i + \beta_{ji}\phi_i^*\right), \tag{6.7}$$

for the new modes in terms of the old ones. Assuming that both the sets are defined in the same spacetime region and using the orthonormalization relations for the mode functions, we see that

$$\begin{aligned}
\left(\bar{\phi}_j, \phi_l\right) &= \sum_i \left(\alpha_{ji}\left(\phi_i, \phi_l\right) + \beta_{ji}\left(\phi_i^*, \phi_l\right)\right) = \sum_i \alpha_{ji}\delta_{il} = \alpha_{jl}, \\
\left(\bar{\phi}_j, \phi_l^*\right) &= \sum_i \left(\alpha_{ji}\left(\phi_i, \phi_l^*\right) + \beta_{ji}\left(\phi_i^*, \phi_l^*\right)\right) = -\sum_i \beta_{ji}\delta_{il} = -\beta_{jl}.
\end{aligned}$$

Consequently, the coefficients in the expansion (6.7) are given by the expressions:

$$\alpha_{ji} = \left(\bar{\phi}_j, \phi_i\right), \quad \beta_{ji} = -\left(\bar{\phi}_j, \phi_i^*\right).$$

In a similar way we may write

$$\phi_i = \sum_j \left(\bar{\alpha}_{ij}\bar{\phi}_j + \bar{\beta}_{ij}\bar{\phi}_j^*\right), \tag{6.8}$$

with the coefficients

$$\begin{aligned}
\left(\phi_i, \bar{\phi}_l\right) &= \sum_j \left(\bar{\alpha}_{ij}\left(\bar{\phi}_j, \bar{\phi}_l\right) + \bar{\beta}_{ij}\left(\bar{\phi}_j^*, \bar{\phi}_l\right)\right) = \bar{\alpha}_{il}, \\
\left(\phi_i, \bar{\phi}_l^*\right) &= \sum_j \left(\bar{\alpha}_{ij}\left(\bar{\phi}_j, \bar{\phi}_l^*\right) + \bar{\beta}_{ij}\left(\bar{\phi}_j^*, \bar{\phi}_l^*\right)\right) = -\bar{\beta}_{il},
\end{aligned}$$

and

$$\bar{\alpha}_{ij} = \left(\phi_i, \bar{\phi}_j\right), \quad \bar{\beta}_{ij} = -\left(\phi_i, \bar{\phi}_j^*\right).$$

By taking into account that

$$\begin{aligned}
\left(\phi_i, \bar{\phi}_j\right) &= -\left(\bar{\phi}_j^*, \phi_i^*\right) = \left(\bar{\phi}_j, \phi_i\right)^*, \\
\left(\phi_i, \bar{\phi}_j^*\right) &= -\left(\bar{\phi}_j, \phi_i^*\right),
\end{aligned}$$

we get the relations

$$\bar{\alpha}_{ij} = \alpha_{ji}^*, \quad \bar{\beta}_{ij} = -\beta_{ji}.$$

Hence, we can write

$$\phi_i = \sum_j \left(\alpha_{ji}^*\bar{\phi}_j - \beta_{ji}\bar{\phi}_j^*\right). \tag{6.9}$$

Inserting this expansion (6.7) into the orthogonality relation $(\bar{\phi}_j, \bar{\phi}_l) = \delta_{jl}$, we can see that

$$\sum_i (\alpha_{ji}\alpha_{li}^* - \beta_{ji}\beta_{li}^*) = \delta_{jl}.$$

In a similar way, from the relation $(\bar{\phi}_j, \bar{\phi}_l^*) = 0$ it follows that

$$\sum_i (\alpha_{ji}\beta_{li} - \beta_{ji}\alpha_{li}) = 0.$$

Now let us consider the relations between the annihilation and creation operators in two constructions based on the modes sets $\{\phi_i(x), \phi_i^*(x)\}$ and $\left\{\bar{\phi}_j(x), \bar{\phi}_j^*(x)\right\}$. By taking into account (6.4), we may expand the two sets of creation and annihilation operators in terms of one another as

$$\begin{aligned} a_l &= (\phi, \phi_l) = \sum_j \left([\bar{a}_j \bar{\phi}_j + \bar{a}_j^+ \bar{\phi}_j^*], \phi_l \right) = \sum_j \bar{a}_j \left(\bar{\phi}_j, \phi_l \right) + \sum_j \bar{a}_j^+ \left(\bar{\phi}_j^*, \phi_l \right) \\ &= \sum_j \left(\bar{a}_j \alpha_{jl} + \bar{a}_j^+ \beta_{jl}^* \right), \end{aligned}$$

$$\begin{aligned} \bar{a}_l &= (\phi, \bar{\phi}_l) = \sum_j \left([a_j \phi_j + a_j^+ \phi_j^*], \bar{\phi}_l \right) = \sum_j a_j \left(\phi_j, \bar{\phi}_l \right) + \sum_j a_j^+ \left(\phi_j^*, \bar{\phi}_l \right) \\ &= \sum_j a_j \alpha_{lj}^* - \sum_j a_j^+ \left(\phi_j, \bar{\phi}_l^* \right)^* = \sum_j \left(a_j \alpha_{lj}^* - a_j^+ \beta_{lj}^* \right), \end{aligned}$$

and, hence,

$$\begin{aligned} a_l &= \sum_j \left(\bar{a}_j \alpha_{jl} + \bar{a}_j^+ \beta_{jl}^* \right), \\ \bar{a}_l &= \sum_i \left(a_i \alpha_{li}^* - a_i^+ \beta_{li}^* \right). \end{aligned} \tag{6.10}$$

Here we have used the relations

$$\alpha_{ji} = \left(\bar{\phi}_j, \phi_i \right), \quad \beta_{ji} = - \left(\bar{\phi}_j, \phi_i^* \right).$$

The relations (6.10) between two sets of annihilation and creation operators are Bogoliubov transformation, and the coefficients α_{ji} and β_{ji} are called the Bogoliubov coefficients.

Based on two sets of modes we have defined two vacuum states, $|0\rangle$ and $|\bar{0}\rangle$, and based on them two different sets of Fock spaces. In order to see the relation between these two sets let us consider the action of the old annihilation operator on the new vacuum state $|\bar{0}\rangle$. One has

$$a_i \, |\bar{0}\rangle = \sum_j \left(\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^+ \right) |\bar{0}\rangle = \sum_j \beta_{ji}^* \, |\bar{1}_j\rangle \neq 0.$$

This shows that if $\beta_{ji} \neq 0$ then the state $|\bar{0}\rangle$ is not a vacuum state for the modes $\{\phi_i(x), \phi_i^*(x)\}$. As a consequence, the Fock spaces based on the two choices of modes $\{\phi_i(x), \phi_i^*(x)\}$ and $\left\{\bar{\phi}_j(x), \bar{\phi}_j^*(x)\right\}$ are different so long as $\beta_{ji} \neq 0$. For the expectation value of the number of particles we get

$$\langle \bar{0} | \, N_i \, |\bar{0}\rangle = \langle \bar{0} | \, a_i^+ a_i \, |\bar{0}\rangle = \sum_{j,l} \beta_{li} \beta_{ji}^* \left\langle \bar{1}_l | \bar{1}_j \right\rangle = \sum_j |\beta_{ji}|^2.$$

This shows that the vacuum of the $\bar{\phi}_j$ modes contains $\sum_j |\beta_{ji}|^2$ particles in the ϕ_i mode.

If the background geometry possesses a timelike Killing vector η, then we can define the positive-frequency (energy) modes ϕ_i for the frequency by the relation

$$\mathcal{L}_\eta \phi_i = -i\omega\phi_i,$$

where $\mathcal{L}_\eta$ stands for the Lie derivative along the direction η. For a static metric $\mathcal{L}_\eta = \partial_t$. The mode functions ϕ_i^* correspond to the negative-frequency modes. Now we see that if $\beta_{ji} = 0$ then the transformations (6.8) and (6.9) between two sets of mode functions do not mix the positive- and negative-frequency modes. In Minkowski spacetime, under Poincaré transformations positive frequency solutions transform to positive frequency solutions and so the concept of particle is the same for inertial observers: all inertial observers agree on the number of particles present. Further the Minkowski vacuum state, defined as the state with no particles present, is invariant under the Poincaré group.

6.3 Notion of particles: Particle detectors

The particles are defined as states of a quantum field obtained from the vacuum state acting by the creation operator. For example a particle carrying the set of quantum numbers i is described by the state $|1_i\rangle = a_i^+ |0\rangle$. The definition of the vacuum state is based on the choice of a complete set of mode functions $\{\phi_i(x), \phi_i^*(x)\}$. The latter is sensitive to both the local and global properties of the background geometry. For example, the mode functions for the background spacetimes having the topologies $R \times R^D$ and $R \times R^{D-1} \times S^1$ are different, though both these spacetimes are flat. Hence, the notions of the vacuum and particle are global. When we speak about the presence or absence of particles, it is necessary to specify the details of the measurement process for detection. Particles may be registered by some detectors but not by others. Even in the Minkowski spacetime, the concept of particle is ambiguous if we do not specify the state of motion of the detector. More objective probes of the state are given by locally defined quantities having a tensorial nature, such as $\langle\psi|\, \phi^2(x)\, |\psi\rangle$ and $\langle\psi|\, T_{\mu\nu}(x)\, |\psi\rangle$. For these characteristics, the outcome of different measuring devices are related by the usual tensor transformation. For example, if $\langle\psi|\, T_{\mu\nu}(x)\, |\psi\rangle = 0$ for one observer, it will vanish for all observers.

6.3.1 Unruh-DeWitt detector

In our discussion we will use a model of a particle detector due to Unruh and De Witt. It consists of an idealized point particle having internal energy levels labelled by E, and coupled with a scalar field ϕ via a monopole interaction. Let the worldline of the detector is given by the functions $x^\mu = x^\mu(\tau)$ with τ being the detector proper time. The part of the Lagrangian describing the interaction of the detector with the scalar field is given by the expression

$$\mathcal{L}_{\text{int}} = cm(\tau)\phi(x^\mu(\tau)),$$

where c is a coupling constant and $m(\tau)$ is the detector's monopole momentum operator. The evolution of $m(\tau)$ is governed by the the Hamilton operator for the detector, H_D, and is given by

$$m(\tau) = e^{iH_D\tau} m(0) e^{-iH_D\tau}.$$

At any given time, the interaction takes place at a point along the trajectory and the detector is called as a point-like detector.

We assume that at initial time τ_0 the detector and field are in the product state $|0, E_0\rangle = |0\rangle\, |E_0\rangle$, where $|E_0\rangle$ is the detector state with energy E_0. We want to know the probability that at a later time $\tau_1 > \tau_0$ the detector is found in state $|E_1\rangle$, regardless of the final state of the field. The cases $E_1 > E_0$ and $E_1 < E_0$ correspond to excitations and de-excitations, respectively. We will work in the interaction picture where all field operators satisfy the free field equations. In this picture the time evolution of the product states is governed by the interaction Hamiltonian ($H_{\text{int}} = -\mathcal{L}_{\text{int}}$)

$$i\frac{d}{d\tau}\, |\psi(\tau)\rangle = -cm(\tau)\phi(x^\mu(\tau))\, |\psi(\tau)\rangle\,.$$

The amplitude for the transition from state $|0, E_0\rangle$ at $\tau = \tau_0$ to state $|\psi, E_1\rangle$ at $\tau = \tau_1$, by the usual interaction picture theory is

$$\langle\psi, E_1|0, E_0\rangle = \langle\psi, E_1|\, T\exp[-i\int_{\tau_0}^{\tau_1} d\tau\, H_{\text{int}}(\tau)]\, |0, E_0\rangle\,,$$

where T is the time ordering operator. Assuming that the interaction is weak (the parameter c is small), to the first order in perturbation theory the expression reads

$$\begin{aligned}\langle\psi, E_1|0, E_0\rangle &= ic\,\langle\psi, E_1|\int_{\tau_0}^{\tau_1} d\tau\, m(\tau)\phi(x^\mu(\tau))]\, |0, E_0\rangle \\ &= ic\,\langle E_1|\, m(0)\, |E_0\rangle \int_{\tau_0}^{\tau_1} d\tau\, e^{i(E_1-E_0)\tau}\, \langle\psi|\, \phi(x^\mu(\tau))\, |0\rangle\,.\end{aligned}$$

The transition probability to all possible states of the field is given by squaring this expression and summing over the complete set $\{|\psi\rangle\}$ of final field states, with the result

$$\sum_{\psi} |\langle\psi, E_1|0, E_0\rangle|^2 = c^2 \, |\langle E_1| \, m(0) \, |E_0\rangle|^2 \int_{\tau_0}^{\tau_1} d\tau \int_{\tau_0}^{\tau_1} d\tau' \, e^{-i(E_1-E_0)(\tau-\tau')} G^+(x^\mu(\tau), x^\mu(\tau')), \tag{6.11}$$

where

$$G^+(x, x') = \langle 0| \, \phi(x)\phi(x') \, |0\rangle$$

is the positive frequency Wightman function for a scalar field $\phi(x)$. Here we have used the relation

$$\langle 0| \, \phi(x)\phi(x') \, |0\rangle = \sum_{\psi} \langle 0| \, \phi(x) \, |\psi\rangle \, \langle\psi| \, \phi(x') \, |0\rangle \, .$$

The expression in the right-hand side of the equation (6.11) contains two parts. The sensitivity $c^2 \, |\langle E_1| \, m(0) \, |E_0\rangle|^2$ depends only on the internal details of the detector. The remaining part, referred as the "response function",

$$F_{\tau_0,\tau_1}(\omega) = \int_{\tau_0}^{\tau_1} d\tau \int_{\tau_0}^{\tau_1} d\tau' \, e^{-i\omega(\tau-\tau')} G^+(x^\mu(\tau), x^\mu(\tau')),$$

where $\omega = E_1 - E_0$ ($\omega > 0$ for excitations and $\omega < 0$ for de-excitations), does not depend on the internal details of the detector and so is common for all such detectors. In the special case $\tau_0 \to -\infty$ and $\tau_1 \to +\infty$ the response function is given by the expression

$$F(\omega) = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \, e^{-i\omega(\tau-\tau')} G^+(x^\mu(\tau), x^\mu(\tau')).$$

In the special case, when the system is invariant under the time translations in the reference frame of the detector ($\tau \to \tau$+const), one has

$$G^+(x^\mu(\tau), x^\mu(\tau')) = G^+(\Delta\tau), \;\; \Delta\tau = \tau - \tau'.$$

This corresponds to the detector in an equilibrium with the field. In this case the number of quanta absorbed or emitted by the detector per unit proper time τ is constant. If the absorbtion or emission rate is nonzero, the transition probability will diverge, as the transition amplitude is computed for an infinite proper time interval. We can consider the transition probability per unit proper time, given by the expression.

$$c^2 \, |\langle E_1| \, m(0) \, |E_0\rangle|^2 \int_{-\infty}^{+\infty} d(\Delta\tau) \; e^{-i(E_1-E_0)\Delta\tau} G^+(\Delta\tau). \tag{6.12}$$

Let us consider examples of the detector motion where the transition probability can be exactly evaluated.

6.3.2 Inertial detector

We start with the case of a massless scalar field in 4-dimensional Minkowski spacetime. For the evaluation of the transition probability we need to have the positive frequency Wightman function. For the latter we have the mode sum

$$G^+(x, x') = \sum_{\alpha} \phi_\alpha(x)\phi^*_\alpha(x').$$

As the mode functions appearing in this expression we take the plane waves with $\alpha = \mathbf{k} = (k_1, k_2, k_3)$ and

$$\phi_{\mathbf{k}}(x) = \frac{e^{-ik\cdot x}}{\sqrt{2(2\pi)^3\omega_k}},\ \omega_k = |\mathbf{k}|.$$

Thi gives

$$\begin{aligned} G^+(x,x') &= \frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\frac{e^{-ik\cdot(x-x')}}{\omega_k} = \frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\frac{e^{-ik(t-t')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega_k} \\ &= \frac{1}{2(2\pi)^2}\int_0^\infty d\omega_k\,\omega_k e^{-i\omega_k(t-t')}\int_0^\pi d\theta\,\sin\theta e^{i\omega_k|\mathbf{x}-\mathbf{x}'|\cos\theta} \\ &= \frac{1}{(2\pi)^2}\int_0^\infty d\omega_k e^{-i\omega_k(t-t'-i\varepsilon)}\frac{\sin(k\,|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}. \end{aligned} \tag{6.13}$$

The integration here can be done explicitly:

$$G^+(x,x') = -\frac{1}{4\pi^2}\left[(t-t'-i\varepsilon)^2 - |\mathbf{x}-\mathbf{x}'|^2\right]^{-1}. \tag{6.14}$$

For an inertial detector moving with the velocity $\mathbf{v}$ one has

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t = \mathbf{x}_0 + \mathbf{v}\tau\gamma, \tag{6.15}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

Substituting into the expression (6.13), one finds

$$G^+(\Delta\tau) = \frac{1}{(2\pi)^2}\int_0^\infty dk\,e^{-k\varepsilon}e^{-ik\gamma\Delta\tau}\frac{\sin(kv\gamma\Delta\tau)}{v\gamma\Delta\tau}.$$

For the integral in (6.12) this gives

$$\begin{aligned} &\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau}G^+(\Delta\tau) = \frac{1}{v\gamma}\int_0^\infty \frac{dk}{(2\pi)^2}\,e^{-k\varepsilon}\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i(\omega+k\gamma)\Delta\tau}\frac{\sin(kv\gamma\Delta\tau)}{\Delta\tau} \\ &= \frac{1}{v\gamma}\int_0^\infty \frac{dk}{2\pi^2}\,e^{-k\varepsilon}\int_0^\infty d(\Delta\tau)\ \cos[(\omega+k\gamma)\Delta\tau]\frac{\sin(kv\gamma\Delta\tau)}{\Delta\tau} \\ &= \frac{1}{8\pi v\gamma}\int_0^\infty dk\,e^{-k\varepsilon}\left[\mathrm{sgn}(\omega+k\gamma+kv\gamma) - \mathrm{sgn}(\omega+k\gamma-kv\gamma)\right]. \end{aligned}$$

If $\omega = E_1 - E_0 > 0$, then the expression in the square brackets is zero and

$$\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau}G^+(\Delta\tau) = 0.$$

Hence, if the detector was in the state with the energy E_0 then it is not excited. No particles are detected.

For $\omega = E_1 - E_0 < 0$ one has

$$\begin{aligned} &\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau}G^+(\Delta\tau) = \frac{1}{8\pi v\gamma}\int_0^\infty dk\,e^{-k\varepsilon} \\ &\times\left[\mathrm{sgn}(\omega + k\sqrt{\frac{1+v}{1-v}}) - \mathrm{sgn}(\omega + k\sqrt{\frac{1-v}{1+v}})\right] \\ &= \frac{1}{4\pi v\gamma}\int_{|\omega|\sqrt{\frac{1-v}{1+v}}}^{|\omega|\sqrt{\frac{1+v}{1-v}}} dk\,e^{-k\varepsilon} = \frac{|\omega|}{4\pi v\gamma}\left(\sqrt{\frac{1+v}{1-v}} - \sqrt{\frac{1-v}{1+v}}\right) = \frac{|\omega|}{2\pi}. \end{aligned}$$

Hence, in this case one gets

$$\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau} G^{+}(\Delta\tau) = \frac{E_0 - E_1}{2\pi},\ E_0 > E_1. \tag{6.16}$$

An alternative approach is based on the expression (6.14) for the positive frequency Wightman function. Let us consider the general case of spatial dimension D. For the corresponding Wightman function one has

$$G^{+}(x, x') = \frac{\Gamma((D-1)/2)}{4e^{i\pi(D-1)/2}\pi^{(D+1)/2}} \left[(t - t' - i\varepsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2\right]^{-(D-1)/2}. \tag{6.17}$$

For an inertial observer with (6.15) this gives

$$G^{+}(\Delta\tau) = \frac{e^{i\pi(1-D)/2}\Gamma((D-1)/2)}{4\pi^{(D+1)/2}(\Delta\tau - i\varepsilon)^{D-1}}.$$

and, hence,

$$\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau} G^{+}(\Delta\tau) = \frac{\Gamma((D-1)/2)}{4\pi^{(D+1)/2}e^{i\pi(D-1)/2}} \int_{-\infty}^{+\infty} dx\ \frac{e^{-i\omega x}}{(x - i\varepsilon)^{D-1}}.$$

For $\omega > 0$ we close the integration contour by the semicircle in the lower half-plane and by the Cauchy theorem the integral is zero. In the case $\omega < 0$, the contour is closed in the upper half-plane and we use the residue theorem:

$$\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau} G^{+}(\Delta\tau) = \frac{\Gamma((D-1)/2)}{4\pi^{(D+1)/2}e^{i\pi(D-1)/2}} \frac{2\pi i}{\Gamma(D-1)} \lim_{x\to 0} \frac{d^{D-2}}{dx^{D-2}} e^{-i\omega x}.$$

The final result is given by the formula

$$\int_{-\infty}^{+\infty} d(\Delta\tau)\ e^{-i\omega\Delta\tau} G^{+}(\Delta\tau) = \frac{\pi^{1-D/2}|\omega|^{D-2}}{2^{D-1}\Gamma(D/2)}.$$

In the special case $D = 3$ this coincides with (6.16).

We see that the inertial detector will not be excited if the field is in the Minkowski vacuum state. For an excited detector, the probability, per unit proper time, to de-excite is given by the expression (in $D = 3$ spatial dimensions)

$$\frac{E_0 - E_1}{2\pi} c^2 \left|\langle E_1 |\, m(0)\, | E_0 \rangle\right|^2. \tag{6.18}$$

Recall that this result is obtained on the base of perturbation theory and the parameter c is assumed to be small.

6.3.3 Uniformly accelerated detector

As another example of the equilibrium case for the interaction of a detector with a quantum field, we consider a detector with uniform proper acceleration α^{-1}. It moves along a hyperbolic trajectory (see Chapter 10 below)

$$x^1 = \sqrt{t^2 + \alpha^2}, \quad x^l = 0, \quad \alpha = \text{const},$$

with $l = 2, 3, \ldots, D$. In this discussion we keep the number of spatial dimensions arbitrary. For the detector's proper time one has

$$d\tau^2 = dt^2 - (dx^1)^2, \quad d\tau = \frac{\alpha dt}{\sqrt{t^2 + \alpha^2}},$$

and the worldline is given parametrically as

$$t = \alpha \sinh(\tau/\alpha), \quad x^1 = \alpha \cosh(\tau/\alpha), \quad x^l = 0. \tag{6.19}$$

As before, we assume that the field $\phi(x)$ is in the Minkowskian vacuum state. For the corresponding positive frequency Wightman function we have the expression (6.17). The factor in the transition probability per unit proper time, depending on the field state is determined by

$$F(\Omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} d\tau \int_{-T}^{T} d\tau' \, e^{-i\Omega(\tau-\tau')} G^+(x^\mu(\tau), x^\mu(\tau')).$$

Plugging the Wightman function and the worldline functions (6.19) we find

$$\begin{aligned}
F(\Omega) &= \frac{e^{i\pi(1-D)/2}\Gamma((D-1)/2)}{4\pi^{(D+1)/2}\alpha^{D-1}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau \int_{-T}^{T} d\tau' \, e^{-i\Omega\Delta\tau} \\
&\quad \times \left[(\sinh(\tau/\alpha) - \sinh(\tau'/\alpha) - i\varepsilon/\alpha)^2 - |\cosh(\tau/\alpha) - \cosh(\tau'/\alpha)|^2\right]^{(1-D)/2} \\
&= \frac{e^{i\pi(1-D)/2}\Gamma((D-1)/2)}{2^{D+1}\pi^{(D+1)/2}\alpha^{D-1}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau \\
&\quad \times \int_{-T}^{T} d\tau' \, \frac{e^{-i\Omega\Delta\tau}}{\left[(\sinh x \cosh y - i\varepsilon/2\alpha)^2 - \sinh^2 x \sinh^2 y\right]^{(D-1)/2}} \\
&= \frac{e^{i\pi(1-D)/2}\Gamma((D-1)/2)}{2^{D+1}\pi^{(D+1)/2}\alpha^{D-1}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau \int_{-T}^{T} d\tau' \, \frac{e^{-i\Omega\Delta\tau}}{\sinh^{D-1}(x - i\epsilon/\alpha)},
\end{aligned}$$

with infinitesimally small $\epsilon > 0$ related to ε and

$$x = \frac{\Delta\tau}{2\alpha}, \quad y = \frac{\tau+\tau'}{2\alpha}.$$

As is seen, the integrand depends on τ and τ' in the form of $\Delta\tau$. This shows that the uniformly accelerated detector is in an equilibrium with the field.

Passing to new integration variables

$$\Delta\tau = \tau - \tau', \quad \tau_2 = \tau + \tau',$$

we can see that

$$F(\Omega) = \frac{e^{i\pi(1-D)/2}\Gamma((D-1)/2)}{2^D\pi^{(D+1)/2}\alpha^{D-2}} \int_{-\infty}^{+\infty} dx \, \frac{e^{-2i\alpha\Omega x}}{\sinh^{D-1}(x - i\epsilon/\alpha)}. \tag{6.20}$$

We consider two separate cases.

For $\Omega > 0$ we close the integration contour by the semicircle of large radius in the lower half-plane. The integrand has poles at

$$x = -i\pi n, \quad n = 1, 2, \ldots.$$

By the residue theorem

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx \, \frac{e^{-2i\alpha\Omega x}}{\sinh^{D-1}(x - i\epsilon/\alpha)} &= -\frac{2\pi i}{\Gamma(D-1)} \sum_{n=1}^{\infty} \lim_{x\to -i\pi n} \frac{d^{D-2}}{dx^{D-2}} \frac{(x+i\pi n)^{D-1} e^{-2i\alpha\Omega x}}{\sinh^{D-1} x} \\
= -\frac{2\pi i}{\Gamma(D-1)} \sum_{n=1}^{\infty} e^{-2\pi n\alpha\Omega} \lim_{y\to 0} \frac{d^{D-2}}{dy^{D-2}} \frac{y^{D-1} e^{-2i\alpha\Omega y}}{\sinh^{D-1}(y - i\pi n)} \\
= -\frac{2\pi i}{\Gamma(D-1)} \sum_{n=1}^{\infty} (-1)^{n(D-1)} e^{-2\pi n\alpha\Omega} \lim_{y\to 0} \frac{d^{D-2}}{dy^{D-2}} \frac{y^{D-1} e^{-2i\alpha\Omega y}}{\sinh^{D-1} y}.
\end{aligned}$$

For the series in this expression one has

$$\sum_{n=1}^{\infty}(-1)^{n(D-1)}e^{-2\pi n\alpha\Omega} = \frac{(-1)^{D-1}}{e^{2\pi\alpha\Omega}+(-1)^D}.$$

Introducing the notation

$$A_D(u) = -e^{i\pi D/2}\lim_{y\to 0}\frac{d^{D-2}}{dy^{D-2}}\frac{y^{D-1}e^{-2iuy}}{\sinh^{D-1}y},$$

for the function $F(\Omega)$ we find the representation

$$F(\Omega) = \frac{2^{3-2D}\pi^{1-D/2}}{\alpha^{D-2}\Gamma(D/2)}\frac{A_D(\alpha\Omega)}{e^{2\pi\alpha\Omega}+(-1)^D}. \tag{6.21}$$

For separate values of D one has

$$A_3(u) = 2u, \quad A_4(u) = 1+4u^2, \quad A_5(u) = 8u\left(1+u^2\right).$$

In particular, for $D=3$ the expression takes the form

$$F(\Omega) = \frac{1}{2\pi}\frac{\Omega}{e^{2\pi\alpha\Omega}-1}. \tag{6.22}$$

Now we turn to the case $\Omega < 0$. In this case we close the integration contour of (6.20) in the upper half-plane. The integrand has poles at $x = i\pi n$, $n = 1, 2, \ldots$, and at $x = i\epsilon/\alpha$. By the calculations similar to the previous case we can see that

$$\begin{aligned} F(\Omega) &= -\frac{\Gamma((D-1)/2)}{2^D\pi^{(D+1)/2}\alpha^{D-2}}\frac{2\pi}{\Gamma(D-1)}\sum_{n=0}^{\infty}(-1)^{n(D-1)} \\ &\quad\times e^{-2\pi n\alpha|\Omega|}e^{-i\pi D/2}\lim_{y\to 0}\frac{d^{D-2}}{dy^{D-2}}\frac{y^{D-1}e^{2i\alpha|\Omega|y}}{\sinh^{D-1}y}. \end{aligned}$$

The final result is given by the formula

$$F(\Omega) = \frac{2^{3-2D}\pi^{1-D/2}}{\Gamma(D/2)\alpha^{D-2}}A_D(\alpha|\Omega|)\left[1+\frac{(-1)^{D-1}}{e^{2\pi\alpha|\Omega|}+(-1)^D}\right]. \tag{6.23}$$

For the special case $D=3$ we find

$$F(\Omega) = \frac{|\Omega|}{2\pi}\left(1+\frac{1}{e^{2\pi\alpha|\Omega|}-1}\right).$$

In this case, for the transition probability per unit proper time we have:

$$c^2\sum_E|\langle E|\,m(0)\,|E_0\rangle|^2\int_{-\infty}^{+\infty}d(\Delta\tau)\;e^{-i(E-E_0)\Delta\tau}G^+(\Delta\tau) = \frac{c^2}{2\pi}\sum_E(E-E_0)\frac{|\langle E|\,m(0)\,|E_0\rangle|^2}{e^{2\pi(E-E_0)\alpha}-1}.$$

The appearance of the Planck factor $[e^{2\pi(E-E_0)\alpha}-1]^{-1}$ indicates that the equilibrium between the accelerated detector and the field in the state $|0_M\rangle$ is the same as that which would have been achieved had the detector remained unaccelerated, but immersed in a bath of thermal radiation at the temperature

$$T = 1/(2\pi\alpha) = \hbar\,\text{acceleration}/(2\pi c).$$

Let us compare the result (6.21) with that for a detector at rest in a thermal bath of temperature $T = 1/\beta$ for general spatial dimensions D (for more detailed discussion see [30]). In the case $\Omega > 0$ the corresponding response functions is given by the expression

$$F_\beta(\Omega) = \frac{2^{1-D}\pi^{1-D/2}}{\Gamma(D/2)} \frac{\Omega^{D-2}}{e^{\beta\Omega} - 1}.$$

For $D = 3$ this coincides with (6.22). The latter is the case for $D = 1$ as well. For other values D, the response function $F_\beta(\Omega)$ is not equal to $F(\Omega)$ with $\beta = 2\pi\alpha$. An interesting feature in (6.21) is that for even values of D the response function is proportional to the For a massive field the response is not identical even in the cases $D = 1, 3$.

6.4 Scalar and fermionic fields in external electromagnetic field

Consider the case where in addition to the gravitational field an external classical electromagnetic field is present with the vector potential $A_\mu(x)$. The corresponding field equations for charged scalar and fermionic fields are obtained from (5.2) and (5.18) by the replacement

$$\nabla_\mu \to D_\mu = \nabla_\mu + ieA_\mu, \tag{6.24}$$

where e is the charge of the field quantum. These equations take the form

$$\left(D_\mu D^\mu + m^2 + \xi R\right)\phi = 0, \tag{6.25}$$

for a complex scalar field $\phi(x)$ and

$$(i\gamma^\mu D_\mu - m)\psi = 0, \tag{6.26}$$

for a fermionic field $\psi(x)$. The field equations are invariant under the local gauge transformations $\phi(x) \to e^{i\alpha(x)}\phi(x)$, $\psi(x) \to e^{i\alpha(x)}\psi(x)$, $A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)/e$.

The respective action functional for scalar field is presented as

$$S[\phi] = \frac{1}{2}\int d^{D+1}x\sqrt{|g|}\left[g^{\mu\nu}D_\mu^*\phi^+ D_\nu\phi - \left(m^2 + \xi R\right)\phi^+\phi\right].$$

For the current density one gets

$$j_\mu = i\phi^+ D_\mu\phi - i\left(D_\mu^*\phi^+\right)\phi,$$

and the expression for the metric energy-momentum tensor takes the form

$$\begin{aligned} T_{\mu\nu} &= D_\mu^*\phi^+ D_\nu\phi + D_\nu^*\phi^+ D_\mu\phi - g_{\mu\nu}D_\rho^*\phi^+ D^\rho\phi + m^2 g_{\mu\nu}\phi^+\phi \\ &\quad -2\xi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\phi^+\phi + 2\xi\left(g_{\mu\nu}\nabla_\rho\nabla^\rho - \nabla_\mu\nabla_\nu\right)\phi^+\phi. \end{aligned}$$

Note that the last term in the expression for the energy-momentum tensor comes from the variation of the Ricci scalar and it contains the standard covariant derivative instead of D_μ. By taking into account that

$$D_\rho^*\phi^+ D^\rho\phi = \frac{1}{2}\nabla_\rho\nabla^\rho\left(\phi^+\phi\right) - \frac{1}{2}\phi^+ D_\rho D^\rho\phi - \frac{1}{2}\left(D_\rho^* D^{\rho *}\phi^+\right)\phi,$$

the energy-momentum tensor can also be written in the form

$$\begin{aligned} T_{\mu\nu} &= D_\mu^*\phi^+ D_\nu\phi + D_\nu^*\phi^+ D_\mu\phi + 2\left[(\xi - 1/4)\, g_{\mu\nu}\nabla_\rho\nabla^\rho - \xi\nabla_\mu\nabla_\nu - \xi R_{\mu\nu}\right]\phi^+\phi \\ &\quad +\frac{1}{2}g_{\mu\nu}\left\{\phi^+\left(D_\rho D^\rho + m^2 + \xi R\right)\phi + \left[\left(D_\rho^* D^{\rho *} + m^2 + \xi R\right)\phi^+\right]\phi\right\}. \end{aligned}$$

Note that on the solutions of the field equation the part in the figure braces vanishes.

For a fermionic field the action functional in an external electromagnetic field is given by the expression

$$S[\psi] = \int d^{D+1}x\sqrt{|g|}\left[\frac{i}{2}\left(\bar{\psi}\gamma^{\mu}D_{\mu}\psi - \left(D_{\mu}^{*}\bar{\psi}\right)\gamma^{\mu}\psi\right) - m\bar{\psi}\psi\right].$$

For the current density operator one has $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$. The expression of the corresponding energy-momentum tensor reads

$$T_{\mu\nu} = \frac{i}{2}\left[\bar{\psi}\gamma_{(\mu}D_{\nu)}\psi - (D_{(\mu}^{*}\bar{\psi})\gamma_{\nu)}\psi\right].$$

The part in the Lagrangian density describing the interaction of the fermionic field with the electromagnetic field is presented as $-ej^{\mu}A_{\mu}$.

In the presence of an external electromagnetic field the canonical quantization procedure follows the same steps described in section 6.1. We will also have a similar consideration for the Bogoliubov transformations. Note that the replacement (6.24) should also be done in the formula for the corresponding scalar product of the mode functions. For example, in the case of a charged scalar field the scalar product takes the form

$$(\phi_1, \phi_2) = -i\int_{\Sigma} d\Sigma^{\mu}\sqrt{|g_{\Sigma}|}\left[\left(D_{\mu}^{*}\phi_2^{+}(x)\right)\phi_1(x) - \phi_2^{+}(x)D_{\mu}\phi_1(x)\right].$$

For a Dirac fermionic field the normalization condition remains in the form (6.6).

Chapter 7

Adiabatic expansion of the Green function

7.1 Divergences and regularization

As it has been discussed before, in general curved backgrounds no natural definition of particle exists. In particular this is related to that the concept of particle is defined globally, by a special choice of the mode functions in the quantization procedure. It is advantageous to study physical quantities that are defined locally and have a tensorial nature. Among the most important objects of this kind is the expectation value of the energy-momentum tensor. In addition to describing the physical structure of the quantum filed at a given point, this expectation value acts as the source of gravity in semiclassical Einstein field equation. It therefore plays an important part in any attempt to model a self-consistent dynamics involving the gravitational field.

A number of quantities of physical interest, such as the action and the energy-momentum tensor, are quadratic in the fields and their derivatives evaluated at a single point. The corresponding expectation values are divergent. In quantum theory of free fields on the Minkowski bulk these divergences are regularized by normal ordering procedure. In curved spacetime, even for free fields, the gravitational interaction introduces additional divergences. Furthermore, vacuum energy must be treated more carefully because it can give rise to gravitational effects.

Various methods have been developed to regularize and renormalize quantities that involve squares and higher powers of fields or their derivatives evaluated at a single point of spacetime. Among them are:

1. Proper-time regularization

2. Dimensional regularization

3. Zeta-function regularization

4. Point splitting regularization

The quantities quadratic in the fields can be expressed in terms of the two-point functions. In order to understand the structure of divergences, it is important to have the behavior of the two-point functions in the coincidence limit of the arguments. This behavior is mainly determined by the local geometry of the spacetime near the point under consideration.

7.2 Two-point functions

7.2.1 Two-point functions in Minkowski spacetime

Vacuum expectation values of various products of free field operators can be identified with various two-point functions. First let us consider the two-point functions in the Minkowski bulk. Let us start with the scalar field. The expectation values

$$\begin{aligned} G^{+}(x,x') &= \langle 0|\,\phi(x)\phi(x')\,|0\rangle\,, \\ G^{-}(x,x') &= \langle 0|\,\phi(x')\phi(x)\,|0\rangle\,, \end{aligned}$$

are called positive and negative frequency Wightman functions, respectively. With these functions, for the commutator or Pauli-Jordan function one has

$$G(x,x') = -i\,\langle 0|\,[\phi(x),\phi(x')]\,|0\rangle = -i\left[G^{+}(x,x') - G^{-}(x,x')\right].$$

The Hadamard function is defined as

$$G^{(1)}(x,x') = -i\left[G^{+}(x,x') - G^{-}(x,x')\right].$$

All these functions obey the homogeneous equation

$$\left(\Box_x + m^2\right)\mathcal{G}(x,x') = 0, \quad \Box_x = \eta^{\mu\nu}\partial_\mu\partial_\nu, \tag{7.1}$$

with $\mathcal{G} = G^{+}, G^{-}, G, G^{(1)}$.

Next we define the Feynman propagator, defined as the time-ordered product of fields

$$iG_F(x,x') = \langle 0|\,T(\phi(x),\phi(x'))\,|0\rangle = \theta(t-t')G^{+}(x,x') + \theta(t'-t)G^{-}(x,x'),$$

where

$$\theta(t) = \begin{cases} 1, & t>0 \\ 0, & t<0, \end{cases}$$

is the Heaviside step function. Finally, the retarded and advanced two-point functions are defined respectively by

$$\begin{aligned} G_R(x,x') &= -\theta(t-t')G(x,x'), \\ G_A(x,x') &= \theta(t'-t)G(x,x'). \end{aligned}$$

These functions obey the equations

$$\begin{aligned} (\Box_x + m^2)G_F(x,x') &= -\delta^{(D+1)}(x-x'), \\ (\Box_x + m^2)G_{R,A}(x,x') &= \delta^{(D+1)}(x-x'). \end{aligned}$$

The Green functions describe the propagation of field perturbations obeying certain boundary conditions.

The two-point functions can be presented in the form of the mode sums. Let us consider for example the positive frequency Wightman function. If $\{\phi_i(x), \phi_i^*(x)\}$ is a complete set of mode functions for a scalar field, for the field operator we have the expansion (6.3). By taking into account that for the vacuum state one $a_i\,|0\rangle = 0$, $\langle 0|\,a_i^{+} = 0$, we have

$$\begin{aligned} G^{+}(x,x') &= \sum_{i,j} \langle 0|\,a_i a_j \phi_i(x)\phi_j(x') + a_i a_j^{+}\phi_i(x)\phi_j^*(x') \\ &\quad + a_i^{+} a_j \phi_i^*(x)\phi_j(x') + a_i^{+} a_j^{+}\phi_i^*(x)\phi_j^*(x')\,|0\rangle \\ &= \sum_{i,j} \phi_i(x)\phi_j^*(x')\,\langle 0|\,a_i a_j^{+}\,|0\rangle = \sum_{i,j} \phi_i(x)\phi_j^*(x')\,\langle 0|\,\delta_{ij} + a_j^{+} a_i\,|0\rangle\,. \end{aligned}$$

By taking into account that $a_i |0\rangle = 0$, the following mode sum is obtained

$$G^+(x,x') = \sum_i \phi_i(x)\phi_i^*(x').$$

Similar representations are obtained for the other two-point functions. For example, for the Hadamard function one has

$$G^{(1)}(x,x') = \sum_i \left[\phi_i(x)\phi_i^*(x') + \phi_i(x')\phi_i^*(x)\right].$$

In particular, in $(D+1)$-dimensional Minkowski spacetime, as the normalized mode functions we can take the plane waves

$$\phi_{\mathbf{k}}(x) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}}{\sqrt{2(2\pi)^D\omega}},\ \omega = \sqrt{|\mathbf{k}|^2+m^2},$$

with the set $i = \mathbf{k}$. For the positive frequency Wightman function this gives

$$G^+(x,x') = \frac{1}{2(2\pi)^D}\int d^D\mathbf{k}\,\frac{1}{\omega}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega(t-t')}.$$

In order to evaluate the integral in this formula we use

$$\int d^D\mathbf{k}\, f(\mathbf{k}^2,\mathbf{k}\cdot\mathbf{y}) = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}\int_0^\infty du\, u^{D-1}\int_0^\pi dv\, \sin^{D-2}v f(u^2,|\mathbf{y}|u\cos v).$$

This gives

$$G^+(x,x') = \frac{2^{-D}\pi^{-(D+1)/2}}{\Gamma((D-1)/2)}\int_0^\infty du\,\frac{u^{D-1}e^{-i\sqrt{u^2+m^2}(t-t')}}{\sqrt{u^2+m^2}}\int_0^\pi dv\, \sin^{D-2}v e^{iu|\mathbf{x}-\mathbf{x}'|\cos v}.$$

For the angular integral one has [37]

$$\begin{aligned}\int_0^\pi dv\, \sin^{D-2}v e^{iu|\mathbf{x}-\mathbf{x}'|\cos v} &= 2\int_0^1 dy\,(1-y^2)^{(D-3)/2}\cos(u|\mathbf{x}-\mathbf{x}'|y) \\ &= 2^{D/2-1}\sqrt{\pi}\Gamma((D-1)/2)\frac{J_{D/2-1}(u|\mathbf{x}-\mathbf{x}'|)}{(u|\mathbf{x}-\mathbf{x}'|)^{D/2-1}},\end{aligned}$$

with $J_\nu(x)$ being the Bessel function.

The Wightman function is presented as

$$G^+(x,x') = \frac{|\mathbf{x}-\mathbf{x}'|^{1-D/2}}{2(2\pi)^{D/2}}\int_0^\infty du\,\frac{u^{D/2}}{\sqrt{u^2+m^2}}J_{D/2-1}(u|\mathbf{x}-\mathbf{x}'|)e^{-i\sqrt{u^2+m^2}(t-t')}. \tag{7.2}$$

By taking into account that [71]

$$\begin{aligned}&\int_0^\infty dx\,\frac{x^{\nu+1}J_\nu(cx)}{\sqrt{x^2+z^2}}\left\{\begin{array}{c}\sin(b\sqrt{x^2+z^2})\\ \cos(b\sqrt{x^2+z^2})\end{array}\right\} \\ &= \pm\sqrt{\frac{\pi}{2}}c^\nu z^{\nu+1/2}(b^2-c^2)^{-(2\nu+1)/4}\left\{\begin{array}{r}\mathrm{sgn}(b)J_{-\nu-1/2}(z\sqrt{b^2-c^2})\\ Y_{-\nu-1/2}(z\sqrt{b^2-c^2})\end{array}\right\} \\ &= \left\{\begin{array}{c}0\\ 1\end{array}\right\}\sqrt{\frac{2}{\pi}}c^\nu z^{\nu+1/2}(c^2-b^2)^{-(2\nu+1)/4}K_{\nu+1/2}(z\sqrt{c^2-b^2}),\end{aligned}$$

in the cases $|b| > c$ and $|b| < c$, respectively, for the integral in (7.2) we get

$$\begin{aligned}\int_0^\infty dx\, \frac{x^{\nu+1}J_\nu(cx)}{\sqrt{x^2+z^2}}e^{-ib\sqrt{x^2+z^2}} &= -\sqrt{\frac{\pi}{2}}\frac{\mathrm{sgn}(b)e^{-\nu\pi i}c^\nu z^{\nu+1/2}}{(b^2-c^2)^{(\nu+1/2)/2}} \\ &\times \begin{cases} H^{(2)}_{\nu+1/2}(z\sqrt{b^2-c^2}), & b>0 \\ -e^{2i\pi\nu}H^{(1)}_{\nu+1/2}(z\sqrt{b^2-c^2}), & b<0 \end{cases},\ |b|>c, \\ &= \sqrt{\frac{2}{\pi}}c^\nu z^{\nu+1/2}\frac{K_{\nu+1/2}(z\sqrt{c^2-b^2})}{(c^2-b^2)^{(\nu+1/2)/2}},\ |b|<c,\end{aligned}$$

where $H^{(1,2)}_\nu(x)$ are the Hankel functions of the first and second kinds and $K_\nu(x)$ is the MacDonald function. As a result, the final expression for the Wightman function takes the form

$$\begin{aligned}G^+(x,x') &= \mathrm{sgn}(t-t')\frac{(2\pi)^{(1-D)/2}i^{-D}m^{(D-1)/2}}{4[(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2]^{(D-1)/4}} \\ &\times\begin{cases} H^{(2)}_{(D-1)/2}(m\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}), & t-t'>0 \\ e^{i\pi D}H^{(1)}_{(D-1)/2}(m\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}), & t-t'<0\end{cases},\end{aligned} \tag{7.3}$$

for $(t-t')^2 > |\mathbf{x}-\mathbf{x}'|^2$ (the point x is inside the light cone of x') and

$$G^+(x,x') = \frac{m^{(D-1)/2}}{(2\pi)^{(D+1)/2}}\frac{K_{(D-1)/2}(m\sqrt{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2})}{[|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2]^{(D-1)/4}}, \tag{7.4}$$

for $(t-t')^2 < |\mathbf{x}-\mathbf{x}'|^2$. The expression (7.4) is obtained from (7.3) by the analytical continuation. For $(t-t')^2 < |\mathbf{x}-\mathbf{x}'|^2$ we write

$$(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2 = e^{-\pi i}\left[|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2\right].$$

By taking into account that $H^{(2)}_{(D-1)/2}(e^{-\pi i/2}x) = (2i/\pi)e^{i\pi(D-1)/4}K_{(D-1)/2}(x)$, from (7.3) one gets the expression (7.4). In the coincidence limit of the arguments, for the leading term in the expansion of (7.4) one gets

$$G^+(x,x') = \frac{\pi^{-(D+1)/2}\Gamma((D-1)/2)}{4\left[|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2\right]^{(D-1)/2}}. \tag{7.5}$$

The leading term does not depend on the mass.

The negative frequency negative function is evaluated in a similar way. The only difference is the replacement $t-t' \to t'-t$. In particular, we see that

$$G^-(x,x') = G^+(x,x'),\ (t-t')^2 < |\mathbf{x}-\mathbf{x}'|^2.$$

This shows that the Pauli-Jordan function $G(x,x')$ vanishes outside the light cone. By taking into account that

$$[\phi(x),\phi(x')] = iG(x,x'),$$

we conclude that the commutator of the operators $\phi(x)$ and $\phi(x')$ vanishes if the points x and x' are separated by a spacelike interval. Physically, this corresponds to that the events at the spacelike separated points are causally independent. In the case $(t-t')^2 > |\mathbf{x}-\mathbf{x}'|^2$, the Pauli-Jordan function is given by the expression

$$G(x,x') = -\frac{i^{-D}m^{(D-1)/2}}{2\,(2\pi)^{(D-1)/2}}\frac{J_{(D-1)/2}(m\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2})}{[(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2]^{(D-1)/4}}.$$

As it is seen from (7.3) and (7.4), the two-point functions diverge on the light cone corresponding to the limit $(x-x')^2 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) \to 0$.

Substituting the mode decomposition of the field and using the equation (7.1), one obtains integral representations for the two-point functions:

$$\mathcal{G}(x,x') = \int \frac{d^{D+1}k}{(2\pi)^{(D+1)}} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-ik^0(t-t')}}{(k^0)^2-\omega^2}, \quad \omega = \sqrt{|\mathbf{k}|^2+m^2}.$$

Considered as a contour integral, the k^0 integration may be performed by deforming the contour around the poles $k^0 = \pm\omega$. The way in which this deformation is performed depends on a specific two-point function (see figure 7.1). For example, the integral corresponding to the Feynman Green function yields

$$G_F(x,x') = \frac{-1}{4(4\pi i)^{(D-1)/2}} \left(\frac{2m^2}{-\sigma+i\varepsilon}\right)^{(D-1)/4} H^{(2)}_{(D-1)/2}((2m^2(\sigma-i\varepsilon))^{1/2}),$$

where

$$\sigma = \frac{1}{2}(x-x')^2 = \frac{1}{2}\eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu).$$

The term $-i\varepsilon$ is added to indicate that $G_F(x,x')$ is really the limiting value of a function analytic in the lower-half plane σ.

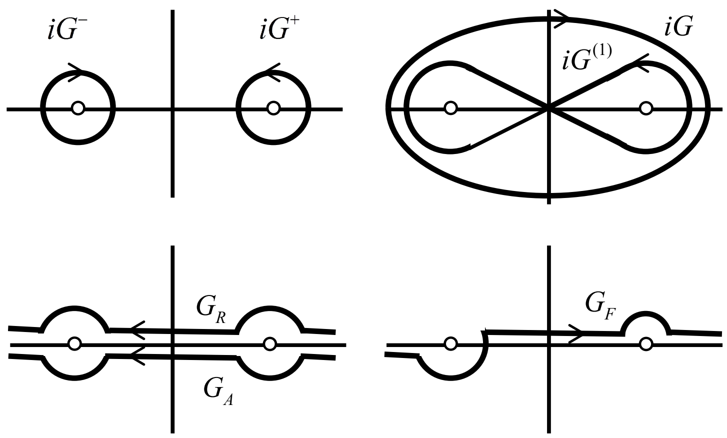

Figure 7.1: Contours of integrations in the complex k^0 plane for various two-point functions. The circles on the real axis correspond to the poles $k^0 = \pm\sqrt{|\mathbf{k}|^2+m^2}$.

For a Dirac spinor field we can introduce two-point functions in a similar way. Let us consider the Feynamn Green function and the Hadamard two-point function defined as

$$\begin{aligned} iS_F(x,x') &= \langle 0|\, T(\psi(x)\bar{\psi}(x'))\, |0\rangle\,. \\ S^{(1)}(x,x') &= \langle 0|\, [\psi(x)\bar{\psi}(x')]\, |0\rangle\,. \end{aligned}$$

These functions obey the equations

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)S_F(x,x') &= \delta^{(D+1)}(x-x'), \\ (i\gamma^\mu\partial_\mu - m)S^{(1)}(x,x') &= 0. \end{aligned}$$

The spinor two-point functions are matrices in the spinor indices of the fields. For example, the Feynman green function is explicitly written as

$$iS_F(x,x')_{ab} = \langle 0|\, \psi_a(x)\bar{\psi}_b(x')\, |0\rangle\, \theta(t-t') - \langle 0|\, \bar{\psi}_b(x')\psi_a(x)\, |0\rangle\, \theta(t'-t).$$

Spinor functions can be expressed in terms of the scalar two point functions. For example,

$$\begin{aligned} S_F(x,x') &= (i\gamma^\mu\partial_\mu + m)G_F(x,x'), \\ S^{(1)}(x,x') &= -(i\gamma^\mu\partial_\mu + m)G^{(1)}(x,x'). \end{aligned}$$

The Feynman propagator for the electromagnetic field is defined by

$$iD_{F\mu\nu}(x,x') = \langle 0| T(A_\mu(x)A_\nu(x')) |0\rangle .$$

Of course, this functions is not gauge invariant. Adding a noninvariant term in the Lagrangian (see (3.22)), from the field equation (3.23) we get the equation for the two-point function

$$\left[\eta_{\mu\rho}\eta^{\beta\sigma}\partial_\beta\partial_\sigma - (1-\alpha^{-1})\partial_\mu\partial_\rho\right] D_F^{\rho\nu}(x,x') = \delta_\mu^\nu \delta^{(D+1)}(x-x'). \tag{7.6}$$

This yields to the integral representation

$$D_{F\mu\nu}(x,x') = -\int \frac{d^{D+1}k}{(2\pi)^{(D+1)}} \frac{\eta_{\mu\nu} + (\alpha-1)k_\mu k_\nu/k^2}{(k^0)^2 - \omega^2} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}') - ik^0(t-t')}. \tag{7.7}$$

In the Feynman gauge one has $\alpha = 1$ and we get a simple relation between the eelctromagnetic and scalar two-point functions

$$D_{F\mu\nu}(x,x') = -\eta_{\mu\nu}G_F(x,x').$$

In the limit when the gauge noninvariant term in the Lagrangian is removed, $\alpha \to \infty$, the expression in the right-hand side of (7.7) tends to infinity. This means that the operator in the left-hand side of (7.6) is not invertible.

7.3 Adiabatic expansion of Green function in curved spacetime

In curved spacetime, having defined the vacuum state and the tower of Fock space, the formal generalization of the two-point functions is straightforward. For a scalar field they obey the homogeneous equation

$$(\Box_x + m^2 + \xi R(x))\mathcal{G}(x,x') = 0,$$

for Pauli-Jordan, Hadamard, positive and negative frequency two-point functions. In the case of Feynman, advanced and retarded two-point functions one had to add to the right-hand side the term $-|g(x)|^{-1/2}\delta^{(D+1)}(x-x')$. For example, the equation for the Feynman propagator reads

$$(\Box_x + m^2 + \xi R(x))G_F(x,x') = -|g(x)|^{-1/2}\delta^{(D+1)}(x-x'). \tag{7.8}$$

The equation for two-point functions does not specify the vacuum state used in the definition of the function. To fix the state, boundary conditions should be imposed on the solution. Compared to the case of the Minkowski bulk, in curved spacetime, the specification of boundary conditions is more complicated.

Expectation values of physical observables bilinear in the field operator are expressed in terms of the two-point functions. The divergences of these functions on the light cones lead to the divergences in the expectation values. For the subtraction of these divergences and the renormalization of the expectation values we need to know the short distance behaviour of the two-point functions. The leading terms are given by the corresponding adiabatic expansion. As a two-point function we will consider the Feynman propagator. For other functions the terms singular in the coincidence limit are the same.

We start our discussion by the choice of the appropriate coordinate system with the origin at x'. As such we will use the Riemann normal coordinates, assuming that for any point x in the neighborhood of x' there is a unique geodesic joining these two points. Let P be an arbitrary point in a neighborhood of the point Q. The Riemann coordinates y^μ of a point P are given by

$$y^\mu = \lambda \xi^\mu,$$

where λ is the value at P of an affine parameter of the geodesic joining Q to P, ξ^μ is the tangent to the geodesic at Q:

$$\xi^\mu = (dx^\mu/d\lambda)_Q.$$

Along any given geodesic through Q, ξ^μ is constant or independent of λ. Hence, the equation of the geodesic is

$$d^2 y^\alpha/d\lambda^2 = 0,$$

which implies that in the Riemann coordinates

$$\Gamma^\alpha_{\ \beta\gamma}(y)\frac{dy^\beta}{d\lambda}\frac{dy^\gamma}{d\lambda} = \Gamma^\alpha_{\ \beta\gamma}(y)\xi^\beta(y)\xi^\gamma(y) = 0 \Rightarrow \Gamma^\alpha_{\ \beta\gamma}(y)y^\beta y^\gamma = 0.$$

At point Q itself, we have $\Gamma^\alpha_{\ \beta\gamma}(Q)\xi^\beta\xi^\gamma = 0$ for ξ^μ pointing along any geodesic through Q. Hence, in these coordinates $\Gamma^\alpha_{\ \beta\gamma}(Q) = 0$. We are free to take $g_{\mu\nu}(Q) = \eta_{\mu\nu}$. Note that $d^2y^\alpha/d\lambda^2 = 0$ with the boundary conditions $y^\alpha(0) = 0$ and $(dy^\alpha/d\lambda)_0 = \xi^\alpha$ is equivalent to $y^\mu = \lambda\xi^\mu$. The relation $\Gamma^\alpha_{\ \beta\gamma}(y)y^\beta y^\gamma = 0$ implies that the coordinate system y^μ is Riemannian.

Now, we can expand the Christoffel symbols about the point Q:

$$\Gamma^\alpha_{\ \beta\gamma}(y) = \Gamma^\alpha_{\ \beta\gamma}(0) + (\partial_\mu\Gamma^\alpha_{\ \beta\gamma})(0)y^\mu + \frac{1}{2!}(\partial_\mu\partial_\nu\Gamma^\alpha_{\ \beta\gamma})(0)y^\mu y^\nu + \cdots$$

Then the condition for Riemann coordinates, that $\Gamma^\alpha_{\ \beta\gamma}(y)y^\beta y^\gamma = 0$ for all y^μ, is equivalent to

$$\Gamma^\alpha_{\ \beta\gamma}(0) = 0,\ \partial_{(\mu}\Gamma^\alpha_{\ \beta\gamma)}(0) = 0, \quad \partial_{(\mu}\partial_\nu\Gamma^\alpha_{\ \beta\gamma)}(0) = 0, \ldots,$$

with the notation

$$A_{(\alpha_1\cdots\alpha_n)} = \frac{1}{n!}\sum_P A_{\alpha_1\cdots\alpha_n},$$

where $\sum_P$ denotes the sum over all permutations of $\alpha_1\cdots\alpha_n$. For the Riemann tensor at the origin, one has

$$R^\alpha_{\ \beta\gamma\delta}(0) = \partial_\delta\Gamma^\alpha_{\ \beta\gamma}(0) - \partial_\gamma\Gamma^\alpha_{\ \beta\delta}(0).$$

From this relation we get

$$\begin{aligned} \partial_{(\beta}\Gamma^\nu_{\ \alpha)\mu}(0) &= -\frac{1}{3}R^\nu_{\ (\alpha\beta)\mu}(0), \\ \partial_{(\gamma}\partial_\beta\Gamma^\nu_{\ \alpha)\mu}(0) &= \frac{1}{2}R_{\mu(\gamma\ \beta;\alpha)}^{\ \ \ \nu}(0). \end{aligned}$$

If $W_{\alpha_1\cdots\alpha_n}$ is any tensor with analytic components in a neighborhood of $y^\mu = 0$, it can be Taylor expanded about the origin, and the ordinary derivatives expressed in terms of covariant derivatives and affine connections. Then, using the above relations, it can be shown that

$$\begin{aligned} W_{\alpha_1\cdots\alpha_n}(y) &= W_{\alpha_1\cdots\alpha_n}(0) + W_{\alpha_1\cdots\alpha_n;\mu}(0)y^\mu \\ &+\frac{1}{2!}\left[W_{\alpha_1\cdots\alpha_n;\mu\omega} - \frac{1}{3}\sum_{k=1}^n R^\nu_{\ \mu\alpha_k\omega}W_{\alpha_1\cdots\alpha_{k-1}\nu\alpha_{k+1}\cdots\alpha_n}\right]_0 y^\mu y^\omega + \cdots \end{aligned}$$

In particular, for the metric tensor we have the expansion:

$$\begin{aligned} g_{\mu\nu}(x) &= \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma \\ &+ \left(\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^{\lambda}_{\ \gamma\nu\delta}\right) y^\alpha y^\beta y^\gamma y^\delta + \cdots , \end{aligned}$$

with the coefficients evaluated at $y = 0$.

Having defined the coordinate system, we introduce the two-point function

$$\mathcal{G}_F(x, x') = |g|^{1/2} G_F(x, x').$$

For this function we can write the Fourier expansion

$$\mathcal{G}_F(x, x') = \int \frac{d^{D+1}k}{(2\pi)^{D+1}}\, e^{-ik\cdot y}\mathcal{G}_F(k), \quad k\cdot y = \eta^{\alpha\beta}k_\alpha y_\beta. \tag{7.9}$$

The space based on $(D + 1)$-vector k can be considered as a local momentum space. Expanding all the quantities in the equation for the Feynman propagator in Riemann coordinates y^μ and substituting (7.9), to the adiabatic order four for the Fourier coefficients we find

$$\begin{aligned} \mathcal{G}_F(k) &\approx (k^2 - m^2)^{-1} - (1/6 - \xi)R(k^2 - m^2)^{-2} + \frac{i}{2}(1/6 - \xi)R_{;\alpha}\frac{\partial}{\partial k_\alpha}(k^2 - m^2)^{-2} \\ &-\frac{1}{2}a_{\alpha\beta}\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\beta}(k^2 - m^2)^{-2} + \frac{(1/6 - \xi)^2R^2 + 2a^{\lambda}_{\lambda}}{(k^2 - m^2)^3}, \end{aligned}$$

where

$$\begin{aligned} a_{\alpha\beta} &= \frac{1}{2}\left(\xi - \frac{1}{6}\right)R_{;\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}(R_{\alpha\beta;\lambda})^{;\lambda} - \frac{1}{30}R^{\lambda}_{\alpha}R_{\lambda\beta} \\ &+\frac{1}{60}R^{\kappa\ \lambda}_{\ \alpha\ \beta}R_{\kappa\lambda} + \frac{1}{60}R^{\lambda\mu\kappa}_{\ \ \ \ \alpha}R_{\lambda\mu\kappa\beta}. \end{aligned}$$

By the inverse Fourier transform, for the coordinate space two-point function we find

$$\mathcal{G}_F(x, x') \approx \int \frac{d^{D+1}k}{(2\pi)^{D+1}}e^{-ik\cdot y}\left[a_0(x, x') - a_1(x, x')\partial_{m^2} + a_2(x, x')\partial^2_{m^2}\right](k^2 - m^2)^{-1},$$

where

$$a_0(x, x') = 1, \tag{7.10}$$

and to adiabatic order 4,

$$\begin{aligned} a_1(x, x') &= (1/6 - \xi)R - \frac{1}{2}(1/6 - \xi)R_{;\alpha}y^\alpha - \frac{1}{3}a_{\alpha\beta}y^\alpha y^\beta, \\ a_2(x, x') &= \frac{1}{2}(1/6 - \xi)^2R^2 + \frac{1}{3}a^{\lambda}_{\lambda}, \end{aligned} \tag{7.11}$$

with all geometrical quantities on the right-hand side evaluated at x'.

For the further transformation we employ the integral representation

$$(k^2 - m^2 + i\varepsilon)^{-1} = -i\int_0^\infty ds\, e^{is(k^2 - m^2 + i\varepsilon)},$$

then

$$\begin{aligned} \mathcal{G}_F(x, x') &\approx -i\int_0^\infty ds\, e^{-im^2s - s\varepsilon}\left[a_0(x, x') + a_1(x, x')is + a_2(x, x')(is)^2\right]\int \frac{d^{D+1}k}{(2\pi)^{D+1}}\, e^{-ik\cdot y + isk^2} \\ &= (4\pi)^{-(D+1)/2}\int_0^\infty ds\,(is)^{-(D+1)/2}e^{-im^2s + \sigma(x,x')/(2is)}F(x, x'; is), \end{aligned} \tag{7.12}$$

where we used the integral

$$\int \frac{d^n k}{(2\pi)^n} e^{isk^2 - ik\cdot y} = \frac{i(is)^{-n/2}}{(4\pi)^{n/2}} e^{\sigma(x,x')/(2is)},$$

with

$$\sigma(x,x') = (1/2)\eta_{\alpha\beta} y^\alpha y^\beta.$$

The latter is one-half of the square of the proper distance between the points x and x'. The function $F(x,x';is)$ in the integrand has the following asymptotic expansion

$$F(x,x';is) \approx a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2 + \cdots \tag{7.13}$$

The relation gives the De Witt-Schwinger representation for the Feynman Green function:

$$G_F^{DS}(x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi)^{(D+1)/2}} \int_0^\infty ds\, (is)^{-(D+1)/2} e^{-im^2 s + \sigma(x,x')/(2is)} F(x,x';is), \tag{7.14}$$

where Δ is the Van Vleck determinant

$$\Delta(x,x') = -\det[\partial_\mu \partial_\nu \sigma(x,x')][g(x)g(x')]^{-1/2}.$$

In the normal coordinates about x' that we are currently using, Δ reduces to $|g(x)|^{-1/2}$. Note that we have the relations

$$\begin{aligned} \partial_\mu \partial_\nu \sigma(x,x')|_{x'=x} &= g(x),\ g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma = 2\sigma, \\ \nabla^\mu(\det[\partial_\alpha\partial_\beta\sigma(x,x')]\partial_\mu\sigma) &= \det[\partial_\alpha\partial_\beta\sigma(x,x'). \end{aligned}$$

The extension of the adiabatic expansion of F to all orders is written as

$$F(x,x';is) \approx \sum_{j=0}^{\infty} a_j(x,x')(is)^j,$$

with $a_0(x,x') = 1$, the other a_j being given by recursion relations which enable their adiabatic expansions to be obtained. Substituting the expansion of the function into the De Witt-Schwinger representation, we find the expansion in the coordinate space

$$G_F^{DS}(x,x') = -\frac{\Delta^{1/2}(x,x')}{4(4\pi i)^{(D-1)/2}} \sum_{j=0}^{\infty} a_j(x,x')(-\partial_{m^2})^j \left[\left(\frac{2m^2}{-\sigma}\right)^{(D-1)/4} H^{(2)}_{(D-1)/2}(m\sqrt{2\sigma}) \right],$$

where a small imaginary part should be subtracted from σ. The same short distance behavior results from almost all choices of vacuum state.

7.4 Divergences and renormalization on curved backgrounds

The expectation values of physical quantities quadratic in the field operator are expressed in terms of two-point functions or their derivatives in the coincidence limit of the arguments. In this limit ultraviolet divergences appear. A well known example is the vacuum expectation value of the field squared $\langle 0|\,\phi^2(x)\,|0\rangle$, expressed as $\langle 0|\,\phi^2(x)\,|0\rangle = \lim_{x'\to x} G^{(1)}(x,x')$. For the leading divergence in the Hadamard function from the De Witt-Schwinger expansion one has $G^{(1)}(x,x') \propto \sigma^{-(D-1)/2}$ as $\sigma \to 0$. In the case of the energy-momentum tensor the corresponding expression contains the derivatives of the two-point function. As a consequence, the divergences are stronger, like $\sigma^{-(D+1)/2}$. In Minkowski bulk this type of divergences are removed, for example, by using the normal ordering procedure. In non-gravitational physics, the energy differences are observable only and we can simply shift the zero point by an infinite amount. In gravitational physics, the energy-momentum tensor acts as the source of the gravity and its shift is not a satisfactory procedure. Another problem is related to that in curved backgrounds, in addition to Minkowski-type divergences, additional divergences are present. Hence, in the presence of the gravitational field more elaborate renormalization procedure should be employed. The latter is among the most important steps in quantum field theory on curved backgrounds.

In fact, the renormalization implies that infinite quantities must be subtracted from diverging expressions. The subtraction can be done in an infinite variety of ways and additional criteria should be imposed for a unique result. If enough physically reasonable restrictions are imposed, then the subtraction procedure might be defined uniquely. These type of conditions for the energy-momentum tensor will be discussed below.

In a semiclassical theory considering quantum fields in curved spacetime, described by General Relativity, the expectation value of the energy-momentum tensor appears in the right-hand side of Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} = -8\pi G_B \langle T_{\mu\nu} \rangle ,$$

with the gravitational and cosmological constants G_B and Λ_B, respectively. Introducing the effective action, W, for quantum fields, these equations are obtained from the minimal action principle for the action functional

$$S = \frac{1}{16\pi G_B} \int d^n x \sqrt{|g|} \, (R - 2\Lambda) + W.$$

The expectation value of the energy-momentum tensor is obtained from the effective action with the help of the standard functional derivative with respect to the metric tensor:

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{|g|}} \frac{\delta W}{\delta g^{\mu\nu}}.$$

For the investigation of the character of divergences it is convenient to assume that the spacetime asymptotically static. In this case we can construct the Fock spaces of states in asymptotic regions. We will denote these spaces as F_{in} and F_{out} and by $|\psi_{\text{in}}\rangle$ and $|\psi_{\text{out}}\rangle$ the corresponding states. For the vacuum states we use the notations $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$. In general, these states are different. This means that the state $|0_{\text{out}}\rangle$ contains particles defined as states of F_{in} (particle creation by the gravitational field). We are interested in the expectation values $\langle \psi_{\text{in}} | T_{\mu\nu} | \psi_{\text{in}} \rangle$ and $\langle \psi_{\text{out}} | T_{\mu\nu} | \psi_{\text{out}} \rangle$. The difference $\langle \psi_{\text{in}} | T_{\mu\nu} | \psi_{\text{in}} \rangle - \langle 0_{\text{in}} | T_{\mu\nu} | 0_{\text{in}} \rangle$ is finite and for the investigation of the structure of divergences it is sufficient to consider the vacuum expectation value $\langle 0_{\text{in}} | T_{\mu\nu} | 0_{\text{in}} \rangle$.

It is convenient to use the path-integral quantization procedure. Consider the generating functional

$$Z[J] = \int \mathcal{D}[\phi] \exp \left\{ i S_m[\phi] + i \int dx^n \, J(x)\phi(x) \right\} ,$$

with a source $J(x)$. It is interpreted as the vacuum persistent amplitude $\langle 0_{\text{out}} | \, 0_{\text{in}} \rangle$. The external current J may lead to the instability of the initial vacuum state $|0_{\text{in}}\rangle$. In flat spacetime, in the absence of external sources, $J = 0$, no particles are produced and the vacuum is stable. This is expressed by the relation $Z[0] = \langle 0_{\text{out}} | \, 0_{\text{in}} \rangle_{J=0} = \langle 0_{\text{in}} | \, 0_{\text{in}} \rangle = 1$. In curved backgrounds, in general, the states $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$ are different, even in the absence of the source currents. Hence, in general, $Z[0] \neq 1$.

Taking the variation of $Z[0]$, we get

$$\delta Z[0] = i \int \mathcal{D}[\phi] \, \delta S_m e^{i S_m[\phi]} = i \langle 0_{\text{out}} | \, \delta S_m \, | 0_{\text{in}} \rangle .$$

From here, by taking into account the definition of the metric energy-momentum tensor, one can write

$$i\left\langle 0_{\rm out}\right| T_{\mu\nu}\left|0_{\rm in}\right\rangle = \frac{2}{\sqrt{|g|}}\frac{\delta Z[0]}{\delta g^{\mu\nu}}.$$

We define the effective action W in accordance with

$$Z[0] = e^{iW},\ W = -i\ln\left\langle 0_{\rm out}\right|\left. 0_{\rm in}\right\rangle.$$

From here we get the relation

$$\frac{\left\langle 0_{\rm out}\right| T_{\mu\nu}\left|0_{\rm in}\right\rangle}{\left\langle 0_{\rm out}\right|\left. 0_{\rm in}\right\rangle} = \frac{2}{\sqrt{|g|}}\frac{\delta W}{\delta g^{\mu\nu}}.$$

7.4.1 Evaluation of the path-integral

Functional Gaussian integrals can be considered as the product of large number of usual integrals. For the simplest one we have

$$G(a) = \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \sqrt{\pi/a}.$$

This can be generalized for the product of N integrals:

$$G(A) = \int dx_1 dx_2 \cdots dx_N\, e^{-x_i a_{ij} x_j},$$

where A is the matrix with the elements a_{ij}. One can write

$$x_i a_{ij} x_j = X^T A X,\ A^T = A.$$

The matrix A can be diagonalized by the rotation:

$$A = R^T D R,\ R^T R = R R^T = 1,$$

where D is a diagonal matrix with the diagonal elements d_1, d_2,..., d_N. We have

$$G(A) = \int dx_1 dx_2 \cdots dx_N\, e^{-X^T R^T D R X} = \int dy_1 dy_2 \cdots dy_N\, e^{-Y^T D Y},\quad Y = RX.$$

The integral is splitted into the product of N gaussian integrals and we find

$$G(A) = \pi^{N/2}(d_1 d_2 \cdots d_N)^{-1/2} = \pi^{N/2}(\det A)^{-1/2}. \qquad (7.15)$$

Formally, the gaussian path-integral can be obtained in the limit $N \to \infty$.

The action functional for a scalar field is given by the expression (5.1). The latter can be written in the form

$$S[\phi] = \frac{1}{2}\int d^{D+1}x \partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\phi\partial_\nu\phi\right) - \frac{1}{2}\int d^{D+1}x\sqrt{|g|}\left(g^{\mu\nu}\phi\nabla_\mu\nabla_\nu\phi + m^2\phi^2 + \xi R\phi^2\right).$$

Omitting the total derivative term we get

$$S[\phi] = -\frac{1}{2}\int d^{D+1}x\sqrt{|g|}\phi(x)\left(\Box_x + m^2\phi + \xi R\right)\phi(x). \qquad (7.16)$$

For the evaluation of the path integral in

$$Z[0] = \int \mathcal{D}[\phi]\exp\left\{iS[\phi]\right\},$$

we will use this form of the action.

By using the property of the Dirac delta function, for the right function $\phi(x)$ we write

$$\phi(x) = \int d^{D+1}y \sqrt{|g(y)|}\, \phi(y)\delta^{(D+1)}(x-y)|g(x)|^{-1/2}.$$

As a consequence the action is rewritten as

$$S[\phi] = -\frac{1}{2}\int d^{D+1}x \sqrt{|g(x)|}\int d^{D+1}y \sqrt{|g(y)|}\phi(x)K_{xy}\phi(y), \tag{7.17}$$

with the operator

$$K_{xy} = \left(\Box_x + m^2 + \xi R(x) - i\varepsilon\right) \delta^{(D+1)}(x-y)|g(y)|^{-1/2}. \tag{7.18}$$

The expression in the right-hand side of (7.17) can be understood as a product of matrices with continuous indices:

$$\begin{aligned}\hat{\phi}^T \hat{K}\, \hat{\phi} \;&=\; \int d^{D+1}x \sqrt{|g(x)|}\int d^{D+1}y \sqrt{|g(y)|}\phi(x) \\ &\quad \times \left(\Box_x + m^2 + \xi R(x) - i\varepsilon\right) \delta^{(D+1)}(x-y)|g(x)|^{-1/2}\phi(y).\end{aligned}$$

This expression is quadratic with respect to the field operator and for the corresponding path integral in

$$Z[0] = \int \mathcal{D}[\phi]\, e^{-i\hat{\phi}^T \hat{K}\, \hat{\phi}/2},$$

we can use the formula for the gaussian integrals (7.15). For the functional integral we find

$$Z[0] \propto (\det \hat{K})^{-1/2},$$

where the proportionality constant is metric-independent and can be ignored.

The operator can be related to the Green function $G_F(x,z)$. In order to see that, we note that by the definition of the inverse matrix one has

$$\int d^{D+1}y \sqrt{|g(y)|}K_{xy}K^{-1}_{yz} = \delta^{(D+1)}(x-z)|g(z)|^{-1/2}.$$

Substituting the expression (7.18) for K_{xy}, we find

$$\begin{aligned}\Rightarrow \quad & \int d^{D+1}y \sqrt{|g(y)|}\left(\Box_x + m^2 + \xi R(x) - i\varepsilon\right) \delta^{(D+1)}(x-y)|g(y)|^{-1/2}K^{-1}_{yz} \\ = \quad & \left(\Box_x + m^2 + \xi R(x) - i\varepsilon\right) K^{-1}_{xz} = \delta^{(D+1)}(x-z)|g(z)|^{-1/2}.\end{aligned}$$

Comparing with (7.8) we see that

$$K^{-1}_{xz} = -G_F(x,z) = -\langle x|\, G_F\, |z\rangle\,.$$

In the last relation G_F is interpreted as an operator which acts on a space of vectors $|x\rangle$, normalized by $\langle x\,|x'\rangle = \delta^{(D+1)}(x-x')|g(x)|^{-1/2}$. Hence, for the effective action one has

$$W = -i\ln Z[0] = -i\ln(\det \hat{K})^{-1/2} = -\frac{i}{2}\ln(\det \hat{K}^{-1}) = -\frac{i}{2}\ln(\det(-G_F)).$$

By using the log-det to tr-log relation for matrices we get the final relation between the effective action and the Green function:

$$W = -\frac{i}{2}\mathrm{tr}[\ln(-G_F)]. \tag{7.19}$$

Trace of the operator M is defined by

$$\operatorname{tr} M = \int d^{D+1}x\,\sqrt{|g(x)|}M_{xx} = \int d^{D+1}x\,\sqrt{|g(x)|}\,\langle x|\,M\,|x\rangle\,.$$

We shall use the DeWitt-Schwinger representation (7.14) of the Green function:

$$G_F(x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi)^{(D+1)/2}}\int_0^\infty ds\,(is)^{-(D+1)/2}e^{-im^2s+\sigma(x,x')/(2is)}F(x,x';is).$$

By using the result obtained before, we can write the following integral formula

$$G_F = -K^{-1} = -i\int_0^\infty ds\,e^{-iKs}.$$

Comparing with the DeWitt-Schwinger representation one gets

$$\langle x|\,e^{-iKs}\,|x'\rangle = i\frac{\Delta^{1/2}(x,x')}{(4\pi)^{(D+1)/2}}(is)^{-(D+1)/2}e^{-im^2s+\sigma(x,x')/(2is)}F(x,x';is).$$

From here it follows that

$$\langle x|\,(is)^{-1}e^{-iKs}\,|x'\rangle = \int_{m^2}^\infty dm^2\,i\frac{\Delta^{1/2}(x,x')}{(4\pi)^{(D+1)/2}}(is)^{-(D+1)/2}e^{-im^2s+\sigma(x,x')/(2is)}F(x,x';is).$$

Assuming that K has a small negative imaginary part, we have the reltion

$$i\int_\Lambda^\infty ds\,(is)^{-1}e^{-iKs} = -\mathrm{Ei}(-i\Lambda K),$$

where Ei is the exponential integral function. For small values of the argument one has the following asymptotic relation

$$\mathrm{Ei}(x) = \gamma + \ln(-x) + O(x),$$

with γ being Euler constant. Taking the limit $\Lambda \to 0$ we find

$$i\int_0^\infty ds\,(is)^{-1}e^{-iKs} = -\ln(K) = \ln(-G_F),$$

up to the addition of a metric-independent constant. Hence, the DeWitt-Schwinger representation can be written in the form

$$\begin{aligned}
\langle x|\ln(-G_F)\,|x'\rangle &= \langle x|\,i\int_0^\infty ds\,(is)^{-1}e^{-iKs}\,|x'\rangle \\
&= -\int_{m^2}^\infty dm^2\int_0^\infty ds\,\frac{\Delta^{1/2}(x,x')}{(4\pi)^{(D+1)/2}}(is)^{-(D+1)/2}e^{-im^2s+\sigma(x,x')/(2is)}F(x,x';is) \\
&= -\int_{m^2}^\infty dm^2\,G_F^{DS}(x,x').
\end{aligned}$$

By making use of (7.19), for the effective action we find

$$W = \frac{i}{2}\int d^{D+1}x\,\sqrt{|g(x)|}\lim_{x'\to x}\int_{m^2}^\infty dm^2\,G_F^{DS}(x,x').$$

Changing the integrations order, in the limit $x' \to x$ one obtains

$$W = \frac{i}{2}\int_{m^2}^\infty dm^2\int d^{D+1}x\,\sqrt{|g(x)|}\,G_F^{DS}(x,x). \tag{7.20}$$

The integral $d^{D+1}x$ is seen the expression corresponding to the one-loop Feynman diagram. Related to this, W is called the one-loop effective action.

On the base (7.20), we may define the effective Lagrangian as

$$L_{\text{eff}}(x) = \frac{i}{2} \lim_{x' \to x} \int_{m^2}^{\infty} dm^2 \, G_F^{DS}(x, x'). \tag{7.21}$$

The expression on the right-hand side is divergent at the lower limit of the integral over s. The convergence in the upper limit is guaranteed by adding $-i\varepsilon$ in the DeWitt-Schwinger representation of Feynman Green function G_F. In four dimensions, the divergences may come from first three terms of the DeWitt-Schwinger expansion

$$L_{\text{div}} = - \lim_{x' \to x} \frac{\Delta^{1/2}(x, x')}{32\pi^2} \int_0^{\infty} \frac{ds}{s^3} e^{-i(m^2 s - \sigma/2s)} [a_0(x, x') + a_1(x, x') is + a_2(x, x')(is)^2].$$

The terms involving the coefficients $a_n(x, x')$ with $n \geqslant 3$ give finite contributions in the limit $x' \to x$. From the expressions (7.11) for the coefficients with $n \leqslant 2$ it follows that the expression in square brackets is entirely geometrical. It is built out of the Riemann tensor $R_{\mu\nu\alpha\beta}$ and its contractions. The divergences are a consequence of the ultraviolet behaviour of the field modes. Short wavelength modes only probe the local geometry and they are not sensitive to the global features of the spacetime, such as the topology. Additionally, they are also independent of the quantum state considered.

Because the divergent part of the effective Lagrangian L_{div} is purely geometrical, it can be considered as a contribution to the gravitational Lagrangian rather than the quantum matter Lagrangian. Although it arises from the action of the quantum matter field, it behaves as a quantity constructed solely from the gravitational field. This will not be true for the remaining, finite part of L_{eff}, which contains the contributions from the long wavelength modes as well.

Chapter 8

Renormalization in the effective action

8.1 Divergences

The further procedure for removing the divergences is similar to that used in quantum field theory in Minkowski spacetime. That is to present the divergent terms in the form $\infty\times$object and to include similar term in the bare Lagrangian. In the case under consideration we require is to display the divergent terms in the form $\infty\times$ geometrical object. From the adiabatic expansion, in $(D+1)$-dimensional spacetime for the effective Lagrangian on has

$$L_{\text{eff}} = \lim_{x'\to x} \frac{\Delta^{1/2}(x,x')}{2(4\pi)^{(D+1)/2}} \sum_{j=0}^{\infty} a_j(x,x') \int_0^\infty ds\, i(is)^{j-(D+3)/2} e^{-i(m^2 s - \sigma/2s)}.$$

In the coincidence limit $\sigma \to 0$, the divergent contributions come from the first $(D+3)/2$ terms. In the dimensional regularization procedure D is treated as a variable which can be analytically continued throughout the complex plane. Considering values of D for which the integrals are convergent, we can take the limit $x' \to x$. This gives

$$\begin{aligned} L_{\text{eff}} &\approx \frac{i}{2}(4\pi)^{-(D+1)/2} \sum_{j=0}^{\infty} a_j(x) \int_0^\infty ds\, (is)^{j-(D+3)/2} e^{-im^2 s} \\ &= \frac{1}{2}(4\pi)^{-(D+1)/2} \sum_{j=0}^{\infty} a_j(x) (m^2)^{(D+1)/2-j} \Gamma(j-(D+1)/2), \end{aligned}$$

with the functions $a_j(x) = a_j(x,x)$ and $a_j(x,x')$ given by (7.10) and (7.11).

In order to keep the dimension for L_{eff} as $(\text{length})^{-4}$ for general values of D, we introduce an arbitrary mass scale μ in the corresponding expression:

$$L_{\text{eff}} \approx -\frac{1}{2}(4\pi)^{-(D+1)/2} (m/\mu)^{D-3} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma(j-(D+1)/2).$$

In this representation, the divergences in the limit $D \to 3$ appear in the form of poles of the gamma functions:

$$\begin{aligned} \Gamma(-(D+1)/2) &= \frac{4}{(D+1)(D-1)}\left(\frac{2}{3-D} - \gamma\right) + \mathcal{O}(D-3), \\ \Gamma(1-(D+1)/2) &= \frac{2}{1-D}\left(\frac{2}{3-D} - \gamma\right) + \mathcal{O}(D-3), \\ \Gamma(2-(D+1)/2) &= \frac{2}{3-D} - \gamma + O(D-3). \end{aligned}$$

By taking into account the expansion

$$(m/\mu)^{D-3} = 1 + \frac{D-3}{2}\ln(m^2/\mu^2) + O((D-3)^2),$$

the part corresponding to the diverging contributions in the effective Lagrangian is presented as

$$\begin{aligned} L_{\rm div} &= -(4\pi)^{-(D+1)/2}\left\{\frac{1}{D-3} + \frac{1}{2}\left[\gamma + \ln(m^2/\mu^2)\right]\right\} \\ &\times \left[\frac{4m^4 a_0(x)}{(D+1)(D-1)} - \frac{2m^3 a_1(x)}{D-1} + a_2(x)\right]. \end{aligned}$$

Here, the functions $a_j(x)$ are given by the expressions

$$\begin{aligned} a_0(x) &= 1,\ a_0(x) = (1/6-\xi)R, \\ a_2(x) &= \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^2. \end{aligned}$$

The effective Lagrangian $L_{\rm eff}$, coming from the scalar field, is a part of the total Lagrangian. The divergences in this part are purely geometrical and we can try to absorb it into the gravitational Lagrangian.

For the total gravitational Lagrangian, including the terms coming from the divergent terms in $L_{\rm eff}$, we get

$$-\left(A + \frac{\Lambda_B}{8\pi G_B}\right) + \left(B + \frac{1}{16\pi G_B}\right)R - \frac{a_2(x)}{(4\pi)^{n/2}}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln(m^2/\mu^2)\right]\right\}, \tag{8.1}$$

where

$$\begin{aligned} A &= \frac{4m^4}{(4\pi)^{(D+1)/2})(D^2-1)}\left\{\frac{1}{D-3} + \frac{1}{2}\left[\gamma + \ln(m^2/\mu^2)\right]\right\}, \\ B &= \frac{2m^2(1/6-\xi)}{(4\pi)^{(D+1)/2}(D-1)}\left\{\frac{1}{D-3} + \frac{1}{2}\left[\gamma + \ln(m^2/\mu^2)\right]\right\}. \end{aligned}$$

As seen, the effect of the quantum scalar field is to to renormalize the cosmological constant from Λ_B to

$$\Lambda \equiv \Lambda_B + \frac{32\pi G_B m^4}{(4\pi)^{(D+1)/2})(D^2-1}\left\{\frac{1}{D-3} + \frac{1}{2}\left[\gamma + \ln(m^2/\mu^2)\right]\right\}.$$

We never see the separate terms in the right-hand side in isolation. A physical observation will only yield the renormalized value presented by Λ. We need not ask about the value Λ_B, nor worry about the fact that the term in curly brackets diverges in the limit $D \to 3$. 'Bare' cosmological constant Λ_B is never observed. The divergent part $L_{\rm div}$ also leads to the renormalization of the gravitational constant:

$$G = G_B/(1 + 16\pi G_B B).$$

The last term in (8.1) is not to be found in the usual Einstein Lagrangian. It gives higher order correction to General relativity. With these corrections in the Lagrangian, the left-hand side of the equations for the gravitational field is modified to

$$R_{\mu\nu} - Rg_{\mu\nu}/2 + \Lambda g_{\mu\nu} + \alpha^{(1)}H_{\mu\nu} + \beta^{(2)}H_{\mu\nu} + \gamma H_{\mu\nu}. \tag{8.2}$$

The new terms are given by the expressions

$$
\begin{aligned}
{}^{(1)}H_{\mu\nu} &= \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^{D+1}x\,\sqrt{|g|}R^2 \\
&= 2\nabla_\nu\nabla_\mu R - 2g_{\mu\nu}\Box R - g_{\mu\nu}R^2/2 + 2RR_{\mu\nu}, \\
{}^{(2)}H_{\mu\nu} &= \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^{D+1}x\,\sqrt{|g|}R_{\alpha\beta}R^{\alpha\beta} \\
&= \nabla_\nu\nabla_\mu R - g_{\mu\nu}\Box R/2 - \Box R_{\mu\nu} - cR_{\alpha\beta}R^{\alpha\beta}/2 + 2R^{\alpha\beta}R_{\alpha\beta\mu\nu},
\end{aligned}
\tag{8.3}
$$

and

$$
\begin{aligned}
H_{\mu\nu} &= \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^{D+1}x\,\sqrt{|g|}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\
&= -\frac{1}{2}g_{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + 2R_{\mu\alpha\beta\gamma}R_\nu^{\alpha\beta\gamma} - 4\Box R_{\mu\nu} \\
&\quad +2\nabla_\nu\nabla_\mu R - 4R_{\mu\alpha}R^\alpha_\nu + 4R^{\alpha\beta}R_{\alpha\mu\beta\nu}.
\end{aligned}
\tag{8.4}
$$

The generalized Gauss-Bonnet theorem states that the quantity

$$\int d^{D+3}x\,\sqrt{|g|}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + R^2 - 4R_{\alpha\beta}R^{\alpha\beta}\right)$$

is a topological invariant (called the Euler number) in $D = 3$. From here it follows that its variation with respect to the metric tensor vanishes identically. From here the following relation is obtained:

$$H_{\mu\nu} = -\,{}^{(1)}H_{\mu\nu} + 4\,{}^{(2)}H_{\mu\nu}.$$

In (8.2), the coefficients α, β and γ all contain divergent terms of the form $(\cdots)/(D-3$. In order to have a possibility for the renormalization, one should introduce terms of adiabatic order 4 into the original gravitational Lagrangian wit bare coefficients a_B, b_B, c_B. The divergent terms involving α, β, γ can be absorbed to yield renormalized coefficients a, b, c. Only two of these coefficients are independent, so we may choose $c = 0$. The values of a and b can only be determined from experiment. In principal there is no reason why these renormalized quantities may not be set equal to zero, thus leading to General Relativity. As seen, quantum field theory indicates that terms involving higher derivatives of the metric are a priori expected.

8.1.1 Renormalized effective Lagrangian

Having removed the divergent contribution from the Lagrangian, the remaining part. For the renormalized effective Lagrangian one has

$$L_{\rm ren} = L_{\rm eff} - L_{\rm div}.$$

In 4-dimensional spacetime the asymptotic expansion of the renormalized Lagrangian consists of all terms with $j \geqslant 3$. Taking the limit $x' \to x$ and putting $D = 3$, this expansion is written as

$$L_{\rm ren} \approx \frac{i}{32\pi^2}\int_0^\infty ds \sum_{j=1}^{3} a_j(x)(is)^{j-3}e^{-im^2 s}. \tag{8.5}$$

Integrating by parts three times, the expression in the right-hand side can be written in the form

$$
\begin{aligned}
&-\frac{i}{64\pi^2}\int_0^\infty ds\,\ln(is)\partial^3_{is}[F(x,x;is)e^{-im^2 s}] \\
&+\frac{i}{64\pi^2}\int_0^\infty ds\,\ln(is)\partial^3_{is}\{[a_0 + a_1(is) + a_2(is)^2]e^{-im^2 s}\}.
\end{aligned}
\tag{8.6}
$$

The second term is of the same form as $L_{\rm div}$. It will lead to the finite renormalization of the constants Λ, G, a, b, c. The renormalized effective Lagrangian is determined up to the terms with the structure constant$\times a_j$, $j = 1, 2, 3$, and hence we can drop the second term in (8.6).

For the same reason we need not worry about the choice of the mass scale μ. Rescaling μ changes $L_{\rm div}$ by a finite amount, but only by altering the coefficients of the geometrical terms a_0, a_1, a_2. In practice, one wood choose a fixed value of μ and use the results one's calculations with this value of μ to calibrate the instruments used to measure the constants Λ, G, a, b. Once these constants have been measured, further calculations using the same value of μ and the measured values of the constants can be used to make predictions about the outcome of experiments using the previously calibrated instruments. If the value of μ is changed one must either recalibrate one's instruments or else change the values Λ, G, a, b. The effect of either of these changes will leave invariant the predictions made about the outcome of experiments. Therefore we may write

$$L_{\rm ren} = -\frac{i}{64\pi^2}\int_0^\infty ds \ln(is)\partial_{is}^3[F(x,x;is)e^{-im^2s}],$$

where it is understood that any finite multiple of a_0, a_1, a_2 may be added to this expression. Having being derived from an asymptotic expansion for F, this expression cannot be regarded as the complete Lagrangian associated with the physical, renormalized $\langle T_{\mu\nu}\rangle$. In principle, the complete Lagrangian could be computed from this expression if the exact expression for F were available.

8.2 Higher spin fields

Higher spin fields can be considered in a similar way. First let us consider the case of a spinor field $\psi(x)$. The corresponding effective action can be written in terms of the bi-spinor Green function G_F defined by

$$S_F(x,x') = [i\gamma^\mu(x)\nabla_\mu^x + m]G_F(x,x').$$

For the effective action one has

$$W_{(1/2)} = \frac{i}{2}{\rm tr}[\ln(-G_F)].$$

In this expression, the trace is taken over spinor indices as well. The difference in sign compared with the scalar case is related to the anticommuting nature of the spinor fields. For a spin 1/2 field, the coefficients in the De Witt-Schwinger expansion for the Feynman Green function G_F are spinors:

$$\begin{aligned} a_0(x) &= \mathbb{I}, \\ a_1(x) &= -\frac{1}{12}R\,\mathbb{I}, \\ a_2(x) &= \left(\frac{1}{288}R^2 + \frac{1}{120}\Box R - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\mathbb{I} \\ &\quad + \frac{1}{48}\Sigma_{\mu\nu}\Sigma_{\rho\sigma}R^{\mu\nu\xi\lambda}R^{\rho\sigma}{}_{\xi\lambda}, \end{aligned} \tag{8.7}$$

where $\mathbb{I}$ is a unit spinor and

$$\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu\gamma_\nu].$$

Taking the trace over spinor indices one gets the following result:

$${\rm tr}\, a_2(x) = \frac{s}{720}\left(\frac{5}{2}R^2 + 6\Box R - \frac{7}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu}\right),$$

where s is the number of spinor components.

In the case of gauge fields the contribution from the gauge fields should be added as well. For example, for the electromagnetic field from the path-integral approach one finds

$$W_{\rm EM} = \frac{i}{2}{\rm tr}[\ln(D_F)] + W_{\rm ghost}.$$

Here, the contribution of the ghost field, $W_{\rm ghost}$, is obtained from the effective action for a minimally coupled scalar field with an additional coefficient -2. This corresponds to two anticommuting scalar ghost fields. For the electromagnetic field the coefficients in the expansion of $D_{F\mu\nu}$ are tensors and are given in the Feynman gauge $\zeta = 1$ by

$$\begin{aligned} a_{0\mu\nu}(x) &= g_{\mu\nu}, \\ a_{1\mu\nu}(x) &= \frac{1}{6}Rg_{\mu\nu} - R_{\mu\nu} \\ a_{2\mu\nu}(x) &= \frac{1}{6}RR_{\mu\nu} - \frac{1}{6}\Box R_{\mu\nu} + \frac{1}{2}R_{\mu\rho}R^{\rho}_{\nu} - \frac{1}{12}R^{\lambda\sigma\rho}{}_{\mu}R_{\lambda\sigma\rho\nu} \\ &+ \left(\frac{1}{72}R^2 - \frac{1}{30}\Box R - \frac{1}{180}R_{\rho\sigma}R^{\rho\sigma} + \frac{1}{180}R_{\rho\sigma\lambda\omega}R^{\rho\sigma\lambda\omega}\right)g_{\mu\nu}. \end{aligned} \tag{8.8}$$

The action for the system of gravitational field plus quantum matter the total action is decomposed as

$$S = S_g + W.$$

We transfer the divergent part of W into a suitably general S_g absorbing the infinities into renormalized coupling constants. Thus the action is rewritten as

$$S = (S_g)_{\rm ren} + W_{\rm ren}.$$

With this action, the semiclassical field equations take the form.

$$R_{\mu\nu} - Rg_{\mu\nu}/2 + \Lambda g_{\mu\nu} + a^{(1)}H_{\mu\nu} + b^{(2)}H_{\mu\nu} = -8\pi G\frac{\langle {\rm out}, 0|\, T_{\mu\nu}\, |0, {\rm in}\rangle}{\langle {\rm out}, 0|0, {\rm in}\rangle}$$

The expectation value of the energy-momentum tensor appears as the source of the gravity.

8.3 Conformal anomalies

8.3.1 Trace anomaly

In this section we will consider some features appearing in theories where the classical classical action is invariant under conformal transformations (4.6). In accordance with the definition of functional differentiation one has

$$S[\bar{g}_{\mu\nu}] = S[g_{\mu\nu}] + \int d^n x\, \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\rho\sigma}(x)}\delta\bar{g}^{\rho\sigma}(x).$$

For the conformal transformation the variation of the metric tensor is written as

$$\delta\bar{g}^{\rho\sigma}(x) = -2\Omega^{-3}(x)\delta\Omega(x)g^{\rho\sigma}(x) = -2\bar{g}^{\rho\sigma}(x)\Omega^{-1}(x)\delta\Omega(x).$$

Hence, for the action we can write

$$S[\bar{g}_{\mu\nu}] = S[g_{\mu\nu}] - \int d^{D+1}x\, \sqrt{-\bar{g}}T^{\rho}_{\rho}[\bar{g}_{\mu\nu}]\Omega^{-1}(x)\delta\Omega(x). \tag{8.9}$$

From this relation it follows that

$$T_{\rho}^{\rho}[g_{\mu\nu}] = -\frac{\Omega(x)}{\sqrt{|g|}} \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\Omega(x)}|_{\Omega=1}.$$

If the classical action is invariant under the conformal transformations, one has $\delta S[\bar{g}_{\mu\nu}]/\delta\Omega(x) = 0$ and consequently the classical energy-momentum tensor is traceless. Examples of this kind are massless scalar field with the curvature coupling parameter $\xi = \xi_c$, with ξ_c given by (5.6), massless fermionic field in arbitrary number of dimensions and electromagnetic field in 4-dimensional spacetime. The conformal transformations may be interpreted as local rescaling of lengths. The presence of a mass leads to a preferable length scale in the theory and, hence, will break the conformal invariance. Therefore, for conformally invariant fields at classical level we are led to the massless limit of the regularization and renormalization procedures discussed before.

The problem which appears in the massless limit is that all higher order ($j > 2$) terms in the DeWitt-Schwinger expansion of the effective Lagrangian are infrared divergent for $D = 3$ in this limit. However, this expansion can be still used to investigate the ultraviolet divergent terms arising from $j = 0, 1, 2$ in 4-dimensional spacetime. In the terms $j = 0, 1$ the substitution $m = 0$ can be done directly and these terms vanish. The only nonvanishing potentially ultraviolet divergent contribution comes from the $j = 2$ term:

$$\frac{1}{2}(4\pi)^{-(D+1)/2}(m/\mu)^{D-3}a_2(x)\Gamma(-(D-3)/2),$$

with the coefficient

$$a_2(x) = \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2.$$

For a conformally coupled field one has $\xi = \xi_c$ and the divergent term in the effective action is presented in the form

$$\begin{aligned} W_{\rm div} &= \frac{1}{2}(4\pi)^{-(D+1)/2}(m/\mu)^{D-3}\Gamma(-(D-3)/2)\int d^{D+1}x\sqrt{|g|}a_2(x) \\ &= \frac{1}{2}(4\pi)^{-(D+1)/2}(m/\mu)^{D-3}\Gamma(-(D-3)/2) \\ &\quad \times \int d^{D+1}x\sqrt{|g|}[\alpha F(x) + \beta G(x)] + \mathcal{O}(D-3), \end{aligned} \tag{8.10}$$

with the functions

$$\begin{aligned} F(x) &= R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - 2R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2, \\ G(x) &= R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2, \end{aligned} \tag{8.11}$$

and the coefficients

$$\alpha = \frac{1}{120}, \quad \beta = -\frac{1}{360}.$$

In the final expression for the divergent part of the effective action we have omitted the terms in $a_2(x)$ containing $\Box R$ and R^2. The first term is a total divergence, whereas the coefficient of the second one contains the factor $(D-3)^2$ when the conformal coupling is taken. In the limit $D \to 3$, this coefficient beats the $(D-3)^{-1}$ singularity coming from the gamma function.

In 4-dimensional spacetime (and only in that dimension), the function $F(x)$ in (8.11) is expressed in terms of the Weyl tensor $C_{\alpha\beta\rho\sigma}$ as

$$F(x) = C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma}.$$

Recall that the Weyl tensor, defined by the relation

$$\begin{aligned} C_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + \frac{1}{6} R\left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right) \\ &+ \frac{1}{2}\left(g_{\alpha\delta} R_{\beta\gamma} + g_{\beta\gamma} R_{\alpha\delta} - g_{\alpha\gamma} R_{\beta\delta} - g_{\beta\delta} R_{\alpha\gamma}\right), \end{aligned} \tag{8.12}$$

is the traceless part of the Riemann tensor. It is invariant under the conformal transformations of the metric tensor. In spacetime dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions $\geqslant 4$, the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension $\geqslant 4$, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. In (8.10), the integral

$$\int d^4x \sqrt{|g|} G$$

is a topological invariant. This quantity remain invariant under conformal transformations as well. Now we see that in 4-dimensional spacetime and for a massless conformally coupled field, the divergent part of the effective action, $W_{\rm div}$, is invariant under conformal transformations.

However, though the total action W is conformally invariant, it should be taken into account that $W_{\rm div}$ is not conformally invariant for $D \neq 3$. We should not relax the regularization and take $D = 3$ before the evaluation of physical quantities of interest. For example, in the evaluation of the energy-momentum tensor we can use the identities

$$\begin{aligned} \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^{D+1}x \sqrt{|g|} F(x) &= -(D-3)(F(x) - \frac{2}{3}\Box R) \\ \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^{D+1}x \sqrt{|g|} G(x) &= -(D-3) G(x). \end{aligned}$$

As a result, for the trace one gets

$$\begin{aligned} \langle T^{\mu}_{\mu} \rangle_{\rm div} &= \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta W_{\rm div}}{\delta g^{\mu\nu}} = \frac{1}{2} (4\pi)^{-(D+1)/2} (m/\mu)^{D-3} (3-D) \\ &\times \Gamma(-(D-3)/2)[\alpha(F - \frac{2}{3}\Box R) + \beta G] + O(D-3). \end{aligned}$$

In the limit $D \to 3$ this gives

$$\langle T^{\mu}_{\mu} \rangle_{\rm div} = \frac{1}{16\pi^2}[\alpha(F(x) - \frac{2}{3}\Box R) + \beta G(x)].$$

This result is local and does not depend on the state of a quantum field. It depends only on the geometry at x.

Because the total effective action W is conformally invariant in the massless conformally coupled limit, the expectation value of the trace of the total energy-momentum tensor vanishes:

$$\langle T^{\mu}_{\mu} \rangle_{m=0,\xi=1/6} = -\frac{\Omega(x)}{\sqrt{|g|}} \frac{\delta W[\bar{g}_{\mu\nu}]}{\delta \Omega(x)} |_{m=0,\xi=1/6,\Omega=1} = 0.$$

As we have shown, the divergent part has acquired the trace. Consequently, the renormalized expectation value should also have trace nonzero trace:

$$\begin{aligned} \langle T^{\mu}_{\mu} \rangle_{\rm ren} &= -\frac{a_2}{16\pi^2} = -\frac{1}{16\pi^2}[\alpha(F - \frac{2}{3}\Box R) + \beta G] \\ &= -\frac{1}{2880\pi^2}\left(R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} - R_{\alpha\beta} R^{\alpha\beta} - \Box R^2\right). \end{aligned} \tag{8.13}$$

Trace appeared in the theory even though the classical energy-momentum tensor is traceless, and even though W and $W_{\rm div}$ remain conformally invariant in four dimensions. This is a consequence, of the nonconformal nature of $W_{\rm div}$ away from $D=3$. This leaves a finite imprint at $D=4$ due to the $1/(D-3)$ divergent nature of $W_{\rm div}$. This result is known as a conformal, or trace, anomaly. We have discussed the anomaly on the base of the dimensional regularization. All the regularization schemes predict the same conformal anomaly for the scalar field.

The trace anomaly can also be generalized to arbitrary dimensions. When D is an even number, the corresponding effective Lagrangian, given by

$$L_{\rm eff} \approx \frac{1}{2}(4\pi)^{-(D+1)/2}\sum_{j=0}^{\infty} a_j(x) m^{(D+1)-2j}\Gamma(j-(D+1)/2),$$

is finite. Consequently, there is no anomaly in odd-dimensional spacetimes. For odd values of the spatial dimension D, equal to D_0, only the first $(D_0+3)/2$ terms are ultraviolet divergent. Among these terms, in the limit $m\to 0$, all but $a_{(D_0+1)/2}$ term vanish at $D=D_0$. The corresponding anomalous trace is given by the expression

$$\langle T^\mu_\mu\rangle_{\rm ren} = -\frac{a_{(D_0+1)/2}}{(4\pi)^{(D_0+1)/2}}.$$

For $D_0=1$ we have

$$\langle T^\mu_\mu\rangle_{\rm ren} = -\frac{R}{24\pi}.$$

Similar results can be obtained for higher spin fields. In the case $D_0=3$, the trace is presented as

$$\langle T^\mu_\mu\rangle_{\rm ren} = -\frac{(-1)^{2A+2B}}{16\pi^2}{\rm tr}\, a_2(A,B),$$

where (A,B) labels the representation of the Lorentz group under which the field transforms. The trace can be written in terms of four parameters as

$$\langle T^\mu_\mu\rangle_{\rm ren} = \frac{1}{2880\pi^2}[aC_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma} + b(R_{\alpha\beta}R^{\alpha\beta} - R^2/3) + c\Box R^2 + dR^2]. \qquad (8.14)$$

The coefficients a, b, c, d are expressed as simple polynomials in A and B. Some of the more important results for fields with spins $\leqslant 2$ are listed in the table. In general, a higher-spin physical field will not correspond to a single representation (A,B) of the Lorentz group, but will be a linear combination of several such representations. For example, the electromagnetic field contains scalar ghost contributions, so to obtain the electromagnetic anomaly one must subtract from the $(1/2,1/2)$ anomaly twice the $(0,0)$ anomaly. In the table we list these various physical combinations for the massless fields with spins $\leqslant 2$.

(A,B)	a	b	c	d
$(0,0)$	-1	-1	$6-30\xi$	$-90(\xi-1/6)^2$
$(1/2,0)$	$-7/4$	$-11/2$	3	0
$(1/2,1/2)$	11	-64	-6	-5
$(1,0)$	-33	27	12	$-5/2$
$(1,1/2)$	$291/4$	$\times$	$\times$	$61/8$
$(1,1)$	-189	$\times$	$\times$	$-747/4$

In the table, for the case of the spin 1/2 the results are given for two-component spinors. In the case of 4-components spinors the coefficients should be multiplied by a factor 2. The crosses show that the consistency conditions for higher spin fields require vanishing of the corresponding geometrical object in the expression for the anomal trace. In the table, the anomal contributions are presented only. In the case of conformally non-invariant field, additional contributions are present. For example, in the case of a scalar field with $\xi \neq 1/6$, one has a non-anomal contribution

$$(6\xi - 1)\left[\langle \nabla_\mu \phi \nabla^\mu \phi \rangle + \xi R \langle \phi^2 \rangle \right].$$

The latter depends on the choice of quantum state.

In the general case, physical fields with higher spins are presented as liner combinations separate representations. For example, the electromagnetic field contains contributions from scalar ghosts and it is necessary to subtract from the anomaly for the field $(1/2, 1/2)$ the doubled anomaly for the field $(0, 0)$. The coefficients in the trace anomalies for physical massless fields having spins $\leqslant 2$ are listed in the table below. For spins 1 and 2 the notation $(0, 0)$ corresponds to a minimally coupled scalar field.

Spin	(A, B)	a	b	c	d
0	$(0, 0)$	-1	-1	$6 - 30\xi$	$-90(\xi - 1/6)^2$
1/2	$(1/2, 0)$	$-7/4$	$-11/2$	3	0
1	$(1/2, 1/2) - 2(0, 0)$	13	-62	-18	0
3/2	$(1, 1/2) - 2(1/2, 0)$	233/4	$\times$	$\times$	61/8
2	$(1, 1) + (0, 0) - 2(1/2, 1/2)$	-212	$\times$	$\times$	$-747/4$

In the theory, conformally invariant in arbitrary number of dimensions, by using the dimensional regularization, we can find consistency relations between the coefficients a, b, c, d. These relations are obtained by comparing (8.13) and (8.14):

$$a = -180\,(\alpha + \beta)\,, \quad b = 360\beta, \quad c = 120\alpha, \quad d = 0.$$

From here we get the constraint

$$2a + b + 3c = 0.$$

$$26 - 62 - 54$$

The values of the coefficients α and β are presented in the table below.

Spin	(A, B)	α	β
0	$(0, 0)$	1/120	$-1/360$
1/2	$(1/2, 0)$	1/40	$-11/720$
1	$(1/2, 1/2) - 2(0, 0)$	$-1/15$	$-31/180$

8.3.2 Energy-momentum tensors in conformally related problems

For a conformally invariant background and for a conformally invariant field, the anomalous trace determines the entire energy-momentum tensor once the quantum state is specified. In order to see that let us use the relation (8.9) for the renormalized effective action:

$$W_{\rm ren}[\bar{g}_{\mu\nu}] = W_{\rm ren}[g_{\mu\nu}] - \int d^{D+1}x\,\sqrt{-\bar{g}}\,\langle T^\rho_\rho[\bar{g}_{\mu\nu}]\rangle_{\rm ren}\,\Omega^{-1}(x)\delta\Omega(x).$$

By taking into account that for the expectation value of the energy-momentum tensor one has

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{|g|}}\frac{\delta W}{\delta g^{\mu\nu}},$$

and using

$$\bar{g}^{\mu\nu}\frac{\delta}{\delta \bar{g}^{\mu\nu}} = g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}},$$

one gets the following relation

$$\langle T^\nu_\mu[\bar{g}_{\mu\nu}]\rangle_{\rm ren} = \sqrt{\frac{g}{\bar{g}}}\,\langle T^\nu_\mu[g_{\mu\nu}]\rangle_{\rm ren} - \frac{2}{\sqrt{|\bar{g}|}}\bar{g}^{\nu\sigma}\frac{\delta}{\delta \bar{g}^{\mu\sigma}}\int d^{D+1}x\,\sqrt{|\bar{g}|}\,\langle T^\rho_\rho[\bar{g}_{\mu\nu}]\rangle_{\rm ren}\,\frac{\delta\Omega(x)}{\Omega(x)}.$$

For conformally invariant theories for the trace in the integrand one has

$$\left\langle T_\rho^\rho[\bar{g}_{\mu\nu}]\right\rangle_{\rm ren} = -\left\langle T_\rho^\rho[\bar{g}_{\mu\nu}]\right\rangle_{\rm div} = \frac{\Omega(x)}{\sqrt{|\bar{g}|}}\frac{\delta W_{\rm div}[\bar{g}_{\mu\nu}]}{\delta\Omega(x)}.$$

It is completely determined by the local geometry and does not depend on the quantum state under consideration. Substituting into the integral and by making use the relation

$$W_{\rm div}[\bar{g}_{\mu\nu}] - W_{\rm div}[g_{\mu\nu}] = \int d^{D+1}x\, \delta W_{\rm div}[\bar{g}_{\mu\nu}],$$

we get

$$\left\langle T_\mu^\nu[\bar{g}_{\mu\nu}]\right\rangle_{\rm ren} = \sqrt{\frac{g}{\bar{g}}}\left\langle T_\mu^\nu[g_{\mu\nu}]\right\rangle_{\rm ren} - \frac{2}{\sqrt{|\bar{g}|}}\bar{g}^{\nu\sigma}\frac{\delta W_{\rm div}[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\sigma}} + \frac{2}{\sqrt{|\bar{g}|}}g^{\nu\sigma}\frac{\delta W_{\rm div}[g_{\mu\nu}]}{\delta g^{\mu\sigma}}. \tag{8.15}$$

This gives a simple relation between the expectation values of the energy-momentum tensor in two conformally related problems.

In a 2-dimensional spacetime, for a conformally coupled field one has $\xi = 0$. By taking into account that

$$\Gamma(-(D-1)/2) = -2/(D-1) + O(1),$$

for the divergent part in the effective action one finds

$$\begin{aligned} W_{\rm div}[g_{\mu\nu}] &= -\frac{1}{4\pi(D-1)}\int d^{D+1}x\sqrt{|g(x)|}a_1(x) \\ &= -\frac{1}{24\pi(D-1)}\int d^{D+1}x\sqrt{|g(x)|}R(x). \end{aligned}$$

Substituting this into (8.15), we get the following relation

$$\left\langle T_\mu^\nu[\bar{g}_{\mu\nu}]\right\rangle_{\rm ren} = \sqrt{\frac{g}{\bar{g}}}\left\langle T_\mu^\nu[g_{\mu\nu}]\right\rangle_{\rm ren} + \frac{\bar{G}_\mu^\nu - G_\mu^\nu}{12\pi(D-1)},$$

where G_μ^ν is the Einstein tensor. By using the relation between the curvature tensors for conformally related metric tensors, this formula may be rewritten in the form

$$\begin{aligned} \left\langle T_\mu^\nu[\bar{g}_{\mu\nu}]\right\rangle_{\rm ren} &= \sqrt{\frac{g}{\bar{g}}}\left\langle T_\mu^\nu[g_{\mu\nu}]\right\rangle_{\rm ren} + \frac{1}{12\pi}\left[g^{\nu\rho}\left(\frac{\nabla_\mu\nabla_\rho\Omega}{\Omega^3} - 2\frac{\nabla_\mu\Omega\nabla_\rho\Omega}{\Omega^4}\right)\right. \\ &\quad \left. +\delta_\mu^\nu g^{\rho\sigma}\left(\frac{3\nabla_\rho\Omega\nabla_\sigma\Omega}{2\Omega^4} - \frac{\nabla_\sigma\nabla_\rho\Omega}{\Omega^3}\right)\right]. \end{aligned} \tag{8.16}$$

This relation is valid for a two-components massless spinor field as well.

All two-dimensional spacetimes are conformally flat and, hence, one can write

$$g_{\mu\nu} = C(x)\eta_{\mu\nu}.$$

Taking in (8.16) $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Omega = C^{1/2}$, we present the expectation value of the energy-momentum tensor in an arbitrary 2-dimensional spacetime in terms of the expectation value in flat spacetime. In particular, writing the flat spacetime metric in in null coordinates,

$$ds^2 = C(u,v)dudv,$$

the relation between the energy-momentum tensors takes the form

$$\left\langle T_\mu^\nu[g_{\mu\nu}]\right\rangle_{\rm ren} = \frac{1}{\sqrt{-g}}\left\langle T_\mu^\nu[\eta_{\rho\sigma}]\right\rangle_{\rm ren} + \theta_\mu^\nu - \frac{R}{48\pi}\delta_\mu^\nu,$$

where

$$\theta_{\mu\nu} = -\frac{C^{1/2}}{12\pi}\,\mathrm{diag}(\partial_u^2 C^{-1/2}, \partial_v^2 C^{-1/2}).$$

If the state in flat spacetime is a vacuum state, then the corresponding state in the curved spacetime is a conformal vacuum. If the flat spacetime vacuum is the Minkowskian vacumm state then $\left\langle T_\mu^\nu[\eta_{\rho\sigma}]\right\rangle_{\mathrm{ren}} = 0$. If the curved spacetime is conformally related to a part of the Minkowski spacetime, then $\left\langle T_\mu^\nu[\eta_{\rho\sigma}]\right\rangle_{\mathrm{ren}}$ can be different from zero.

In 4-dimensional spacetime one has

$$W_{\mathrm{div}} = -\frac{1}{16\pi^2(D-3)}\int d^{D+1}x\sqrt{-g}[\alpha F(x) + \beta G(x)] + O(1).$$

Taking as $g_{\mu\nu}$ flat spacetime metric and performing the functional differentiation we get the following relation

$$\left\langle T_\mu^\nu[\bar{g}_{\mu\nu}]\right\rangle_{\mathrm{ren}} = \left(\frac{g}{\bar{g}}\right)^{1/2}\left\langle T_\mu^\nu[g_{\mu\nu}]\right\rangle_{\mathrm{ren}} - \frac{1}{16\pi^2}\left[\frac{\alpha}{9}\,{}^{(1)}H_\mu^\nu + 2\beta\,{}^{(3)}H_\mu^\nu\right], \tag{8.17}$$

with

$$\begin{aligned} {}^{(3)}H_\mu^\nu &= \frac{1}{12}R^2 g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu} \\ &= R_\mu^\rho R_{\rho\nu} - \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \frac{1}{4}R^2 g_{\mu\nu}. \end{aligned}$$

For a scalar field, by taking into account the values for the coefficients α and β, the relation between the energy-momentum tensors take the form

$$\left\langle T_\mu^\nu[\bar{g}_{\mu\nu}]\right\rangle_{\mathrm{ren}} = \left(\frac{g}{\bar{g}}\right)^{1/2}\left\langle T_\mu^\nu[g_{\mu\nu}]\right\rangle_{\mathrm{ren}} - \frac{1}{2880\pi^2}\left[\frac{1}{6}\,{}^{(1)}H_\mu^\nu - {}^{(3)}H_\mu^\nu\right]. \tag{8.18}$$

Recall that in conformally flat spacetimes the Weyl tensor vanishes and the Riemann tensor is completely determined by the Ricci tensor (see the relation (8.12)). In addition, in conformally flat spacetimes the locally conserved tensors ${}^{(1)}H_\mu^\nu$ and ${}^{(2)}H_\mu^\nu$ are not independent:

$${}^{(2)}H_\mu^\nu = \frac{1}{3}\,{}^{(1)}H_\mu^\nu.$$

Note that the tensor ${}^{(3)}H_\mu^\nu$ is locally conserved in conformally flat spacetimes only.

8.3.3 Examples

Let us consider some examples for the relation between conformally related problems. We start with FRW spacetimes given by the line element (4.16). In terms of the conformal time coordinate η, the corresponding line element is written in the form (4.18) with the scale factor $C(\eta) = a(t)$. In accordance with the symmetry of the problem for the expectation value of the energy-momentum tensor we have

$$\left\langle T_1^1\right\rangle = \left\langle T_2^2\right\rangle = \left\langle T_3^3\right\rangle, \quad \left\langle T_\mu^\nu\right\rangle = 0, \quad \mu \neq \nu.$$

For the components of the tensors appearing in the relation for conformally related problems one has

$$
\begin{aligned}
{}^{(1)}H_{00} &= \frac{1}{C}\left(-9HH'' + \frac{9}{2}H'^2 + \frac{27}{8}H^4 + 9kH^2 - 18k^2\right), && (8.19)\\
{}^{(1)}H_{11} &= \frac{u}{C}\left(6H''' - 3HH'' + \frac{3}{2}H'^2 - 9H^2H' + \frac{9}{8}H^4 - 12kH' + 3kH^2 - 6k^2\right),\\
{}^{(3)}H_{00} &= \frac{1}{C}\left(\frac{3}{16}H^4 + \frac{3}{2}kH^2 + 3k^2\right),\\
{}^{(3)}H_{11} &= \frac{u}{C}\left(-\frac{1}{2}H^2H' + \frac{1}{16}H^4 - 2kH' + \frac{1}{2}kH^2 + k^2\right), && (8.20)
\end{aligned}
$$

where $H = C'/C$, $u = (1 - kr^2)^{-1}$ and the prime stands for the derivative with respect to the conformal time η.

For the Einstein universe one has $C = a^2 = \text{const}$ and, hence, $H = 0$. The corresponding vacuum state is conformally related to the Minkowski vacuum and in (8.17) $\langle T_\mu^\nu[g_{\mu\nu}]\rangle_{\text{ren}} = 0$. From the general formulas (8.20) one gets

$$
{}^{(1)}H_0^0 = -18/a^4,\ {}^{(1)}H_1^1 = 6/a^4,\ {}^{(3)}H_0^0 = 3/a^4,\ {}^{(3)}H_1^1 = -1/a^4. \tag{8.21}
$$

Plugging into (8.17) we find

$$
\langle T_\mu^\nu\rangle_{\text{ren}} = \frac{\alpha - 3\beta}{8\pi^2 a^4}\text{diag}(1, -1/3, -1/3, -1/3).
$$

By taking into account the values of the coefficients, for the vacuum expectation value of the energy-momentum tensor one gets

$$
\langle T_\mu^\nu\rangle = \frac{p(s)}{2\pi^2 a^4}\text{diag}\left(1, -1/3, -1/3, -1/3\right), \tag{8.22}
$$

where the spin dependent coefficient is given by

$$
p(0) = \frac{1}{240},\quad p(1/2) = \frac{17}{960},\quad p(1) = \frac{11}{120}.
$$

For a closed de Sitter spacetime and for stationary de Sitter spacetime the scale factors are given by $a(t) = \alpha\cosh(t/\alpha)$ and $a(t) = e^{t/\alpha}$, respectively. The corresponding problems are conformally related to the ones in Minkowski spacetime and for the vacuum energy-momentum tensor we have

$$
\langle T_\mu^\nu\rangle = \frac{q(s)\delta_\mu^\nu}{960\pi^2\alpha^4}, \tag{8.23}
$$

with the coefficients $q(0) = 1$, $q(1/2) = 11/2$, and $q(1) = 62$. Note that in these cases both the background spacetime and the conformal vacuum are maximally symmetric and, as a result, $\langle T_\mu^\nu\rangle$ has the form (8.23) in all coordinate systems.

In some cases the conformal vacuum is related to the Rindler vacuum in flat spacetime (for a discussion of quantum fields in Rindler spacetime see below). In this case $\langle T_\mu^\nu[g_{\mu\nu}]\rangle_{\text{ren}} \neq 0$. The line element in Rindler coordinates is given by the expression

$$
ds^2 = \rho^2 d\tau^2 - d\rho^2 - dy^2 - dz^2,
$$

with $0 < \rho < \infty$. For a field with the spin s, the expectation value of the energy-momentum tensor in the Rinlder vacuum has the form [31]

$$
\langle T_\mu^\nu\rangle = \frac{h(s)}{2\pi^2\rho^4}\int_0^\infty dx\frac{x(x^2+s^2)}{e^{2\pi x} - (-1)^{2s}}\text{diag}\left(-1, 1/3, 1/3, 1/3\right), \tag{8.24}
$$

where $h(s)$ is the number of helicity states. One has $h(0) = 1$, $h(1/2) = 2$, $h(1) = 2$. Note that we have the following relation between the coefficients $p(s)$ and $h(s)$:

$$p(s) = h(s) \int_0^\infty dx \frac{x(x^2+s^2)}{e^{2\pi x} - (-1)^{2s}}. \tag{8.25}$$

Making a coordinate transformation

$$\begin{aligned} \rho &= \frac{u}{1 - ru\cos\theta}, \\ y &= \frac{ru\sin\theta\cos\varphi}{1 - ru\cos\theta}, \\ z &= \frac{ru\sin\theta\sin\varphi}{1 - ru\cos\theta}, \end{aligned}$$

with $u = 1/\sqrt{1+r^2}$, the line element is written as

$$ds^2 = \rho^2 \left[d\tau^2 - u^2 dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]. \tag{8.26}$$

This shows that the Rindler spacetime is conformally related to the Robertson-Walker spacetime with negative spatial curvature, $k = -1$. Hence, having the result (8.24), we can find the vacuum expectation value of the energy-momentum tensor for conformally coupled fields in the Robertson-Walker spacetime with the help of the relation (8.17). Note that, in this transformation, the tensor (8.24) should be transformed to the coordinates (r, θ, φ). Let us consider a special case of the static Robertson-Walker spacetime with negative curvature space:

$$ds^2 = a^2 \left[d\tau^2 - u^2 dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

where $a =$ const. The geometrical contributions ${}^{(1)}H_\mu^\nu$ and ${}^{(3)}H_\mu^\nu$ are the same as those for the static Einstein universe and are given by (8.21). Hence, the second term in the right-hand side of (8.17) coincides with (8.22). For the vacuum expectation values of the energy-momentum tensor in the static Robertson-Walker spacetime with negative curvature space one gets

$$\langle T_\mu^\nu \rangle = (\rho/a)^4 \langle T_\mu^\nu \rangle_{\text{Rind}} + \langle T_\mu^\nu \rangle_{\text{Einst}} = 0,$$

and, hence, the corresponding energy-momentum tensor vanishes. This result is derived in [50] on the base of the direct evaluation by using the point-splitting method.

Another example of the spacetime conformally related to the Rindler one is the static de Sitter universe. The corresponding line element is given by (see below)

$$ds^2 = (1 - r^2/\alpha^2)dt^2 - \frac{dr^2}{1 - r^2/\alpha^2} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Introducing a new dimensionless radial coordinate r' in accordance with

$$\frac{r}{\alpha} = \frac{r'}{\sqrt{1+r'^2}},$$

the line element is rewritten as

$$ds^2 = (\alpha^2 - r^2) \left[\alpha^{-2} dt^2 - u^2 dr^2 - r'^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

with $u = 1/\sqrt{1+r'^2}$. In this form, we see the conformal relation with the Rindler line element (8.26). The geometrical part in the expectation value of the energy-momentum tensor is completely determined by the local geometry and, at has been mentioned before, is given by (8.23) in all coordinate systems. Hence, from (8.17) we get

$$\begin{aligned}\langle T_\mu^\nu\rangle &= \frac{\rho^4\left\langle T_\mu^\nu\right\rangle_{\rm Rind}}{(\alpha^2-r^2)^2}+\frac{q(s)\delta_\mu^\nu}{960\pi^2\alpha^4}\\ &= -\frac{p(s)}{2\pi^2(\alpha^2-r^2)^2}{\rm diag}\,(1,-1/3,-1/3,-1/3)+\frac{q(s)\delta_\mu^\nu}{960\pi^2\alpha^4}.\end{aligned}$$

Note that the expecttaion value diverges on the horizon $r=\alpha$. Though the background geometry is maximally symmetric, this is not the case for the vacuum state under consideration. For that reason the vacuum energy-momentum tensor is not of the form const $\cdot\,\delta_\mu^\nu$.

8.4 Renormalization of the energy-momentum tensor

The renormalized expectation value of the energy-momentum tensor is among the most important quantities in quantum field theory on curved space-time. It appears as the source of gravity in semiclassical Einstein equations and is responsible for the backreaction of the quantum field on the spacetime geometry. The energy-momentum tensor involves products of quantum operators at the same spacetime point. Consequently, the corresponding expectation value is divergent and requires renormalization. In the discussion above we have shown that the formally divergent expectation value $\langle T_{\mu\nu}\rangle$ can be rendered finite by renormalization of coupling constants in the gravitational action. Divergent part is purely geometrical and can be regarded as part of the gravitational dynamics. In considering specific problems we need the finite part of the expectation value $\langle T_{\mu\nu}\rangle$. As we have demonstrated before, in conformally trivial systems the renormalized expectation value $\langle T_{\mu\nu}\rangle_{\rm ren}$ may be computed on the base of the trace anomaly. In more complicated problems special techniques are required.

Unlike to the divergent part, the renormalized expectation value $\langle T_{\mu\nu}\rangle_{\rm ren}$ depends on the global structure of the spacetime and on the quantum state under consideration. The formal discussion of renormalization we have presented above was based on the action functional. However, in practical calculations it is difficult to proceed in that way. The reason is that for the functional differentiation of renormalized effective action $W_{\rm ren}$ with respect to $g_{\mu\nu}$ it is generally necessary to know $W_{\rm ren}$ for all geometries $g_{\mu\nu}$. An exception is the conformally trivial case, where the energy-momentum tensor is determined by the scaling behaviour alone. Hence, in general, it is necessary to work directly with $\langle T_{\mu\nu}\rangle$. Exact analytical results can be obtained for highly symmetric background geometries only where the wave equations are exactly solvable. Examples of maximally symmetric spacetimes will be discussed below.

For the regularization and subsequent renormalization of the expectation values of physical quantities involving squares or higher powers of field operators or their derivatives, various methods have been developed. They include point-splitting regularization, dimensional regularization, zeta-function regularization, and adiabatic regularization in homogeneous spacetimes.

In practical calculations, among the most efficient regularization techniques is the point-splitting method. It is convenient to work with two-point functions, for example, the Hadamard one, $G^{(1)}(x,x')$. Renormalization may be carried out, by subtracting from $\langle T_{\mu\nu}\rangle$ terms up to adiabatic order $D+1$ in the corresponding DeWitt-Schwinger expansion $\langle T_{\mu\nu}\rangle_{\rm DS}$ which is formed by differentiation of $G^{(1)}_{\rm DS}(x,x')$. This is equivalent to renormalization of constants in the generalized Einstein action. Alternatively, one can obtain $\langle T_{\mu\nu}\rangle_{\rm ren}$ by acting on

$$G^{(1)}_{\rm ren}(x,x')=G^{(1)}(x,x')-\,{}^{(D+1)}G^{(1)}_{\rm DS}(x,x'),$$

with a differential operator obtained from the form of the classical energy-momentum tensor. In ${}^{(D+1)}G^{(1)}_{\rm DS}(x,x')$, only those terms should be kept which give a contribution to the energy-momentum tensor of adiabatic order $\leqslant D+1$. Thus the procedure for computing the renormalized energy-momentum tensor using point splitting can be summarized as follows:

1. Solve the field equation for a complete set of normal modes from which particle states may be defined.

2. Construct $G^{(1)}(x,x')$ as a mode sum.

3. Form $G^{(1)}_{\rm ren}(x,x') = G^{(1)}(x,x') - {}^{(D+1)}G^{(1)}_{\rm DS}(x,x')$ truncating the expansion of $G^{(1)}_{\rm DS}(x,x')$ at order $D+1$.

4. Operate on $G^{(1)}_{\rm ren}(x,x')$ to form $\langle 0|\, T_{\mu\nu}(x,x')\,|0\rangle_{\rm ren}$ discarding any terms of adiabatic order greater than $D+1$ which have appeared from differentiation of terms in ${}^{(D+1)}G^{(1)}_{\rm DS}(x,x')$.

5. Let $x' \to x$ and display the finite result $\langle T_{\mu\nu}(x)\rangle_{\rm ren}$.

The vacuum state $|0\rangle$ depends on the choice of the modes in the step 1. In some problems the function $G^{(1)}(x,x')$ can be found by solving directly the corresponding differential equation with appropriate boundary conditions. In the step 4, the differentiation of $G^{(1)}(x,x')$ is generally a complicated procedure. Formally one has

$$\langle T_{\mu\nu}(x)\rangle = \lim_{x\to x'} \mathcal{D}_{\mu\nu}(x,x')G^{(1)}(x,x'), \tag{8.27}$$

where the differential operator $\mathcal{D}_{\mu\nu}(x,x')$ in the right-hand side is determined by the form of the classical energy-momentum tensor. By taking into account that the Hadamard function $G^{(1)}(x,x')$ is not a scalar function of x but a bi-scalar of the two spacetime points x, x', we see that the differential operator $\mathcal{D}_{\mu\nu}(x,x')$ is a non-local operator. In problems with higher spin fields, bi-spinors, bi-vectors are involved. For example, in the case of a scalar field, the term

$$\langle 0|\, \nabla_\mu\phi(x)\nabla_\nu\phi(x)\,|0\rangle$$

appears in the energy-momentum tensor. The part in the right-hand side of (8.27) corresponding to this term has the form

$$\lim_{x\to x'} \frac{1}{2}(\nabla_\mu\nabla_{\nu'} + \nabla_{\mu'}\nabla_\nu)G^{(1)}(x,x').$$

This shows that the resulting object is not a tensor, but a bi-vector.

The renormalized expectation value $\langle T_{\mu\nu}(x)\rangle_{\rm ren}$ should be a tensor to maintain general covariance. To construct a tensor from a bi-vector it is necessary to parallel transport the derivative vector back to the same spacetime point, which could be the midpoint between x, x', one of the end points or somewhere else. For curved backgrounds differences between parallel-transported and non transported results will arise, even when the points x, x' are made to coincide, from a σ^{-1} factor in the expansion of $G^{(1)}(x,x')$ multiplying by a σ-order transport correction. These complicated corrections have been worked out once and for all. It should be noted, however, that if $G^{(1)}$ is renormalized first, then all σ^{-1} terms are in any case removed, so any transport corrections are of order σ and vanish when we let $\sigma \to 0$ at the end of calculation. Thus, only if one insists on first constructing an unrenormalized EMT will the effects of parallel transport need to be taken into account in a practical calculation.

8.5 Wald axioms

In the discussion above we have shown how to extract from the one-loop divergent expressions finite results for the expectation value of the energy-momentum tensor. Of course, a number of questions remain. Can one consistently neglect higher order contributions? Wether the semiclassical theory makes sense at all. A possible approach to the physical significance of $\langle T_{\mu\nu}\rangle$ is to ask that if the semiclassical theory is to make physical sense, what criteria might one wish $\langle T_{\mu\nu}\rangle$ to satisfy? The approach of attempting to define a unique $\langle T_{\mu\nu}\rangle$ purely by imposing physical criteria (axioms) has been developed by Wald. Wald proposes that any physically meaningful $\langle T_{\mu\nu}\rangle$ should satisfy four reasonable conditions:

1. Covariant conservation
2. Causality
3. Standard results for off-diagonal elements
4. Standard results in Minkowski spacetime

Covariant conservation, $\nabla_\mu \langle T^\mu_\nu\rangle = 0$, is necessary if $\langle T_{\mu\nu}\rangle$ is to appear in the right-hand side of the gravitational field equations, as the left hand side is divergenceless. The precise statement of the causality axiom is: For a fixed 'in' state, $\langle T_{\mu\nu}\rangle$ at a given point depends only on the spacetime geometry to the causal part of the spacetime point. By this it is meant that the changes in in the metric structure of the spacetime outside the past null cone will not effect $\langle T_{\mu\nu}\rangle$. A time-reversed statement then applies to fixed 'out' states and changes in the geometry outside the future null cone. The condition 3 is simply the observation that as $\langle\Phi| T_{\mu\nu} |\Psi\rangle$ is finite for orthogonal states $\langle\Phi|\Psi\rangle$, the value of this quantity ought to be the usual one. Condition 4 means that the normal ordering procedure in Minkowski spacetime should be valid. It can be proved that if $\langle T_{\mu\nu}\rangle$ satisfies the first three of conditions, then it is unique to within a local conserved tensor.

Chapter 9

Quantum effects from topology and boundaries

9.1 2-dimensional spacetime with compact dimension

There has been a large interest to the physical problems with compact spatial dimensions. Several models of this sort appear in high energy physics, in cosmology and in condensed matter physics. In particular, many of high energy theories of fundamental physics, including supergravity and superstring theories, are formulated in spacetimes having extra compact dimensions which are characterized by extremely small length scales. These theories provide an attractive framework for the unification of gravitational and gauge interactions.

In the models with compact dimensions, the nontrivial topology of background space can have important physical implications in classical and quantum field theories, which include instabilities in interacting field theories, topological mass generation and symmetry breaking. The periodicity conditions imposed on fields along compact dimensions allow only the normal modes with suitable wavelengths. As a result of this, the expectation values of various physical observables are modified. In particular, many authors have investigated the effects of vacuum or Casimir energies and stresses associated with the presence of compact dimensions (for reviews see Refs. [32, 33, 34]). The topological Casimir effect is a physical example of the connection between quantum phenomena and global properties of spacetime. The Casimir energy of bulk fields induces a non-trivial potential for the compactification radius of higher-dimensional field theories providing a stabilization mechanism for the corresponding moduli fields and thereby fixing the effective gauge couplings. The Casimir effect has also been considered as a possible origin for the dark energy in both Kaluza-Klein type models and in braneworld scenario.

In this chapter we discuss some simple examples of exactly solvable problems of quantum field theory in non-Minkowksian backgrounds. First we consider examples for 2-dimensional spacetime with compact spatial sections (topology $R^1 \times S^1$). The line element coincides with that for Minkowski spacetime, but the spatial points x and $x + L$ are identified, where L is the periodicity length:

$$ds^2 = dt^2 - dx^2.$$

For a massless scalar field a complete set of mode functions obeying periodic boundary conditions along x ($\phi(t, x + L) = \phi(t, x)$) is given by the expression

$$\phi_k = (2L\omega)^{-1/2} e^{i(kx-\omega t)},\ k = 2\pi n/L,\ \omega = |k|,\ n = 0, \pm 1, \pm 2, \ldots$$

Because the filed modes are changed compared with the case of trivial topology, the expectation values of physical observables are modified. For a minimally coupled field the separate components of the energy-momentum tensor have the form

$$T_{00} = T_{11} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2,\ T_{01} = \frac{1}{2}\left(\partial_t\phi\partial_x\phi + \partial_x\phi\partial_t\phi\right)$$

We shall evaluate the vacuum expectation value $\langle 0_L | T_{\mu\nu} | 0_L\rangle$, where $|0_L\rangle$ is the vacuum associated with the modes given above. By using the mode sum formula

$$\langle 0_L | T_{\mu\nu} | 0_L\rangle = \sum_{n=-\infty}^{+\infty} T_{\mu\nu}\{\phi_k, \phi_k^*\},$$

for the energy-density we find

$$\langle 0_L | T_{00} | 0_L\rangle = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(\partial_t\phi_k\partial_t\phi_k^* + \partial_x\phi_k\partial_x\phi_k^*\right) = \frac{1}{2L}\sum_{n=-\infty}^{+\infty}\omega = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n$$

and the off-diagonal component vanishes. The energy density is clearly divergent. This was expected, as the spacetime under consideration suffers from the same ultraviolet divergence properties as Minkowski space. We renormalize the vacuum expectation value by subtracting corresponding quantity in Minkowski spacetime. For the mode functions in Minkowski spacetime one has

$$\phi_k^{(\mathrm{M})} = (4\pi\omega)^{-1/2} e^{i(kx-\omega t)},\ -\infty < k < +\infty,\ \omega = |k|$$

By using the mode sum, for the corresponding energy density we find

$$\begin{aligned}\langle 0_{\mathrm{M}} | T_{00} | 0_{\mathrm{M}}\rangle &= \frac{1}{2}\int_{-\infty}^{+\infty} dk (\partial_t\phi_k^{(\mathrm{M})}\partial_t\phi_k^{(\mathrm{M})*} + \partial_x\phi_k^{(\mathrm{M})}\partial_x\phi_k^{(\mathrm{M})*}) \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\, |k| = \frac{1}{2\pi}\int_0^{\infty} dk\, k\end{aligned}$$

For the renormalized vacuum expectation value we have

$$\langle T_{00}\rangle_{\mathrm{ren}} = \langle 0_L | T_{00} | 0_L\rangle - \langle 0_{\mathrm{M}} | T_{00} | 0_{\mathrm{M}}\rangle .$$

Various regularization procedures may be used to make finite the diverging quantities in the right-hand side. First let us consider the introduction of the cutoff function $e^{-\alpha\omega}$ with the cutoff parameter $\alpha > 0$. We write

$$\langle T_{00}\rangle_{\mathrm{ren}} = \lim_{\alpha\to 0}\left[\frac{2\pi}{L^2}\sum_{n=0}^{\infty} n e^{-2\pi\alpha n/L} - \frac{1}{2\pi}\int_0^{\infty} dk\, k e^{-\alpha k}\right].$$

By using the results

$$\begin{aligned}\sum_{n=0}^{\infty} n e^{-\beta n} &= \frac{e^{-\beta}}{(1-e^{-\beta})^2} = \frac{1}{\beta^2} - \frac{1}{12} + \mathcal{O}(\beta^2), \\ \int_0^{\infty} dk\, k e^{-\alpha k} &= \alpha^{-2}\end{aligned}$$

we find

$$\langle T_{00}\rangle_{\mathrm{ren}} = \lim_{\alpha\to 0}\left[\frac{2\pi}{L^2}\left(\frac{1}{(2\pi\alpha/L)^2} - \frac{1}{12} + \mathcal{O}(\alpha^2)\right) - \frac{1}{2\pi\alpha^2}\right] = -\frac{\pi}{6L^2}.$$

The total renormalized vacuum energy:

$$E_{\mathrm{ren}} = L\,\langle T_{00}\rangle_{\mathrm{ren}} = -\frac{\pi}{6L}.$$

For the effective pressure we find

$$p = \langle T_{11} \rangle_{\rm ren} = -\frac{\pi}{6L^2}.$$

The energy-momentum tensor is traceless.

We can also use the zeta function technique. For the vacuum energy density we have

$$\langle 0_L | \, T_{00} \, | 0_L \rangle = \frac{2\pi}{L^2} \sum_{n=0}^{\infty} n = \frac{2\pi}{L^2} \zeta(-1),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. The latter is an analytic function in the complex plane s except the simple pole at $s = 1$. In particular, $\zeta(-1) = -1/12$. This leads to the vacuum energy derived before by using the cutoff function. Another way for the renormalization is based on the Abel-Plana summation formula:

$$\sum_{n=0}^{\infty}{}' f(n) = \int_0^{\infty} dx \, f(x) + i \int_0^{\infty} dx \, \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} \tag{9.1}$$

where the prime on the summation sign means that the term with $n = 0$ should be multiplied by 1/2. If we apply to the series in the expression for the energy density, it can be seen that the first term corresponds to the energy density in the Minkowski spacetime:

$$\begin{aligned} \frac{2\pi}{L^2} \sum_{n=0}^{\infty} n &= \frac{2\pi}{L^2} \int_0^{\infty} dx \, x - \frac{4\pi}{L^2} \int_0^{\infty} dx \, \frac{x}{e^{2\pi x} - 1} \\ &= \frac{1}{2\pi} \int_0^{\infty} dk \, k - \frac{1}{\pi L^2} \int_0^{\infty} dy \, \frac{y}{e^y - 1} \\ &= \frac{1}{2\pi} \int_0^{\infty} dk \, k - \frac{1}{\pi L^2} \int_0^{\infty} dy \, \frac{y}{e^y - 1} = \frac{1}{2\pi} \int_0^{\infty} dk \, k - \frac{\pi}{6L^2} \end{aligned}$$

One can also consider imposing antiperiodic boundary condition

$$\phi(t, x + L) = -\phi(t, x).$$

The corresponding field is called as twisted scalar field. The mode functions are given by

$$\phi_k = (2L\omega)^{-1/2} e^{i(kx - \omega t)}, \; k = 2\pi(n + 1/2)/L, \; \omega = |k|, \; n = 0, \pm 1, \pm 2, \ldots$$

The evaluation of the energy density is similar to that for the periodic case. For the evaluation of the series over half integer values we can use the trick

$$\sum_{n=-\infty}^{+\infty} f(2n+1) = \sum_{n=-\infty}^{+\infty} f(n) - \sum_{n=-\infty}^{+\infty} f(2n).$$

For the energy density we find

$$\begin{aligned} \langle 0_L | \, T_{00} \, | 0_L \rangle &= \frac{\pi}{2L^2} \sum_{n=-\infty}^{+\infty} |2n+1| e^{-\pi\alpha|2n+1|/L} \\ &= \frac{\pi}{L^2} \sum_{n=0}^{\infty} n e^{-\pi\alpha n/L} - \frac{2\pi}{L^2} \sum_{n=0}^{\infty} n e^{-2\pi\alpha n/L} \\ &= -\frac{2\pi}{6(2L)^2} + \frac{\pi}{6L^2} = \frac{\pi}{12L^2} \end{aligned}$$

For a spin 1/2 field the vacuum energy density in the topology $R^1 \times S^1$ is given by the expressions

$$\begin{aligned}\langle 0_L | T_{00} | 0_L \rangle &= -\frac{8\pi}{L^2}\sum_{n=0}^{\infty} n \,, \text{ for untwisted field} \\ \langle 0_L | T_{00} | 0_L \rangle &= -\frac{2\pi}{L^2}\sum_{n=-\infty}^{\infty} |2n+1| \,, \text{ for twisted field}\end{aligned}$$

In both cases the result is simply minus four times the corresponding scalar result. Having the mode functions we can also evaluate various two-point functions.

We can consider a more general condition along the compact dimension

$$\phi(t, x+L) = e^{2\pi i\alpha}\phi(t,x),\ 0 \leqslant \alpha \leqslant 1,$$

with a constant phase α. The corresponding mode functions $n = 0, \pm1, \pm2, \ldots$are given by the expressions

$$\phi_k = (2L\omega)^{-1/2} e^{i(kx-\omega t)},\ \omega = |k|,$$

with the momentum eigenvalues

$$k = k_n = 2\pi(n+\alpha)/L,\ n = 0, \pm1, \pm2, \ldots \tag{9.2}$$

For the energy density we get

$$\langle 0_L | T_{00} | 0_L \rangle = \frac{1}{2L}\sum_{n=-\infty}^{+\infty} |k_n|.$$

For the further evaluation of the energy density we apply to the series over n the Abel-Plana-type summation formula [35, 36]

$$\frac{2\pi}{L}\sum_{n=-\infty}^{\infty} f(|k_n|) = 2\int_0^{\infty} dz\, f(z) + i\int_0^{\infty} dz\, [f(iz) - f(-iz)] \sum_{\lambda=\pm1} \frac{1}{e^{zL+2\pi\lambda i\alpha} - 1}, \tag{9.3}$$

where k_n is given by Eq. (9.2). In the special case $\alpha = 0$ this formula is reduced to the standard Abel-Plana formula (9.1). The contribution of the first term in the right hand side of (9.3) gives the corresponding energy density in the geometry with trivial topology $R^1 \times R^1$. The renormalization is reduced to the subtraction of this contribution. As a result, for the renormalized energy density we get the expression

$$\langle 0_L | T_{00} | 0_L \rangle = -\frac{1}{2\pi}\int_0^{\infty} dz \sum_{\lambda=\pm1} \frac{z}{e^{zL+2\pi\lambda i\alpha} - 1}.$$

Next, we expand the integrand by using the formula

$$\frac{1}{e^{zL+2\pi\lambda i\alpha} - 1} = \sum_{n=1}^{\infty} e^{-znL-2\pi\lambda in\alpha}. \tag{9.4}$$

This leads to the result

$$\langle 0_L | T_{00} | 0_L \rangle = -\frac{1}{\pi L^2}\sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n^2}.$$

For the series in the last expression one has

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n^2} = \pi^2 B_2(\alpha),\ 0 \leqslant \alpha \leqslant 1,$$

where $B_l(\alpha)$ is the Bernoulli polynomial. For the polynomial $B_2(\alpha)$ we have the expression

$$B_2(\alpha) = \frac{1}{6} - \alpha + \alpha^2. \tag{9.5}$$

Finally, for the vacuum energy density one gets the following expression

$$\langle 0_L | T_{00} | 0_L \rangle = -\frac{\pi}{L^2}\left(\frac{1}{6} - \alpha + \alpha^2\right).$$

In the special cases $\alpha = 0$ and $\alpha = 1/2$ we recover the results for untwisted and twisted fields. The energy density vanishes for $\alpha = (1 \pm 1/\sqrt{3})/2$.

9.2 Higher-dimensional spaces with a compact dimension

9.2.1 Hadamard function

We consider the quantum scalar field $\phi(x)$ on background of $(D+1)$ dimensional flat spacetime with spatial topology $R^{D-1} \times S^1$ (for a recent discussion of quantum effects in spacetimes with toroidal spatial dimensions see [34]). For the Cartesian coordinates along uncompactified dimensions we use the notation $\mathbf{x}_{D-1} = (x^1, ..., x^{D-1})$. The length of the compact dimension we denote as L. Hence, for coordinates one has $-\infty < x^l < \infty$ for $l = 1, .., D-1$, and $0 \leqslant x^D \leqslant L$. In the presence of a gauge field A_μ the field equation has the form

$$\left(g^{\mu\nu} D_\mu D_\nu + m^2\right)\phi = 0, \tag{9.6}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and e is the charge associated with the field. One of the characteristic features of field theory on backgrounds with nontrivial topology is the appearance of topologically inequivalent field configurations. The boundary conditions should be specified along the compact dimension for the theory to be defined. We assume that the field obeys generic quasiperiodic boundary condition,

$$\phi(t, \mathbf{x}_{D-1}, x^D + L) = e^{2\pi i\alpha}\phi(t, \mathbf{x}_{D-1}, x^D), \tag{9.7}$$

with a constant phas $|\alpha| \leqslant \pi$. The condition (9.7) includes the periodicity conditions for both untwisted and twisted scalar fields as special cases with $\alpha = 0$ and $\alpha = \pi$, respectively.

In the discussion below we will assume a constant gauge field A_μ. Though the corresponding field strength vanishes, the nontrivial topology of the background spacetime leads to the Aharonov-Bohm-like effects on physical observables. In the case of constant A_μ, by making use of the gauge transformation

$$\phi(x) = e^{-ie\chi}\phi'(x),\ A_\mu = A'_\mu + \partial_\mu\chi, \tag{9.8}$$

with $\chi = A_\mu x^\mu$ we see that in the new gauge one has $A'_\mu = 0$ and the vector potential disappears from the equation for $\phi'(x)$. For the new field we have the periodicity condition

$$\phi'(t, \mathbf{x}_{D-1}, x^D + L) = e^{2\pi i\tilde{\alpha}}\phi'(t, \mathbf{x}_{D-1}, x^D), \tag{9.9}$$

where

$$\tilde{\alpha} = \alpha + eA_D L/2\pi. \tag{9.10}$$

In what follows we will work with the field $\phi'(x)$ omitting the prime. Note that for this field $D_\mu = \partial_\mu$. As it is seen from Eq. (9.10), the presence of a constant gauge field shifts the phases in the periodicity conditions along compact dimensions. In particular, a nontrivial phase is induced for special cases of twisted and untwisted fields. Note that the term in Eq. (9.10) due to the gauge field may be written as

$$eA_D L/2\pi = \Phi_D/\Phi_0, \tag{9.11}$$

where Φ_D is a formal flux enclosed by the circle corresponding to the compact dimension and $\Phi_0 = 2\pi/e$ is the flux quantum.

The complete set of positive- and negative-energy solutions for the problem under consideration can be written in the form of plane waves:

$$\phi_{\mathbf{k}}^{(\pm)}(x) = C_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} \mp i\omega t},\ \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \tag{9.12}$$

where $\mathbf{k} = (\mathbf{k}_{D-1}, k_D)$, $\mathbf{k}_{D-1} = (k_1, \dots, k_{D-1})$, with $-\infty < k_i < +\infty$ for $i = 1, \dots, D-1$. For the momentum component along the compact dimension the eigenvalues are determined from the conditions (9.9):

$$k_D = 2\pi\left(n + \tilde{\alpha}\right)/L, \quad n = 0, \pm 1, \pm 2, \dots \tag{9.13}$$

From Eq. (9.13) it follows that the physical results will depend on the fractional part of $\tilde{\alpha}$ only. The integer part can be absorbed by the redefinition of n. Hence, without loss of generality, we can assume that $|\tilde{\alpha}| \leqslant \pi$. The normalization coefficient in (9.12) is found from the orthonormalization condition

$$\int d^D x\, \phi_{\mathbf{k}}^{(\lambda)}(x)\phi_{\mathbf{k}'}^{(\lambda')*}(x) = \frac{1}{2\omega_{\mathbf{k}}}\delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'}, \tag{9.14}$$

where $\delta_{\mathbf{k}\mathbf{k}'} = \delta(\mathbf{k}_{D-1} - \mathbf{k}'_{D-1})\delta_{n_D, n'_D}$. Substituting the functions (9.12), for the normalization coefficient we find

$$|C_{\mathbf{k}}|^2 = \frac{1}{2(2\pi)^{D-1}L\omega_{\mathbf{k}}}, \tag{9.15}$$

with

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}_{D-1}^2 + k_D^2 + m^2}. \tag{9.16}$$

The expectation values of the physical quantities bilinear in the field operator can be evaluated by using the Hadamard function

$$G^{(1)}(x, x') = \left\langle \phi(x)\phi^{+}(x') + \phi^{+}(x')\phi(x) \right\rangle. \tag{9.17}$$

In order to evaluate the expectation value in this expression we expand the field operator over a complete set of solutions:

$$\phi(x) = \int d\mathbf{k}_{D-1} \sum_{n=-\infty}^{+\infty} [\hat{a}_{\mathbf{k}}\phi_{\mathbf{k}}^{(+)}(x) + \hat{b}_{\mathbf{k}}^{+}\phi_{\mathbf{k}}^{(-)}(x)]. \tag{9.18}$$

Substituting the expansion (9.18) into (9.17), for the Hadamard function we get

$$G^{(1)}(x, x') = \int d\mathbf{k}_{D-1} \sum_{n=-\infty}^{+\infty} \sum_{s=\pm} \phi_{\mathbf{k}}^{(s)}(x)\phi_{\mathbf{k}}^{(s)*}(x'). \tag{9.19}$$

By using the expressions (9.12) for the mode functions and the expansion

$$(e^y - 1)^{-1} = \sum_{n=1}^{\infty} e^{-ny},$$

the mode sum for the Hadamard function is written in the form

$$G^{(1)}(x, x') = \frac{1}{L}\int \frac{d\mathbf{k}_{D-1}}{(2\pi)^{D-1}} e^{i\mathbf{k}_{D-1}\cdot\Delta\mathbf{x}_{D-1}} \sum_{n=-\infty}^{+\infty} \frac{e^{ik_D \Delta x^D}}{\omega_{\mathbf{k}}} \cos(\omega_{\mathbf{k}}\Delta t), \tag{9.20}$$

where $\Delta\mathbf{x}_{D-1}=\mathbf{x}_{D-1}-\mathbf{x}'_{D-1}$, $\Delta x^D=x^D-x^{D\prime}$, $\Delta t=t-t'$. For the evaluation of the Hadamard function we apply to the series over n the summation formula [35, 36]

$$\frac{2\pi}{L}\sum_{n=-\infty}^{\infty}g(k_D)f(|k_D|)=\int_0^\infty dz[g(z)+g(-z)]f(z)$$
$$+i\int_0^\infty dz\,[f(iz)-f(-iz)]\sum_{\lambda=\pm1}\frac{g(i\lambda z)}{e^{zL+2\pi i\lambda\tilde{\alpha}}-1}, \tag{9.21}$$

where k_D is given by (9.13). For the Hadamard function we find the expression

$$G^{(1)}(x,x')=G_0^{(1)}(x,x')+\frac{1}{\pi}\int\frac{d\mathbf{k}_{D-1}}{(2\pi)^{D-1}}e^{i\mathbf{k}_{D-1}\cdot\Delta\mathbf{x}_{D-1}}$$
$$\times\int_{\omega_{D-1}}^{\infty}dz\frac{\cosh(\Delta t\sqrt{z^2-\omega_{D-1}^2})}{\sqrt{z^2-\omega_{D-1}^2}}\sum_{\lambda=\pm1}\frac{e^{-\lambda z\Delta x^D}}{e^{zL+2\pi\lambda i\tilde{\alpha}}-1}, \tag{9.22}$$

where

$$\omega_{D-1}=\sqrt{\mathbf{k}_{D-1}^2+m^2}. \tag{9.23}$$

The first term in the right-hand side of Eq. (9.22), $G_0^{(1)}(x,x')$, comes from the first term on the right of Eq. (9.21) and it is the Hadamard function for the trivial topology R^D.

For the further transformation of the expression (9.22) we use the expansion

$$\frac{e^{-\lambda z\Delta x^D}}{e^{zL+\lambda i\tilde{\alpha}}-1}=\sum_{l=1}^{\infty}e^{-z(lL+\lambda\Delta x^D)-\lambda il\tilde{\alpha}}. \tag{9.24}$$

With this expansion the z-integral is expressed in terms of the Macdonald function of the zeroth order. Then the integral over $\mathbf{k}_{D-1}$ is evaluated by using the formula

$$\int d\mathbf{k}_{D-1}\,e^{i\mathbf{k}_{D-1}\cdot\Delta\mathbf{x}_{D-1}}F(|\mathbf{k}_{D-1}|)=\frac{(2\pi)^{(D-1)/2}}{|\Delta\mathbf{x}_{D-1}|^{(D-3)/2}}\int_0^\infty d|\mathbf{k}_{D-1}|\,|\mathbf{k}_{D-1}|^{(D-1)/2}$$
$$\times F(|\mathbf{k}_{D-1}|)J_{(D-3)/2}(|\mathbf{k}_{D-1}||\Delta\mathbf{x}_{D-1}|).$$

For the Hadamard function we arrive to the final expression

$$G^{(1)}(x,x')=\frac{2m^{D-1}}{(2\pi)^{(D+1)/2}}\sum_{n=-\infty}^{\infty}e^{in\tilde{\alpha}}f_{(D-1)/2}\left(m\sqrt{|\Delta\mathbf{x}_{D-1}|^2+(\Delta x^D-nL)^2-(\Delta t)^2}\right), \tag{9.25}$$

where

$$f_\nu(x)=x^{-\nu}K_\nu(x). \tag{9.26}$$

Note that the $n=0$ term in Eq. (9.25) corresponds to the function $G_0^{(1)}(x,x')$. Hence, the part of the Hadamard function in Eq. (9.25) with $n\neq0$ is induced by the compactification of the D-th direction to a circle with the length L.

An alternative expression for the Hadamard function is obtained directly from Eq. (9.20). We first integrate over the angular part of $\mathbf{k}_{D-1}$ and then the integral over $|\mathbf{k}_{D-1}|$ is expressed in terms of the Macdonald function. The corresponding expression is written in terms of the function (9.26) as

$$G^{(1)}(x,x')=\frac{2}{(2\pi)^{D/2}L}\sum_{n=-\infty}^{\infty}e^{ik_D\Delta x^D}\omega_n^{D-2}f_{D/2-1}(\omega_n\sqrt{|\Delta\mathbf{x}_{D-1}|^2-(\Delta t)^2}), \tag{9.27}$$

with the notation

$$\omega_n=\sqrt{k_D^2+m^2}, \tag{9.28}$$

and k_D is given by (9.13).

9.2.2 Vacuum expectation values of the field squared and energy-momentum tensor

Having the Hadamard function we can evaluate the vacuum expectation values of the bilinear combinations of the field operator. We start with the expectation value of the field squared. For the corresponding renormalized expectation value one has

$$\langle \phi(x)\phi^{+}(x)+\phi^{+}(x)\phi(x)\rangle_{\rm ren}=\lim_{x'\to x}G^{(1)}_{\rm sub}(x,x'),$$

where

$$G^{(1)}_{\rm sub}(x,x')=G^{(1)}(x,x')-G^{(1)}_{0}(x,x').$$

The subtraction is reduced to omitting the $n=0$ term in (9.25) and we get

$$\langle \phi(x)\phi^{+}(x)+\phi^{+}(x)\phi(x)\rangle_{\rm ren}=\frac{4m^{D-1}}{(2\pi)^{(D+1)/2}}\sum_{n=1}^{\infty}\cos(2\pi n\tilde{\alpha})f_{(D-1)/2}\left(nmL\right).$$

For large values of the mass, $mL\gg 1$, the effects induced by nontrivial topology are suppressed by the factor e^{-mL}. In the massless limit, by taking into account that $f_{\nu}(x)\approx 2^{\nu-1}\Gamma(\nu)/x^{2\nu}$ for $x\to 0$, one finds

$$\langle \phi(x)\phi^{+}(x)+\phi^{+}(x)\phi(x)\rangle_{\rm ren}=\frac{\Gamma((D-1)/2)}{\pi^{(D+1)/2}L^{D-1}}\sum_{n=1}^{\infty}\frac{\cos(2\pi n\tilde{\alpha})}{n^{D-1}}.$$

For odd values of $D=2l+1$, for the series in this expression one has [37]

$$\sum_{n=1}^{\infty}\frac{\cos(2\pi n\tilde{\alpha})}{n^{2l}}=\frac{(-1)^{l-1}}{2(2l)!}(2\pi)^{2l}B_{2l}(\tilde{\alpha}),\quad 0\leqslant\tilde{\alpha}\leqslant 1, \tag{9.29}$$

and, hence,

$$\begin{aligned}\langle \phi(x)\phi^{+}(x)+\phi^{+}(x)\phi(x)\rangle_{\rm ren}&=\frac{(2\pi)^{2l}\Gamma(l)}{\pi^{l+1}L^{2l}}\frac{(-1)^{l-1}}{2\Gamma(2l+1)}B_{2l}(\tilde{\alpha})\\&=\frac{(-1)^{(D+1)/2}\pi^{D/2-1}B_{D-1}(\tilde{\alpha})}{(D-1)\Gamma(D/2)L^{D-1}}.\end{aligned} \tag{9.30}$$

As we see, the vacuum expectation value is a periodic function of the magnetic flux with the period equal to the flux quantum. For the Bernoulli polynomials in (9.30) one has

$$\begin{aligned}B_4(x)&=-1/30+x^2-2x^3+x^4,\\B_6(x)&=1/42-x^2/2+5x^4/2-3x^5+x^6,\end{aligned}$$

and $B_2(x)$ is given by (9.5).

Now we turn to the vacuum expectation values of the energy-momentum tensor. It can be evaluated by using the formula

$$\langle T_{\mu\nu}\rangle=\frac{1}{2}\lim_{x'\to x}\partial_{\mu}\partial'_{\nu}G^{(1)}(x,x')+\frac{1}{2}\left[\left(\xi-\frac{1}{4}\right)g_{\mu\nu}\nabla_{\sigma}\nabla^{\sigma}-\xi\nabla_{\mu}\nabla_{\nu}\right]G^{(1)}(x,x). \tag{9.31}$$

Here we have used the expression of the classical energy-momentum tensor for a charged scalar field which differs from the standard one by the term which vanishes on the solutions of the field equation. This term does not contribute to the topological part in the vacuum expectation value. The renormalization is reduced to the replacement $G^{(1)}(x,x')\to G^{(1)}_{\rm sub}(x,x')$ in (9.31).

The vacuum expectation value

$$G^{(1)}_{\rm sub}(x,x) = \left\langle \phi(x)\phi^{+}(x) + \phi^{+}(x)\phi(x)\right\rangle_{\rm ren}$$

does not depend on the spacetime point. Consequently, the second term in the right-hand side of (9.31) will not contribute to the vacuum energy-momentum tensor and

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{1}{2}\lim_{x'\to x}\partial_\mu\partial'_\nu G^{(1)}_{\rm sub}(x,x').$$

It is easy to see that this tensor is diagonal and (no summation over μ)

$$\langle T^{\mu}_{\mu}\rangle_{\rm ren} = \langle T^{0}_{0}\rangle_{\rm ren},$$

with $\mu = 1,\ldots,D-1$. The latter relation could be directly obtained by taking into account the boost invariance along uncompact dimensions. For the renormalized energy density one finds

$$\langle T^{0}_{0}\rangle_{\rm ren} = \frac{2m^D}{(2\pi)^{(D+1)/2}}\sum_{n=1}^{\infty}\frac{\cos(2\pi n\tilde{\alpha})}{nL}f'_{(D-1)/2}\left(nmL\right).$$

By taking into account that

$$f'_\nu(x) = -xf_{\nu+1}(x), \tag{9.32}$$

the final expression is presented as

$$\langle T^{0}_{0}\rangle_{\rm ren} = -\frac{2m^{D+1}}{(2\pi)^{(D+1)/2}}\sum_{n=1}^{\infty}\cos(2\pi n\tilde{\alpha})f_{(D+1)/2}\left(nmL\right).$$

For the stress along the compact dimension we get

$$\langle T^{D}_{D}\rangle_{\rm ren} = \frac{2m^{D+1}}{(2\pi)^{(D+1)/2}}\sum_{n=1}^{\infty}\cos(2\pi n\tilde{\alpha})f''_{(D-1)/2}\left(nmL\right).$$

Again, by using the relation (9.32), this expression takes the form

$$\langle T^{D}_{D}\rangle_{\rm ren} = \langle T^{0}_{0}\rangle_{\rm ren} + \frac{2m^{D+1}}{(2\pi)^{(D+1)/2}}\sum_{n=1}^{\infty}\cos(2\pi n\tilde{\alpha})(nmL)^2 f_{(D+3)/2}(nmL).$$

The vacuum energy density and stresses are even periodic functions of the magnetic flux with the period equal to flux quantum.

For a massless field, taking the limit $m\to 0$, we find

$$\langle T^{\nu}_{\mu}\rangle_{\rm ren} = -\frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}L^{D+1}}\sum_{n=1}^{\infty}\frac{\cos(2\pi n\tilde{\alpha})}{n^{D+1}}{\rm diag}(1,1,\ldots,1,-D).$$

In this case the vacuum energy-momentum tensor is traceless. For odd values of the spatial dimension $D = 2l-1$, the series is summed with the help of the formula (9.29). This gives

$$\langle T^{\nu}_{\mu}\rangle_{\rm ren} = \frac{(-1)^{(D+1)/2}\pi^{D/2+1}B_{D+1}(\tilde{\alpha})}{(D+1)\Gamma(D/2+1)L^{D+1}}{\rm diag}(1,1,\ldots,1,-D).$$

Note that the vacuum effective pressure along the μth spatial dimension is given by $p_\mu = -\langle T^{\mu}_{\mu}\rangle_{\rm ren}$. Hence, we have the following equations of state:

$$\begin{aligned} p_\mu &= -\varepsilon,\ \mu = 1,\ldots,D-1,\\ p_D &= D\varepsilon. \end{aligned}$$

As is seen, in the uncompact subspace the vacuum energy-momentum tensor is a source of the cosmological constant type. The latter is the case for a massive field as well.

9.2.3 Vacuum currents

For charged fields another important characteristic, bilinear in the field, is the expectation value of the current density in a given state. Having the Hadamard function we can evaluate the expectation value for the current density

$$j_l(x) = ie[\phi^+(x)\partial_l\phi(x) - (\partial_l\phi^+(x))\phi(x)],$$

$l = 0, 1, \ldots, D$, by using the formula

$$\langle j_l(x)\rangle = \frac{i}{2}e \lim_{x'\to x}(\partial_l - \partial_l')G^{(1)}(x, x'). \tag{9.33}$$

It can be easily seen, the vacuum expectation values of the charge density and of the components of the current density along the uncompactified dimensions vanish: $\langle j_l\rangle = 0$ for $l = 0, 1, \ldots, D-1$. By making use of Eq. (9.33) and the expression (9.25) of the Hadamard function, for the current density along the compact dimension we get:

$$\left\langle j^D\right\rangle = \frac{4eLm^{D+1}}{(2\pi)^{(D+1)/2}}\sum_{n=1}^{\infty} n\sin(2\pi n\tilde{\alpha})f_{(D+1)/2}(nmL).$$

It is an odd periodic function of the magnetic flux through the compact dimension with the period equal to flux quantum. The current vanishes for $\tilde{\alpha} = 0, 1/2, 1$. In the absence of the magnetic flux this corresponds to untwisted and twisted scalar fields. For large values of the length of the compact dimension compared with the Compton wavelength m^{-1}, $mL \gg 1$, the current density is exponentially suppressed.

The vacuum current density for a massless field is obtained in the limit $m \to 0$:

$$\left\langle j^D\right\rangle = \frac{2e\Gamma((D+1)/2)}{\pi^{(D+1)/2}L^D}\sum_{n=1}^{\infty}\frac{\sin(2\pi n\tilde{\alpha})}{n^D}.$$

For odd values of the spatial dimension $D = 2l+1$, the series in this expression is summed in terms of the Bernoulli polynomials:

$$\sum_{n=1}^{\infty}\frac{\sin(2\pi n\tilde{\alpha})}{n^D} = \frac{(-1)^{(D+1)/2}(2\pi)^D}{2\Gamma(D+1)}B_D(\tilde{\alpha}),\ 0 < \tilde{\alpha} < 1.$$

For the current density this yields

$$\begin{aligned}\left\langle j^D\right\rangle &= \frac{e\Gamma((D+1)/2)}{\pi^{(D+1)/2}L^D}\frac{(-1)^{l-1}(2\pi)^D}{\Gamma(D+1)}B_D(\tilde{\alpha}) \\ &= e\frac{(-1)^{(D+1)/2}\pi^{D/2}B_D(\tilde{\alpha})}{\Gamma(D/2+1)L^D}.\end{aligned}$$

For the Bernoulli polynomials in this expression one has

$$\begin{aligned}B_3(x) &= x/2 - 3x^2/2 + x^3, \\ B_5(x) &= -x/6 + 5x^3/3 - 5x^4/2 + x^5.\end{aligned}$$

The zeros of these polynomials in the interval $0 \leqslant x \leqslant 1$ are the points $x = 0, 1/2, 1$.

9.3 Boundary-induced quantum effects: Casimir effect

We have discussed the effects of the vacuum fluctuations of quantum fields resulting from the compactification of spatial dimensions. The periodicity conditions along compact dimension modify the spectrum of fluctuations and, as a result of this, the vacuum expectation values of physical observables are shifted from the values they had in absence of the compactification. Another type of boundary conditions, imposed on a field operator, arise in problems with boundaries. In the presence of boundaries on which the field obeys some prescribed boundary conditions, the mode function differ from those in the geometry when the boundaries are absent. As a result the expectation values of physical observables are changed. This effect is known as the Casimir effect [32], [38]-[44]. We start with the simplest problem of a massless scalar field in two-dimensional spacetime. The field obeys Dirichlet boundary conditions on the boundaries located at $x = 0$ and $x = a$. We consider the region between the boundaries $0 \leqslant x \leqslant a$. The field equation and the BC are

$$\left(\partial_t^2 - \partial_x^2\right)\phi = 0,\ \phi(t,0) = \phi(t,a) = 0.$$

The mode functions have the form

$$\phi_k = \frac{e^{-i\omega t}}{\sqrt{a\omega}} \sin(\omega x),\ \omega = \pi n/a,\ n = 1, 2, \ldots$$

For a minimally coupled field the separate components of the energy-momentum tensor have the form

$$T_{00} = T_{11} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2, \quad T_{01} = \frac{1}{2}\left(\partial_t\phi\partial_x\phi + \partial_x\phi\partial_t\phi\right)$$

Substituting the mode functions into the mode sum formulas for the components, it can be seen that the off-diagonal component vanishes. For the diagonal components one has

$$\langle 0_a | T_{\mu\nu} | 0_a \rangle = \frac{1}{2a}\sum_{n=1}^{\infty} \omega\delta_{\mu\nu} = \frac{\pi}{2a^2}\sum_{n=1}^{\infty} n\delta_{\mu\nu}$$

The evaluation of the sum was presented for the example with non-trivial topology. The final result is

$$\langle T_{\mu\nu}\rangle_{\rm ren} = -\frac{\pi}{24a^2}\delta_{\mu\nu}.$$

The corresponding energy for a single boundary is obtained in the limit $a \to \infty$ and vanishes. The total energy is given by

$$E = a\,\langle T_{00}\rangle_{\rm ren} = -\frac{\pi}{24a}.$$

The energy is related to the vacuum stress by the usual thermodynamical relation

$$dE = -Pda,$$

where $P = -\left\langle T_1^1\right\rangle_{\rm ren}$ is the vacuum stress (effective pressure).

For two parallel plates in 4D spacetime, located at $x^3 = 0$ and $x^3 = a$, the mode functions in the region between the plates are given by the expression

$$\phi_\alpha = \frac{e^{i\mathbf{k}_\parallel\cdot\mathbf{x}_\parallel - i\omega t}}{2\pi\sqrt{\omega a}} \sin(\pi n x^3/a)$$

with $\alpha = (k_1, k_2, n)$, $-\infty < k_1, k_2 < +\infty$, $n = 1, 2, \ldots$ and

$$\omega^2 = k_1^2 + k_2^2 + (\pi n/a)^2$$

For a conformally coupled massless scalar field the renormalized energy-momentum tensor is given by the expression

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{\pi^2}{1440a^4}{\rm diag}(-1,1,1,-3)$$

For the Neumann boundary condition the result is the same. For a minimally coupled scalar field

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{\pi^2}{1440a^4}{\rm diag}(-1,1,1,-3) \pm \frac{\pi^2}{48a^4}\frac{1+2\cos^2(\pi x^3/a)}{\sin^4(\pi x^3/a)}{\rm diag}(-1,1,1,0)$$

where the upper/lower sign correspond to Dirichlet/Neumann BC. Here there are divergences on the boundaries. However, the forces acting on boundaries are determined by $\langle T_{33}\rangle_{\rm ren}$ and they are finite everywhere.

For the electromagnetic field in the geometry of two parallel conducting plates the problem is reduced to the corresponding problem for a set of two massless scalar fields with Dirichlet and Neumann boundary conditions. The renormalized vacuum expectation value in the region between the plates is given by the relation

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{\pi^2}{720a^4}{\rm diag}(-1,1,1,-3)$$

For the plates with surface area S, the total vacuum energy is

$$E = Sa\,\langle T_{00}\rangle_{\rm ren} = -\frac{\pi^2 S}{720a^3}.$$

The attraction force between the plates:

$$F = -\partial_a E = -\frac{\pi^2\hbar c S}{240a^4}$$

This force is measured in the experiments with high accuracy. For $S = 1{\rm cm}^2$ and $a = 0.5$ micrometer $F \approx 0.2$ dyn. For a conducting sphere with radius a the Casimir energy is positive and the corresponding force is repulsive:

$$E = \frac{0.046}{a}.$$

9.4 Casimir effect for Robin boundary conditions

In this section we consider a scalar field $\phi(x)$ in $(D+1)$-dimensional flat spacetime with the line element $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ in the presence of two parallel plates located at $x^D = y = a_1 = 0$ and $y = a_2 = a$, on which the field operator obeys the Robin boundary condition

$$\left(\tilde{A} + \tilde{B}n^i\nabla_i\right)\phi(x) = 0, \tag{9.34}$$

where $\tilde{A}$ and $\tilde{B}$ are constants, and n^i is the unit inward-pointing normal (with respect to the region under consideration) to the boundary. Robin type conditions are an extension of Dirichlet and Neumann boundary conditions and appear in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields, spinor and gauge field theories, quantum gravity, supergravity and braneworld scenarios. In some geometries, Robin boundary conditions may be useful for depicting the finite penetration of the field into the boundary with the 'skin-depth' parameter related to the Robin coefficient.

In the region between the plates the eigenfunctions are presented in the form

$$\varphi_\sigma = \beta(k_y)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega(k_y)t}\cos\left(k_y y + \alpha(k_y)\right), \tag{9.35}$$

where $\omega(k_y) \equiv \sqrt{k_y^2 + k^2 + m^2}$, the vector $\mathbf{x} = (x^1, x^2, \ldots, x^{D-1})$ specifies the spatial dimensions parallel to the plates and $\alpha(k_y)$ is defined by the relation

$$e^{2i\alpha(k_y)} \equiv \frac{i\beta_1 k_y - 1}{i\beta_1 k_y + 1}, \quad \beta_j = (-1)^{j-1}\tilde{B}_j/\tilde{A}_j. \tag{9.36}$$

The corresponding eigenvalues for k_y are obtained from the boundary conditions and are solutions of the following transcendental equation:

$$F(z) \equiv \left(1 - b_1 b_2 z^2\right) \sin z - (b_1 + b_2)\, z \cos z = 0, \; z = k_y a, \; b_j = \beta_j/a. \tag{9.37}$$

The expression for the coefficient $\beta(k_y)$ in (9.35) is obtained from the normalization condition:

$$\beta^{-2}(k_y) = (2\pi)^{D-1} a\omega \left[1 + \frac{\sin(k_y a)}{k_y a} \cos(k_y a + 2\alpha(k_y))\right]. \tag{9.38}$$

The eigenvalue equation (9.37) has an infinite set of real zeros which we will denote by $k_y = \lambda_n/a$, $n = 1, 2, \ldots$. In addition, depending on the values of the coefficients in the boundary conditions, this equation has two or four complex conjugate purely imaginary zeros $\pm i y_l$, $y_l > 0$ (see [45]).

Substituting eigenfunctions (9.35) into mode-sum formula for the positive-frequency Wightman function, in the region between two plates one finds

$$\begin{aligned} G^+(x, x') &= \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \sum_{k_y = \lambda_n/a, i y_l/a} \beta^2(k_y) e^{-i\omega(k_y)\Delta t} \\ &\quad \times \cos(k_y y + \alpha(k_y)) \cos(k_y y' + \alpha(k_y)), \end{aligned} \tag{9.39}$$

where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$, $\Delta t = t - t'$. For the summation over the eigenvalues we will use the formula [45]

$$\begin{aligned} \sum_{z=\lambda_n, i y_l} \frac{\pi f(z)}{1 + \cos(z + 2\alpha) \sin z / z} &= \frac{\pi f(0)/2}{b_1 + b_2 - 1} + \int_0^\infty dz f(z) \\ &\quad + i \int_0^\infty dt \frac{f(te^{\pi i/2}) - f(te^{-\pi i/2})}{\frac{(b_1 t - 1)(b_2 t - 1)}{(b_1 t + 1)(b_2 t + 1)} e^{2t} - 1} \\ &\quad - \frac{\theta(b_1)}{2b_1} \left[h(e^{\frac{\pi i}{2}}/b_1) + h(c_1 e^{-\frac{\pi i}{2}}/b_1)\right], \end{aligned} \tag{9.40}$$

where $h(z) \equiv \left(b_1^2 z^2 + 1\right) f(z)$.

As a function $f(z)$ in (9.40) we take

$$f(z) \equiv \frac{e^{-i\omega(z/a)\Delta t}}{a\omega(z/a)} \cos(zy/a + \alpha(z/a)) \cos(zy'/a + \alpha(z/a)), \tag{9.41}$$

with first-order poles at $z = \pm i/b_j$. By making use of the definition for $\alpha(k)$ we see that $e^{2i\alpha(0)} = -1$, and hence $\cos(2\alpha(0)) = -1$, which implies that $f(0) = 0$. The resulting Wightman function from (9.39) is found to be

$$\begin{aligned} G^+(x, x') &= G_0^+(x, x') + \frac{4}{(2\pi)^D} \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_{k_m}^\infty du \\ &\quad \times \frac{\cosh(yu + \tilde{\alpha}(u)) \cosh(y'u + \tilde{\alpha}(u))}{\frac{(\beta_1 u - 1)(\beta_2 u - 1)}{(\beta_1 t + 1)(\beta_2 t + 1)} e^{2au} - 1} \frac{\cosh\left[\Delta t \sqrt{u^2 - k_m^2}\right]}{\sqrt{u^2 - k_m^2}}, \end{aligned} \tag{9.42}$$

where $k_m = \sqrt{k^2 + m^2}$ and the function $\tilde{\alpha}_j(t)$ is defined by the relation

$$e^{2\tilde{\alpha}(u)} \equiv \frac{\beta_1 u - 1}{\beta_1 u + 1}. \tag{9.43}$$

In formula (9.42),

$$\begin{aligned} G_0^+(x,x') &= G_M^+(x,x') + \int \frac{d\mathbf{k}}{(2\pi)^D} e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_0^\infty du \frac{e^{-i\omega(u)\Delta t}}{\omega(u)} \cos\left[u(y+y') + 2\alpha(u)\right] \\ &+ \frac{\theta(\beta_1) e^{-(y+y')/\beta_1}}{(2\pi)^{D-1}\beta_1} \int d\mathbf{k} \frac{\exp(i\mathbf{k}\cdot\Delta\mathbf{x} - i\Delta t\sqrt{k_m^2 - 1/\beta_1^2})}{\sqrt{k_m^2 - 1/\beta_1^2}}, \end{aligned} \tag{9.44}$$

is the Wightman function for a single plate located at $y = 0$ and $G_M^+(x,x')$ is the Wightman function in the Minkowski spacetime without boundaries. The last term on the right comes from the bound state present in the case $\beta_1 > 0$. To escape the instability of the vacuum state, we will assume that $m\beta_1 \geqslant 1$. On taking the coincidence limit, for the vacuum expectation value of the field squared we obtain the formula

$$\langle\varphi^2\rangle_{\rm ren} = \langle\varphi^2\rangle_{\rm ren}^{(0)} + 4\frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt \frac{(t^2 - m^2)^{D/2-1}}{\frac{(\beta_1 t - 1)(\beta_2 t - 1)}{(\beta_1 t + 1)(\beta_2 t + 1)} e^{2at} - 1} \cosh^2(ty + \tilde{\alpha}(t)), \tag{9.45}$$

where

$$\langle\varphi^2\rangle_{\rm ren}^{(0)} = \frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt\,(t^2 - m^2)^{D/2-1} e^{-2ty} \frac{\beta_1 t + 1}{\beta_1 t - 1}, \tag{9.46}$$

is the corresponding vacuum expectation value in the region $y > 0$ for a single plate at $y = 0$. The surface divergences on the plate at $y = 0$ are contained in this term. The second term on the right of formula (9.45) is finite at $y = 0$ and is induced by the second plate located at $y = a$. This term diverges at $y = a$. The corresponding divergence is the same as that for the geometry of a single plate located at $y = a$. Note that in obtaining (9.46) from (9.44) we have written the cos function in the second integral term on the right of (9.44) as a sum of exponentials and have rotated the integration contour by the angle $\pi/2$ and by $-\pi/2$ for separate exponentials. For $\beta_1 > 0$ the corresponding integrals have poles $\pm i/\beta_1$ on the imaginary axis and the contribution from these poles cancel the part coming from the last term on the right of (9.44).

In special cases of Dirichlet and Neumann boundary conditions, from (9.45) we get

$$\langle\phi^2\rangle_{\rm ren} = \langle\phi^2\rangle_{\rm ren}^{(0)} + \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)} \int_m^\infty dx \frac{(x^2 - m^2)^{D/2-1}}{e^{2ax} - 1} \left[1 \mp \cosh(2xy)\right], \tag{9.47}$$

with the single plate part

$$\langle\phi^2\rangle_{\rm ren}^{(0)} = \frac{\mp m^{D-1}}{(2\pi)^{(D+1)/2}} \frac{K_{(D-1)/2}(2m|y|)}{(2m|y|)^{(D-1)/2}}. \tag{9.48}$$

Here, the upper/lower signs correspond to Dirichlet/Neumann boundary conditions. For a massless scalar field, taking the limit $m \to 0$, from (9.48) one gets

$$\langle\phi^2\rangle_{\rm ren}^{(0)} = \frac{\mp\Gamma((D-1)/2)}{(4\pi)^{(D+1)/2}|y|^{D-1}}.$$

The expression on the right-hand side gives the leading term in the asymptotic expansion of the $\langle\phi^2\rangle_{\rm ren}$ for a massive field near the plate at $y = 0$.

The vacuum expectation value of the energy-momentum tensor is evaluated by the formula

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \lim_{x'\to x} \partial_\mu \partial'_\nu G^+(x,x') + \left[\left(\xi - \frac{1}{4}\right) g_{\mu\nu}\nabla_\sigma\nabla^\sigma - \xi\nabla_\mu\nabla_\nu\right] \langle 0|\phi^2(x)|0\rangle. \tag{9.49}$$

By taking into account formulae (9.42), (9.45), for the region between the plates one finds

$$\langle T_\mu^\nu\rangle_{\rm ren} = \langle T_\mu^\nu\rangle_{\rm ren}^{(0)} + 2\delta_\mu^\nu \frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt\, \frac{(t^2-m^2)^{D/2-1}}{\frac{(\beta_1 t-1)(\beta_2 t-1)}{(\beta_1 t+1)(\beta_2 t+1)} e^{2at}-1} f_\mu(t,y), \tag{9.50}$$

where

$$f_\mu(t,x) = \left[4t^2\left(\zeta_D - \zeta\right) + m^2/D\right] \cosh\left(2ty + 2\tilde{\alpha}(t)\right) - (t^2-m^2)/D, \tag{9.51}$$

for $\mu = 0,1,\ldots,D-1$, and $f_D(t,y) = t^2$. In formula (9.50),

$$\begin{aligned} \langle T_\mu^\nu\rangle_{\rm ren}^{(0)} &= \delta_\mu^\nu \frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt (t^2-m^2)^{D/2-1} e^{-2yt} \\ &\times \frac{\beta_1 t+1}{\beta_1 t-1}\left[4\left(\xi_c - \xi\right)t^2 + m^2/D\right], \end{aligned} \tag{9.52}$$

for $\mu = 0,1,\ldots,D-1$, and $\langle T_D^D\rangle_{\rm ren}^{(0)} = 0$, are the vacuum expectation values in the region $y > 0$ induced by a single plate at $y = 0$, and the second term on the right is the part of the energy-momentum tensor induced by the presence of the second plate. For a conformally coupled massless scalar field the vacuum energy-momentum tensor is uniform and traceless. Note that in this case the single plate parts vanish. We have investigated the vacuum densities in the bulk. For Robin boundary conditions in addition to this part there is a contribution to the energy-momentum tensor located on the plates.

For Dirichlet and Neumann boundary conditions the vacuum expectation value of the energy-momentum tensor id further simplified to (no summation over μ)

$$\begin{aligned} \langle T_\mu^\mu\rangle_{\rm ren} &= \langle T_\mu^\mu\rangle_{\rm ren}^{(0)} - \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)} \int_m^\infty dx\, \frac{(x^2-m^2)^{D/2-1}}{e^{2ax}-1} \\ &\times \left\{\frac{x^2-m^2}{D} \mp \left[4(\zeta-\zeta_D)x^2 - m^2/D\right]\cosh(2xy)\right\}, \end{aligned} \tag{9.53}$$

$$\langle T_D^D\rangle_{\rm ren} = \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)} \int_m^\infty dx\, x^2 \frac{(x^2-m^2)^{D/2-1}}{e^{2ax}-1}, \tag{9.54}$$

with $\mu = 0,1,\ldots,D-1$ and single plate part $\langle T_D^D\rangle_{\rm J,ren}^{(0)} = 0$,

$$\langle T_\mu^\mu\rangle_{\rm ren}^{(0)} = \mp\frac{4m^{D+1}(\xi-\xi_c)}{(2\pi)^{(D+1)/2}}\left[f_{(D+1)/2}(2m|y|) - (2m|y|)^2 f_{(D+3)/2}(2m|y|)\right] + \frac{m^2}{D}\langle\phi^2\rangle_{\rm ren}^{(0)}. \tag{9.55}$$

For a conformally coupled massless field the single plate part vanishes.

Vacuum forces acting on the plates are determined by $\langle T_D^D\rangle_{\rm ren}$. This component is uniform and, hence, is finite on the plates. The latter property is a consequence of the high symmetry of the problem and is not valid for curved boundaries. In dependence of the values for the coefficients β_j the vacuum forces can be either attractive or repulsive. For the vacuum pressure on the plates one has $P = -\langle T_D^D\rangle_{\rm ren}$. It is given by the formula

$$P = -2\frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt\, \frac{t^2(t^2-m^2)^{D/2-1}}{\frac{(\beta_1 t-1)(\beta_2 t-1)}{(\beta_1 t+1)(\beta_2 t+1)} e^{2at}-1}.$$

The corresponding Casimir force is attractive for $P < 0$ and repulsive for $P > 0$. For scalar fields with Dirichlet and Neumann boundary conditions the Casimir pressure is presented as

$$P = -2\frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt \, \frac{t^2(t^2 - m^2)^{D/2-1}}{e^{2at} - 1}. \tag{9.56}$$

This corresponds to attractive forces between the plates. In the case of Dirichlet boundary condition on one plate and Neumann boundary condition on the other the vacuum forces are repulsive. For a massless field from (9.56) one gets

$$P = -\frac{D\zeta_{\rm R}(D+1)}{(4\pi)^{(D+1)/2} a^{D+1}} \Gamma\left(\frac{D+1}{2}\right), \tag{9.57}$$

where $\zeta_{\rm R}(x)$ is the Riemann zeta function.

Chapter 10

Quantum fields in Rindler spacetime

10.1 Worldline for a uniformly accelerated observer

One of the classical examples for the investigation of the quantum field theoretical effects in non-Minkowskian geometries is the quantum field theory in uniformly accelerated reference frames. We start our consideration with the corresponding background geometry. Uniformly accelerated motion is defined as motion when the acceleration in the proper reference frame (the inertial frame that is instantaneously comoving) remains constant. For the 4-velocity one has

$$u^i = (\gamma, \gamma \mathbf{v}) , \quad \gamma = 1/\sqrt{1 - v^2}.$$

For the corresponding 4-acceleration this gives

$$w^i = \frac{du^i}{ds} = \frac{1}{\gamma}\frac{du^i}{dt} = \frac{1}{\gamma}\left(\partial_t \gamma, \partial_t \gamma \mathbf{v} + \gamma \mathbf{a}\right),$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}, \quad \partial_t \gamma = \gamma^3 \mathbf{v} \cdot \mathbf{a}.$$

Hence, for the 4-acceleration we get

$$w^i = \left(\gamma^2 \mathbf{v} \cdot \mathbf{a}, \mathbf{a} + \gamma^2 (\mathbf{v} \cdot \mathbf{a}) \mathbf{v}\right).$$

In the proper reference frame one has $\mathbf{v} = 0$ and the 4-acceleration is given by $w^i = (0, \mathbf{a}_0)$. The uniformly accelerated motion corresponds to $\mathbf{a}_0 = \text{const}$.

In order to find the worldline of the uniformly accelerated point particle let us consider a rectilinear motion along the x axis: $\mathbf{a}_0 = (w, 0, 0)$. The condition for a uniformly accelerated motion is written in a relativistically invariant form as

$$w_i w^i = \text{const} = -a^2,$$

where we have denoted by a the proper acceleration. From here it can be seen that

$$v = \frac{at}{\sqrt{1 + a^2 t^2}},$$

where we have assumed that $v = 0$ for $t = 0$. The integration of this relation gives

$$x = \frac{1}{a}\sqrt{1 + a^2 t^2},$$

with the initial condition $x(0) = 1/a$. For the proper time of the uniformly accelerated particle one has $d\eta = \int dt\sqrt{1-v^2}$, with

$$\eta = \frac{1}{a}\text{arcsinh}(at).$$

Hence, the worldline for a uniformly accelerated particle is written in the parametric form as

$$t = \frac{1}{a}\sinh(a\eta), \quad x = \frac{1}{a}\cosh(a\eta). \tag{10.1}$$

This corresponds to the hyperbola

$$x^2 - t^2 = a^{-2}. \tag{10.2}$$

For the velocity we have $v = t/x = \tanh(a\eta)$.

We could follow another way. One has the relations

$$\frac{dx^i}{d\eta} = u^i, \ \frac{du^i}{d\eta} = w^i, \ u_i u^i = 1, \ u_i w^i = 0, \ w_i w^i = -a^2.$$

with $x^i = (t, x^1, 0, 0)$, $u^i = (u^0, u^1, 0, 0)$.

$$u^0 w^0 = u^1 w^1, \quad \left(w^0\right)^2 - \left(w^1\right)^2 = -a^2 \Rightarrow \left(w^0\right)^2 = a^2\left(u^1\right)^2, \quad \left(w^1\right)^2 = a^2\left(u^0\right)^2.$$

We get the following equations

$$w^1 = \frac{du^1}{d\eta} = au^0, \quad w^0 = \frac{du^0}{d\eta} = au^1.$$

With an appropriate choice of the initial conditions, the solutions is given by (10.1).

10.2 Rindler coordinates

Consider $(D+1)$-dimensional Minkwoski spacetime. For an inertial observer the line element in Cartesian coordinates is written as $ds^2 = dt^2 - \left(dx^1\right)^2 - d\mathbf{x}^2$, where $\mathbf{x} = (x^2, \dots, x^D)$. We introduce the *Rindler coordinates* $(\tau, \rho, \mathbf{x})$ by the relations

$$t = \rho\sinh\tau, \quad x^1 = \rho\cosh\tau. \tag{10.3}$$

In these coordinates, by taking into account that

$$\begin{aligned} dt^2 - \left(dx^1\right)^2 &= d\left(t - x^1\right)d\left(t + x^1\right) = -d\left(\rho e^{-\tau}\right)d\left(\rho e^{\tau}\right) \\ &= -\left(d\rho - \rho d\tau\right)\left(d\rho + \rho d\tau\right) = \rho^2 d\tau^2 - d\rho^2, \end{aligned}$$

the line element takes the form

$$ds^2 = \rho^2 d\tau^2 - d\rho^2 - d\mathbf{x}^2. \tag{10.4}$$

For a worldline $\rho, \mathbf{x} = \text{const}$ one has $(x^1)^2 - t^2 = \rho^2$. Comparing with (10.2), we see that this worldline describes an observer with constant proper acceleration ρ^{-1} with the proper time $t_s = \rho\tau$. The coordinates $(\tau, \rho, \mathbf{x})$ cover the part of the Minkowski spacetime corresponding to $x^1 > |t|$. This part is called the *right Rindler wedge* and is denoted by the letter R. The metric corresponding to (10.4) is static, admitting the Killing vector field $\partial/\partial\tau$. The Fulling-Rindler vacuum is the vacuum state determined by choosing positive-frequency modes to have positive frequency with respect to this Killing vector.

In a similar way, we can introduce the coordinates

$$t = -\rho\sinh\tau, \quad x^1 = -\rho\cosh\tau. \tag{10.5}$$

with the same line element (10.4) (note that if we would change the sign of x^1 only, the transformation Jacobian would be negative). These coordinates cover the part of the Minkowski spacetime specified by $x^1 < -|t|$. This region is called the *left Rindler wedge* and is denoted the letter L. Remaining wedges of the Minkwoski spacetime, namely, $t < -|x^1|$ and $t > |x^1|$ are called as contracting and expanding Kasner universes, respectively. All the regions are separated by the hyper planes $x^1 = \pm t$.

Let us introduce new coordinates

$$\tau = a\eta, \quad \rho = \frac{1}{a}e^{a\xi}, \tag{10.6}$$

in terms of which the interval is rewritten as

$$ds^2 = e^{2a\xi}\left(d\eta^2 - d\xi^2\right) - d\mathbf{x}^2. \tag{10.7}$$

Now the part of the line element with the coordinates (η, ξ) is conformally related to the line element of the Minkowski spacetime in Cartesian coordinates. The new coordinates are related to the inertial ones by

$$t = \pm\frac{1}{a}e^{a\xi}\sinh(a\eta), \quad x^1 = \pm\frac{1}{a}e^{a\xi}\cosh(a\eta). \tag{10.8}$$

The proper acceleration for an observer $\xi = \text{const}$ is given by $ae^{-\xi}$ and for the proper time one has $t_s = \pm\eta e^{a\xi}$.

The importance of the Rindler coordinates from the point of view of the gravitational physics is that the near horizon and large mass limit the black hole geometry may be approximated by the Rindler-like manifold. The line element for the geometry of $(D+1)$-dimensional topological black hole is described by the line element

$$ds^2 = A_H(r)dt^2 - \frac{dr^2}{A_H(r)} - r^2 d\Sigma_{D-1}^2, \tag{10.9}$$

where $d\Sigma_{D-1}^2$ is the line element for the space with constant curvature, $A_H(r) = k + r^2/l^2 - r_0^D/l^2 r^n$, $n = D-2$, and the parameter k classifies the horizon topology, with $k = 0, -1, 1$ corresponding to flat, hyperbolic, and elliptic horizons, respectively. The parameter l is related to the bulk cosmological constant and the parameter r_0 depends on the mass of the black hole. In the non extremal case the function $A_H(r)$ has a simple zero at $r = r_H$, and in the near horizon limit, introducing new coordinates τ and ρ in accordance with

$$\tau = A'_H(r_H)t/2, \quad r - r_H = A'_H(r_H)\rho^2/4, \tag{10.10}$$

the line element is written in the form

$$ds^2 = \rho^2 d\tau^2 - d\rho^2 - r_H^2 d\Sigma_{D-1}^2, \tag{10.11}$$

Note that for a $(D+1)$-dimensional Schwarzschild black hole one has $A_H(r) = 1 - (r_H/r)^n$ and, hence, $A'_H(r_H) = n/r_H$.

10.3 Massless scalar field in 2-dimensional Rindler spacetime

In 2-dimensional Rindler spacetime we have

$$ds^2 = e^{2a\xi}\left(d\eta^2 - d\xi^2\right). \tag{10.12}$$

This is conformally related to the Minkowskian line element: $ds^2 = e^{2a\xi}ds_M^2$, $ds_M^2 = d\eta^2 - d\xi^2$. As the massless scalar field is conformally invariant in 2-dimensional spacetime, the latter property essentially simplifies the quantization procedure in Rindler coordinates.

So, we consider a massless scalar field $\phi(x)$ in (1+1)-dimensional spacetime obeying the Klein-Gordon equation

$$\Box\phi(x) = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\right)\phi(x) = 0.$$

In the coordinates (10.12), $x^\mu = (\eta, \xi)$, this equation reads

$$\left(\partial_\eta^2 - \partial_\xi^2\right)\phi(x) = 0.$$

It has the same form as in the inertial coordinates. The corresponding positive-energy solutions have the form

$$\phi_k = \frac{e^{ik\xi \pm i\omega\eta}}{\sqrt{4\pi\omega}},\ -\infty < k < +\infty,\ \omega = |k|, \tag{10.13}$$

where the upper and lower signs correspond to the L and R Rindler wedges respectively. Recall that for the proper time of an observer with a fixed ξ one has $t_s = \pm\eta e^{a\xi}$. The different signs in front of ω in (10.13) for the modes in L- and R-regions is related to the different signs in the relation for the proper time.

Now we can introduce the modes $\{\phi_k^{(L)}, \phi_k^{(R)}\}$ that form a complete set in both the Rindler wedges. The separate parts are defined as

$$\phi_k^{(L)} = \begin{cases} e^{ik\xi+i\omega\eta}/\sqrt{4\pi\omega}, & \text{in } L\text{-region} \\ 0, & \text{in } R\text{-region} \end{cases},$$

$$\phi_k^{(R)} = \begin{cases} e^{ik\xi-i\omega\eta}/\sqrt{4\pi\omega}, & \text{in } R\text{-region} \\ 0, & \text{in } L\text{-region} \end{cases}.$$

These modes can also be analytically continued to the remaining wedges of the Minkowski spacetime (see D.G. Boulware, Phys. Rev. D **11**, 1404 (1975); **12**, 350 (1975)). The field operator can be expanded over this complete set of modes as

$$\phi = \int_{-\infty}^{+\infty} dk \sum_{P=L,R}\left[b_k^{(P)}\phi_k^{(P)} + b_k^{(P)\dagger}\phi_k^{(P)*}\right], \tag{10.14}$$

with $b_k^{(P)}$ and $b_k^{(P)\dagger}$ being the creation and annihilation operators. The vacuum state for a uniformly accelerated observer, the Fulling-Rindler vacuum state, $|0\rangle_R$, is defined as a state of the quantum field ϕ obeying the conditions

$$b_k^{(P)}|0\rangle_R = 0,\ P = L, R.$$

Alternatively, we could expand the field operator over the complete set of the Minkowskian modes

$$\phi_K^{(M)} = \frac{e^{iKx^1 - i\Omega t}}{\sqrt{4\pi\Omega}},\ -\infty < K < +\infty,\ \Omega = |K|. \tag{10.15}$$

The corresponding expansion has the form

$$\phi = \int_{-\infty}^{+\infty} dK\left[b_K^{(M)}\phi_K^{(M)} + b_K^{(M)\dagger}\phi_K^{(M)*}\right], \tag{10.16}$$

with the Minkowskian vacuum $|0\rangle_M$ defined in accordance with

$$b_k^{(M)}|0\rangle_M = 0.$$

In order to find the Bogoliubov coefficients let us consider the right Rindler wedge. In this wedge the expansion (10.14) is reduced

$$\phi = \int_{-\infty}^{+\infty} dk\left[b_k^{(R)}\phi_k^{(R)} + b_k^{(R)\dagger}\phi_k^{(R)*}\right].$$

From other side, for the same operator, we have the expansion (10.16) over the Minkowskian modes. The expansion over the Rindlerian modes can be rewritten as

$$\phi = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{d\omega}{\sqrt{\omega}}\left[b_\omega^{(R)}e^{i\omega u} + b_\omega^{(R)\dagger}e^{-i\omega u} + b_{-\omega}^{(R)}e^{-i\omega v} + b_{-\omega}^{(R)\dagger}e^{i\omega v}\right], \tag{10.17}$$

where

$$u = \xi - \eta, \quad v = \xi + \eta$$

are the corresponding light-cone coordinates. In the same coordinates, the Minkowskian expansion is presented in the form

$$\phi = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left[b_\Omega^{(M)}e^{i\frac{\Omega}{a}e^{au}} + b_\Omega^{(M)\dagger}e^{-i\frac{\Omega}{a}e^{au}} + b_{-\Omega}^{(M)}e^{-i\frac{\Omega}{a}e^{av}} + b_{-\Omega}^{(M)\dagger}e^{i\frac{\Omega}{a}e^{av}}\right]. \tag{10.18}$$

The right-hand sides in (10.17) and (10.18) should be equal to each other for arbitrary u and v. As the latter can be varied independently, we conclude that

$$\begin{aligned}
\int_0^\infty \frac{d\omega}{\sqrt{\omega}}\left[b_\omega^{(R)}e^{i\omega u} + b_\omega^{(R)\dagger}e^{-i\omega u}\right] &= \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left[b_\Omega^{(M)}e^{i\frac{\Omega}{a}e^{au}} + b_\Omega^{(M)\dagger}e^{-i\frac{\Omega}{a}e^{au}}\right], \\
\int_0^\infty \frac{d\omega}{\sqrt{\omega}}\left[b_{-\omega}^{(R)}e^{-i\omega v} + b_{-\omega}^{(R)\dagger}e^{i\omega v}\right] &= \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left[b_{-\Omega}^{(M)}e^{-i\frac{\Omega}{a}e^{av}} + b_{-\Omega}^{(M)\dagger}e^{i\frac{\Omega}{a}e^{av}}\right].
\end{aligned} \tag{10.19}$$

We want to find the relation between the Rindlerian and Minkowskian creation and annihilation operators. Let us multiply the first equation in (10.19) by $e^{-i\omega' u}$, with $\omega' > 0$, and integrate over u in the range $(-\infty, +\infty)$:

$$b_\omega^{(R)} = \frac{\sqrt{\omega}}{2\pi}\int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left[b_\Omega^{(M)}\int_{-\infty}^{+\infty} du\, e^{i\frac{\Omega}{a}e^{au} - i\omega u} + b_\Omega^{(M)\dagger}\int_{-\infty}^{+\infty} du\, e^{-i\frac{\Omega}{a}e^{au} - i\omega u}\right],$$

where we have omitted the prime in ω'. Let us consider the integral

$$\mathcal{I}_\pm = \int_{-\infty}^{+\infty} du\, e^{\pm i\frac{\Omega}{a}e^{au} - i\omega u}.$$

First we introduce a new integration variable $x = e^{au}$, $u = \ln(x)/a$, $du = dx/(ax)$:

$$\mathcal{I}_\pm = \frac{1}{a}\int_0^\infty dx\, x^{-i\omega/a - 1}e^{\pm i\frac{\Omega}{a}x}.$$

Now we rotate the integration contour in the complex plane x by the angle $\pi/2$ for the upper sign and by the angle $-\pi/2$ for the lower sign. That gives

$$\begin{aligned}
\mathcal{I}_\pm &= \frac{1}{a}\int_0^{\pm i\infty} dx\, x^{-i\omega/a - 1}e^{i\frac{\Omega}{a}x} = \frac{1}{a}\left(e^{\pm i\pi/2}\right)^{-i\omega/a}\int_0^\infty dy\, y^{-i\omega/a - 1}e^{-\frac{\Omega}{a}y} \\
&= \frac{1}{a}e^{\pm\pi\omega/2a}\left(\frac{\Omega}{a}\right)^{i\omega/a}\int_0^\infty dz\, z^{-i\omega/a - 1}e^{-z} = \frac{1}{a}e^{\pm\pi\omega/2a}\left(\frac{\Omega}{a}\right)^{i\omega/a}\Gamma\left(-i\omega/a\right).
\end{aligned}$$

Hence, we get

$$b_\omega^{(R)} = \frac{\sqrt{\omega}}{2\pi a}\Gamma\left(-i\omega/a\right)\int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left(\frac{\Omega}{a}\right)^{i\omega/a}\left[b_\Omega^{(M)}e^{\pi\omega/2a} + b_\Omega^{(M)\dagger}e^{-\pi\omega/2a}\right].$$

For the Hermitian conjugate we find

$$b_\omega^{(R)\dagger} = \frac{\sqrt{\omega}}{2\pi a}\Gamma\left(i\omega/a\right)\int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left(\frac{\Omega}{a}\right)^{-i\omega/a}\left[b_\Omega^{(M)}e^{-\pi\omega/2a} + b_\Omega^{(M)\dagger}e^{\pi\omega/2a}\right]. \tag{10.20}$$

Comparing this with the general relation

$$a_l = \sum_j \left(\bar{a}_j \alpha_{jl} + \bar{a}_j^+ \beta_{jl}^*\right),$$

for the Bogoliubov coefficients one finds the expressions

$$\begin{aligned} \alpha_{\Omega\omega} &= \frac{\sqrt{\omega/\Omega}}{2\pi a}\Gamma\left(-i\omega/a\right)\left(\frac{\Omega}{a}\right)^{i\omega/a} e^{\pi\omega/2a}, \\ \beta_{\Omega\omega} &= \frac{\sqrt{\omega/\Omega}}{2\pi a}\Gamma\left(i\omega/a\right)\left(\frac{\Omega}{a}\right)^{-i\omega/a} e^{-\pi\omega/2a}. \end{aligned} \tag{10.21}$$

For the VEV determining the number of the Rindler particles in the Minkowskian vacuum one obtains

$$\begin{aligned} \langle 0| b_{\omega'}^{(R)\dagger} b_{\omega}^{(R)} |0\rangle_M &= \frac{\sqrt{\omega\omega'}}{(2\pi a)^2}\Gamma\left(i\omega'/a\right)\Gamma\left(-i\omega/a\right)e^{-\pi(\omega+\omega')/2a}\int_0^\infty \frac{d\Omega d\Omega'}{\sqrt{\Omega\Omega'}} \\ &\times \left(\frac{\Omega}{a}\right)^{i\omega/a}\left(\frac{\Omega'}{a}\right)^{-i\omega'/a} \langle 0| b_{\Omega'}^{(M)} b_{\Omega}^{(M)\dagger} |0\rangle_M . \end{aligned}$$

By taking into account that $\langle 0| b_{\Omega'}^{(M)} b_{\Omega}^{(M)\dagger} |0\rangle_M = \delta(\Omega' - \Omega)$, this expression is transformed to

$$\langle 0| b_{\omega'}^{(R)\dagger} b_{\omega}^{(R)} |0\rangle_M = \frac{\sqrt{\omega\omega'}}{(2\pi a)^2}\Gamma\left(i\omega'/a\right)\Gamma\left(-i\omega/a\right)e^{-\pi(\omega+\omega')/2a}\int_0^\infty \frac{d\Omega}{\Omega}\left(\frac{\Omega}{a}\right)^{i(\omega-\omega')/a}.$$

The integral in the right hind side is reduced to

$$\int_0^\infty dx\, x^{i(\omega-\omega')/a-1} = \int_0^\infty dx\,\frac{1}{x}e^{i(\omega-\omega')\ln(x)/a} = \int_{-\infty}^\infty dy\, e^{i(\omega-\omega')y/a} = 2\pi a\delta(\omega-\omega'),$$

with the result

$$\langle 0| b_{\omega'}^{(R)\dagger} b_{\omega}^{(R)} |0\rangle_M = \frac{\omega}{2\pi a}|\Gamma\left(i\omega/a\right)|^2 e^{-\pi\omega/a}\delta(\omega-\omega').$$

By using the formula

$$|\Gamma\left(iy\right)|^2 = \frac{\pi}{y\sinh(\pi y)},$$

we come to the final expression

$$\langle 0| b_{\omega'}^{(R)\dagger} b_{\omega}^{(R)} |0\rangle_M = \frac{\delta(\omega-\omega')}{e^{2\pi\omega/a}-1}.$$

Let us compare this result with the similar quantity for a scalar field in Minkowski spacetime in the thermodynamical equilibrium at temperature T. For the corresponding expectation value one has

$$\left\langle b_{\Omega'}^{(M)\dagger} b_{\Omega}^{(M)} \right\rangle = \mathrm{tr}[\hat{\rho} b_{\Omega'}^{(M)\dagger} b_{\Omega}^{(M)}], \tag{10.22}$$

where $\langle \cdots \rangle$ means the ensemble average and $\hat{\rho}$ is the density matrix. The latter is given by

$$\hat{\rho} = Z^{-1}e^{-\hat{H}/T}, \tag{10.23}$$

where $Z = \mathrm{tr}[e^{-\hat{H}/T}]$ is the canonical partition function. For (10.22) one has

$$\mathrm{tr}[\hat{\rho} b_{\Omega'}^{(M)\dagger} b_{\Omega}^{(M)}] = \frac{\delta(\Omega-\Omega')}{e^{\Omega/T}-1}. \tag{10.24}$$

This can be shown in the following way. We define

$$b_K^{(M)}(\beta) = e^{\beta\hat{H}} b_K^{(M)} e^{-\beta\hat{H}}, \quad b_K^{(M)\dagger}(\beta) = e^{-\beta\hat{H}} b_K^{(M)\dagger} e^{\beta\hat{H}},$$

where $\beta = 1/T$. One has the following relations

$$\partial_\beta b_K^{(M)}(\beta) = -[b_K^{(M)}(\beta), \hat{H}], \quad \partial_\beta b_K^{(M)\dagger}(\beta) = [b_K^{(M)\dagger}(\beta), \hat{H}]. \tag{10.25}$$

By taking into account the expression for the Hamilton operator,

$$\hat{H} = \int_{-\infty}^{+\infty} dK\, \Omega(b_K^{(M)\dagger} b_K^{(M)} + 1/2),$$

for the commutator one gets

$$\begin{aligned} [b_K^{(M)}, \hat{H}] &= \int_{-\infty}^{+\infty} dK'\, \Omega' [b_K^{(M)}, b_{K'}^{(M)\dagger}] b_{K'}^{(M)} = \Omega b_K^{(M)}, \\ [b_K^{(M)\dagger}, \hat{H}] &= \int_{-\infty}^{+\infty} dK'\, \Omega' b_{K'}^{(M)\dagger} [b_K^{(M)\dagger}, b_{K'}^{(M)}] = -\Omega b_K^{(M)\dagger}, \end{aligned}$$

where the commutation relation $[b_K^{(M)}, b_{K'}^{(M)\dagger}] = \delta(K - K')$ was used. With these relations, (10.25) gives

$$\partial_\beta b_K^{(M)}(\beta) = -\Omega b_K^{(M)}(\beta), \quad \partial_\beta b_K^{(M)\dagger}(\beta) = -\Omega b_K^{(M)\dagger}(\beta),$$

having the solutions

$$b_K^{(M)}(\beta) = e^{-\Omega\beta} b_K^{(M)}, \quad b_K^{(M)\dagger}(\beta) = e^{-\Omega\beta} b_K^{(M)\dagger}.$$

We can now write:

$$\begin{aligned} \mathrm{tr}[\hat{\rho} b_K^{(M)\dagger} b_{K'}^{(M)}] &= \mathrm{tr}[\hat{\rho} b_K^{(M)\dagger} e^{\beta\widehat{H}} e^{-\beta\widehat{H}} b_{K'}^{(M)}] = \mathrm{tr}[e^{-\beta\widehat{H}} b_{K'}^{(M)} \hat{\rho} b_K^{(M)\dagger} e^{\beta\widehat{H}}] \\ &= \frac{1}{Z} \mathrm{tr}[e^{-\beta\widehat{H}} b_{K'}^{(M)} e^{-\beta\widehat{H}} b_K^{(M)\dagger} e^{\beta\widehat{H}}] = \frac{1}{Z} \mathrm{tr}[e^{-\beta\widehat{H}} b_{K'}^{(M)} b_K^{(M)\dagger}(\beta)] \\ &= \mathrm{tr}[\hat{\rho} b_{K'}^{(M)} b_K^{(M)\dagger}(\beta)] = \mathrm{tr}[\hat{\rho} b_{K'}^{(M)} b_K^{(M)\dagger}] e^{-\Omega\beta} \\ &= \mathrm{tr}[\hat{\rho} \left(b_K^{(M)\dagger} b_{K'}^{(M)} + \delta(K - K') \right)] e^{-\Omega\beta} \\ &= \delta(K - K') e^{-\Omega\beta} + \mathrm{tr}[\hat{\rho} b_K^{(M)\dagger} b_{K'}^{(M)}] e^{-\Omega\beta}, \end{aligned}$$

and, hence,

$$\mathrm{tr}[\hat{\rho} b_K^{(M)\dagger} b_{K'}^{(M)}] = \frac{\delta(K - K')}{e^{\Omega\beta} - 1}.$$

For $K = \Omega$, $K' = \Omega'$ this relation is reduced to (10.24).

As a result, we come to an important conclusion: In the Minkowskian vacuum, a uniformly accelerated observer detects particles (Rindler particles) which are distributed in a way that coincides with the distribution of particles in an inertial frame at thermal equilibrium at temperature T_a related to the acceleration by the formula

$$T_a = \frac{a}{2\pi} \left(= \frac{\hbar a}{2\pi c} \text{ in standard units} \right).$$

This is the Unruh effect.

Chapter 11

Quantum fields in de Sitter spacetime

11.1 De Sitter spacetime

De Sitter spacetime is one of the simplest and most interesting spacetimes allowed by general relativity. Quantum field theory in this background has been extensively studied during the past two decades. Much of the early interest was motivated by the questions related to the quantization of fields on curved backgrounds. dS spacetime has a high degree of symmetry, and numerous physical problems are exactly solvable on this background. The importance of this theoretical work increased by the appearance of the inflationary cosmology scenario [46]. In most inflationary models, an approximately dS spacetime is employed to solve a number of problems in standard cosmology. During an inflationary epoch, quantum fluctuations in the inflaton field introduce inhomogeneities which play a central role in the generation of cosmic structures from inflation. More recently, astronomical observations of high redshift supernovae, galaxy clusters, and cosmic microwave background indicate that at the present epoch the Universe is accelerating and can be well approximated by a world with a positive cosmological constant [47]. If the Universe were to accelerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. Hence, the investigation of physical effects in dS spacetime is important for understanding both the early Universe and its future. Another motivation for investigations of de Sitter-based quantum theories is related to the holographic duality between quantum gravity on de Sitter spacetime and a quantum field theory living on a boundary identified with the timelike infinity of de Sitter spacetime [48].

11.2 Maximally symmetric solutions of Einstein equations with a cosmological constant

Let us consider the Einstein equations with the source corresponding to the vacuum energy-momentum tensor. For the latter one has

$$T_{\mu\nu}^{(\mathrm{vac})} = \rho g_{\mu\nu},\ \rho = \mathrm{const}, \tag{11.1}$$

where ρ is the energy density. From the Einstein equations one has

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\varkappa^2 \rho g_{\mu\nu}.$$

Taking the trace we get

$$R = 2\varkappa^2 \rho \frac{D+1}{D-1},$$

and, hence,

$$R_{\mu\nu} = \frac{2\varkappa^2 \rho}{D-1} g_{\mu\nu}.$$

The constant Λ defined as

$$\Lambda = \varkappa^2 \rho = 8\pi G \rho$$

is the cosmological constant.

In terms of the cosmological constant, for the Ricci scalar and Ricci tensor one has

$$R = 2\Lambda \frac{D+1}{D-1}, \quad R_{\mu\nu} = 2\Lambda \frac{g_{\mu\nu}}{D-1}.$$

The corresponding Riemann tensor is given by the expression

$$R_{\mu\nu\alpha\beta} = 2\Lambda \frac{g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}}{D(D-1)}.$$

Hence, for the maximally symmetric solutions of the Einstein equations with a cosmological constant we have

$$R = 2\Lambda \frac{D+1}{D-1}, \quad R_{\mu\nu} = 2\Lambda \frac{g_{\mu\nu}}{D-1}, \quad R_{\mu\nu\alpha\beta} = 2\Lambda \frac{g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}}{D(D-1)}. \tag{11.2}$$

The corresponding spacetimes are

$$\begin{aligned} \text{Minkowski spacetime for } \Lambda &= 0, \\ \text{de Sitter (dS) spacetime for } \Lambda &> 0, \\ \text{anti-de Sitter (AdS) spacetime for } \Lambda &< 0. \end{aligned}$$

The number of the Killing vectors for all these spacetimes is equal to $(D+1)(D+2)/2$ as is should be for $(D+1)$-dimensional maximally symmetric spaces.

11.3 Geometry of de Sitter spacetime and the coordinate systems

$(D+1)$-dimensional de Sitter space is represented as the hyperboloid

$$-\eta_{AB} z^A z^B = -(z^0)^2 + (z^1)^2 \cdots + (z^{D+1})^2 = \alpha^2, \quad A, B = 0, 1, \ldots, D+1, \tag{11.3}$$

with the parameter α having the dimension of length, embedded in $(D+2)$-dimensional Minkowski space with the line element

$$ds^2_{D+2} = \eta_{AB} dz^A dz^B.$$

The symmetry group of de Sitter space is the $(D+1)(D+2)/2$ parameter group $SO(1, D+1)$ of homogeneous Lorentz transformations in the $(D+2)$-dimensional embedding space known as the dS group. de Sitter group of the symmetries on dS space is fundamental to the discussion of quantization.

By using

$$z^2_{D+1} = \alpha^2 + \eta_{\mu\nu} z^\mu z^\nu,$$

we can exclude the coordinate z_{D+1} from the expression for ds^2_{D+2}:

$$dz^{D+1} = \pm \frac{\eta_{\mu\nu} z^\mu dz^\nu}{\sqrt{\alpha^2 + \eta_{\mu\nu} z^\mu z^\nu}},$$

$$ds^2 = \left(\eta_{\mu\nu} - \frac{\eta_{\mu\alpha} \eta_{\nu\beta} z^\alpha z^\beta}{\alpha^2 + \eta_{\alpha\beta} z^\alpha z^\beta} \right) dz^\mu dz^\nu.$$

From here it follows that the induced metric tensor on the hyperboloid is given by

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{\eta_{\mu\alpha} \eta_{\nu\beta} z^\alpha z^\beta}{\alpha^2 + \eta_{\alpha\beta} z^\alpha z^\beta}. \tag{11.4}$$

For the contravariant components of the metric tensor one obtains

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{\alpha^2} z^\mu z^\nu .$$

Indeed, we have

$$g_{\mu\nu} g^{\nu\rho} = \left(\eta_{\mu\nu} - \frac{\eta_{\mu\alpha}\eta_{\nu\beta} z^\alpha z^\beta}{\alpha^2 + \eta_{\alpha\beta} z^\alpha z^\beta} \right) \left(\eta^{\nu\rho} + \frac{1}{\alpha^2} z^\nu z^\rho \right) = \delta^\rho_\mu .$$

Evaluating the curvature tensor for the metric tensor (11.4) and comparing with (11.2), we obtain the relation between the constant α and the cosmological constant:

$$\Lambda = \frac{D(D-1)}{2\alpha^2}. \tag{11.5}$$

For the Ricci scalar one has

$$R = \frac{D(D+1)}{\alpha^2}.$$

The first step in the quantization procedure is to solve the field equation. To accomplish this, the coordinates in de Sitter space should be specified. We will consider the three most widely used coordinatizations of the de Sitter hyperboloid.

11.3.1 Global coordinates

From the relation (11.3) it follows that for a given z^0 the space described by the spatial coordinates $(z^1, z^2, \ldots, z^{D+1})$ presents a sphere with the radius $\sqrt{\alpha^2 + (z^0)^2}$. On the base of this we can write

$$z^0 = \alpha \sinh(t/\alpha), \quad z^i = \alpha \omega^i \cosh(t/\alpha), \ -\infty < t < +\infty,$$

with $i = 1, \ldots, D+1$ and

$$\sum_{i=1}^{D+1} (\omega^i)^2 = 1.$$

The latter equation describes D-dimensional unit sphere. Consequently, we can introduce the angular coordinates θ_1,θ_2,...,θ_D in accordance with

$$\begin{aligned} \omega^1 &= \cos\theta_1, \quad \omega^2 = \sin\theta_1 \cos\theta_2, \ldots, \\ \omega^D &= \sin\theta_1 \cos\theta_2 \cdots \sin\theta_{D-1} \cos\theta_D, \\ \omega^{D+1} &= \sin\theta_1 \cos\theta_2 \cdots \sin\theta_{D-1} \sin\theta_D, \end{aligned} \tag{11.6}$$

where $0 \leqslant \theta_i < \pi$ for $i = 1, \ldots, D-1$, and $0 \leqslant \theta_D < 2\pi$. In terms of the coordinates $(t, \theta_1, \theta_2, \ldots, \theta_D)$ the dS line element is written in the form

$$ds^2 = dt^2 - \alpha^2 \cosh^2(t/\alpha) d\Omega_D^2, \tag{11.7}$$

where $d\Omega_D^2$ is the line element on a unit D-dimensional sphere:

$$\begin{aligned} d\Omega_D^2 &= (d\theta_1)^2 + \sin^2\theta_1 (d\theta_2)^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{D-1} (d\theta_D)^2 \\ &= (d\theta_1)^2 + \sum_{j=2}^{D} \left(\prod_{i=1}^{j-1} \sin^2\theta_i \right) (d\theta_j)^2. \end{aligned}$$

The coordinate system $(t, \theta_1, \theta_2, \ldots, \theta_D)$ covers entire $(D+1)$-dimensional hyperboloid and they are referred as global coordinates. In the global coordinates, the spatial hypersurfaces with fixed time t, correspond to D-dimensional spheres of radius $\alpha\cosh(t/\alpha)$. The radius is infinitely large at $t=-\infty$, decreases to the minimum radius α at $t=0$, and then increases to infinite size as $t\to\infty$.

Let us introduce a new time coordinate T, $-\pi/2 < T/\alpha < \pi/2$, in accordance with

$$\begin{aligned} dt &= \cosh(t/\alpha)dT \Rightarrow dT/\alpha = \frac{dt/\alpha}{\cosh(t/\alpha)} = \frac{d\sinh(t/\alpha)}{\sinh^2(t/\alpha)+1}, \\ T/\alpha &= \arctan(\sinh(t/\alpha)) \Rightarrow \sinh(t/\alpha) = \tan(T/\alpha) \Rightarrow \cosh^2(t/\alpha) = \cos^{-2}(T/\alpha), \end{aligned}$$

the line element is presented in the form

$$ds^2 = \cos^{-2}(T/\alpha)\left(dT^2 - \alpha^2 d\Omega_D^2\right), \tag{11.8}$$

The corresponding metric tensor is conformally static.

11.3.2 Planar or inflationary coordinates

The planar or inflationary coordinates, $(t, x^1, \ldots, x^D)$, are most appropriate for cosmological applications. They are related to the coordinates z^μ by the expressions

$$\begin{aligned} z^0 &= \alpha\sinh(t/\alpha) + \frac{e^{t/\alpha}}{2\alpha}\sum_{l=1}^{D}(x^l)^2, \\ z^l &= x^l e^{t/\alpha},\ l = 1, \ldots, D, \\ z^{D+1} &= \alpha\cosh(t/\alpha) - \frac{e^{t/\alpha}}{2\alpha}\sum_{l=1}^{D}(x^l)^2. \end{aligned} \tag{11.9}$$

They cover the half of hyperboloid (11.3) with $z^0 + z^{D+1} > 0$.

$$\begin{aligned} (z^0)^2 - (z^{D+1})^2 - \sum_{l=1}^{D}(z^l)^2 &= \left[-\alpha e^{-t/\alpha} + \frac{e^{t/\alpha}}{\alpha}\sum_{l=1}^{D}(x^l)^2\right]\alpha e^{t/\alpha} - e^{2t/\alpha}\sum_{l=1}^{D}(x^l)^2 \\ &= -\alpha^2. \end{aligned}$$

In planar coordinates the dS line element takes the form

$$ds^2 = dt^2 - e^{2t/\alpha}\sum_{l=1}^{D}(dx^l)^2. \tag{11.10}$$

In addition to the synchronous time coordinate, t, we may use the conformal time, τ, defined as $\tau = -\alpha e^{-t/\alpha}$, $-\infty < \tau < 0$. In terms of this coordinate the line element takes conformally flat form:

$$ds^2 = \alpha^2\tau^{-2}\left[d\tau^2 - \sum_{l=1}^{D}(dx^l)^2\right]. \tag{11.11}$$

The line element is conformally related to the part of the Minkowski spacetime (τ, x^l) defined by $-\infty < \tau < 0$.

11.3.3 Static coordinates

Let us introduce the radial coordinate r in accordance with

$$r^2 = (z^1)^2 \cdots + (z^D)^2.$$

The equation of the hyperboloid is reduced to

$$(z^{D+1})^2 - (z^0)^2 = \alpha^2 - r^2.$$

On the base of this relation we consider the coordinates (t, r, ω^i) in accordance with

$$\begin{aligned} z^0 &= \sqrt{\alpha^2 - r^2}\sinh(t/\alpha), \\ z^i &= r\omega^i,\ i = 1, 2, \ldots, D, \\ z^{D+1} &= \sqrt{\alpha^2 - r^2}\cosh(t/\alpha). \end{aligned}$$

These coordinates cover the region $r \leqslant \alpha$. The sphere $r = \alpha$ presents the horizon. Introducing angular coordinates $(\theta_1, \theta_2, \ldots, \theta_{D-1})$ instead of ω^i in accordance with (11.6), the line element for de Sitter spacetime is presented as

$$ds^2 = (1 - r^2/\alpha^2)dt^2 - \frac{dr^2}{1 - r^2/\alpha^2} - r^2 d\Omega_{D-1}^2, \tag{11.12}$$

with the line element for unit $(D-1)$-dimensional sphere

$$\begin{aligned} d\Omega_{D-1}^2 &= (d\theta_1)^2 + \sin^2\theta_1 (d\theta_2)^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{D-2}(d\theta_{D-1})^2 \\ &= (d\theta_1)^2 + \sum_{j=2}^{D-1}\left(\prod_{i=1}^{j-1}\sin^2\theta_i\right)(d\theta_j)^2. \end{aligned}$$

Hence, in the coordinates $(t, r, \theta_1, \theta_2, \ldots, \theta_{D-1})$ the metric tensor does not depend on time.

11.4 Scalar field mode functions in dS spacetime

Here we consider complete set of solutions for the Klein-Gordon equation in different coordinate systems of dS spacetime.

11.4.1 Planar coordinates

In planar coordinates for the metric tensor one has

$$g_{\mu\nu} = \alpha^2 \tau^{-2} \eta_{\mu\nu}.$$

Taking the spatial coordinate dependence of the mode functions in the form $e^{i\mathbf{k}\cdot\mathbf{z}}$ with $\mathbf{z} = (x^1, \ldots, x^D)$, we get the equation for the time-dependent part. the latter is solved in terms of the cylindrical functions and the mode functions are presented as

$$\varphi_\sigma(x) = \eta^{D/2} \sum_{j=1,2} c_j H_\nu^{(j)}(k\eta) e^{i\mathbf{k}\cdot\mathbf{z}},\ \eta = |\tau|, \tag{11.13}$$

where $k = |\mathbf{k}|$, $H_\nu^{(j)}(z)$, $j = 1, 2$, are the Hankel functions with the order

$$\nu = \left[D^2/4 - D(D+1)\xi - m^2\alpha^2\right]^{1/2}. \tag{11.14}$$

One of the coefficients c_j is determined by the normalization condition. Different choices of the other coefficient correspond to different choices of the vacuum state in de Sitter spacetime. The choice of the vacuum state is among the most important steps in construction of a quantum field theory in a fixed classical gravitational background. de Sitter spacetime is a maximally symmetric space and it is natural to choose a vacuum state having the same symmetry. In fact, there exists a one-parameter family of maximally symmetric quantum states (see, for instance, Ref. [49] and references therein). Here we will assume that the field is prepared in the de Sitter-invariant Bunch-Davies vacuum state [50] for which $c_2 = 0$. Among the set of de Sitter-invariant quantum states the Bunch-Davies vacuum is the only one for which the ultraviolet behavior of the two-point functions is the same as in Minkowski spacetime.

Hence, for the mode functions realizing the Bunch-Davies vacuum state one has

$$\varphi_{\mathbf{k}}(x) = c_1 \eta^{D/2} H_\nu^{(1)}(k\eta) e^{i\mathbf{k}\cdot\mathbf{z}}. \tag{11.15}$$

The constant c_1 is determined from the condition (6.1):

$$\begin{aligned}
&\alpha^D c_1^2 \int d^D x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{z}} \left[H_\nu^{(1)}(k\eta)\partial_t (H_\nu^{(1)}(k'\eta))^* - (H_\nu^{(1)}(k'\eta))^* \partial_t H_\nu^{(1)}(k\eta)\right] \\
&= -k(\eta/\alpha)(2\pi)^D \alpha^D c_1^2 \delta(\mathbf{k}-\mathbf{k}') \left[H_\nu^{(1)}(k\eta)\partial_{k\eta}(H_\nu^{(1)}(k\eta))^* - (H_\nu^{(1)}(k\eta))^* \partial_{k\eta} H_\nu^{(1)}(k\eta)\right] \\
&= \frac{4i}{\pi\eta k} e^{-i(\nu-\nu^*)\pi/2} k(\eta/\alpha)(2\pi)^D \alpha^D c_1^2 \delta(\mathbf{k}-\mathbf{k}') = i\delta(\mathbf{k}-\mathbf{k}'),
\end{aligned}$$

where we have used the Wronskian relation

$$H_\nu^{(1)}(k\eta) H_{\nu^*}^{(2)\prime}(k\eta) - H_{\nu^*}^{(2)}(k\eta) H_\nu^{(1)\prime}(k\eta) = -\frac{4i}{\pi\eta k} e^{-i(\nu-\nu^*)\pi/2}.$$

For the normalization constant we get

$$c_1^2 = \frac{e^{i(\nu-\nu^*)\pi/2}}{8(2\pi\alpha)^{D-1}}. \tag{11.16}$$

The information on the properties of the vacuum state is contained in two-point functions. Here we consider the Wightman function, $G^+(x,x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$, where $|0\rangle$ stands for the Bunch-Davies vacuum state. In addition to the vacuum expectation values of various physical observables, this function determines the response of the Unruh-DeWitt particle detector. Having the mode functions, the Wightman function may be evaluated by making use of the mode sum formula

$$G^+(x,x') = \int d^D\mathbf{k}\, \varphi_\sigma(x)\varphi_\sigma^*(x'). \tag{11.17}$$

Substituting the mode functions we find

$$G^+(x,x') = \frac{e^{i(\nu-\nu^*)\pi/2}(\eta\eta')^D}{8(2\pi\alpha)^{D-1}} \int d^D\mathbf{k}\; H_\nu^{(1)}(k\eta)[H_\nu^{(1)}(k\eta')]^* e^{i\mathbf{k}\cdot\Delta\mathbf{z}},$$

where $\Delta\mathbf{z} = \mathbf{z} - \mathbf{z}'$. First we use the formula

$$\int d^D\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{z}} F(k) = (2\pi)^{D/2} \int_0^\infty dk\, k^{D-1} F(k) \frac{J_{D/2-1}(k|\Delta\mathbf{z}|)}{(k|\Delta\mathbf{z}|)^{D/2-1}},$$

for a given function $F(k)$. This gives

$$G^+(x,x') = \frac{e^{i(\nu-\nu^*)\pi/2}(\eta\eta')^D}{8(2\pi)^{D/2-1}\alpha^{D-1}|\Delta\mathbf{z}|^{D/2-1}} \int_0^\infty dk\, k^{D/2} H_\nu^{(1)}(k\eta)[H_\nu^{(1)}(k\eta')]^* J_{D/2-1}(k|\Delta\mathbf{z}|).$$

As the next step, we write the product of the Hankel functions in terms of the Macdonald function:

$$e^{i(\nu-\nu^*)\pi/2} H_\nu^{(1)}(k\eta)[H_\nu^{(1)}(k\eta')]^* = \frac{4}{\pi^2} K_\nu(-ik\eta) K_\nu(ik\eta'), \tag{11.18}$$

and use the integral representation [51]

$$K_\nu(Z)K_\nu(z) = \frac{1}{4}\int_{-\infty}^{+\infty} dy \int_0^\infty \frac{dw}{w} e^{-\nu y - Zzw^{-1}\cosh y} \exp\left(-\frac{w}{2} - \frac{Z^2+z^2}{2w}\right) \tag{11.19}$$

for the product of the Macdonald functions. In this way one finds

$$\begin{aligned} G^+(x,x') &= \frac{\alpha^{1-D}(\eta\eta')^{D/2}}{2(2\pi)^{D/2+1}|\Delta\mathbf{z}|^{D/2-1}} \int_0^\infty dk\, k^{D/2} \int_{-\infty}^{+\infty} dy \int_0^\infty \frac{dw}{w} \\ &\quad \times e^{-\nu y - k^2\eta\eta' w^{-1}\cosh y} e^{-w/2+k^2\left(\eta^2+\eta'^2\right)/(2w)} J_{D/2-1}(k|\Delta\mathbf{z}|) \\ &= \frac{\alpha^{1-D}(\eta\eta')^{D/2}}{2(2\pi)^{D/2+1}|\Delta\mathbf{z}|^{D/2-1}} \int_{-\infty}^{+\infty} dy \int_0^\infty \frac{dw}{w} e^{-\nu y - w/2} \\ &\quad \times \int_0^\infty dk\, k^{D/2} e^{-\gamma k^2/(2w)} J_{D/2-1}(k|\Delta\mathbf{z}|), \end{aligned}$$

where

$$\gamma = 2\eta\eta'\cosh y - \eta^2 - \eta'^2.$$

By using the integral

$$\int_0^\infty dk\, k^{\nu+1} e^{-\gamma k^2/(2w)} J_\nu(\beta x) = (w/\gamma)^{\nu+1}\beta^\nu e^{-w\beta^2/2\gamma},$$

we find

$$\begin{aligned} G^+(x,x') &= \frac{\alpha^{1-D}(\eta\eta')^{D/2}}{2(2\pi)^{D/2+1}} \int_{-\infty}^{+\infty} dy\, \gamma^{-D/2} e^{-\nu y} \int_0^\infty dw\, w^{D/2-1} e^{-w\left(1+|\Delta\mathbf{z}|^2/\gamma\right)/2} \\ &= \frac{\Gamma(D/2)(\eta\eta')^{D/2}}{4\pi^{D/2+1}\alpha^{D-1}} \int_{-\infty}^{+\infty} dy\, \frac{e^{-\nu y}}{(\gamma+|\Delta\mathbf{z}|^2)^{D/2}}. \end{aligned}$$

Introducing a new integration variable $z = e^y$, the integral is reduced to

$$G^+(x,x') = \frac{\Gamma(D/2)}{4\pi^{D/2+1}\alpha^{D-1}} \int_0^\infty dz \frac{z^{D/2-\nu-1}}{[z^2 - 2u(x,x')z + 1]^{D/2}}, \tag{11.20}$$

with the notation

$$u(x,x') = 1 + \frac{(\Delta\eta)^2 - |\Delta\mathbf{z}|^2}{2\eta\eta'}.$$

In deriving Eq. (11.20) we have assumed that $|u| < 1$. Let us denote by $\mu(x,x')$ the distance along the shortest geodesic from x to x'. If the geodesic is parametrized by λ, $x^i = x^i(\lambda)$, when

$$\mu(x,x') = \int_0^1 d\lambda\, [-g_{\sigma\rho}\partial_\lambda x^\sigma(\lambda)\partial_\lambda x^\rho(\lambda)]^{1/2},\ x^\sigma(0) = x,\ x^\sigma(1) = x'. \tag{11.21}$$

We have the relation

$$u(x,x') = \cos[\mu(x,x')/\alpha]. \tag{11.22}$$

The integral in Eq. (11.20) is expressed in terms of the associated Legendre function of the first kind $P_{\nu-1/2}^{(1-D)/2}(u(x,x'))$ (see [37]). Expressing this function through the hypergeometric function $F(a,b;c;z)$, after some transformations, we get the final expression for the Wightman function in dS spacetime (for two-point functions in de Sitter spacetime see [52, 53]):

$$\begin{aligned} G^+(x,x') &= \frac{\alpha^{1-D}}{(4\pi)^{(D+1)/2}} \frac{\Gamma(D/2+\nu)\Gamma(D/2-\nu)}{\Gamma((D+1)/2)} \\ &\quad \times F\left(\frac{D}{2}+\nu, \frac{D}{2}-v; \frac{D+1}{2}; \frac{1+u(x,x')}{2}\right). \end{aligned} \tag{11.23}$$

Note that, if we denote by $Z(x)$ the coordinates in the higher-dimensional embedding space for dS spacetime, then one can write $u(x,x') = 1 + [Z(x) - Z(x')]^2/(2\alpha^2)$. The property that the Wightman function depends on spacetime points through $[Z(x)-Z(x')]^2$ is related to the maximal symmetry of the Bunch-Davies vacuum state. Note that the hypergeometric function in (11.23) can also be written in terms of a new function by using the relation

$$F\left(\frac{D}{2}+\nu, \frac{D}{2}-v; \frac{D+1}{2}; \frac{1+u(x,x')}{2}\right) = F\left(\frac{D+2\nu}{4}, \frac{D-2\nu}{4}; \frac{D+1}{2}; 1-u^2(x,x')\right).$$

Let us consider the leading term in the asymptotic expansion of the Wightman function in the coincidence limit of the arguments, which corresponds to $u(x,x') \to 1$. We will use the relation

$$\begin{aligned} F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-z). \end{aligned}$$

for the hypergeometric functions. In our case, $a+b-c = (D-1)/2$ and for $D>1$ the second term in the right-hand dominates. By taking into account the relation (11.22) to the leading order we get

$$G^+(x,x') \approx \frac{\Gamma((D-1)/2)}{4\pi^{(D+1)/2}\mu^{D-1}(x,x')},$$

with the geodesic distance $\mu(x,x')$. This shows that the Bunch-Davies vacuum is a Hadamard state.

Assuming $\mathrm{Re}((1-D)/2) > 0$, the hypergeometric function in the expression (11.23) for the Wightman function is finite in the coincidence limit:

$$G^+(x,x) = \frac{\alpha^{1-D}}{(4\pi)^{(D+1)/2}} \frac{\Gamma(D/2+\nu)\Gamma(D/2-\nu)}{\Gamma(1/2-\nu)\Gamma(1/2+\nu)}\Gamma((1-D)/2). \tag{11.24}$$

For even values this expression is finite. For odd values the gamma function has poles. In this case we can write

$$\begin{aligned} G^+(x,x) &= \frac{\alpha^{1-D}}{(4\pi)^{(D+1)/2}}((D/2-1)^2-\nu^2)\cdots((1/2)^2-\nu^2)\Gamma((1-D)/2) \\ &= \frac{\alpha^{1-D}\Gamma((1-D)/2)}{(4\pi)^{(D+1)/2}} \prod_{l=0}^{(D-3)/2}[(l+1/2)^2-\nu^2]. \end{aligned}$$

The renormalized vacuum expectation value of the energy-momentum tensor in de Sitter spacetime will be discussed below in section 13.5.

11.4.2 Global coordinates

In the global coordinates the line element is given by (11.7). For a scalar field $\varphi(x)$ with curvature coupling parameter ξ, the corresponding field equation

$$\left(\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu) + m^2 + \xi R\right)\varphi = 0,$$

in background of dS_{D+1} takes the form

$$\left[\frac{\partial_0(\cosh^D(t/\alpha)\partial_0)}{\cosh^D(t/\alpha)} - \frac{\Delta_\vartheta}{\alpha^2\cosh^2(t/\alpha)} + m^2 + \xi R\right]\varphi = 0, \tag{11.25}$$

where Δ_θ is the Laplace operator on a D-dimensional sphere with unit radius. The solution of this equation can be presented in the decomposed form

$$\varphi = A(t/\alpha)Y(m_k;\vartheta), \tag{11.26}$$

where $\vartheta = (\theta_1, \theta_2, \ldots, \theta_D)$, $m_k = (m_0 = l, m_1, \ldots, m_{D-1})$ and $m_1, \ldots, m_{D-1}$ are integers such that

$$0 \leqslant m_{D-2} \leqslant m_{D-3} \leqslant \cdots \leqslant m_1 \leqslant l, \ -m_{D-2} \leqslant m_{D-1} \leqslant m_{D-2},$$

$Y(m_k;\vartheta)$ is the surface harmonic of degree l (see [54], Sec. 11.2). The latter can be expressed through the Gegenbauer or ultraspherical polynomial $C_p^q(x)$ of degree p and order q as

$$Y(m_k;\vartheta) = e^{m_{D-1}\theta_D} \prod_{k=1}^{D-1} (\sin\theta_k)^{|m_k|} C_{m_{k-1}-|m_k|}^{|m_k|+(D-1)/2-k/2}(\sin\theta_k). \tag{11.27}$$

The surface harmonic obeys the equation

$$\Delta_\vartheta Y(m_k;\vartheta) = -l(l+D-1)Y(m_k;\vartheta).$$

The corresponding normalization integral is in the form

$$\int d\Omega\, |Y(m_k;\vartheta)|^2 = N(m_k).$$

The explicit form for $N(m_k)$ is given in [54], Sec. 11.3, and will not be necessary for the following consideration. From the addition theorem [54], Sec. 11.4, one has

$$\sum_{m_k} Y(m_k;\vartheta)Y^*(m_k;\vartheta') = \frac{2l+D-1}{(D-1)S_D} C_l^{(D-1)/2}(\cos\theta),$$

where $S_D = 2\pi^{(D+1)/2}\Gamma(D+1)/2$, θ is the angle between directions ϑ and ϑ'.

Substituting (11.26) into (11.25) we get the equation for the function $A(t)$:

$$\left[\frac{(\cosh^D x A'(x))'}{\cosh^D x} + \frac{l(l+D-1)}{\cosh^2 x} + \alpha^2\left(m^2 + \xi R\right)\right]\varphi = 0.$$

The solution of this equation is given by

$$A(x) = \left(1-u^2\right)^{D/4}\left[c_1 P_{l+D/2-1}^{\nu}(u) + c_2 Q_{l+D/2-1}^{\nu}(u)\right],$$

where $P_\mu^\nu(u)$ and $Q_\mu^\nu(u)$ are the associated Legendre functions, and

$$u = \tanh(t/\alpha).$$

11.4.3 Static coordinates

The de Sitter line element in static coordinates is given by (11.12). The Klein-Gordon equation

$$\left[\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu) + m^2 + \xi R\right]\varphi = 0,$$

is rewritten as

$$\left[\partial_0((1-r^2/\alpha^2)^{-1}\partial_0) - \frac{1}{r^{D-1}}\partial_r(r^{D-1}(1-r^2/\alpha^2)\partial_r) - \frac{1}{r^2}\Delta_\vartheta + m^2 + \xi R\right]\varphi = 0,$$

where Δ_ϑ is the Laplace operator on a $(D-1)$-dimensional sphere. The solution is decomposed as

$$\varphi = R(r/\alpha)Y^{(D-1)}(m_k;\vartheta)e^{-i\omega t},$$

where $Y^{(D-1)}(m_k;\vartheta)$ are the spherical harmonics on a $(D-1)$-dimensional sphere. The formulas for the latter are obtained from those in the previous section for the function $Y(m_k;\vartheta)$ by the replacement $D \to D-1$. By taking into account that

$$\Delta_\vartheta Y^{(D-1)}(m_k;\vartheta) = -l(l+D-2)Y^{(D-1)}(m_k;\vartheta),$$

for the radial function we get the equation

$$\frac{1}{x^{D-1}}\partial_x(x^{D-1}(1-x^2)R'(x)) + \left[\frac{\alpha^2\omega^2}{1-x^2} - \frac{l(l+D-2)}{x^2} - m^2\alpha^2 - D(D+1)\xi\right]R(x) = 0.$$

Introducing

$$x = \sin u,\ 0 \leqslant u \leqslant \pi/2,$$

the equation is written as

$$\frac{\partial_u(\sin^{D-1}u\cos u\partial_u R)}{\sin^{D-1}u\cos u} + \left[\frac{\alpha^2\omega^2}{\cos^2 u} - \frac{l(l+D-2)}{\sin^2 u} - m^2\alpha^2 - D(D+1)\xi\right]R = 0.$$

The linearly independent solutions of this equation are the functions (see)

$$p = 1,\ q = D-1,\ r = i\alpha\omega,\ n(n+D) = -m^2\alpha^2 - D(D+1)\xi,$$

$$\begin{aligned}
R_1 &= \tan^l(u)\cos^n(u)F\left(\frac{l-n+i\alpha\omega}{2},\frac{l-n-i\alpha\omega}{2};l+D/2;-\tan^2 u\right),\\
R_2 &= \cot^{l+D-2}(u)\cos^n(u)\\
&\quad\times F\left(1-\frac{l+n+D-i\alpha\omega}{2},1-\frac{l+n+D+i\alpha\omega}{2};2-l-D/2;-\tan^2 u\right),
\end{aligned}$$

where $F(a,b;c;y)$ is the hypergeometric function.

$$(n+D/2)^2 = D^2/4 - m^2\alpha^2 - D(D+1)\xi \Rightarrow n = -D/2 \pm \nu.$$

By using the the properties of the hypergeometric function it can be seen that both the signs lead to the same solutions. We take the lower sign, $n = -\nu - D/2$, with the solutions

$$\begin{aligned}
R_1(x) &= \frac{\tan^l(u)}{\cos^{D/2+\nu}(u)}F\left(\frac{l+D/2+\nu+i\alpha\omega}{2},\frac{l+D/2+\nu-i\alpha\omega}{2};l+D/2;-\tan^2 u\right),\\
R_2(x) &= \frac{\cot^{l+D-2}(u)}{\cos^{D/2+\nu}(u)}\\
&\quad\times F\left(1-\frac{l+D/2-\nu-i\alpha\omega}{2},1-\frac{l+D/2-\nu+i\alpha\omega}{2};2-l-D/2;-\tan^2 u\right).
\end{aligned}$$

Note that for real ν both these solutions are real. Introducing a new variable

$$y = \tan u = \frac{x}{\sqrt{1-x^2}},$$

the solutions are wrtten in the form

$$
\begin{aligned}
R_1(x) &= y^l\left(1+y^2\right)^{(D+2\nu)/4} F\left(\frac{l+D/2+\nu+i\alpha\omega}{2}, \frac{l+D/2+\nu-i\alpha\omega}{2}; l+D/2; -y^2\right), \\
R_2(x) &= \frac{\left(1+y^2\right)^{(D+2\nu)/4}}{y^{l+D-2}} \\
&\quad \times F\left(1-\frac{l+D/2-\nu-i\alpha\omega}{2}, 1-\frac{l+D/2-\nu+i\alpha\omega}{2}; 2-l-D/2; -y^2\right).
\end{aligned}
$$

The solution $R_2(x)$ diverges at the origin, $y \to 0$. An equivalent form for the radial functions is obtained by using the linear transformation formula

$$
F(a,b;c;z) = (1-z)^{-a} F(a, c-b; c; \frac{z}{z-1}).
$$

For the function $R_1(u)$ this gives

$$
\begin{aligned}
R_1(x) &= \frac{\tan^l(u)}{\cos^{D/2+\nu}(u)} F\left(\frac{l+D/2+\nu+i\alpha\omega}{2}, \frac{l+D/2+\nu-i\alpha\omega}{2}; l+D/2; \frac{x^2}{x^2-1}\right) \\
&= x^l\left(1-x^2\right)^{i\alpha\omega/2} F\left(\frac{l+D/2+\nu+i\alpha\omega}{2}, \frac{l+D/2-\nu+i\alpha\omega}{2}; l+D/2; x^2\right).
\end{aligned}
$$

Hence, the solutions to the Klein-Gordon equation in static coordinates, regular at the origin, are given by the expression

$$
\varphi_\sigma = C R_1(r/\alpha) Y^{(D-1)}(m_k; \vartheta) e^{-i\omega t}, \tag{11.28}
$$

with C being the normalization constant and the modes are specified by the set $\sigma = (\omega, m_k) = (\omega, l, m_1, \ldots, m_{D-2})$. It is of interest to see the behavior of the solution near the de Sitter horizon, corresponding to $x \to 1$ and, hence, $y \to +\infty$. By using the linear transformation formula for the hypergeometric function [55],

$$
\begin{aligned}
F(a,b;c;-z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{-a} F(a, 1-c+a; 1-b+a; 1/z) \\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} z^{-b} F(b, 1-c+b; 1-a+b; 1/z),
\end{aligned}
$$

we see that for $z \to +\infty$ one has

$$
F(a,b;c;-z) = \Gamma(c)\left[\frac{\Gamma(b-a) z^{-a}}{\Gamma(b)\Gamma(c-a)} + \frac{\Gamma(a-b) z^{-b}}{\Gamma(a)\Gamma(c-b)}\right].
$$

For the function $R_1(x)$ this gives

$$
\begin{aligned}
R_1(x) &\approx 2\Gamma(l+D/2)\,\mathrm{Re}\left[\frac{\Gamma(i\alpha\omega) y^{i\alpha\omega}}{\Gamma\left(\frac{l+D/2+\nu+i\alpha\omega}{2}\right)\Gamma\left(\frac{l+D/2-\nu+i\alpha\omega}{2}\right)}\right] \\
&\approx A_\omega (1-x^2)^{-i\alpha\omega/2} + A_\omega^* (1-x^2)^{i\alpha\omega/2},
\end{aligned} \tag{11.29}
$$

with $y \approx (1-x^2)^{-1/2}$and

$$
A_\omega = \frac{\Gamma(l+D/2)\Gamma(i\alpha\omega)}{\Gamma\left(\frac{l+D/2+\nu+i\alpha\omega}{2}\right)\Gamma\left(\frac{l+D/2-\nu+i\alpha\omega}{2}\right)}.
$$

The constant C in (11.28) is determined from the normalization condition (6.1). In the problem at hand, the condition is reduced to

$$\int d^D x \, \varphi_\sigma \varphi^*_{\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\omega}.$$

The normalization integral is reduced to

$$C^2 \int_0^\alpha dr \frac{r^{D-1} R_{1\omega}(x) R^*_{1\omega'}(x)}{1 - r^2/\alpha^2} \int d\Omega |Y(m_k;\vartheta|^2 = \alpha^D C^2 N(m_k) \int_0^1 dx \frac{x^{D-1} R_{1\omega}(x) R^*_{1\omega'}(x)}{1-x^2}$$
$$= \frac{1}{2}\alpha^D C^2 N(m_k) \int_0^1 du \frac{u^{D/2-1} R_\omega(\sqrt{u}) R^*_{\omega'}(\sqrt{u})}{1-u}.$$

This integral is divergent near the upper limit for $\omega = \omega'$, and the main contribution comes from the region near that limit. So we can use the asymptotic (11.29):

$$\begin{aligned} R_\omega(\sqrt{u}) R^*_{\omega'}(\sqrt{u}) \quad \approx \quad & A_\omega A_{\omega'} (1-u)^{-i\alpha(\omega+\omega')/2} + A_\omega A^*_{\omega'} (1-u)^{-i\alpha(\omega-\omega')/2} \\ & + A^*_\omega A_{\omega'} (1-u)^{i\alpha(\omega-\omega')/2} + A^*_\omega A^*_{\omega'} (1-u)^{i\alpha(\omega+\omega')/2}. \end{aligned}$$

We have the following integrals

$$\int_0^1 du \frac{u^{D/2-1}(1-u)^{-i\alpha(\omega\pm\omega')/2}}{1-u} \approx \int_0^1 d\ln(1-u) e^{-i\alpha(\omega\pm\omega')/2 \ln(1-u)} = \int_0^\infty dz e^{-iz\alpha(\omega\pm\omega')/2}$$
$$= \int_0^\infty dz \cos\left[z\alpha\left(\omega\pm\omega'\right)/2\right] - i\int_0^\infty dz \sin\left[z\alpha\left(\omega\pm\omega'\right)/2\right]$$
$$= \pi\delta\left(\alpha\left(\omega\pm\omega'\right)/2\right) - i\int_{\pi/\alpha(\omega\pm\omega')}^\infty dz \cos\left[z\alpha\left(\omega\pm\omega'\right)/2\right]$$
$$= \pi(1-i)\delta\left(\alpha\left(\omega\pm\omega'\right)/2\right) + \text{finite terms.}$$

$$\int_0^1 du \frac{u^{D/2-1} R_\omega(\sqrt{u}) R^*_{\omega'}(\sqrt{u})}{1-u} \sim A_\omega A_{\omega'} \pi(1-i)\delta\left(\alpha\left(\omega+\omega'\right)/2\right)$$
$$+ A_\omega A^*_{\omega'} \pi(1-i)\delta\left(\alpha\left(\omega-\omega'\right)/2\right) + A^*_\omega A_{\omega'} \pi(1+i)\delta\left(\alpha\left(\omega-\omega'\right)/2\right)$$
$$+ A^*_\omega A^*_{\omega'} \pi(1+i)\delta\left(\alpha\left(\omega+\omega'\right)/2\right) + \text{finite terms}$$
$$= 2|A_\omega|^2 \pi\delta\left(\alpha\left(\omega-\omega'\right)/2\right) = \frac{4\pi}{\alpha}|A_\omega|^2 \delta\left(\omega-\omega'\right).$$

From the normalization condition we have

$$\frac{1}{2}\alpha^D C^2_{\omega l} N(m_k) \int_0^1 du \frac{u^{D/2-1} R_\omega(\sqrt{u}) R^*_{\omega'}(\sqrt{u})}{1-u}$$
$$= \frac{1}{2}\alpha^D C^2_{\omega l} N(m_k) \frac{4}{\alpha} |A_\omega|^2 \pi\delta\left(\omega-\omega'\right) = \frac{1}{2\omega}\delta\left(\omega-\omega'\right).$$

From here it follows that

$$C^2 = \frac{\alpha^{1-D}|A_\omega|^{-2}}{4\pi\omega N(m_k)}.$$

Chapter 12

Quantum fields in anti-de Sitter spacetime

12.1 Introduction

Anti-de Sitter spacetime is one of the simplest and most interesting spacetimes allowed by general relativity. It is the unique maximally symmetric solution of the vacuum Einstein equations with a negative cosmological constant (for geometrical properties of anti-de Sitter space and its uses see, e.g., [56]). Quantum field theory in anti-de Sitter background has long been an active field of research for a variety of reasons. First of all, anti-de Sitter spacetime has a high degree of symmetry and, because of this, numerous physical problems are exactly solvable in this geometry. The maximal symmetry of anti-de Sitter simplifies many aspects of the study of quantum fields and the investigations of the corresponding field-theoretical effects may help to develop the research tools and insights to deal with more complicated geometries. Much of early interest to quantum field theory on anti-de Sitter bulk was motivated by principal questions of the quantization of fields on curved backgrounds. The lack of global hyperbolicity and the presence of both regular and irregular modes give rise to a number of new features which have no analogues in quantum field theory on the Minkowski bulk. Being a constant negative curvature manifold, anti-de Sitter space provides a convenient infrared regulator in interacting quantum field theories [57]. Its natural length scale can be used to regularize infrared divergences without reducing the symmetries. The importance of this theoretical research was increased by the natural appearance of anti-de Sitter spacetime as a ground state in supergravity and Kaluza-Klein theories and also as the near horizon geometry of the extremal black holes and domain walls.

A further increase of interest is related to the crucial role of the anti-de Sitter geometry in two exciting developments of the past decade such as the AdS/CFT correspondence and the braneworld scenario with large extra dimensions. The AdS/CFT correspondence [58] (see also [59]) represents a realization of the holographic principle and relates string theories or supergravity in the anti-de Sitter bulk with a conformal field theory living on its boundary. It has many interesting consequences and provides a powerful tool for the investigation of gauge field theories. Among the recent developments of the AdS/CFT correspondence is the application to strong-coupling problems in condensed matter physics (familiar examples include holographic superconductors, quantum phase transitions and topological insulators). Moreover, the correspondence between the theories on anti-de Sitter and Minkowski bulks may be used to derive new results in mathematical physics, in particular, in the theory of special functions (see, for instance, [60] and references therein). The braneworld scenario (for reviews see [61]) offers a new perspective on the hierarchy problem between the gravitational and electroweak mass scales. The main idea to resolve the large hierarchy is that the small coupling of four-dimensional gravity is generated by the large physical volume of extra dimensions. Braneworlds naturally appear in string/M-theory context and present intriguing possibilities to solve or to address from a different point of view various problems in particle physics and cosmology.

12.2 AdS spacetime: Geometry and coordinate systems

Anti-de Sitter spacetime is a maximally symmetric solution of the Einstein equations with the negative cosmological constant. The corresponding curvature characteristics have the form (11.2). Anti-de Sitter space-time can be visualised geometrically as the hyperboloid

$$(z^0)^2 - (z^1)^2 - \cdots - (z^D)^2 + (z^{D+1})^2 = \alpha^2,$$

embedded in a flat $(D+2)$-dimensional space with the line element

$$ds^2 = (dz^0)^2 - (dz^1)^2 - \cdots - (dz^D)^2 + (dz^{D+1})^2.$$

The latter has two time-like coordinates z^0 and z^{D+1}. The parameter α is related to the cosmological constant by

$$\alpha = \sqrt{-\frac{D(D-1)}{2\Lambda}},$$

and for the scalar curvature one has

$$R = -\frac{D(D+1)}{\alpha^2}.$$

The isometry group of AdS is $SO(D,2)$ which is simply the "Lorentz" group of the $(D+2)$-dimensional embedding space. The conformal group is $SO(D+1,2)$, as for Minkowski space, which is of relevance when considering conformally invariant field equations. AdS has the topology $S^1 \times R^D$ and hence contains closed timelike curves. "Unwrapping" the S^1 one gets the universal covering space (CAdS) with the topology of R^{D+1}. The latter contains no closed timelike curves. First we consider the most frequently used coordinate systems in anti-de Sitter spacetime.

12.2.1 Global coordinates

First we introduce the coordinates t and r in accordance with

$$z^0 = \alpha \cosh r \sin(t/\alpha), \quad z^{D+1} = \alpha \cosh r \cos(t/\alpha). \tag{12.1}$$

The equation for the hyperboloid is written as

$$(z^1)^2 + \cdots + (z^D)^2 = \alpha^2 \sinh^2 r.$$

This relation determines $(D-1)$-dimensional sphere with the radius $\alpha \sinh r$. We introduce angular coordinates $(\theta_1, \theta_2, \ldots, \theta_{D-1})$ in accordance with

$$\begin{aligned} z^1 &= \alpha \sinh r \cos\theta_1, \quad z^2 = \alpha \sinh r \sin\theta_1 \cos\theta_2, \ldots, \\ z^{D-1} &= \alpha \sinh r \sin\theta_1 \cos\theta_2 \cdots \sin\theta_{D-2} \cos\theta_{D-1}, \\ z^D &= \alpha \sinh r \sin\theta_1 \cos\theta_2 \cdots \sin\theta_{D-2} \sin\theta_{D-1}, \end{aligned} \tag{12.2}$$

where $0 \leqslant \theta_i < \pi$ for $i = 1, \ldots, D-2$, and $0 \leqslant \theta_{D-1} < 2\pi$. The coordinates $(t, r, \theta_1, \theta_2, \ldots, \theta_{D-1})$ cover the entire hyperboloid (global coordinates). The corersponding line element has the form

$$ds^2 = \cosh^2 r dt^2 - \alpha^2(dr^2 + \sinh^2 r d\Omega_{D-1}^2). \tag{12.3}$$

The time coordinate t is periodic with the period $2\pi\alpha$ and anti-de Sitter space time has the topology $S^1 \times R^D$, where S^1 corresponds to the time coordinate t. The periodicity of t is not evident from (12.3) and it is more natural to take $-\infty < t < +\infty$. Such a range of coordinates corresponds to an infinite number of turns around the hyperboloid. We can unwrap the circle S^1 and extend it to R^1 instead, without reference to the parametrisation (12.1). In this way, one gets a universal covering space of the anti-de Sitter spacetime with topology R^4 which does not contain any closed timelike curves. Introducing a new radial coordinate χ in accordance with

$$\tan\chi = \sinh r, \quad 0 \leqslant \chi < \pi/2,$$

the line element is rewritten as

$$ds^2 = \frac{1}{\cos^2\chi}\left[dt^2 - \alpha^2(d\chi^2 + \sin^2\chi d\Omega^2_{D-1})\right]. \tag{12.4}$$

This shows that the whole anti-de Sitter spacetime is conformal to the region $0 \leq \chi < \pi/2$ of the Einstein static universe

$$ds^2_{\mathrm{ES}} = dt^2 - \alpha^2(d\chi^2 + \sin^2\chi d\Omega^2_{D-1}). \tag{12.5}$$

with the conformal factor $1/\cos\chi$. For the Einstein static universe one has $0 \leq \chi < \pi$ and the the spatial sections are D-dimensional spheres, S^D. The Einstein static universe is therefore a manifold with the topology $R^1 \times S^D$. The conformal infinity of anti-de Sitter spacetime is located at the boundary $\chi = \pi/2$ corresponding to $r = \infty$.

Another coordinate system is obtained by introducing a new coordinate R in accordance with $R = \alpha\sinh r$ with $0 \leqslant R < \infty$. For the points on the hyperboloid we have the parametrization

$$\begin{aligned}
z^0 &= \sqrt{\alpha^2 + R^2}\sin(t/\alpha), \\
z^1 &= R\cos\theta_1, \quad z^2 = R\sin\theta_1\cos\theta_2, \ldots, \\
z^{D-1} &= R\sin\theta_1\cos\theta_2\cdots\sin\theta_{D-2}\cos\theta_{D-1}, \\
z^D &= R\sin\theta_1\cos\theta_2\cdots\sin\theta_{D-2}\sin\theta_{D-1}, \\
z^{D+1} &= \sqrt{\alpha^2 + R^2}\cos(t/\alpha).
\end{aligned} \tag{12.6}$$

The coordinates $(t, R, \theta_1, \theta_2, \ldots, \theta_{D-1})$ cover entire hyperboloid. The corresponding line element takes the form

$$ds^2 = (1 + R^2/\alpha^2)dt^2 - \frac{dR^2}{1 + R^2/\alpha^2} - R^2 d\Omega^2_{D-1}. \tag{12.7}$$

These coordinates are analog of the static coordinates in dS spacetime. In the ant-de Sitter case there are no event horizons.

12.2.2 Poincaré coordinates

The third coordinate system we want to consider are the Poincaré coordinates $(t, x^1, x^2, \ldots, x^{D-1}, z)$ defined as

$$\begin{aligned}
z^0 &= \frac{\alpha^2 - u}{2z}, \quad z^1 = \frac{\alpha^2 + u}{2z}, \\
z^l &= \alpha\frac{x^{l-1}}{z}, \ l = 2, \ldots, D, \quad z^{D+1} = \alpha\frac{t}{z},
\end{aligned} \tag{12.8}$$

$$\frac{-\alpha^2}{z^2}\left(t^2 - \sum_{l=1}^{D-1}(x^l)^2 - z^2\right) - \frac{\alpha^2}{z^2}\sum_{l=1}^{D-1}(x^l)^2 + \alpha^2\frac{t^2}{z^2} = \alpha^2,$$

with $-\infty < t, x^l < +\infty$ and $u = t^2 - \sum_{l=1}^{D}(x^l)^2$, $x^D = z$. The corresponding line element takes the form

$$ds^2 = \frac{\alpha^2}{z^2}\left(dt^2 - \sum_{l=1}^{D}(dx^l)^2\right). \tag{12.9}$$

For the corresponding geometry $z = 0$ and $z = \infty$ are coordinate singularities. These hypersurfaces are called the anti-de Sitter boundary and anti-de Sitter horizon, respectively. Introducing the coordinate y in accordance with

$$z = \pm\alpha e^{y/\alpha}$$

in the regions $z > 0$ and $z < 0$ for the upper and lower signs, respectively, the line element is rewritten as

$$ds^2 = e^{-2y/\alpha}\left[dt^2 - \sum_{l=1}^{D-1}(dx^l)^2\right] - dy^2. \tag{12.10}$$

The Poincaré coordinates cover a part of the whole anti-de Sitter spacetime.

12.2.3 FRW coordinates

Another class of coordinates, $(t, \rho, \theta_1, \theta_2, \ldots, \theta_{D-1})$, $-\infty < t < +\infty$, $0 \leqslant \rho < +\infty$, is introduced by the relations

$$\begin{aligned}
z^0 &= \alpha\sin(t/\alpha), \quad z^{D+1} = \alpha\cos(t/\alpha)\cosh\rho, \\
z^1 &= \alpha\cos(t/\alpha)\sinh\rho\cos\theta_1, \quad z^2 = \alpha\cos(t/\alpha)\sinh\rho\sin\theta_1\cos\theta_2, \ldots, \\
z^{D-1} &= \alpha\cos(t/\alpha)\sinh\rho\sin\theta_1\cos\theta_2\cdots\sin\theta_{D-2}\cos\theta_{D-1}, \\
z^D &= \alpha\cos(t/\alpha)\sinh\rho\sin\theta_1\cos\theta_2\cdots\sin\theta_{D-2}\sin\theta_{D-1}.
\end{aligned}$$

These coordinates cover only a l part of the complete manifold. The corresponding line element takes the form

$$ds^2 = dt^2 - \alpha^2\cos^2(t/\alpha)\left(d\rho^2 + \sinh^2\rho d\Omega_{D-1}^2\right). \tag{12.11}$$

This line element is a particular case of FRW line element with a negative spatial curvature $k = -1$.

Anti-de Sitter spacetime is an example of a non-globally hyperbolic manifolds. It possesses both closed timelike curves and a timelike boundary at spatial infinity through which data can propagate. The latter property is also possessed by the universal covering space ("CAdS") and is the prime cause of the lack of hyperbolicity. The surface $\chi = \pi/2$ in coordinates (12.4) (spatial infinity) is timelike (i.e., the metric pulled back to the surface has signature $(+, -, \cdots, -)$). As a consequence, the information may be lost to, or gained from, spatial infinity in finite coordinate time. It is this loss and gain of information which has the most disruptive effect on the Cauchy problem.

This problem is similar to that encountered when considering quantization in a box in Minkowski space-time. For the transparent box the information may escape or be thrown in from outside, and the Cauchy data within the box at a given time does not uniquely determine that at other times. One needs to additionally specify boundary data on the surface of the box. When dealing with boxes one usually imposes boundary conditions on the walls, so that information is reflected and not lost. The time evolution of the Cauchy data is then unique. For a transparent box, one way of establishing a well-defined Cauchy problem is simply to accept that the box constitutes an incomplete manifold, and require that Cauchy data be specified on a Cauchy surface of the surrounding spacetime, not just within the box. But unlike the box, anti-de Sitter is complete and there is no such surrounding space-time.

The absence of a global Cauchy surface in anti-de Sitter can be seen from the equation $dt = \alpha d\chi$ for radial null geodesics. One sees that information propagates from $\chi = 0$ to spatial infinity

$\chi = \pi/2$ in finite time $t = \alpha\pi/2$. This can be viewed as information crossing the equator of S^D in the conformal extension to Einstein static universe, which itself is globally hyperbolic because S^D is compact. The Cauchy problem in anti-de Sitter becomes well-defined if suitable boundary conditions are imposed on the equator.

12.3 Scalar field mode functions in anti-de Sitter spacetime

In order to find the mode functions foe a scalar field we should solve the Klein-Gordon equation on anti-de Sitter bulk. Of course, the modes depend on the coordinate system. We start with global coordinates (12.4). Introducing the dimensionless time coordinate $\tau = t/\alpha$, the line element is written as

$$ds^2 = \frac{\alpha^2}{\cos^2\chi}\left(d\tau^2 - d\chi^2 - \sin^2\chi d\Omega_{D-1}^2\right).$$

We want to solve the Klein-Gordon equation

$$\left[\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu) + m^2 + \xi R\right]\varphi = 0,$$

on this bulk. The solution is decomposed as

$$\varphi = P(\chi)Y^{(D-1)}(m_k;\vartheta)e^{-i\omega\tau},$$

with the spherical function $Y^{(D-1)}(m_k;\vartheta)$. By taking into account the equation for the latter, we obtain the following equation for the function $P(\chi)$:

$$\left(\partial_\chi^2 + \frac{D-1}{\cos^2\chi\tan\chi}\partial_\chi - \frac{l(l+D-2)}{\sin^2\chi} - \frac{\alpha^2\left(m^2+\xi R\right)}{\cos^2\chi} + \omega^2\right)P(\chi) = 0.$$

Introducing a new independent variable $y = \sin^2\chi$, this equation is rewritten as

$$\left[4y(1-y)\partial_y^2 + 4\left(D/2-y\right)\partial_y - \frac{l(l+D-2)}{y} - \frac{\alpha^2\left(m^2+\xi R\right)}{1-y} + \omega^2\right]P = 0. \tag{12.12}$$

In the limit $y \to 0$ this equation is reduced to

$$\left[4y\partial_y^2 + 2D\partial_y - \frac{l(l+D-2)}{y}\right]P = 0,$$

with the solutions

$$P \sim y^{l/2} \text{ and } P \sim y^{1-(l+D)/2}.$$

From the regularity condition at $y = 0$ ($\chi = 0$) it follows that we have to take the solution with $P \sim y^{l/2}$ for $y \to 0$. In the limit $y \to 1$ one has

$$\left[4u\partial_u^2 - 4\left(D/2-1\right)\partial_u - \frac{\alpha^2\left(m^2+\xi R\right)^2}{u}\right]P = 0,\ u = 1-y,$$

with the solutions

$$P \sim (1-y)^{b_\pm/2},\ b_\pm = D/2 \pm \nu.$$

On the base of these asymptotics we present the solution of the equation (12.12) in the form

$$P(y) = y^{l/2}(1-y)^{b_\pm/2}R(y).$$

For the function $R(y)$ we obtain the following equation

$$y(1-y)R''(y) + [l + D/2 - (l+1+b_\pm)\,y]\,R'(y) + \frac{1}{4}\left[\omega^2 - (l+b_\pm)^2\right]R(y) = 0. \tag{12.13}$$

This coincides with the hypergeometric equation with the parameters

$$a = (c \pm \nu - \omega)/2,\ b = (c \pm \nu + \omega)/2,\ c = l + D/2. \tag{12.14}$$

As linearly independent solutions for $R(y)$ we can take the hypergeometric functions

$$F(a,b;c;y) \text{ and } y^{1-c}F(a-c+1,b-c+1;2-c;y).$$

In the limit $y \to 0$ the second function behaves as $y^{1-l-D/2}$ and the corresponding solution for $P(y)$ behaves like $y^{1-l/2-D/2}$. The regularity condition at $y=0$ excludes this solution. Hence, for the regular solution of (12.13) one has $R(y) = \text{const}\cdot F(a,b;c;y)$ and

$$P(y) = \text{const}\cdot y^{l/2}(1-y)^{b_\pm/2}F(a,b;c;y)). \tag{12.15}$$

Now let us consider the normalization condition. The normalization integral has the form

$$\begin{aligned}(\phi_1,\phi_2) &= 2\omega\int_\Sigma d^Dx\,\sqrt{|g|}g^{00}\phi_1\phi_2^* = 2\alpha^{D-1}\omega N(m_k)\int_0^{\pi/2}d\chi\,\tan^{D-1}\chi P_1P_2^* \\ &= \alpha^{D-1}\omega N(m_k)\int_0^1 dy\,\frac{y^{D/2-1}}{(1-y)^{D/2}}P_1P_2^*. \end{aligned}\tag{12.16}$$

In order to see the convergence properties of the integrand at the upper limit of the integral we need the asymptotic of the hypergeometric function in (12.15) in the limit $y \to 1$. For the upper sign in (12.14), to the leading order, one has

$$F(a,b;c;y) \approx (1-y)^{-\nu}\frac{\Gamma(c)\Gamma(\nu)}{\Gamma(a)\Gamma(b)},$$

and for the lower sign

$$F(a,b;c;y) \approx \frac{\Gamma(c)\Gamma(\nu)}{\Gamma(c-a)\Gamma(c-b)}.$$

For both cases the integrand in (12.16) near the upper limit behaves as $(1-y)^{-\nu}$ and the integral is divergent for $\nu \geqslant 1$. Hence, for both the signs the normalization integral is divergent on the AdS boundary. In order to escape the divergence we have to cut the hypergeometric series taking $a = -n$ with $n = 0, 1, \ldots$. By taking into account (12.14), this condition leads to the quantization of the eigenvalues for the energy:

$$\omega = \omega_n = l + D/2 + 2n \pm \nu.$$

The corresponding function $P(y)$ takes the form

$$P(y) = \text{const}\cdot y^{l/2}(1-y)^b F(-n, l+2b+n; l+D/2; y).$$

This solution can also be written in terms of the Jacobi's polynomials

$$P(y) = \text{const}\cdot\frac{n!}{(l+D/2)_n}y^{l/2}(1-y)^{(D/2\pm\nu)/2}P_n^{(l+D/2-1,\pm\nu)}(1-2y),$$

or, by taking into account that $y = \sin^2\chi$,

$$P(\chi) = C\sin^l(\chi)\cos^{D/2\pm\nu}(\chi)P_n^{(l+D/2-1,\pm\nu)}(\cos(2\chi)).$$

With this function, the normalization condition takes the form

$$(\phi,\phi) = \frac{2|C|^2}{2^{l+D/2-1\pm\nu}}\alpha^{D-1}\omega N(m_k)\int_{-1}^{1} dx\,(1-x)^{l+D/2-1}(1+x)^{\pm\nu}\left[P_n^{(l+D/2-1,\pm\nu)}(x)\right]^2 = 1.$$

The integral in this formula is evaluated by using the result from [71] and for the normalization coefficent we find

$$|C|^2 = \frac{\Gamma(l+D/2+n\pm\nu)}{4\alpha^{D-1}\Gamma(n+1\pm\nu)}\frac{n!}{N(m_k)\Gamma(l+D/2+n)}.$$

Finally the mode functions are given by the expression

$$\varphi_\sigma = C\sin^l(\chi)\cos^{D/2\pm\nu}(\chi)P_n^{(l+D/2-1,\pm\nu)}(\cos(2\chi))Y^{(D-1)}(m_k;\vartheta)e^{-i\omega_n\tau}. \quad (12.17)$$

For the set of quantum numbers specifying the solutions one has $\sigma = (n, m_k)$. As is seen, we have two sets of modes with the upper and lower signs. These set correspond to two different quantization schemes in AdS spacetime described by global coordinates

Chapter 13

Two-point functions in maximally symmetric spaces

13.1 Maximally symmetric bitensors

We have considered mode functions of a scalar field in maximally symmetric spaces. The corresponding two-point function may be derived by direct summation of the related mode sums. Alternatively, for maximally symmetric vacuum states the two-point functions can be derived solving the corresponding differential equation. This elegant method has been used in [62] for scalar and vector fields. In the first part of this chapter, following [62], the scalar two-point functions are obtained for de Sitter and anti-de Sitter spacetime by using the symmetry arguments and solving the equation obeyed by these functions. In the second part, by using the two-point function, the renormalized vacuum expectation values of the field squared and of the energy-momentum tensor are evaluated on the base of the general renormalization procedure discussed before.

Consider a $(D+1)$-dimensional maximally symmetric spaces. As it has been mentioned in section 4.4, these spaces have the maximal number of independent Killing vectors, equal $(D+1)(D+2)/2$. First we will discuss some relations for bitensors in these spaces. Let us consider the shortest geodesic from x to x'. We will denote by $n^i(x,x')$ and $n^{i'}(x,x')$ the unit tangents to the geodesic at x and x'. The parallel propagator along the geodesic will be denoted by $g^i{}_{k'}(x,x') = g^i{}_{k'}$. If the geodesic is parametrized by λ, $x^\sigma = x^\sigma(\lambda)$, then the distance $\mu(x,x')$ along the shortest geodesic is given by the expression (11.21). In the pseudo-Riemannian case not all pairs of points can be connected by a geodesic. However, these geometric objects have unique analytic extensions to such pairs. These type of tensors are called bitensors with unprimed and primed indices living in the tangent spaces at x and x'. These indices are raised with g^{ik} and $g^{i'k'}$, respectively. From the definition of the parallel propagator one has $g_i{}^{l'}(x,x')g_{l'k}(x',x) = g_{ik}(x)$ and similarly for other combinations. We also introduce the vectors

$$n_i(x,x') = \nabla_i\mu(x,x'),\ n_{i'}(x,x') = \nabla_{i'}\mu(x,x'). \tag{13.1}$$

They have unit length, $n^i n_i = -1$, and point away from each other $g_i{}^{k'}n_{k'} = -n_i$.

Any maximally symmetric bitensor can be expressed as a sum of products of g_{ik}, $g_{i'k'}$, μ, n_i, $n_{i'}$, and $g_{ik'}$. The corresponding coefficients are functions of $\mu(x,x')$ alone. The covariant derivatives of maximally symmetric bitensors define maximally symmetric bitensors. Consequently, we can write

$$\nabla_i n_k = A(\mu)g_{ik} + B(\mu)n_i n_k.$$

Let us determine the coefficients in these formulas. From $n^k n_k = -1$ it follows that $n^k\nabla_i n_k = 0$ and, hence, $B = A$. This gives

$$\nabla_i n_k = A(\mu)\left(g_{ik} + n_i n_k\right).$$

Contracting this relation yields

$$\nabla^i n_i = DA(\mu). \tag{13.2}$$

From (13.1) we get

$$A(\mu) = \frac{1}{D}\nabla^i\nabla_i\mu(x,x').$$

The biscalar $\nabla^i\nabla_i\mu(x,x')$ is again maximally symmetric, hence it must depend only on μ. Let us consider three special cases, R^{D+1} a, S^{D+1}, and H^{D+1}, which are maximally symmetric spaces of constant (zero, positive and negative) scalar curvature.

For R^{D+1} in spherical coordinates we have $\mu = r$ and $\nabla^i\nabla_i = -r^{-D}\partial_r\left(r^D\partial_r\right)$. Hence, $\nabla^i\nabla_i\mu(x,x') = -D/\mu$ and $A(\mu) = -1/\mu$. For S^{D+1} with radius α, in coordinates centered about x' the line element is $ds^2 = -\alpha^2(d\theta^2 + \sin^2\theta d\Omega_D^2)$. The geodesic distance is $\mu = r_0\theta$ and the Laplacian on a function of μ is

$$\nabla^i\nabla_i = -\alpha^{-2}(\sin\theta)^{-D}\partial_\theta\left(\sin^D\theta\partial_\theta\right).$$

This gives $\nabla^i\nabla_i\mu = -(D/\alpha)\cot(\mu/\alpha)$. Hence, for the sphere one has $A(\mu) = -\cot(\mu/\alpha)/\alpha$. The corresponding result for H^{D+1} is obtained by letting $\alpha = i|\alpha|$ and $A(\mu) = -\coth(\mu/|\alpha|)/|\alpha|$.

13.2 Scalar two-point function

Let us consider the scalar two-point function

$$G(x,x') = \langle\psi|\phi(x)\phi(x')|\psi\rangle,$$

where $|\psi\rangle$ is a maximally symmetric state. The function $G(x,x')$ depends only upon the geodesic distance $\mu(x,x')$. The two-point function obeys the equation

$$\left(\nabla^i\nabla_i + m^2 + \xi R\right)G(x,x') = 0,$$

for $x \neq x'$. Denoting the derivative with respect to μ by the prime, $\partial_\mu G(x,x') = G'(\mu)$, and by taking into account (13.2) we see that

$$\begin{aligned}\nabla^i\nabla_i G(x,x') &= \nabla^i\left[G'(\mu)n_i\right] = G''(\mu)n^i n_i + G'(\mu)\nabla^i n_i \\ &= -G''(\mu) + DA(\mu)G'(\mu).\end{aligned}$$

The equation for the two-point function takes the form

$$G''(\mu) - DA(\mu)G'(\mu) - \left(m^2 + \xi R\right)G(\mu) = 0.$$

Introducing a new variable

$$y = \cos^2(\mu/2\alpha), \tag{13.3}$$

the equation becomes

$$y(1-y)\partial_y^2 G(\mu) + (D+1)(1/2-y)\partial_y G(\mu) - \alpha^2\left(m^2 + \xi R\right)G(\mu) = 0. \tag{13.4}$$

This coincides with the hypergeometric equation

$$y(1-y)\partial_y^2 G(\mu) + (c-(a+b+1)y)\partial_y G(\mu) - abG(\mu) = 0.$$

with

$$a = \frac{D}{2} + \nu, \quad b = \frac{D}{2} - \nu, \quad c = \frac{D+1}{2},$$

and

$$\nu = \sqrt{D^2/4 - \alpha^2\left(m^2 + \xi R\right)}.$$

In the Riemanian case $\mu^2 > 0$ and one has $0 \leqslant y < 1$. In the Lorentzian case, we have in addition the timelike intervals with $\mu^2 < 0$, corresponding to $1 < y < \infty$. Thus in the Riemannian case $y \in [0, 1]$, and in the Lorentzian case $y \in [0, \infty)$.

Note that the parameters obey the relation $a+b+1 = c$. From here it follows that the equation (13.4) is invariant under the replacement $y \to 1 - y$. Consequently, two independent solutions are the hypergeometric functions $F(a, b; c; y)$ and $F(a, b; c; 1-y)$. These solutions are singular at $y = 1$ and $y = 0$, respectively. They have different singular points and, hence, are lineraly independent. From (13.3) it follows that $y = 1$ corresponds to $\mu(x, x') = 0$. For Riemanian space this woud imply $x = x'$. In Lorentzian spaces this corresponds to null-related points. For the antipodal points x and $\bar{x}$ one has $\mu(x, \bar{x}) = \pi\alpha$ and one gets $y = 0$. This correspond to the second possible singularity of the two-point function.

The choice of a particular solution for the equation (13.4) depends on the maximally symmetric state $|\psi\rangle$ and on the two-point function $G(x, x')$. The following points are essential in the choice of the solution:

1. Short distance behavior as $\mu \to 0$,
2. Long distance behavior as $\mu \to \infty$,
3. Location of singular points,
4. Location of branch cuts.

We consider appropriate solutions for de Sitter and anti-de Sitter spaces.

13.3 Scalar two-point function in dS spacetime

For de Sitter spacetime the parameter α^2 is positive. In this geometry there exists a one complex parameter family of de Sitter-invariant vacuum states [63]. Each one determines a particular solution $G(y)$. The Euclidean or Bunch-Davies vacuum [50, 64] is a special member of this family. It is the only one whose two-point function $G(x, x')$ (a) has only one singular point, at $\mu(x, x') = 0$, and (b) in the limit $\mu \to 0$ has the same strength singularity as in flat space. For the Bunch-Davies vacuum state the two-point function regular at $\mu(x, \bar{x}) = \pi\alpha$. One may obtain the two-point functions for any other de Sitter invariant vacuum from from that for Bunch-Davies vacuum.

From the condition (a) it follows that for the Bunch-Davies vacuum one has

$$G(\mu) = \text{const}\, F(a, b; c; y).$$

The constant is determined from the condition (b). By taking into account that in the limit $\mu \to 0$ for the two-point function in flat space one has

$$G(\mu) \sim \frac{\Gamma((D+1)/2)\mu^{1-D}}{2(D-1)\pi^{(D+1)/2}},$$

and the relation

$$F(a, b; c; y) \sim \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)(1-y)^{a+b-c}},\ y \to 1,$$

with $1 - z \approx (\mu/2\alpha)^2$, for the two-point function we get the final expression

$$G(x, x') = \frac{\Gamma(D/2+\nu)\Gamma(D/2-\nu)}{(4\pi)^{(D+1)/2}\Gamma((D+1)/2)\alpha^{D-1}} F\left(\frac{D}{2}+\nu, \frac{D}{2}-\nu; \frac{D+1}{2}; y\right). \tag{13.5}$$

This formula gives the two-point function for spacelike intervals $0 \leqslant y < 1$. For timelike intervals $y > 1$. The formula (13.5) coincides with (11.23) previously obtained by using the direct summation over a complete set of modes in planar coordinates. Generically the hypergeometric function $F(a, b; c; y)$ has a branch cut along the real axis in that region. The Feynman function is the limiting value $G(y + i0)$ approaching the branch cut from above. The symmetric function is the average value across the cut, $G(y + i0) + G(y - i0)$. The commutator function is given by $\epsilon(x, x')\Delta G(y)$, where $\epsilon(x, x') = (+1, -1, 0)$ if x and x' are (future, past, spacelike) separated, and $\Delta G(y) = G(y + i0) - G(y - i0)$. For some values of mass and spacetime dimension the branch cut is absent. However, there is a pole at $y = 1$, and the same $i0$ limiting prescription holds.

13.4 Scalar two-point function in anti-de Sitter spacetime

Anti-de Sitter spacetime is not globally hyperbolic and the Cauchy problem is not well posed. Boundary conditions, controlling the flow of information through a timelike surface at spatial infinity, are therefore required to define a quantum field theory [65]. The possible states and and the corresponding two-point functions are determined by the boundary conditions. We will select a vacuum by requiring that the two-point function (a) falls off as fast as possible at spatial infinity $\mu^2 \to \infty$, and (b) in the limit $\mu \to 0$ has the same strength singularity as in flat space. These requirements correspond to the "reflecting" Dirichlet boundary conditions of [65]. For higher spin fields, the condition (a) seems to be required in order that the state be stable against small fluctuations (for other reasons see [66]). In the scalar case, however, condition (a) is not the only possibility.

In AdS case one has $y = \cosh^2(\mu/2|\alpha|)$. Timelike intervals correspond to $0 < y < 1$ on the real axis y, and spacelike intervals correspond to $y > 1$. Spatial infinity $\mu \to \infty$ corresponds to $z \to \infty$. As linearly independent solutions of the equation (13.4) it is convenient to take the functions

$$y^{-a}F(a, a - c + 1; a - b + 1; 1/y) \text{ and } y^{-b}F(b, b - c + 1; b - a + 1; 1/y).$$

In the limit $y \to \infty$ they decay as y^{-a} and y^{-b} respectively. By taking into account that in the problem at hand $0 < b < a$, from the condition (a) one gets

$$G(y) = \text{const} \cdot y^{-a}F(a, a - c + 1; a - b + 1; 1/y).$$

The constant is determined from the condition (b) and for the two-point function we get the expression

$$G(x, x') = \frac{\Gamma(D/2 + \nu)\Gamma(\nu + 1/2)}{(4\pi)^{(D+1)/2}\Gamma(2\nu)|\alpha|^{D-1}y^{D/2+\nu}} F(D/2 + \nu, \nu + 1/2; 2\nu + 1; 1/y). \tag{13.6}$$

The standard branch of the hypergeometric function is cut along the real y axis from 1 to ∞. The function y^{-a} defined as $e^{-a \ln y}$ where $\ln y$ is cut along the negative real axis. Hence $G(y)$ is cut for $y \leqslant 1$. In particular, it is cut along the timelike region $0 \leqslant y < 1$. As before, the Feynman function is obtained as the limiting value $G(y + i0)$ above this cut. The symmetric function is the average value $G(y+i0)+G(y-i0)$ across the cut. The commutator function is $\epsilon(x, x')[G(y+i0)-G(y-i0)]$. The two-point function (13.6) has also been obtained by summation over the regular modes [67]. Two-point functions and the vacuum expectation values of the energy-momentum tensor have been discussed in [65, 68, 69, 70].

13.5 Renormalized energy-momentum tensor in de Sitter spacetime

As an application of the two-point function obtained above and the general renormalization procedure discussed before, let us evaluate the renormalizaed vacuum expectation value of the energy-momentum tensor in de Sitter spacetime. We assume that the field is prepared in the Bunch-Davies vacuum state. From the maximally symmetry of the background geometry and of the vacuum state it follows that

$$\langle T_\mu^\nu \rangle = \text{const} \cdot \delta_\mu^\nu,$$

and, hence, it is sufficient to evaluate the trace of the energy-momentum tensor. By taking into account that the field operator obeys the equation (5.2), from (5.5) for the trace one has the relation

$$T_\mu^\mu = D\,(\xi - \xi_c)\,\nabla_\rho \nabla^\rho \phi^2 + m^2\phi^2. \tag{13.7}$$

As a consequence, the vacuum expectation value of the trace can be written in terms of the Hadamard function as

$$\langle T_\mu^\mu \rangle = \frac{D}{2}\,(\xi - \xi_c)\,\nabla_\rho \nabla^\rho G^{(1)}(x,x) + \frac{m^2}{2} G^{(1)}(x,x).$$

In the case under consideration $G^{(1)}(x,x)$ does not depend upon the point x and this relation is simplified to

$$\langle T_\mu^\mu \rangle = \frac{m^2}{2} G^{(1)}(x,x). \tag{13.8}$$

For spacelike intervals $0 \leqslant y < 1$, the expression for the Hadamard function is directly obtained from (13.5):

$$G^{(1)}(x,x') = \frac{2\Gamma(D/2+\nu)\Gamma(D/2-\nu)}{(4\pi)^{(D+1)/2}\Gamma((D+1)/2)\alpha^{D-1}} F\left(\frac{D}{2}+\nu, \frac{D}{2}-\nu; \frac{D+1}{2}; y\right), \tag{13.9}$$

where ν is given by the expression (11.14). Of course, $G^{(1)}(x,x)$ is divergent. We can use the dimensional regularization. By taking into account that for $c-a-b>0$ one has

$$F(a,b;c;y) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-y) + \mathcal{O}((1-y)^{c-a-b}),$$

we see that for $D < 1$

$$G^{(1)}(x,x) = \frac{2\Gamma(D/2+\nu)\Gamma(D/2-\nu)\Gamma((1-D)/2)}{(4\pi)^{(D+1)/2}\alpha^{D-1}\Gamma(1/2-\nu)\Gamma(1/2+\nu)}. \tag{13.10}$$

The expression in the right-hand side has a simple pole at $D = 3$. For removing the divergences we should subtract the corresponding De Witt-Schwinger expansion of the Hadamard function, $G_{DS}^{(1)}(x,x')$, truncated at the adiabatic order 4 and expand the remaining expression near $D = 3$. The expression in the right-hand side (13.8) does not contain derivatives of the two-point function and it is sufficient to truncate the expansion at order 4. The truncated expansion will be denoted by ${}^{(4)}G_{DS}^{(1)}(x,x)$. This subtraction is equivalent to the renormalization of the effective Lagrangian discussed in section 8.1.

By taking into account (7.12) and (7.13), after the integration one gets

$$\begin{aligned} {}^{(4)}G_{DS}^{(1)}(x,x) &= \frac{2m^{D-3}}{(4\pi)^{(D+1)/2}} \left[m^2 a_0(x)\Gamma((1-D)/2)\right. \\ &\quad \left. + a_1(x)\Gamma((3-D)/2) + m^{-2}a_2(x)\Gamma((5-D)/2)\right]. \end{aligned} \tag{13.11}$$

The coefficient of the term $a_2(x)$ is finite at $D = 3$ and we need to know the corresponding expression for $D = 3$:

$$a_2(x) = \frac{2(1-6\xi)^2 - 1/15}{\alpha^4}.$$

For the remaining coefficients we need the expressions for general D:

$$a_0(x) = 1, \quad a_1(x) = \left(\frac{1}{6} - \xi\right)\frac{D(D+1)}{\alpha^2}.$$

Subtracting for (13.10) the truncated expansion (13.11) and expanding the result near $D = 3$, we get

$$\begin{aligned} G^{(1)}(x,x) - {}^{(4)}G^{(1)}_{DS}(x,x) &= \frac{1}{8\pi^2\alpha^2}\left\{m^2\alpha^2 - 2/3 - (\alpha/m)^2 a_2 + \left(m^2\alpha^2 + 12\xi - 2\right)\right. \\ &\quad \left.\times\left[\psi(3/2+\nu) + \psi(3/2-\nu) - 2\ln(m\alpha) - 1\right]\right\} + \mathcal{O}(D-3). \end{aligned}$$

Finally, by using the relation (13.8), for the vacuum expectation value of the energy-momentum tensor we find [52, 53, 50]

$$\begin{aligned} \langle T_l^k\rangle_{\rm ren} &= \frac{\delta_l^k}{32\pi^2\alpha^4}\left\{m^2\alpha^2\left(m^2\alpha^2/2 + 6\xi - 1\right)\left[\psi\left(3/2+\nu\right) + \psi\left(3/2-\nu\right) - \ln\left(m^2\alpha^2\right)\right]\right. \\ &\quad \left. - \left(6\xi - 1\right)^2 + 1/30 + (2/3 - 6\xi)m^2\alpha^2\right\}, \qquad (13.12) \end{aligned}$$

where $\nu = \left(9/4 - 12\xi - m^2\alpha^2\right)^{1/2}$. For a conformally coupled massless field we return to the expression (8.23).

By using the asymptotic expansion of the function $\psi(x)$ for large values of the argument it can be seen that for large values of the parameter $m\alpha$ from (13.12) one has

$$\langle T_l^k\rangle_{\rm ren} \approx \frac{\delta_l^k}{32\pi^2 m^2\alpha^6}\left(\frac{7}{12} - \frac{58\xi}{5} + 72\xi^2 - 144\xi^3\right), \ m\alpha \gg 1. \qquad (13.13)$$

For a conformally coupled scalar field the coefficient in braces is equal $-1/60$. The energy-momentum tensor (13.12) is a gravitational source of the cosmological constant type. Due to the problem symmetry this will be the case for general values D. As a result, in combination with the initial cosmological constant Λ, the one-loop effects lead to the effective cosmological constant

$$\Lambda_{\rm eff} = D(D-1)/2\alpha^2 + 8\pi G\langle T_0^0\rangle_{\rm ren}, \qquad (13.14)$$

where G is the Newton gravitational constant. Another important local characteristic of the vacuum state is the vacuum expectation value of the field squared. In de Sitter spacetime, for the Bunch-Davies vacuum state it is given by the expression

$$\begin{aligned} \langle\varphi^2\rangle_{\rm ren} &= \frac{1}{8\pi^2\alpha^2}\left\{\left(m^2\alpha^2/2 + 6\xi - 1\right)\left[\psi\left(\frac{3}{2}+\nu\right) + \psi\left(\frac{3}{2}-\nu\right) - \ln\left(m^2\alpha^2\right)\right]\right. \\ &\quad \left. + \frac{1/30 - (6\xi-1)^2}{m^2\alpha^2} - 6\xi + \frac{2}{3}\right\}. \qquad (13.15) \end{aligned}$$

This expression is directly obtained from the relation between the expectation values of the energy-momentum tensor and the field squared, by taking into account that $\langle\varphi^2\rangle = G^{(1)}(x,x)/2$.

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