Quantum field theory in AdS Space

(Higher Spins)

Lecture Course

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March 2017

Contents

Introduction

The consideration and construction of *higher spin gauge field* theories has always been considered an important task during the last forty years (See [1]-[8] and ref. there). The complications and difficulties which accompany any serious attempt to solve the essential problems in this area always attracted interest but activity intensified after discovering the important role Higher Spin Fields plays in AdS/CFT correspondence. Particular attention caused the holographic duality between the $O(N)$ sigma model in three dimensional space and HSF gauge theory living in the four dimensional space with negative constant curvature [9]. This case of holography is singled out by the existence of two conformal points of the boundary theory and the possibility to describe them by the same higher spin gauge theory with the help of spontaneously breaking of higher spin gauge symmetry and mass generation by a corresponding Higgs mechanism. All these complicated physical tasks necessitate *quantum loop* calculations for higher spin field theory [10]-[15] in Anti de Sitter space and therefore information about manifest, off-shell and Lagrangian formulation of possible interactions for higher spin field in AdS. Then the successful results on the quantum level can be controlled by comparison with the boundary $O(N)$ model results checking the AdS/CFT correspondence conjecture on the loop level [10], [11], [13]. This theory is interesting also becouse here we do nt need supersymmetry to establish and check AdS/CFT correspondence which means that in this case complicated tasks lead to development of *quantum field theory in AdS space*. In this manuscript we try to formulate 16 starting lectures about geometry of AdS space, construction and investigation of the balk to balk and bulk to boundary propagators for higher spin gauge theories and construction of the some trilinear interaction vertices for higher spin gauge fields and scalar fields in AdS space. The main goal of this lecture course to focus PhD (and graduate) students on the problem of real *one loop* calculations in area of higher spin and scalar fields in the space with constant negative curvature.

Acknowledgment

This Lecture Course is result of many years collaboration with Professor Werner Rühl. I would like to express him my gratitude and note his continuous support during long years. His important suggestions and various valuable discussions always gave me route to new ideas and hopefully improved my knowledge in quantum field theory.

1 Geometry of AdS_D

In real Minkowski space $\mathbf{R}_{D-1,2}$ with a metric

$$(x,y) = x^0y^0 + x^Dy^D - \sum_{k=1}^{D-1} x^ky^k \tag{1.1}$$

in Cartesian coordinates, we can specify a manifold as an AdS_D space by requiring

$$AdS_D = \{x \in \mathbf{R}_{D-1,2} : (x,x) = x^2 = L^2\} \tag{1.2}$$

If not specified otherwise, we set $L = 1$. The coordinates x used in (2.1),(2.2) will be denoted "embedding space Cartesian coordinates" (e.s.C.c.).

The orthogonal group $SO(D-1,2)$ of $\mathbf{R}_{D-1,2}$ and its identity component $SO_0(D-1,2)$ act transitively on AdS_D which is connected. A global coordinate system can be defined by the diffeomorphism

$$\mathbf{S}_1 \times \mathbf{R}_{D-1} \ni (t,\vec{x}) \quad \longrightarrow \quad (\sqrt{1+\vec{x}^2})\sin t, \vec{x}, \sqrt{1+\vec{x}^2}\cos t) \tag{1.3}$$

$$\vec{x} \quad = \quad \{x^1, x^2, ...x^{D-1}\} \in \mathbf{R}_{D-1} \tag{1.4}$$

The universal covering space $AdS_D^{(c)}$ is obtained by extending $\mathbf{S}_1$ to $\mathbf{R}_1$

$$\mathbf{S}_1 = \mathbf{R}_1/2\pi\mathbf{Z} \tag{1.5}$$

An often used coordinate system are the Poincare coordinates u and y in the domain $x^{D-1} + x^D > 0$

$$x^\mu = \frac{y^\mu}{u}, \quad \mu \in \{0,1,2,...D-2\} \tag{1.6}$$

$$x^{D-1} = \frac{1-u^2+y^2}{2u}, \quad y^2 = (y^0)^2 - \sum_{k=1}^{D-2}(y^k)^2 \tag{1.7}$$

$$x^D = \frac{1+u^2-y^2}{2u} \tag{1.8}$$

In these coordinates the metric is

$$ds^2_{AdS} = u^{-2}\{(dy^0)^2 - du^2 - \sum_{k=1}^{D-2}(dy^k)^2\} \tag{1.9}$$

The "chordal distance" squared measured in the embedding space is

$$d(x,x')^2 = \frac{1}{2}(x-x')^2 = 1-(x,x') = 1-\zeta \tag{1.10}$$

In later applications to field theory we shall use "Euclidean AdS spaces". These are obtained by complexification of AdS_D to AdS_D^{compl}

$$AdS_D^{compl} = \{w = x_{re} + ix_{im} \in \mathbf{C}^{D+1} : Re(w^2) = x_{re}^2 - x_{im}^2 = 1\} \tag{1.11}$$

In this complex space we restrict x_{re} and x_{im} to

$$AdS_D^{(E)} = \{w = x_{re} + ix_{im} : x_{re}^0 = 0, x_{im}^k = 0 \quad \text{for} \quad \text{all } k \text{ except} \quad k = 0\} \tag{1.12}$$

and call this space the Euclidean space AdS_D^E. This manifold is a two-sheeted hyperboloid and we consider only the upper half $x^D \geq 1$. The Poincare coordinates for this space are

$$\begin{aligned} z = (z_0, \vec{z}) &= (u, y^1, y^2, ...y^{D-2}, x_{im}^0) && (1.13) \\ ds^2 &= \frac{dz_0^2 + d\vec{z}^2}{z_0^2} && (1.14) \end{aligned}$$

This measure is wrapped Euclidean.

Because the metric of AdS^E has a particularly simple form in Poincare coordinates, these are well suited for many calculations. From

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu \tag{1.15}$$

we obtain

$$\sqrt{g} = \frac{L^D}{(z_0)^D} \tag{1.16}$$

$$[\nabla_\mu, \nabla_\nu]V_\lambda^\rho = R_{\mu\nu\lambda}{}^\sigma V_\sigma^\rho - R_{\mu\nu\sigma}{}^\rho V_\lambda^\sigma \tag{1.17}$$

$$R_{\mu\nu\lambda}{}^\rho = -L^{-2}[g_{\mu\lambda}(z)\delta_\nu^\rho - g_{\nu\lambda}(z)\delta_\mu^\rho] \tag{1.18}$$

$$R_{\mu\lambda}(z) = -\frac{D-1}{L^2}g_{\mu\lambda}(z) \tag{1.19}$$

The last property characterizes AdS as an Einsteinian space. Finally its constant curvature is expressed by

$$R(z) = -\frac{D(D-1)}{L^2} \tag{1.20}$$

From the last equation follows also that the AdS spaces are maximally symmetric. This has important consequences lateron.

The causal structure of AdS_D is crucial for the application of physical concepts, such as the motion of classical point particles ("observers") or classical and quantum fields. Due to the transitive action of the isometry group $SO_0(D-1,2)$ we can select the point

$$e_D = \{0, 0, ...0, 1\} \tag{1.21}$$

given in e.s.C.c. Two points $z_1(x_1), z_2(x_2)$ have the chordal distance squared (2.10)

$$d(z_1, z_2) = \frac{1}{2}(x_1 - x_2)^2 \tag{1.22}$$

This distance squared permits us to study the finite causal properties.

The tangent hyperplane at an arbitrary reference point $x \in AdS$ is defined by

$$\{x + y : (x, y) = 0\} \tag{1.23}$$

(this means that all vectors y of the tangential hyperplane are orthogonal to the "radius" x). In this hyperplane we distinguish the light cone vectors satsfying

$$(y, y) = 0 \tag{1.24}$$

This set of vectors is identical with the intersection of the hyperplane with the manifold of AdS_D. The timelike vectors are defined by

$$(y, y) > 0 \tag{1.25}$$

and the spacelike vectors by

$$(y, y) < 0 \tag{1.26}$$

At the reference point e_D (1.21) the tangential hyperplane is a $D-1$ dimensional Minkowski space with vectors

$$\{y^0, \vec{y}, 0\}, y^D = 0 \tag{1.27}$$

The double cone of timelike vectors can be decomposed in a future light cone $y^0 > 0$ (positive timelike) and a past lightcone $y^0 < 0$ (negative timelike).

By means of the chordal distance (1.10),(1.22) we can characterize the relative position of pairs of points on AdS_D as spacelike if $d(z_1, z_2)^2 < 0$, as timelike if $d(z_1, z_2)^2 > 0$, and lightlike if $d(z_1, z_2)^2 = 0$. With the reference point $z_1 = e_D$ (2.21) these three sets are respectively decribed by

$$\begin{aligned} \Gamma_a(e_D) &= \{x \in \mathbf{R}_{D-1,2} : x^D > 1\} &(1.28)\\ \Gamma_b(e_D) &= \{x \in \mathbf{R}_{D-1,2} : x^D < 1\} &(1.29)\\ \Gamma_c(e_D) &= \{x \in \mathbf{R}_{D-1,2} : x^D = 1\} &(1.30) \end{aligned}$$

where Γ_c coincides with the set of lightlike vectors on the tangential hyperplane at e_D. The set Γ_b can be further decomposed in

$$\Gamma_b(e_D) = \bigcup_{\kappa \in \{+,-,ex\}} \Gamma_\kappa(e_D) \tag{1.31}$$

and each subset is defined by

$$\begin{aligned} \Gamma_+(e_D) &= \{x \in \mathbf{R}_{D-1,2} : -1 < x^D < +1, x^0 > 0\} &(1.32)\\ \Gamma_-(e_D) &= \{x \in \mathbf{R}_{D-1,2} : -1 < x^D < +1, x^0 < 0\} &(1.33)\\ \Gamma_{ex}(e_D) &= \{x \in \mathbf{R}_{D-1,2} : x^D < -1\} &(1.34) \end{aligned}$$

$\Gamma_\pm(e_D)$ are the positive (negative) timelike sets. The last set Γ_{ex} is denoted "exotic" and obtained from $\Gamma_a(e_D)$ by the inversion

$$x \to -x. \tag{1.35}$$

2 Geodesics, trajectories, and one-parameter subgroups

A geodesic is the conical section of AdS_D with a two-plane in $\mathbf{R}_{D-1,2}$ containing the origin. Since such two-planes can be characterized by one-parameter subgroups of $SO_0(D-1,2)$ both concepts are closely related. Planar trajectories are obtained by affine two-planes intersecting AdS_D which are translated off the origin by a fixed vector. This vector can be chosen orthogonal to the two-plane.

Consider two points x_1, x_2 on a geodesic γ. Then the corresponding one-parameter subgroup (1P subgroup) mapping x_1 onto x_2 defines an angle φ seen from the origin of $\mathbf{R}_{D-1,2}$. This angle appears also in the chordal distance $d(x_1, x_2)$ either as a trigonometric or as a hyperbolic angle. A typical example is the two-plane spanned by e_0, e_D. Then in e.s.C.c. the matrices of the 1P-subgroup are

$$\mathbf{\Omega} = \begin{pmatrix} \cos\varphi & 0 & \dots & 0 & -\sin\varphi \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ \sin\varphi & 0 & \dots & 0 & \cos\varphi \end{pmatrix}$$

Moreover the chordal distance is expressed by

$$d(x_1, x_2)^2 = (x_1, x_2) - 1 \tag{2.1}$$

$$(x_1, x_2) = \cos(\varphi(x_1) - \varphi(x_2)) \tag{2.2}$$

In e.s.C.c. φ is extended to the whole real axis yielding the covering space $AdS_D^{(c)}$ and the vector e_0 is mapped on the time axis by application of all $\mathbf{\Omega}$.

Another example of interest is the case of a two-plane spanned by e_0, e_1. The corresponding 1P-subgroup is in e.s.C.c. represented by matrices

$$\Upsilon = \begin{pmatrix} \cosh\eta & \sinh\eta & \dots & 0 & 0 \\ \sinh\eta & \cosh\eta & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and the chordal distance for two points x_1, x_2 on such geodesic is

$$d(x_1, x_2)^2 = (x_1, x_2) - 1 \tag{2.3}$$

$$(x_1, x_2) = \cosh(\eta(x_1) - \eta(x_2)) \tag{2.4}$$

Any trajectory is denoted timelike respectively spacelike if the tangent vector at each point is timelike or spacelike. In the case of geodesics it suffices to know the behaviour at one point, since the corresponding 1P-subgroup translates this property to all other points. The same is then true for planar trajectories since the orthogonal vector v is left invariant under the 1P-subgroup.

We consider the case of the two-plane through e_0, e_D. Then a planar trajectory is given by

$$x = v + \hat{x} \tag{2.5}$$

$$\hat{x} = R\{\sin\varphi, 0\dots, \cos\varphi\} \tag{2.6}$$

Since

$$(v, \hat{x}) = 0 \tag{2.7}$$

for all φ, we have

$$(v + \hat{x})^2 = v^2 + \hat{x}^2 = 1 \tag{2.8}$$

$$v^2 = 1 - R^2 \leq 0 \tag{2.9}$$

as $R \geq 1$ necessarily. The whole class of such plane trajectories is elliptic because of (2.6) and timelike because the tangent vector at $\varphi = 0$ is timelike.

Next we consider the two-plane through e_0, e_1. From the ansatz

$$x = v + \hat{x} \tag{2.10}$$

$$\hat{x} = R\{\cosh\eta, \sinh\eta, 0\ldots\} \quad R \text{ is} \quad \text{arbitrary} \quad \text{real}) \tag{2.11}$$

follows that

$$v^2 = 1 - R^2 = (v^D)^2 - \sum_{k=2}^{D-1}(v^k)^2 < 1 \tag{2.12}$$

These are spacelike hyperbolic planar trajectories since the tangent vector at $\eta = 0$ is spacelike. However, the choice

$$\hat{x} = R\{\sinh\eta, \cosh\eta, 0\ldots\} \tag{2.13}$$

implies timelike hyperbolic planar trajectories with

$$v^2 = 1 + R^2 > 1 \tag{2.14}$$

Thus the property of being spacelike or timelike is encoded in the value of v^2.

The timelike planar trajectories are of physical interest since they can be looked upon as trajectories of a pointlike observer. We can identify the angle variable φ or η with the eigentime t of the observer after the normalization

$$t = R\varphi(R\eta) \tag{2.15}$$

We want to prove now that the pointlike observers are uniformly accelerated (the square of the acceleration is constant).

As in special relativity we differentiate the trajectory $x(t)$ with respect to the eigentime t

$$u(t) = \frac{d}{dt}x(t) \tag{2.16}$$

$$w(t) = \frac{d^2}{dt^2}x(t) \tag{2.17}$$

Then the following identities hold

$$(x(t), u(t)) = 0 \tag{2.18}$$

$$(x(t), w(t)) = -1 \tag{2.19}$$

$$(u(t), u(t)) = 1 \tag{2.20}$$

$$(u(t), w(t)) = 0 \tag{2.21}$$

By definition $x(t)$ lies in the affine two-plane

$$x(t) \in v + \Pi \tag{2.22}$$

and has a component in Π denoted $\hat{x}(t)$

$$x(t) = v + \hat{x}(t) \tag{2.23}$$

$u(t)$ lies in Π and the tangential hyperplane at AdS_D through $x(t)$. The acceleration can be projected on this hyperplane by subtraction of $x(t)$

$$w(t) = \hat{w}(t) - x(t) \tag{2.24}$$

so that

$$(x(t), \hat{w}(t)) = 0 \tag{2.25}$$

which proves that $\hat{w}(t)$ lies in the tangential hyperplane, too. It is the physical acceleration.

Now $u(t)$ lies in the two-plane Π and

$$(u(t), \hat{x}(t)) = (u(t), w(t)) = 0 \tag{2.26}$$

On the other hand $\hat{x}(t)$ and $w(t)$ are neither zero

$$(\hat{x}(t), w(t)) = -1 \tag{2.27}$$

It follows that in the two-plane Π they are collinear

$$w(t) = \lambda(t)\hat{x}(t) \tag{2.28}$$

with

$$\lambda(t) = \frac{(\hat{x}(t), w(t))}{(\hat{x}(t), \hat{x}(t))} = (v^2 - 1)^{-1} \tag{2.29}$$

This implies finally that

$$(\hat{w}(t), \hat{w}(t)) = (w(t), w(t)) - 1 = \lambda(t)^2(1 - v^2) - 1 = \frac{v^2}{1 - v^2} \tag{2.30}$$

Thus the acceleration is uniform indeed.

We define the acceleration by

$$\hat{a} = \{-(\hat{w}(t), \hat{w}(t))\}^{\frac{1}{2}} \tag{2.31}$$

and obtain for both timelike planar trajectories

$$\hat{a} = \sqrt{\frac{v^2}{v^2 - 1}} \quad \text{in the hyperbolic case} \tag{2.32}$$

$$\hat{a} = \sqrt{\frac{|v^2|}{|v^2| + 1}} \quad \text{in the elliptic case} \tag{2.33}$$

Following a fixed planar timelike trajectory $x(t)$ in a quantum field $\psi(x(t))$, the generator of the trajectory is a certain element of the Lie algebra $s(D-1, 2)$ of the isometry. It is a kind of a Hamiltonian which by physical arguments has a spectrum bounded from below. This implies that $\psi(x(t))$ after insertion into a Green's function can be continued analytically in the group parameter, which is the eigentime of the observer on the trajectory. The result is a strip of analyticity in the complex t-plane

$$\{t : -\infty < \Re t < +\infty, \quad 0 < \Im t < \beta\} \tag{2.34}$$

with cut singularities along both boundaries and a β-periodic repetition. This analyticity is a KMS property implying that the observer perceives a temperature T

$$T = \beta^{-1} = \frac{1}{2\pi R} \tag{2.35}$$

Here R is the radius of the circle (2.6) or hyperbola (2.13) which can be expressed by the acceleration $\hat{a}$ in either case as

$$T = \frac{1}{2\pi}\sqrt{|\ \hat{a}^2 - 1\ |} \tag{2.36}$$

This is the AdS "Unruh effect".

3 The Cauchy problem and the antipodal map

In classical field theory on Minkowski space retarded Green functions permit to determine field configurations at a time t_2 from sources at an earlier time t_1. A related problem is to continue a given field configuration at t_1 to later times t_2 by means of Green functions. This is made possible by the structure of the field equations which for integral spin are second order in the time and by the topology of the underlying space-time. The space-time must allow us to define submanifolds $t = const$ on which a set of initial values for the fields can be given that are complete. This means that these initial values are consistent with respect to the field equations and determine future field configurations uniquely. Necessary and sufficient is that the space is globally hyperbolic and this property excludes the possibility that energy, momentum or charges enter or leave the space at spatial infinity.

In the present context it is of interest that anti-de Sitter space is not globally hyperbolic in contrast to Minkowski space, rendering it a toy model for studying the Cauchy initial value problem. In fact we will learn that for classical fields of integer spin and AdS-dimension the Cauchy problem has a solution and is connected with the existence of an "antipodal map". Of course our discussion is based on the universal covering space $AdS^{(c)}$ so that closed timelike trajectories are excluded.

The idea behind this solution is to consider AdS_D space as a "box" embedded in a larger globally hyperbolic space, namely Einstein's static universe (Esu_D). We are in fact interested only in the even dimension $D = 4$. In AdS spaces of even dimension the Green's functions of massless fields (mass is not uniquely defined on AdS spaces, sometimes conformal masses are distinguished from straight masses) have support only on the light cone, we call this "Huygens phenomenon" (the Huygens phenomenon is typical for conformally massless fields). For intuitive understanding the AdS and Esu spaces, we refer to the illustrations.

The AdS_4 space allows a global coordinate frame (if we neglect polar singularities) which separates the Laplace-Beltrami operator and is given in terms of e.s.C.c. by

$$
\begin{aligned}
x^0 &= \frac{\cos\tau}{\cos\rho} && (3.1)\\
x^4 &= \frac{\sin\tau}{\cos\rho} && (3.2)\\
x^1 &= \tan\rho\cos\theta && (3.3)\\
x^2 &= \tan\rho\sin\theta\cos\phi && (3.4)\\
x^3 &= \tan\rho\sin\theta\sin\phi && (3.5)
\end{aligned}
$$

where

$$0 \leq \rho < \pi/2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \tag{3.6}$$

The free scalar conformally massless field obeys the equation

$$(\Box - 1/6R))\psi = 0 \tag{3.7}$$

where

$$R = D(D-1)|_{D=4} = 12 \tag{3.8}$$

The Laplace-Beltrami operator is

$$\Box = \cos^2\rho\frac{\partial^2}{\partial\tau^2}\psi - \cot^2\rho[\cos^2\rho\frac{\partial}{\partial\rho}(\tan^2\rho\frac{\partial}{\partial\rho}\psi) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}\psi) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi] \tag{3.9}$$

The space Esu_4 has the structure $\mathbf{R} \times \mathbf{S^3}$ where the spherical factor comes from

$$\sum_{k=1}^{4}(x^k)^2 = 1 \tag{3.10}$$

in e.s.C.c., whereas y^0 is the time coordinate. Again we introduce polar coordinates on the sphere in terms of the e.s.C.c.

$$
\begin{aligned}
x^0 &= \tau && (3.11)\\
x^4 &= \cos\rho && (3.12)\\
x^1 &= \sin\rho\cos\theta && (3.13)\\
x^2 &= \sin\rho\sin\theta\cos\phi && (3.14)\\
x^3 &= \sin\rho\sin\theta\sin\phi && (3.15)
\end{aligned}
$$

The metric is then

$$(ds^E)^2 = d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.16}$$

yielding the Laplace-Beltrami operator for the field ψ^E

$$\begin{aligned}\Box^E = & \frac{\partial^2}{\partial\tau^2}\psi^E - \frac{1}{\sin^2\rho}[\frac{\partial}{\partial\rho}(\sin^2\rho\frac{\partial}{\partial\rho}\psi^E) + \\ & + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}\psi^E) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi^E]\end{aligned} \tag{3.17}$$

The free scalar conformally massless field equation comes out as

$$(\Box^E - 1/6R^E)\psi^E = (\Box^E + 1)\psi^E = 0 \tag{3.18}$$

where we used

$$R^E = -(D-1)(D-2)|_{D=4} = -6 \tag{3.19}$$

Solutions ψ^E of the scalar equation (3.89) at a fixed time τ form a Hilbert space $\mathcal{H}^E$ with the scalar product

$$< \psi_1^E, \psi_2^E >= i\int_{\tau=const}(\bar{\psi}_1^E\partial_\tau\psi_2^E - \psi_2^E\partial_\tau\bar{\psi}_1^E)d\Omega \tag{3.20}$$

where $d\Omega$ denotes the uniform measure on $\mathbf{S}^3$. This product is independent of τ. A basis in this space can be easily found as

$$\psi_{\omega lm}^E = N_{\omega l}\exp(-i\omega\tau)(\sin\rho)^l C_{\omega-l-1}^{l+1}(\cos\rho)Y_l^m(\theta,\phi) \tag{3.21}$$

where

$$\omega, l, m \in \mathbf{Z}, \quad \omega - 1 \geq l \geq |m| \tag{3.22}$$

and $C_{\omega-l-1}^{l+1}$ are Gegenbauer polynomials whereas Y_l^m are spherical harmonics.

Classical massless particles move along trajectories that are periodic in τ with period 2π. One can say even more. If such trajectory goes through

$$x^0, x^1, x^2, x^3, x^4 \Rightarrow \tau, \rho, \theta, \phi \tag{3.23}$$

it also passes through

$$x^0 + \pi, -x^1, -x^2, -x^3, -x^4 \Rightarrow \tau + \pi, \pi - \rho, \pi - \theta, \phi + \pi \tag{3.24}$$

Such pairs of points are called "antipodal". A time translation

$$\tau \to \tau + \pi \tag{3.25}$$

followed by an inversion in the space components of $\mathbf{R}_{3,2}$ is denoted an "antipodal map".

Next we map AdS_4 into Esu_4 by identifying the polar coordinates but restricting ρ to the interval $< 0, \pi/2)$. Moreover we apply a conformal map relating the metrics by

$$g^E_{\mu\nu} = \omega^2 g_{\mu\nu}, \quad \omega = \cos\rho \tag{3.26}$$

Since both massless equations are conformally massless, they go into each other under this injection if we set

$$\psi^E = \omega^{-1}\psi \tag{3.27}$$

If we give Cauchy data on the manifold

$$\Sigma^E = Esu_4|_{\tau=0} \tag{3.28}$$

the part Σ_1, namely $0 \leq \rho < \pi/2$, is covered by AdS_4 under the injection whereas the part Σ_2, namely $\pi/2 < \rho \leq \pi$, is left over. The point $\rho = \pi/2$ can be neglected if we apply the Hilbert space norm to the initial values.

On the other hand the Cauchy data on Σ_2 reappear after a time shift of π on the interval $\rho \in < 0, \pi/2)$. Denote the time shifted manifold Σ_2 by $\hat{\Sigma}_2$. Complete sets of initial values can therefore be obtained for AdS_4, by taking initial values on Σ_1 and $\hat{\Sigma}_2$. The support of these initial values is invariant under the antipodal map. It is therefore suggestive to consider initial values that are even respectively odd under the antipodal map. This is best done with the help of the basis. We give this basis after application of the conformal map (3.98).

The even antipodal parity elements are

$$\psi^{(+)}_{\omega lm} = N_{\omega l} \exp(-i\omega\tau) \cos\rho (\sin\rho)^l C^{l+1}_{2n+1}(\cos\rho) Y^m_l(\theta,\phi) \tag{3.29}$$

where

$$\frac{1}{\cos\rho}\psi^{(+)}_{\omega lm} \to 0 \quad \text{as} \quad \rho \to \frac{\pi}{2} \tag{3.30}$$

The eigenvalue ω can be expressed by

$$\omega = l + 2n + 2, \quad n \in \mathbf{N} \tag{3.31}$$

The odd antipodal parity elements are analogously

$$\psi^{(-)}_{\omega lm} = N_{\omega l} \exp(-i\omega\tau) \cos\rho (\sin\rho)^l C^{l+1}_{2n}(\cos\rho) Y^l_m(\theta,\phi) \tag{3.32}$$

where

$$\frac{\partial}{\partial\rho}(\frac{1}{\cos\rho}\psi^{(-)}_{\omega lm}) \to 0 \quad \text{as} \quad \rho \to \frac{\pi}{2} \tag{3.33}$$

The eigenvalue ω can now be expressed by

$$\omega = l + 2n + 1, \quad n \in \mathbf{N} \tag{3.34}$$

The Hilbert space of initial values decomposes correspondingly into a direct orthogonal sum

$$\mathcal{H}^E = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)} \tag{3.35}$$

To each antipodal parity class the Cauchy problem is well defined. The initial values have to be square integrable with respect to the right measure (the AdS_4 and Esu_4 measures are related by the conformal injection). An analogous construction can be carried over to massive scalar equations if the mass belongs to a discrete spectrum corresponding to integer AdS dimesnions.

We shall return to the antipodal map when we discuss two-point functions in the next Sections.

4 Two-point functions, The basic assumptions

As we shall see later, quantum field theory on AdS spaces and conformal field theory on Minkowski or Euclidean spaces are closely related. Such conformal field theories on Minkowski spaces are in practice often defined by a series of axioms known as "Wightman axioms" [18]. So we use them also for AdS field theories. Wightman functions for all $n \in \mathbf{N}_0$

$$W_n(x_1, x_2, \ldots, x_n) =< 0|\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n))|0 > \tag{4.1}$$

are tempered distributions acting on test function spaces which are C^∞ on the embedding space $\mathbf{R}_{D-1,2}$ and strongly decreasing at the boundary of AdS_D. They are invariant with respect to the isometry group $S_0(D-1,2)$. The vacuum state $|0>$ generates a Hilbert space $\mathcal{H}$ that carries a unitary representation $U_g, g \in SO_0(D-1,2)$ which leaves the vacuum invariant

$$\begin{aligned} U_g|0> &= |0> && (4.2)\\ U_g^{-1}\varphi_k(x)U_g &= (T_g^{(k)}\varphi_k)(x) && (4.3)\end{aligned}$$

where the possible action T_g on a field operator φ_k, its covariance, corresponds to a representation of the isometry group and is discussed below.

Let $(T_g)^\dagger$ be the adjoint equal the inverse representation of T_g and $f_k(x)$ a test function so that

$$\int_{AdS} (f_k(x)^\dagger, \varphi_k(x))dx = (f_k, \varphi_k) \tag{4.4}$$

where on the l.h.s. the sesquilinear form $(f_k(x)^\dagger, \varphi_k(x))$ goes into $(f_k(g^{-1}x)^\dagger, \varphi(g^{-1}x))$ under action respectively of T_g and dx is invariant

$$dx = d(g^{-1}x) \tag{4.5}$$

Then

$$|(f, \varphi)|^2 \geq 0 \tag{4.6}$$

implies the positivity of the distribution kernel

$$W_{1,1}(x_1, y_2) =< 0|\varphi(x_1)^\dagger, \varphi(y_2)|0 > \tag{4.7}$$

Correspondingly the vacuum expectation value of

$$\sum_{n=0}^{N} \int dx_1 dx_2 \dots dx_n f^{(n)}(x_1, x_2 \dots x_n)^\dagger \varphi_1(x_1)\varphi_2(x_2) \dots \varphi_n(x_n) \tag{4.8}$$

and its adjoint leads to a positivity constraint on the set of Wightman functions

$$W_{n,m}(x_1, x_2 \dots x_n; y_1, y_2 \dots y_m) \quad \text{with} \quad n + m \leq N \tag{4.9}$$

In this formulation we make also use of the hermiticity

$$\overline{< 0|\varphi_1(x_1), \varphi_2(x_2) \dots \varphi_n(x_n)|0 >} =< 0|\varphi_n(x_n)^\dagger, \varphi_{n-1}(x_{n-1})^\dagger \dots \varphi_1(x_1)^\dagger|0 > \tag{4.10}$$

We emphasize that the positivity of W_2 restricts the possible use of representations for the covariance.

Any timelike planar trajectory is generated by a Lie algebra element which can be viewed upon as a Hamiltonian. Its spectrum must be positive for physical reasons. This special property implies that Wightman distributions are boundary values of analytic functions in "tuboid domains". This is formulated as a spectral axiom.

We would also like to derive uniqueness of the vacuum state from a cluster property. However, the fall off of the Wightman functions at infinity is always powerlike. Nevertheless a sufficient cluster decomposition theorem can be formulated. If all these axioms are fulfilled for the Wightman functions W_n, then the Hilbert space and field operators generating it cyclically from the vacuum can be reconstructed by the Gelfand-Naimark-Segal theorem.

We have already earlier (Section 3) mentioned the AdS-version of the Unruh theorem. It can also be shown that a version of Osterwalder-Schrader theorem exists, which admits the Wick rotation technique and the transition between a Minkowskian quantum field theory and an Euclidean stochastic theory.

Let us now turn to a detailed study of the covariance axiom. We prefer to deal with covariance in the Euclidean framework. The de Sitter space was introduced by eqs.(1.11)-(1.20). The isometry is $G = SO_0(D,1)$. We impose an Iwasawa decomposition on G

$$G = \tilde{N}AK \tag{4.11}$$

where K is the maximal compact subgroup $SO(D)$. The coset space G/K is the de Sitter space. $\tilde{N}$ is the group of $D-1$ Euclidean translations and A is the abelian group of dilations. In matrix form

$$\tilde{n}_z = \begin{pmatrix} \delta_{ij} & -z_i & z_i \\ z_j & 1-\frac{1}{2}\vec{z}^2 & \frac{1}{2}\vec{z}^2 \\ z_j & -\frac{1}{2}\vec{z}^2 & 1+\frac{1}{2}\vec{z}^2 \end{pmatrix} \in \tilde{N}$$

$$a = \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & \cosh\eta & \sinh\eta \\ 0 & \sinh\eta & \cosh\eta \end{pmatrix} \in A$$

$$k = \begin{pmatrix} k_{ij} & k_{i,D} & 0 \\ k_{Dj} & k_{DD} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K$$

where the labels i,j run over $1,2\ldots D-1$ and $z_0 = \exp\eta$.

Given $g \in SO_0(D,1)$ its $(D+1)$st column fixes the point $z \in dS_D$ uniquely. Namely

$$g_{i,D+1} = x_i = \frac{z_i}{z_0}, \quad i \in \{1,2,\ldots D-1\} \tag{4.12}$$

$$g_{D,D+1} = x_D = \frac{z_0^2+\vec{z}^2-1}{2z_0} \tag{4.13}$$

$$g_{D+1,D+1} = x_{D+1} = \frac{z_0^2+\vec{z}^2+1}{2z_0} \tag{4.14}$$

These equations yield the e.s.C.c. x in terms of Poincare's coordinates $z_0, \vec{z}$ known from eqs. (2.6)- (2.8). Also the quadric of dS_D inside $\mathbf{R}_{D,1}$ is easily checked. We concentrate only on the connected hyperboloid shell in the half space $x_{D+1} \geq 0$. In this half space

$$x_{D+1} - x_D = \frac{1}{z_0} > 0 \tag{4.15}$$

so that only this shell is covered by the Poincare coordinates. Spatial infinity is reached at $z_0 \to 0$.

Let σ denote an arbitrary irreducible representation of K acting on a vector space U_σ. Then the representation space

$$\tilde{C}_\sigma = \{\phi \in C^\infty(\mathbf{R}_d \times \mathbf{R}_{>0}, U_\sigma)\} \tag{4.16}$$

carries the representation induced from K

$$(\tilde{T}_g^\sigma \phi)(z_0, \vec{z}) = \tilde{D}^\sigma(k)\phi(z_0', \vec{z}') \tag{4.17}$$

The Iwasawa decomposition is used to calculate z'

$$g^{-1}\tilde{n}_z a = \tilde{n}_{z'} a' k^{-1}, \quad \tilde{n}_z, \tilde{n}_{z'} \in \tilde{N}, \quad a, a' \in A, \quad k \in K \tag{4.18}$$

and $\tilde{D}^\sigma(k)$ is the representation matrix of k.

The representation $\tilde{T}^\sigma$ is in general reducible, it reduces into elementary representations. On the other hand not all elementary representations can be obtained from the reduction of these $\tilde{T}^\sigma$. In fact massless scalar conformal free fields belong to such representations. At the end of this chapter we will give a review over all elementary representations and those which are known not to arise from the induction with respect to K. V. K. Dobrev [38] has proposed to apply the Casimir operators of $SO_0(D, 1)$ to reduce $\tilde{T}^\sigma$. This amounts to submit the functions ϕ to field equations. In section 5 we shall derive two-point functions where the projection on an irreducible component of $\tilde{T}^\sigma$ is already included in the construction.

5 Scalar two-point functions

If all n-point functions W_n can be reduced to sums of products of two-point functions by the combinatorics applied also in Wick's theorem, we say that our fields are generalized free fields. Such two-point functions can be reduced to sums or integrals over free field two-point functions which we call Kallen-Lehmann representations.

One can derive two-point functions from field equations with boundary conditions imposed. This is practical if the field equations are simple as in the case of scalar fields. If this is not so, methods of representation theory can also be applied. In the case of conformal field theory and field theories on AdS spaces these methods are very restrictive and lead to explicit results soon. In these cases two-point functions are intertwining integral operators between elementary representations. This comes about as follows.

If $\varphi(x)$ transforms as an elementary representation $\chi = [\Delta, \mu]$, where Δ is the conformal dimension and μ an irreducible representation of $SO(d)$, then any non-vanishing two-point function

$$W_2(x_1, x_2) =< 0|\varphi_1(x_1)\varphi_2(x_2)|0 > \tag{5.1}$$

is invariant by the axioms. Let the first factor be covariant as χ_1 and the second as χ_2. Consider then a pair of test functions which have the dual covariance $\chi_{1,d}, \chi_{2,d}$. Then

$$\int dx_1 dx_2 f_1(x_1) f_2(x_2) W_2(x_1, x_2) \tag{5.2}$$

is invariant. But this means that

$$\int dx_2 f_2(x_2) W_2(x_1, x_2) \tag{5.3}$$

is covariant as χ_1 and W_2 is an intertwining kernel from the representation $\chi_{2,d}$ to χ_1.

For scalar fields the method of field equations is standard and most comfortable. We start from the Klein-Gordon equation

$$\Box\varphi + m^2\varphi = 0 \tag{5.4}$$

From now on we use the shorthands

$$\mu = \frac{d}{2} \tag{5.5}$$

$$\nu = \sqrt{\mu^2 + m^2} \tag{5.6}$$

$$\Delta_\pm = \mu \pm \nu \tag{5.7}$$

$$\Delta_+ + \Delta_- = 2\mu \tag{5.8}$$

Then (5.4) possesses two solutions

$$\begin{aligned} W_2(z,z') &= w_{\pm\nu}(\zeta) && (5.9)\\ w_\nu(\zeta) &= \frac{\exp -i\pi(\mu-1/2)}{(2\pi)^{\mu+1/2}}(\zeta^2-1)^{-\frac{\mu-1/2}{2}}Q^{\mu-1/2}_{\nu-1/2}(\zeta) && (5.10)\end{aligned}$$

where Q is a Legendre function of the second kind ([19], equ, 8.771.2). Another way of presentation is

$$w_\nu(\zeta) = \frac{\Gamma(\Delta)}{2^{\Delta+1}\pi^\mu\Gamma(\Delta-\mu+1)}\zeta^{-\Delta}F(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta-\mu+1;\zeta^{-2}) \tag{5.11}$$

Positivity of the scalar two-point function in conformal field theory (unitarity of the corresponding elementary representation) necessitates that

$$\Delta \geq \mu - 1. \tag{5.12}$$

However, from eq. (5.11) we see that the lower limit is excluded, and we have on AdS the stronger inequality

$$\Delta > \mu - 1 \tag{5.13}$$

Thus here we found the first case that an elementary representation is not realized on AdS space. It is not necessary that m^2 is positive, but the "Breitenlohner-Freedman" bound

$$m^2 + \mu^2 \geq 0 \tag{5.14}$$

is necessary to make the dimension real (and the representation in Minkowski space unitary). If this bound is fulfilled then (5.13) is automatically fulfilled for Δ_+. On the other hand Δ_- satisfies the bound (5.13) if

$$0 \leq \nu < 1 \tag{5.15}$$

which amounts to

$$-\mu^2 \leq m^2 < -\mu^2 + 1 \tag{5.16}$$

The solutions that fulfill the constraints of positivity of the two-point function (3.31) and the reality of the dimension (5.14), (5.16) finally are all square integrable.

We can then return to the Cauchy initial value problem which as we saw possesses a solution if Δ is an integer. Let

$$\Delta_+ = M, \quad \Delta_- = d - M \tag{5.17}$$

and

$$\nu = |M - \mu| \tag{5.18}$$

For the regular solutions based on Δ_+ infinitely many integer M are possible

$$M \in [\mu] + \mathbf{N}_0 \tag{5.19}$$

whereas for the irregular solutions based on Δ_- there is one solution possible with

$$0 \leq (M - \mu)^2 < 1 \tag{5.20}$$

For these M we obtain even respectively odd antipodal parity solutions. The antipodal mapping can be applied to ζ

$$\zeta(x, x') \to \zeta(x, -x') = -\zeta(x, x') \tag{5.21}$$

so that even or odd parity corresponds to even respectively odd Δ.

By (5.9), (5.10) we have obtained analytic functions for the two-point function. The function $w_\nu(\zeta)$ is analytic on the universal covering of the ζ-plane cut along the interval

$< -1, +1 >$ for irrational Δ. For rational Δ this infinite covering reduces to a finite one which is a single sheet if Δ is integer. On the interval $< -1, +1 >$ $w_\nu(\zeta)$ is defined as the boundary value from above or below. The pair x, x' is timelike separated. The difference of the boundary values gives the commutator function (Pauli-Jordan function).

It is remarkable that this commutator function is the same for the regular as for the irregular solution. This follows from the formula

$$w_{-\nu} - w_\nu = \frac{\sin \pi\nu}{(2\pi)^{\mu+1/2}} \Gamma(\Delta_+)\Gamma(\Delta_-)(\zeta^2 - 1)^{-\frac{d-1}{4}} P^{-\mu+1/2}_{-\nu-1/2}(\zeta) \tag{5.22}$$

where the Legendre function of the first kind $P^{-\mu+1/2}_{-\nu-1/2}(\zeta)$ is analytic in a circle $|\zeta| < 1$ ([19], eq. 8.771)

$$P^\mu_\nu(\zeta) = \frac{\exp -i\frac{\pi\mu}{2}}{\Gamma(1-\mu)} (\frac{1+\zeta}{1-\zeta})^{\mu/2} F(-\nu, \nu+1; 1-\mu; \frac{1-\zeta}{2}) \tag{5.23}$$

and can be made uniform by cutting the ζ plane from $-\infty$ to -1 and from $+1$ to $+\infty$. At infinity both functions $w_{\pm\nu}(\zeta)$ differ by their asymptotic behaviour.

6 The boundary limit

In the limit $z_0 \to 0$ the AdS space tends to its boundary manifold which is a Minkowski space $\mathbf{R}_{d-1,1}$ (dS tends correspondingly to $\mathbf{R}_d$). The universal covering space of the Minkowski space closed by an infinite light cone carries the elementary representations of the universal covering group of the conformal group $G = SO_0(d,2)$. We want to derive these representations by induction from appropriate subgroups of G. These elementary representations supply us with all representations of interest for quantum field theory, since a theorem of Langlands-Knapp-Zukerman asserts that every irreducible admissable representation of a real connected semisimple Lie group G with finite center is equivalent to a subrepresentation of an elementary representation of G. On the other hand elementary representations are generically irreducible except in singular cases that describe e.g. gauge fields and conserved currents which may be reducible indecomposable.

The elementary representations can be constructed from induction of a certain coset (Bruhat) decomposition of G, namely

$$G = \tilde{N}MAN \tag{6.1}$$

where (from now on for the Minkowskian case) $M = SO(d)$ is the maximal compact subgroup, N is the subgroup of special conformal transformations, and A is the abelian subgroup of dilations. $\tilde{N}$ are the translations which are identified with Minkowski space $\mathbf{R}_{d-1,1}$

$$\tilde{N} = G/MAN \tag{6.2}$$

This coset space is globally not defined, but the exceptional set, which amounts to the infinite light cone, is of measure zero. Since the elements of the C^∞ space have a fixed asymptotic behaviour at infinity, the exceptional set does not carry any additional degree of freedom.

Let V_μ be the finite dimensional carrier space of the representation μ of M. Then

$$C_\chi = \{f \in C^\infty(\mathbf{R}_d, V_\mu)\} \tag{6.3}$$

carries the elementary representation T^χ of G

$$(T_g^\chi f)(x) = |a|^{-\Delta} D^\mu(m) f(x') \tag{6.4}$$

with

$$g^{-1}\tilde{n}_x = \tilde{n}_{x'} m^{-1} a^{-1} n^{-1}, (g \in G, \tilde{n}_x, \tilde{n}_{x'} \in \tilde{N}, m \in M, a \in A, n \in N) \tag{6.5}$$

This representation is characterized by the label $\chi = [\Delta, \mu]$.

The corresponding two-point function of conformal field theory is an intertwining operator kernel

$$\begin{aligned} G_\chi(x_1, x_2) &= \hat{G}_\chi(x_1 - x_2) && (6.6) \\ \hat{G}_\chi(x) &= \frac{\gamma_\chi}{(x^2)^\Delta} D^\mu(r(x)) && (6.7) \end{aligned}$$

with

$$r(x) = \begin{pmatrix} \hat{r}(x) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\hat{r}(x) = \left(2\frac{x_i x_j}{x^2} - \delta_{ij}\right) \tag{6.8}$$

γ_χ is a normalization constant, $\hat{r}(x)$ is an inversion times a reflection along x.

This intertwiner G_χ maps the representation χ on the dual representation $[d-\Delta, \mu_d]$ where μ_d is the mirror image representation of μ. Then G_χ has the property $G_\chi : C_{\chi_d} \to C_\chi$ with

$$\begin{aligned} T_g^\chi G_\chi &= G_\chi T_g^{\chi_d} \quad \text{for all g} && (6.9) \\ (G_\chi f)(x) &= \int dx' \hat{G}_\chi(x - x') f(x') && (6.10) \end{aligned}$$

The representations χ, χ_d are generically irreducible and then inverse to each other, so that by adjusting $\gamma_\chi, \gamma_{\chi_d}$

$$G_\chi G_{\chi_d} = 1 = G_{\chi_d} G_\chi \tag{6.11}$$

It has been Dobrev's idea [38] to use this concept to construct an intertwiner between an elementary representation on the boundary of AdS and an elementary subrepresentation on the bulk of AdS. We start from an ansatz $L^\sigma_\chi : \tilde{C}_\sigma \to C_\chi$ so that

$$(L^\sigma_\chi \varphi)(\vec{z}) = \lim_{z_0 \to 0} |z_0|^{-\Delta} \Pi^\sigma_\mu \varphi(z_0, \vec{z}) \tag{6.12}$$

where Π^σ_μ is the standard projection operator from the representation space U_σ of K to the representation space V_μ of M

$$\Pi^\sigma_\mu \hat{D}^\sigma(k) = D^\mu(m(k)) \Pi^\sigma_\mu \hat{D}^\sigma(k_x) \tag{6.13}$$

Here the matrix k appears decomposed

$$k = m(k) k_x \quad (k \in SO(d+1), \quad m(k) \in SO(d), \quad m(k) \in M) \tag{6.14}$$

where

$$m(k) = \begin{pmatrix} \tilde{m}(k) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$k_x = \begin{pmatrix} \tilde{k}_x & 0 \\ 0 & 1 \end{pmatrix} \in K$$

$$\tilde{k}_x = \begin{pmatrix} \delta_{ij} - \frac{2x_i x_j}{1+x^2} & -\frac{2x_i}{1+x^2} \\ +\frac{2x_j}{1+x^2} & \frac{1-x^2}{1+x^2} \end{pmatrix}$$

where $x \in \mathbf{R}_d$ and

$$x_i = \frac{k_{d+1,i}}{1 + k_{d+1,d+1}} \tag{6.15}$$

$$x^2 = \frac{1 - k_{d+1,d+1}}{1 + k_{d+1,d+1}} \tag{6.16}$$

and $1 + k_{d+1,d+1} \neq 0$ has been presumed.

Reducing the representation σ on the subgroup $SO(d)$ the representation μ should appear at least once and the projection operator Π^σ_μ chooses any one. For practical reasons σ should be chosen "minimal". The case that μ equals symmetric traceless tensors of rank l is of greatest interest for us. In this case we choose σ also as symmetric traceless tensors of the same rank. The general case has, however, also a simple solution.

Let μ be the set of labels

$$\mu = [l_1, l_2 \ldots l_{[d/2]}] \tag{6.17}$$

where l_1 may be negative only if d is even, and moreover

$$|l_1| \leq l_2 \leq \ldots \leq l_{[d/2]} \tag{6.18}$$

To embed this representation into the representation σ of $SO(d+1)$

$$\sigma = [l'_1, l'_2, \ldots l'_{[\frac{d+1}{2}]}] \tag{6.19}$$

it is necessary that if d is odd

$$|l'_1| \leq l_1 \leq \ldots \leq l_{[d/2]} \leq l'_{[\frac{d+1}{2}]} \tag{6.20}$$

and if d is even

$$-l'_1 \leq l_1 \leq l'_1 \leq l_2 \leq \ldots \leq l_{[d/2]} \leq l'_{[\frac{d+1}{2}]} \tag{6.21}$$

In the latter case $[d/2] = [\frac{d+1}{2}]$ of course.

Dobrev has proven that L^σ_χ is an intertwiner indeed

$$L^\sigma_\chi \tilde{T}^\sigma_g = T^\chi_g L^\sigma_\chi \quad \text{for all} \quad g \in G \tag{6.22}$$

and the operator Π^σ_μ acts in a truncated fashion: For $x = \vec{z}$ in the limit $z_0 \to 0$ the factor $\hat{D}^\sigma(k_{\vec{z}})$ is integrated into the function φ.

Next we follow Dobrev [38] in constructing an inverse intertwiner $\tilde{L}^\sigma_\chi : C_\chi \to \tilde{C}_\sigma$

$$\tilde{T}^\sigma_g \tilde{L}^\sigma_\chi = \tilde{L}^\sigma_\chi T^\chi_g \tag{6.23}$$

in the form of an integral kernel (now in Euclidean form)

$$(\tilde{L}^\sigma_\chi f)(z_0, \vec{z}) = \int_{\mathbf{R}_d} K^\sigma_\chi(z_0, \vec{z}; x) f(x) dx \tag{6.24}$$

With a normalization constant N^σ_χ the solution is

$$K^\sigma_\chi(z_0, \vec{z}; 0) = N^\sigma_\chi (\frac{z_0}{z_0^2 + \vec{z}^2})^{d-\Delta} \tilde{D}^\sigma(\rho(z_0, \vec{z}))\Pi^\sigma_\mu \tag{6.25}$$

with

$$\rho(z_0, \vec{z}) = \begin{pmatrix} \delta_{ij} - 2\frac{z_i z_j}{(z_0^2+\vec{z}^2} & +2\frac{z_0 z_i}{z_0^2+\vec{z}^2} \\ -2\frac{z_0 z_j}{z_0^2+\vec{z}^2} & \frac{z_0^2-\vec{z}^2}{z_0^2+\vec{z}^2} \end{pmatrix} \in SO(d+1)$$

The normalization constant N^σ_χ can be fixed by the requirement that

$$L^\sigma_\chi \tilde{L}^\sigma_\chi = 1 \quad \text{on} \quad C_\chi \tag{6.26}$$

From Dobrev's work we obtain for the scalar case

$$N^\sigma_\chi = \frac{\Gamma(d-\Delta)}{\pi^\mu \Gamma(\mu - \Delta)} \tag{6.27}$$

which is not defined for $\Delta = d + k, k \in \mathbf{Z}_{\geq 0}$ when d is odd. For even d it vanishes for $\Delta \in \{\mu, \mu+1, \dots d-1\}$. For the tensorial case he gets with $\mu = d/2, \hat{\mu} = \frac{d+1}{2}$

$$N^\sigma_\chi = N_0 \frac{\Gamma([\hat{\mu}] - \Delta)}{\Gamma([\mu] - \Delta)} \prod_{k=1}^{[\mu]} (m_k + \mu - \Delta) \tag{6.28}$$

where

$$m_k = |l_k + k - 1 + \mu - [\mu]| \tag{6.29}$$

In order to construct a two-point function on the bulk of AdS one could think that the product $\tilde{L}^\sigma_\chi L^\sigma_\chi$ could be an appropriate operator. However, the right factor is not defined yet as an integral operator (see (6.12)) and we have to transform it into such shape. Let us consider tentatively (the maesure is Lebesgue times wrapping factor)

$$\int_{z_0 \geq 0} z_0^{-D} d^D z K^\sigma_\chi(z_0, \vec{z}) \varphi(z) \tag{6.30}$$

for any $\varphi \in \tilde{C}_\sigma$. It is crucial now how $\varphi(z)$ depends on z_0 at $z_0 \to 0$. If it is a proper test function which at the boundary of AdS goes faster to zero than any power, there is no problem with the definition of the integral (6.30). If it has a power behaviour $z_0^{\Delta-d}$ in dual correspondence to $\sigma = [\Delta, \ldots]$ in K, then a logarithmic divergence arises at $z_0 = 0$. In this case we have to regularize the integral by $z_0^D \to z_0^{D-\epsilon}$ and extract the residue of the pole in ϵ

$$res_{\epsilon=0} \int d^D z \delta(z_0) K_\chi^\sigma(z;x) \hat{\varphi}(\vec{z}) \tag{6.31}$$

where

$$\hat{\varphi}(\vec{z}) = \lim_{z_0 \to 0} (z_0)^{d-\Delta} \varphi(z) \tag{6.32}$$

and we are apparently back to the limit intertwiner (6.12). Therefore we conclude that bulk-to-bulk two-point functions applicable to test functions falling off rapidly at the boundary of AdS can be found from the convolution of two intertwiners

$$\int dx K_\chi^\sigma(z;x) K_{\chi_d}^{\sigma_d}(z';x) \tag{6.33}$$

which we will study in the next sections. Such fall off was postulated for the test functions applicable to Wightman functions as distributions.

7 The algebra of two-point functions

A tensor field of rank l at a point $z \in AdS(dS)$ corresponds to a representation of the group $SO(d)$ realized by tensor products of vectors of the tangential hyperplane at z. The symmetry is labelled by integers $l_1, l_2, \dots l_{[D/2]}$ as explained in the preceding section. Two-point functions (or propagators which differ only by the way the boundary value is taken on the ζ interval $< -1, +1 >$) are bitensors, i.e. tensors at either point z, z'. Thanks to the maximal symmetry of $AdS(dS)$ a basis for the bitensors can be constructed by four basic bitensors, which are all deduced from the derivatives of the chordal distance variable ζ or the metric tensor.

For an arbitrary tensor we introduce vectors from $TAdS$, namely $\{a_1, a_2, \dots a_l\}$ at z and $\{c_1, c_2, \dots c_l\}$ at z'. We contract these tangential vectors with the tensors in an invariant fashion and submit them to the symmetrization required for the tensor. Since we shall discuss symmetric tensors only, it is sufficient for us to use a single vector a at z and another c at z'. We define the invariants

$$I_1 = (a\partial_1)(c\partial_2)\zeta \tag{7.1}$$

$$I_{a1} = (a\partial_1)\zeta \tag{7.2}$$

$$I_{c2} = (c\partial_2)\zeta \tag{7.3}$$

$$I_2 = I_{a1}I_{c2} \tag{7.4}$$

In order to extract traces the following invariants are indispensable

$$I_3 = a_1^2 I_{c2}^2 + c_2^2 I_{a1}^2 \tag{7.5}$$

$$I_4 = a_1^2 c_2^2 \tag{7.6}$$

where all scalar products are invariant, e.g.

$$a_1^2 = \sum_{i,j=0}^{d} g_{ij}(z_1) a^i a^j \tag{7.7}$$

A symmetric traceless tensor field has a two-point function of the form

$$\Psi^{(l)}[F] = \sum_{l_i \in \mathbf{N}_0} I_1^{l_1} I_2^{l_2} I_3^{l_3} I_4^{l_4} F_{l_1 l_2 l_3 l_4}(\zeta) \tag{7.8}$$

where the sum extends only over

$$l_1 + l_2 + 2(l_3 + l_4) = l \tag{7.9}$$

Instead we can also use the form

$$\Psi^{(l)}[F] = \sum_{k=0}^{l} I_1^{l-k} I_2^k F_k(\zeta) + \text{trace terms} \tag{7.10}$$

The trace terms containing I_3 and I_4 can be reconstructed from the functions F_k by imposing

$$\frac{\partial}{\partial a^\mu} \frac{\partial}{\partial a_\mu} \Psi^{(l)}[F] = \Box_a \Psi^{(l)}[F] = 0 \tag{7.11}$$

Tracelessness with respect to c follows then automatically.

The derivatives

$$\frac{\partial}{\partial a^\mu} I_{a1} \frac{\partial}{\partial a_\mu} I_{a1} = \zeta^2 - 1 \tag{7.12}$$

$$\frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_{a1} = \zeta I_{c2} \tag{7.13}$$

$$\frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = c_2^2 + I_{c2}^2 \tag{7.14}$$

and other ones containing I_3, I_4 e.g

$$\frac{\partial}{\partial a^\mu} I_{a1} \frac{\partial}{\partial a_\mu} I_3 = 2 I_{a1}(I_{c2}^2 + (\zeta^2 - 1) c_2^2) \tag{7.15}$$

$$\frac{\partial}{\partial a^\mu} I_{a1} \frac{\partial}{\partial a_\mu} I_4 = 2 I_{a1} c_2^2 \tag{7.16}$$

$$\frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_3 = 2(I_1 I_{c2}^2 + \zeta c_2^2 I_2) \tag{7.17}$$

$$\frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_4 = 2 I_1 c_2^2 \tag{7.18}$$

yield

$$\Box_a \Psi^{(l)}[F] = I_{c2}^2 \Psi_1^{(l-2)}(Tr_1 F) + c_2^2 \Psi_2^{(l-2)}(Tr_2 F) \tag{7.19}$$

Both Tr_1F and Tr_2F are linear combinations of the functions $\{F_k\}$, and vanishing of the trace amounts to

$$
\begin{aligned}
Tr_1F &= 0 && (7.20)\\
Tr_2F &= 0 && (7.21)
\end{aligned}
$$

Most often used are the nontrace terms of Tr_1F

$$
(Tr_1F)_k = (l-k)(l-k-1)F_k+2(k+1)(l-k-1)\zeta F_{k+1}+(k+2)(k+1)(\zeta^2-1)F_{k+2} \quad (7.22)
$$

The general formulae for Tr_1F and Tr_2F can be found in next sections. Solving the two constraints (7.20),(7.21) is a linear problem involving an overcomplete but consistent system of equations. A general formula for the result is not known.

Symmetric not necessarily traceless bitensors $\Psi^{(l)}[F]$ form an algebra if summation is introduced in the trivial way by

$$
\Psi[F] = \sum_{l=0}^{N} \alpha_l \Psi^{(l)}[F] \quad (7.23)
$$

In this algebra we have the operations of multiplication, bigradient

$$
(a^\mu \nabla_{1,\mu})(c^\nu \nabla_{2,\nu})\Psi[F] \quad (7.24)
$$

and bidivergence

$$
(\frac{\partial}{\partial a_\mu}\nabla_{1,\mu})(\frac{\partial}{\partial c_\nu}\nabla_{2,\nu})\Psi[F] \quad (7.25)
$$

We can understand the tensor rank l as the "grade" in this algebra so that the bigradient raises the grade and the bidivergence lowers the grade by one.

8 Tensor field propagators

As explained in previous sections we want to study the kernel

$$A^{(l)}_{\Delta,\mu} = \int dx < a^l_\otimes, K^{(l)}_\lambda(z;\vec{x}) >_{i_1 i_2 \ldots i_l} < c^l_\otimes, K^{(l)}_\Delta(z';\vec{x}) >_{i_1 i_2 \ldots i_l} \qquad (\lambda = d - \Delta) \tag{8.1}$$

For a symmetric traceless tensor representation we have

$$< a^l_\otimes, K^{(l)}_\Delta(z;x)\vec{b}^l_\otimes >= N^\sigma_\chi \frac{z_0^{\Delta-l}}{(z_0^2 + (\vec{z}-\vec{x})^2)^\Delta}[< a, \rho(z-\vec{x})\vec{b} >^l - \text{trace terms}] \tag{8.2}$$

We make use of the fact that in Poincare coordinates the metric is wrapped Euclidean with wrapping factor z_0^{-2} (see (1.14)). We discard the last column in ρ (after (6.25)) as commanded by the projection operator Π^μ_σ and using the Euclidean scalar product $<,>$ we get $((z-\vec{x})^2 = z_0^2 + (\vec{z}-\vec{x})^2)$

$$A_j = a_j - 2 < a, z - \vec{x} > \frac{(\vec{z}-\vec{x})_j}{(z-\vec{x})^2} \tag{8.3}$$

$$C_j = c_j - 2 < c, z - \vec{x} > \frac{(\vec{z}-\vec{x})_j}{(z-\vec{x})^2} \tag{8.4}$$

Making (8.2) traceless in $\vec{b}$ we have

$$\frac{l!}{2^l(\mu-1)_l}(|\vec{A}||\vec{b}|)^l C_l^{\mu-1}(\frac{\vec{A}\vec{b}}{|\vec{A}||\vec{b}|}) \tag{8.5}$$

But this expression is traceless in $\vec{A}$ as well. Therefore after the contraction of the two kernels we must obtain tracelessness in $\vec{A}$ and $\vec{C}$ or

$$\frac{l!}{2^l(\mu-1)_l}(|\vec{A}||\vec{C}|)^l C_l^{\mu-1}(\frac{\vec{A}\vec{C}}{|\vec{A}||\vec{C}|}) \tag{8.6}$$

and this has to be integrated over

$$\int dx \frac{z_0^{\lambda-l}}{(z_0^2 + (\vec{z}-\vec{x})^2)^\lambda} \frac{(z_0')^{\Delta-l}}{((z_0')^2 + (\vec{z}'-\vec{x})^2)^\Delta} \cdots \tag{8.7}$$

To do this integral we go to a special coordinate system where $\vec{z} = \vec{z}' = 0$:

$$\xi = z - \vec{x} = \{z_0, -\vec{x}\} \tag{8.8}$$

$$\eta = z' - \vec{x} = \{z_0', -\vec{x}\} \tag{8.9}$$

We integrate first over the angle Ω of $\vec{x}$. To achieve this we expand

$$\vec{A}^2 = \alpha_2(\vec{a}\vec{x})^2 + \alpha_1(\vec{a}\vec{x}) + \alpha_0 \tag{8.10}$$

$$\vec{C}^2 = \beta_2(\vec{c}\vec{x})^2 + \beta_1(\vec{c}\vec{x}) + \beta_0 \tag{8.11}$$

$$\vec{A}\vec{C} = \gamma_{11}(\vec{a}\vec{x})(\vec{c}\vec{x}) + \gamma_{10}(\vec{a}\vec{x}) + \gamma_{01}(\vec{c}\vec{x}) + \gamma_{00} \tag{8.12}$$

introduce the shorthands

$$\vec{x}^2 = r^2 \tag{8.13}$$

$$\xi^2 = z_0^2 + r^2 \tag{8.14}$$

$$\eta^2 = z_0'^2 + r^2 \tag{8.15}$$

and get

$$\alpha_0 = 4(a_0 z_0)^2 \frac{r^2}{\xi^4} + \vec{a}^2 \tag{8.16}$$

$$\alpha_1 = 4(a_0 z_0)(1 - 2\frac{r^2}{\xi^2}) \tag{8.17}$$

$$\alpha_2 = \frac{4}{\xi^2}(\frac{r^2}{\xi^2} - 1) \tag{8.18}$$

The coefficients β_n are analogous with ξ replaced by η and a by c. The coefficients γ_{nm} are

$$\gamma_{00} = 4a_0 c_0 z_0 z_0' \frac{r^2}{\xi^2\eta^2} + \vec{a}\vec{c} \tag{8.19}$$

$$\gamma_{01} = 2\frac{a_0 z_0}{\xi^2}(1 - 2\frac{r^2}{\eta^2}) \tag{8.20}$$

$$\gamma_{10} = 2\frac{c_0 z_0'}{\eta^2}(1 - 2\frac{r^2}{\xi^2}) \tag{8.21}$$

$$\gamma_{11} = -2\frac{z_0^2 z_0'^2}{\xi^2 + \eta^2} \tag{8.22}$$

We introduce (8.10) - (8.12) into the Gegenbauer polynomial

$$\sum_{k=0}^{[l/2]} \frac{(-\frac{l}{2})_k(\frac{1-l}{2})_k}{k!(2-\mu-l)_k}(\vec{A}\vec{C})^{l-2k}(\vec{A}^2\vec{C}^2)^k = \sum_{m,n} \sigma_{mn}(r^2)(\vec{a}\vec{x})^n(\vec{c}\vec{x})^m \tag{8.23}$$

Integration over Ω gives zero whenever $m+n$ is odd

$$\int d\Omega(\vec{a}\vec{x})^n(\vec{c}\vec{x})^m = J(\mu)\frac{(1/2)_\nu}{(\mu)_\nu} r^{n+m} f_{nm} \tag{8.24}$$

where $n+m$ is equal to 2ν and $J(\mu)$ is the area of the unit sphere $\mathbf{S}_{d-1}$

$$J(\mu) = \frac{2\pi^{\mu}}{\Gamma(\mu)} \tag{8.25}$$

and the coefficients f_{nm} depend only on $\vec{a}, \vec{c}$

$$f_{nm} = \binom{n+m}{n}^{-1} \sum_k \binom{\nu}{\frac{n-k}{2}, \frac{m-k}{2}, k} (\vec{a}^2)^{\frac{n-k}{2}} (\vec{c}^2)^{\frac{m-k}{2}} (2\vec{a}\vec{c})^k \tag{8.26}$$

There remains the integral over r

$$z_0^{\lambda-l} z_0'^{\Delta-l} \sum_{nm} \int_0^\infty dr r^{d+n+m-1} \sigma_{nm}(r^2)(\xi^2)^{-\lambda}(\eta^2)^{-\Delta} \tag{8.27}$$

It has been found by computer (for all $l \leq 8$) that

$$\sum_{nm} r^{n+m} f_{nm}\sigma_{nm}(r^2) = \sum_{s=0}^{l} B_s^{(l)}(z_0, z_0')\left(\frac{r^2}{\xi^2\eta^2}\right)^s \tag{8.28}$$

The remaining integral

$$\int_0^\infty dr r^{d+2s-1}(\xi^2)^{-\lambda-s}(\eta^2)^{-\Delta-s} = 1/2(z_0^2)^{-\lambda-s}(z_0'^2)^{-\Delta+\mu} \int_0^\infty dt t^{\mu+s-1}(1+t)^{-\Delta-s}(1+\rho t)^{-\lambda-s} \tag{8.29}$$

is Gaussian hypergeometric (B is the beta function)

$$1/2(z_0^2)^{-\mu-s} B(\mu+s, \lambda-\mu) F(\Delta+s, \mu+s; \Delta-\mu+1; \rho) + \{\Delta \leftrightarrow \lambda\} \tag{8.30}$$

where ρ is

$$\rho = \left(\frac{z_0'}{z_0}\right)^2 \tag{8.31}$$

By a quadratic transformation we can introduce ζ

$$\zeta = \frac{z_0^2 + z_0'^2}{2z_0 z_0'} = \frac{1+\rho}{2\sqrt{\rho}} \tag{8.32}$$

and obtain (up to a normalization)

$$(z_0 z_0')^{-l} \sum_{s=0}^{l} B_s^l(z_0, z_0')\left[\frac{(\mu)_s}{(\lambda)_s}\Lambda_{\Delta,s}(\zeta) + \frac{(\mu)_s}{(\Delta)_s}\Lambda_{\lambda,s}(\zeta)\right] \tag{8.33}$$

where we made use of a type of Legendre function of the second kind

$$\Lambda_{\Delta,s}(\zeta) = (2\zeta)^{-\Delta-s} F\left(\frac{\Delta+s}{2}, \frac{\Delta+s+1}{2}; \Delta-\mu+1; \zeta^{-2}\right) \tag{8.34}$$

The coefficients $B_s^{(l)}$ are in fact polynomials in ζ of degree s. We were able to show that only the combinations

$$Q_k(\zeta) = (\mu)_k \sum_{s=0}^{l-k} \frac{(-l+k)_s (k+1)_s}{(\lambda)_{s+k} s!} (2\zeta)^s \Lambda_{\Delta,s+k}(\zeta), \quad k \in \{0,1,\ldots l\} \tag{8.35}$$

Then the regular part of (8.33) can be brought into the final form

$$A^{(l)}_{\Delta,\mu} = \kappa_l(\mu,\Delta) \sum_{k=0}^{l} Q_k(\zeta) \sum_{r_1 r_2 r_3 r_4} R^{(l,k)}_{r_1 r_2 r_3 r_4}(\mu) L_1^{r_1} L_2^{r_2} L_3^{r_3} L_4^{r_4} \tag{8.36}$$

The irregular part containing $\Lambda_{\lambda,s}(\zeta)$ is discarded. We used a tensor basis $\{L_i\}$ which is related with the $\{I_i\}$ by

$$L_1 = -I_1 - \zeta(1-\zeta^2)^{-1} I_2 \tag{8.37}$$

$$L_2 = (1-\zeta^2)^{-1} I_2 \tag{8.38}$$

$$L_3 = -(1-\zeta^2)^{-1} I_3 \tag{8.39}$$

$$L_4 = I_4 \tag{8.40}$$

It remains to determine the coefficients $R^{(l,k)}_{r_1 r_2 r_3 r_4}(\mu)$. These are rational functions of μ with integer coefficients. They satisfy

$$R_{l,0,0,0} = 1 \tag{8.41}$$

$$R_{r_1 r_2 r_3 r_4} = 0 \tag{8.42}$$

if $r_1 + r_2 + 2(r_3 + r_4) \neq l$ or $l - k - r_1 \notin 2\mathbf{N}_0$, moreover

$$(-1)^{1/2(l-k-r_1)+r_3} R^{(l,k)}_{r_1 r_2 r_3 r_4} \geq 0 \tag{8.43}$$

$$R^{(l,0)}_{l-2k,0,0,k} = \frac{l!}{2^{2k} k! (l-2k)! (2-\mu-l)_k} \tag{8.44}$$

Closed explicit expressions other than (8.44) are unknown.

We close this section with the remark that irreducibility of these propagators is guaranteed by the projection on one elementary representation. Then the result is also unique. However, in the singular cases where the elementary representation is itself not irreducible, that is for conserved currents or gauge fields, usually one exploits the singular structure of analytic normalization factors to analyze these cases. This has not been done yet. In any case we mention the normalization here

$$\lim_{z_0,z_0'\to 0}(z_0z_0')^{-\Delta+l}A^{(l)}_{\Delta,\mu}|_{a_0=c_0=0} = 1/2N^{\sigma}_{\chi}N^{\sigma_d}_{\chi_d}J(\mu)\frac{\lambda-1}{\lambda+l-1}$$

$$B(\mu,\mu-\Delta)\{(\vec{a}\vec{c}-2\frac{(\vec{a}(\vec{z}-\vec{z}'))(\vec{c}(\vec{z}-\vec{z}'))}{(\vec{z}-\vec{z}')^2})^l - \text{trace terms}\} \tag{8.45}$$

In section 12 we consider direct way of construction of the bulk-to-bulk propagator for Higher Spin Gauge field using singular solution of the equation of mouton in different gauges.

9 Currents coupled to (conformal) higher spin fields in AdS

In any known model the AdS/CFT correspondence is an unproven hypothesis still. If such model is derived from string theory as the standard case of AdS_5 supergravity and $SYM_4(\mathcal{N}{=}4)$, supersymmetry permits geometric arguments based on representation theory that support AdS/CFT correspondence and these arguments look quite convincing indeed.

But in models of the type of higher spin gauge fields (HS(d+1)) there is no supersymmetry a priori and the correspondence can be proved only by dynamical calculations both in AdS_{d+1} and CFT_d cases. Since in these models perturbative expansions with small coupling constants are mapped on each other, such calculations are technically feasible and the holographic mapping is order by order.We shall start such calculation for $HS(4)$ and the 3-dimensional conformal $O(N)$ sigma model now.

We concentrate on three-point function of two scalar and one higher spin field

	AdS_4	CFT_3
Scalar	$\sigma(z)$	$\alpha(x)$
HSF	$h^{(\ell)}(z)$	$\mathcal{J}^{(\ell)}(x)$

where $\alpha(x)$ is the "auxiliary" or "Lagrange multiplier" field and $\mathcal{J}^{(\ell)}(x)$ an almost conserved current, which is a traceless symmetric tensor. In the sigma model case the coupling constant is $O(\frac{1}{\sqrt{N}})$. In the higher spin field theory the coupling constant for $\sigma\sigma h^{(\ell)}$ interaction is g_ℓ, so that we expect

$$g_\ell = C^{(\ell)} \frac{1}{\sqrt{N}} \ . \tag{9.1}$$

We determine $C^{(\ell)}$ first in an ad hoc wave function normalization such that

$$\langle\alpha(x)\,\alpha(0)\rangle_{CFT} = \left(x^2\right)^{-\beta} \tag{9.2}$$

$$\langle\sigma(z_1)\,\sigma(z_2)\rangle_{AdS} = (2\zeta)^{-\beta}\, F\left[\begin{matrix}\frac{1}{2}\beta, \frac{1}{2}(\beta+1) \\ \beta-\mu+1\end{matrix}\,;\ \zeta^{-2}\right] \tag{9.3}$$

$$\zeta = \frac{(z_1^0)^2 + z_2^0)^2 + (\vec{z}_1 - \vec{z}_2)^2}{2z_1^0 z_2^0} \quad , \quad \mu = \frac{1}{2}d\,, \tag{9.4}$$

so that (9.2) is obtained from (9.3) by a "simple" boundary limit

$$\lim_{z_1^0\to 0}\lim_{z_2^0\to 0}\left(z_1^0 z_2^0\right)^{-\beta}\langle\sigma(z_1)\,\sigma(z_2)\rangle_{AdS} = \langle\alpha(\vec{z}_1)\,\alpha(\vec{z}_2)\rangle_{CFT}\,. \tag{9.5}$$

The higher spin fields are assumed to be normalized in the same fashion. At the end we renormalize the higher spin field such that $C^{(\ell)}$ is replaced by one.

We shall treat two versions of the minimal $O(N)$ sigma model. In the "free" case we have as a scalar field

$$\alpha_f(x) = \frac{1}{\sqrt{2N}}\phi_i(x)\;\phi_i(x)\,, \tag{9.6}$$

where $\phi_i(x), i = 1, 2, \ldots N$ is the $O(N)$ vector and space-time scalar field normalized so that

$$\langle\phi_i(x)\,\phi_j(x)\rangle_{CFT} = \left(x^2\right)^{-\delta}\delta_{ij} \quad , \quad \delta = \mu - 1 \tag{9.7}$$

and (9.2) follows from (9.7) and (9.6) with

$$\beta_f = 2(\mu-1) = d-2\,. \tag{9.8}$$

In the "interacting" sigma model we have an interaction

$$\mathbf{z}^{1/2}\int dx\,\phi_i(x)\;\phi_i(x)\,\alpha(x) \tag{9.9}$$

and the interaction constant $\mathbf{z}$ is expanded

$$\mathbf{z} = \sum_{k=1}^{\infty}\frac{\mathbf{z}_k}{N^k}\,. \tag{9.10}$$

The "free" theory is unstable and by renormalization flow approaches the stable "interacting" theory. The conformal scalar field $\sigma(z)$ on AdS_{d+1} is massive (tachyonic due to conformal coupling with the AdS metric) and has two boundary values from the two roots of the dimension formula

$$\Delta = \mu \pm \left(\mu^2 + m^2\right)^{\frac{1}{2}} \ , \tag{9.11}$$

where for $d = 3$

$$m^2 = \begin{cases} -2 \quad \text{in the free case} \\ -2 + O(\frac{1}{N}) \quad \text{in the interacting case} \end{cases} \tag{9.12}$$

so that

$$\Delta(d=3) = \begin{cases} \ \beta_f = 1 \quad \text{from (9.8)} \\ \beta = 2 + O(\frac{1}{N}) \quad \text{from (9.9)} \ . \end{cases} \tag{9.13}$$

We assume that the interaction of a spin ℓ gauge field $h^{(\ell)}$ and two scalar fields $\sigma(z)$ is local and mediated by a current $\Psi^{(\ell)}$

$$\int \frac{dz}{(z^0)^{d+1}} Tr\left\{\Psi^{(\ell)}(z)\, h^{(\ell)}(z)\right\} \ . \tag{9.14}$$

$\Psi^{(\ell)}$ and $h^{(\ell)}$ are symmetric tensors of rank ℓ. If we postulate that the covariant divergence of $\Psi^{(\ell)}$ is a trace term, the interaction is gauge invariant. Namely a gauge transformation of $h^{(\ell)}$, being of the form (classical)

$$h^{(\ell)} \rightarrow h^{(\ell)} + \nabla\Lambda^{(\ell-1)} \ , \tag{9.15}$$

where $\Lambda^{(\ell-1)}$ is a symmetric traceless tensor and $\nabla\Lambda^{(\ell-1)}$ is symmetrized, leads to the zero gauge variation of (9.14)

$$Tr\left\{\nabla\Psi^{(\ell)}\Lambda^{(\ell-1)}\right\} = 0 \ . \tag{9.16}$$

This consideration is in agreement with the so called "Fronsdal" theory [24] of higher spin with double-traceless gauge fields and currents. Truncation of this higher spin theory to the *conformal higher spin theory* can be observed if we consider the corresponding double-traceless current and gauge field as a sum of two traceless objects, namely

$$\Psi^{(\ell)} = J^{(\ell)} + g^{(2)}\psi^{(\ell-2)} \ , \tag{9.17}$$

where $J^{(\ell)}$ and $\psi^{(\ell-2)}$ are now the *traceless* tensors, $g^{(2)}$ is the $D = d+1$ dimensional AdS metric and symmetrization is assumed. Following [27] we call $\psi^{(\ell-2)}$ the compensator field. It is easy to see that $\psi^{(\ell-2)}$ plays the role of the traceless trace of the double-traceless current $\Psi^{(\ell)}$ and has to decouple in the conformal limit of higher spin theory. In other words we will assume that at the d dimensional boundary $M_d = \partial AdS_D$, Ψ^ℓ behaves as a conformal tensor field. Now we will consider the general structure of conformal higher spin currents in the AdS_D space constructed from the conformally coupled scalar field $\sigma(z)$ with the corresponding on-shell condition

$$\Box\sigma(z) = \nabla \cdot \nabla\sigma(z) = \frac{D(D-2)}{4L^2}\sigma(z) \ . \tag{9.18}$$

The tachyonic mass here (we use in this section the mainly minus signature of the AdS metric [1]) arises as a result of conformal coupling of the conformal scalar $\sigma(z)$ with the AdS curvature $S = \int d^D z\sqrt{-g}\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{D-2}{4(D-1)}R\sigma^2\right)$. For the investigation of the conservation and tracelessness conditions for general spin ℓ symmetric conformal current $J^{(\ell)}_{\mu_1\mu_2\ldots\mu_\ell}$ we contract it with the ℓ-fold tensor product of a vector a^μ and make

[1]We will use AdS conformal flat metric, curvature and covariant derivatives comutation rules of the type (similar to Euclidian Ads matric (1.15)-(1.20))

$$\begin{aligned}
&ds^2 = g_{\mu\nu}dz^\mu dz^\nu = \frac{L^2}{(z^0)^2}\eta_{\mu\nu}dz^\mu dz^\nu, \quad \eta_{z^0z^0} = -1, \sqrt{-g} = \frac{1}{(z^0)^{d+1}} \ , \\
&[\nabla_\mu, \nabla_\nu]\, V^\rho_\lambda = R_{\mu\nu\sigma}{}^\rho V^\sigma_\lambda - R_{\mu\nu\lambda}{}^\sigma V^\rho_\sigma \ , \\
&R_{\mu\nu\lambda}{}^\rho = -\frac{1}{(z^0)^2}\left(\eta_{\mu\lambda}\delta^\rho_\nu - \eta_{\nu\lambda}\delta^\rho_\mu\right) = -\frac{1}{L^2}\left(g_{\mu\lambda}\delta^\rho_\nu - g_{\nu\lambda}\delta^\rho_\mu\right) \ , \\
&R_{\mu\nu} = -\frac{D-1}{(z^0)^2}\eta_{\mu\nu} = -\frac{D-1}{L^2}g_{\mu\nu} \quad , \quad R = -\frac{D(D-1)}{L^2} \ .
\end{aligned} \tag{9.19}$$

the ansatz including a first curvature correction in contrast to the free flat case [26]

$$
\begin{aligned}
J^{(\ell)}(z;a) &= \frac{1}{2}\sum_{p=0}^{\ell} A_p\,(a\nabla)^{\ell-p}\,\sigma(z)\,(a\nabla)^p\,\sigma(z) \\
&+\frac{a^2}{2}\sum_{p=1}^{\ell-1} B_p\,(a\nabla)^{\ell-p-1}\,\nabla_\mu\sigma(z)\,(a\nabla)^{p-1}\,\nabla^\mu\sigma(z) \\
&+\frac{a^2}{2L^2}\sum_{p=1}^{\ell-1} C_p\,(a\nabla)^{\ell-p-1}\,\sigma(z)\,(a\nabla)^{p-1}\,\sigma(z) + O(a^4) + O(\frac{1}{L^4})\ ,
\end{aligned}
\tag{9.20}
$$

where $A_p = A_{\ell-p}, B_p = B_{\ell-p}, C_p = C_{\ell-p}$ and $A_0 = 1$. Now we try to define the set of unknown constants A_p, B_p and C_p using the current conservation condition

$$
\nabla\cdot\partial_a J^{(\ell)}(z;a) = \nabla^\mu\frac{\partial}{\partial a^\mu}J^{(\ell)}(z;a) = 0 \tag{9.21}
$$

and the tracelessness condition connected with the conformal nature of our scalar field $\sigma(z)$

$$
\Box_a J^{(\ell)}(z;a) = \frac{\partial^2}{\partial a_\mu\partial a^\mu}J^{(\ell)}(z;a) = 0\ . \tag{9.22}
$$

Using the following basic relations

$$
[\nabla_\mu,(a\nabla)^p]\,\sigma = \frac{p(p-1)}{2L^2}\left(a_\mu\,(a\nabla)^{p-1}\,\sigma - a^2\,(a\nabla)^{p-2}\,\nabla_\mu\sigma\right)\ , \tag{9.23}
$$

$$
\begin{aligned}
[\nabla_\mu,(a\nabla)^p]\,\nabla_\nu\sigma &= \frac{p(p-1)}{2L^2}\left(a_\mu\,(a\nabla)^{p-1}\,\nabla_\nu\sigma - a^2\,(a\nabla)^{p-2}\,\nabla_\mu\nabla_\nu\sigma\right) \\
&+\frac{p}{L^2}\left(g_{\mu\nu}\,(a\nabla)^p\,\sigma - a_\nu\left(a\nabla^{p-1}\nabla_\mu\sigma\right)\right)\ ,
\end{aligned}
\tag{9.24}
$$

$$
\begin{aligned}
\frac{\partial}{\partial a^\mu}(a\nabla)^p\,\sigma &= p\,(a\nabla)^{p-1}\,\nabla_\mu\sigma \\
&+\frac{p(p-1)(p-2)}{6L^2}\left(a_\mu\,(a\nabla)^{p-2}\,\sigma - a^2\,(a\nabla)^{p-3}\,\nabla_\mu\sigma\right)\ ,
\end{aligned}
\tag{9.25}
$$

$$
\begin{aligned}
\nabla\cdot\frac{\partial}{\partial a}(a\nabla)^p\,\sigma &= \frac{1}{L^2}\Bigg[\frac{1}{4}pD(D-2) \\
&+p(p-1)\left(D+\frac{2}{3}p-\frac{7}{3}\right)\Bigg](a\nabla)^{p-1}\,\sigma + O(\frac{1}{L^4})\ ,
\end{aligned}
\tag{9.26}
$$

$$
\begin{aligned}
\Box_a\,(a\nabla)^p\,\sigma &= \frac{1}{L^2}\Bigg[\frac{1}{4}p(p-1)D(D-2) \\
&+\frac{1}{3}p(p-1)(p-2)(p+2D-5)\Bigg](a\nabla)^{p-2}\,\sigma + O(\frac{1}{L^4})
\end{aligned}
\tag{9.27}
$$

we can derive recursion relations for A_p, B_p and C_p coming from conservation condition

(9.21)

$$pA_p + (\ell - p + 1)A_{p-1} + 2B_p + 2B_{p-1} = 0, \tag{9.28}$$

$$s_3(p)A_{p+1} + s_2(p, \ell, D)A_p + s_2(\ell - p + 1, \ell, D)A_{p-1}$$

$$+s_3(\ell - p + 1)A_{p-2} + 2C_p + 2C_{p-1} = 0, \tag{9.29}$$

$$s_2(p, \ell, D) = \frac{1}{4}pD(D-2) + p(p-1)(D + \frac{1}{2}\ell + \frac{1}{6}p - \frac{7}{3}), \tag{9.30}$$

$$s_3(p) = \frac{1}{6}(p+1)p(p-1)\ . \tag{9.31}$$

The relation (9.28) relates A_p and B_p recursively as in the flat case [26]. The next relation (9.29) arises from the $\frac{1}{L^2}$ correction and relates recursively C_p and A_p coefficients from our ansatz (9.20). From the other side the tracelessness condition (9.22) gives us two further relations between these coefficients

$$B_p = -\frac{p(\ell - p)}{(D + 2\ell - 4)}A_p, \tag{9.32}$$

$$C_p = \frac{-1}{2(D + 2\ell - 4)}\left[s_t(p+1, \ell, D)A_{p+1} + s_t(\ell - p + 1, \ell, D)A_{p-1}\right], \tag{9.33}$$

$$s_t(p, \ell, D) = \frac{1}{4}p(p-1)D(D-2) + \frac{1}{3}p(p-1)(p-2)(\ell + 2D - 5)\ . \tag{9.34}$$

Again the relation (9.32) is the same as in the flat case and leads the Eq. (9.28) to the recursion

$$A_p = -s_1(p, \ell, D)A_{p-1}, \tag{9.35}$$

$$s_1(p, \ell, D) = \frac{(\ell - p + 1)(2\ell - 2p + D - 2)}{p(D + 2p - 4)}\ . \tag{9.36}$$

From this we can obtain the same solution for the A_p coefficients [26] as in the flat case

$$A_p = (-1)^p \frac{\binom{\ell}{p}\binom{\ell+D-4}{p+\frac{D}{2}-2}}{\binom{\ell+D-4}{\frac{D}{2}-2}}\ . \tag{9.37}$$

For the important case $D = 4$ this formula simplifies to

$$A_p = (-1)^p \binom{\ell}{p}^2\ . \tag{9.38}$$

It means that if our ansatz (9.20) and our consideration for the $\frac{1}{L^2}$ correction are right, the recursion relation for the A_p coefficients obtained by substituting the C_p coefficients in (9.29) by those of the $\frac{1}{L^2}$ tracelessness condition (9.33) must be consistent with (9.35). Indeed using (9.33) and (9.35) we can rewrite the relation (9.29) in the form

$$(9.29) = A_p s_f(p,\ell,D) + A_{p-1} s_f(\ell-p+1,\ell,D) = 0 \tag{9.39}$$

$$\begin{aligned} s_f(p,\ell,D) &= \Bigg[s_2(p,\ell,D) - \frac{s_t(p,\ell,D)}{D+2\ell-4} \\ &- s_1(p+1,\ell,D)\left(s_3(p) - \frac{s_t(p+1,\ell,D)}{D+2\ell-4} \right) \Bigg] \\ &= \frac{(\ell+D-3)(2\ell+D-2)p(D+2p-4)}{4(D+2\ell-4)} \; . \end{aligned} \tag{9.40}$$

It is easy to see that the relation (9.39) coincides with (9.35) because

$$\frac{s_f(p,\ell,D)}{s_f(\ell-p+1,\ell,D)} = s_1(p,\ell,d) \; . \tag{9.41}$$

So we obtain a result that the structure of the conformal higher spin currents constructed from the conformal coupled scalar field in the fixed AdS background remains the same as in the free flat space case. We prove that our ansatz with $\frac{1}{L^2}$ correction connected with the difference between the traces in flat and AdS case does not violate the conservation condition (recursion relation (9.35)) for the coefficients A_p if they obey the tracelessness condition (9.33) for the currents. It means that the traceless conserved higher spin current constructed from conformal scalar field in AdS can be obtained from the flat space expression replacing usual derivatives with covariant ones and adding corresponding curvature corrections to the expression for the traces. For completeness we present in the Appendix an explicit derivation of the conformal conserved current in the case $\ell = 4, D = 4$ in all orders of $\frac{1}{L^2}$.

This phenomenon we can explain now in the following way: The conformal group for D- dimensional flat(with $SO(D-1,1)$ isometry) and AdS space (with $SO(D-1,2)$ isometry) is the same -$SO(D,2)$ [2]. So we can say that the conformal primaries or the traceless conserved currents are the same due to the $\frac{1}{L^{2k}}$ corrections. But these originate from the curvature corrections to the flat space equation of motion and non-

[2] Note that this is about conformal group of AdS space-not boundary

commutativeness of the covariant derivatives. Then because all currents are *traceless* we get the cancellation of all $\frac{1}{L^{2k}}$ accompanying terms coming from these two sources of deformation of the flat case relations in the conservation condition (9.21).

Now we will fix the coefficients A_p from the CFT consideration. We assume that on the boundary ∂AdS_{d+1}, $\Psi^{(\ell)}$ behaves as a conformal tensor field (the trace is decoupled). Moreover this conformal tensor must be local bilinear in $\alpha(x)$ of rank ℓ and of dimension

$$2\beta + \ell + O(\frac{1}{N}) \ . \tag{9.42}$$

For this purpose we evaluate the 3-point function

$$\langle \alpha(x_1)\, \alpha(x_2) \frac{1}{2} \sum_{p=0}^{\ell} A_p \langle a \cdot \partial \rangle^p \alpha(x_3)\, \langle a \cdot \partial \rangle^{\ell - p} \alpha(x_3) \rangle_{CFT_3} \ , \tag{9.43}$$

where $\langle a \cdot \partial \rangle = a^i \partial_i, \ i = 1, 2, 3$.

From the propagator (9.2) for $\alpha(x)$ we obtain for (9.43)

$$2^\ell \sum_{p=0}^{\ell} A_p (\beta)_p (\beta)_{\ell-p} \frac{\left(x_{13}^2 x_{23}^2\right)^{-\beta}}{x_{13}^{2p} x_{23}^{2(\ell-p)}} \langle a \cdot x_{13} \rangle^p \langle a \cdot x_{23} \rangle^{(\ell - p)} \tag{9.44}$$
$$+ \quad \text{trace terms} \ ,$$

where we define the Pochhammer symbols $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$.

As a 3-point function of a conformal tensor is unique up to normalization

$$\mathcal{C} \left(x_{13}^2 x_{23}^2\right)^{-\beta} \left\{ \langle a \cdot \xi \rangle^\ell + \quad \text{trace terms} \right\} , \tag{9.45}$$

$$\xi^i = \frac{x_{13}^i}{x_{13}^2} - \frac{x_{23}^i}{x_{23}^2} \ , \tag{9.46}$$

it follows

$$A_p = \frac{\mathcal{C} (-1)^p \binom{\ell}{p}}{2^\ell (\beta)_p (\beta)_{\ell-p}} \ . \tag{9.47}$$

This expression, for $\beta = 1$, is in agreement with the previous one (9.38) obtained from AdS_4 consideration, if we will normalize in (9.45) $\mathcal{C} = 2^\ell \ell!$.

For $\beta = 2$ we have to change the constraints imposed on our currents. For that we turn from conformal higher spins to Fronsdal's [24] formulation where gauge fields and currents are double traceless only

$$S^{(\ell)}_{int} = \frac{1}{\ell}\int d^4x\sqrt{g}h^{(\ell)\mu_1\dots\mu_\ell}\Psi^{(\ell)}_{\mu_1\dots\mu_\ell}, \tag{9.48}$$

$$h^{(\ell)\alpha\beta}_{\alpha\beta\mu_5\dots\mu_\ell} = 0 \quad , \quad \Psi^{(\ell)\alpha\beta}_{\alpha\beta\mu_5\dots\mu_\ell} = 0, \tag{9.49}$$

$$\delta_0 h^{(\ell)}_{\mu_1\dots\mu_\ell} = \partial_{(\mu_1}\epsilon_{\mu_2\dots\mu_\ell)}, \quad \epsilon^{\alpha}_{\alpha\mu_4\dots\mu_\ell} = 0, \quad \left[\nabla^{\mu_1}\Psi^{(\ell)}_{\mu_1\mu_2\dots\mu_\ell}\right]^{traceless} = 0 \tag{9.50}$$

and the conservation condition looks a little bit different from the usual one due to the double-tracelessness of the gauge field and current. Then we can realize the double-traceless current $\Psi^{(\ell)}$ using two traceless (but not conserved) currents $J^{(\ell)}$, $\Theta^{(\ell-2)}$ with the same dimension $\ell+2\beta+O(\frac{1}{N})$ on the boundary [27]. It means that the expansions for these fields start from the following series

$$J^{(\ell)}(z;a) \;=\; \frac{1}{2}\sum_{p=0}^{\ell} A^{\ell}_p \,(a\nabla)^{\ell-p}\,\phi(z)\,(a\nabla)^p\,\phi(z)+\dots, \tag{9.51}$$

$$\Theta^{(\ell-2)}(z;a) \;=\; \frac{1}{2}\sum_{p=1}^{\ell-1} B^{\ell-2}_p\,(a\nabla)^{\ell-1-p}\,\nabla_\mu\phi(z)\,(a\nabla)^{p-1}\,\nabla^\mu\phi(z)+\dots. \tag{9.52}$$

The Fronsdal field $\Psi^{(\ell)}$ we can present then as

$$\Psi^{(\ell)}(z;a) = J^{(\ell)}(z;a) + \frac{a^2}{2(D+2\ell-4)}\Theta^{(\ell-2)}(z;a), \tag{9.53}$$

$$Tr\Psi^{(\ell)}(z;a) = \Box_a\Psi^{(\ell)}(z;a) = \Theta^{(\ell-2)}(z;a) \tag{9.54}$$

The conservation condition (9.50) in this representation is

$$\nabla^\mu\frac{\partial}{\partial a^\mu}\Psi^{(\ell)}(z;a) = \frac{a^2}{2(D+2\ell-6)}Tr\nabla^\mu\frac{\partial}{\partial a^\mu}\Psi^{(\ell)}(z;a) \tag{9.55}$$

or

$$\nabla^\mu\frac{\partial}{\partial a^\mu}J^{(\ell)}(z;a) + \frac{(a\nabla)\Theta^{(\ell-2)}(z;a)}{(D+2\ell-4)} = \frac{a^2\nabla^\mu\frac{\partial}{\partial a^\mu}\Theta^{(\ell-2)}(z;a)}{(D+2\ell-6)(D+2\ell-4)}. \tag{9.56}$$

From this we can read off a restriction on the coefficients in (9.51) and (9.52)

$$p(D+2p-4)A^{\ell}_p + (\ell-p+1)(D+2\ell-2p-2)A^{\ell}_{p-1} + B^{\ell-2}_p + B^{\ell-2}_{p-1} = 0. \tag{9.57}$$

For $D=4$ we get

$$2p^2A^{\ell}_p + 2(\ell-p+1)^2A^{\ell}_{p-1} + B^{\ell-2}_p + B^{\ell-2}_{p-1} = 0. \tag{9.58}$$

Then after using (9.47) for $\beta = 2$ we obtain

$$B^{\ell-2}_p + B^{\ell-2}_{p-1} = \frac{\mathcal{C}^\ell \ell!}{2^{\ell-1}} \frac{(-1)^p(\ell - 2p + 1)}{(p-1)!(p+1)!(\ell-p)!(\ell-p+2)!} \quad . \tag{9.59}$$

The solution of this equation fulfilling the boundary conditions

$$B^{\ell-2}_0 = B^{\ell-2}_\ell = 0 \tag{9.60}$$

is

$$B^{\ell-2}_p = \frac{\mathcal{C}^\ell \ell!}{2^{\ell-1}}(-1)^p \sum_{k=1}^{p} \frac{(\ell - 2k + 1)}{(k-1)!(k+1)!(\ell-k)!(\ell-k+2)!} \quad . \tag{9.61}$$

The latter sum can be proceeded using Pascal's formula for binomials. The result is very elegant

$$B^{\ell-2}_p = \frac{\mathcal{C}^\ell (-1)^p}{2^{\ell-1}(\ell+1)!} \binom{\ell}{p-1} \binom{\ell}{p+1}. \tag{9.62}$$

So we show that in contrast to the $\beta = 1$ case where the interaction includes the traceless conformal higher spin currents, the $\beta = 2$ boundary condition necessitates the interaction with the double trace higher spin currents. The connection between these two types of interaction can be described adding local Weyl (in the spin two case) and generalized "Weyl" invariants realizing the conformal coupling of the scalar with the higher spin fields.

Now as **Exercise** We will construct directly the traceless fourth rank tensor constructed from four dimensional on-shell scalar field $\sigma(z^\mu)$ in the following way

$$\begin{aligned} T^{\text{traceless}}_{\mu\nu\lambda\rho} &= T_{\mu\nu\lambda\rho} - \frac{3}{8}\left(g_{\mu(\nu}T_{\lambda\rho)} + T_{\mu(\nu}g_{\lambda\rho)}\right) + \frac{1}{16} g_{\mu(\nu}g_{\lambda\rho)}T \ , \\ T_{\mu\nu} &= T^\alpha_{\alpha\mu\nu} \quad , \quad T = T^{\mu\nu}_{\mu\nu} \ . \end{aligned} \tag{9.63}$$

The conservation law which we will check below is

$$\nabla^\mu T^{\text{traceless}}_{\mu\nu\lambda\rho} = \nabla^\mu T_{\mu\nu\lambda\rho} - \frac{3}{8}\left(\nabla_{(\nu}T_{\lambda\rho)} + \nabla^\mu T_{\mu(\nu}g_{\lambda\rho)}\right) + \frac{1}{16} g_{(\nu\lambda}\nabla_{\rho)}T = 0 \ . \tag{9.64}$$

Finally we list here the most important on-shell relations (some of them are due to

$\Box\sigma(z) = \frac{2}{L^2}\sigma(z))$ we will use

$$[\Box, \nabla_\mu]\,\sigma(z) = \frac{3}{L^2}\nabla_\mu\sigma(z)\ , \tag{9.65}$$

$$\left[\nabla_\mu, \nabla^3_{(\nu\lambda\rho)}\right]\sigma(z) = \frac{3}{L^2}g_{\mu(\nu}\nabla^2_{\lambda\rho)}\sigma(z) - \frac{3}{L^2}g_{(\nu\lambda}\nabla_{\rho)}\nabla_\mu\sigma(z)\ , \tag{9.66}$$

$$\nabla^3_{(\mu\lambda\rho)}\sigma(z) = \nabla^2_{(\lambda\rho)}\nabla_\mu\sigma + \frac{1}{3L^2}g_{\mu(\rho}\nabla_{\lambda)}\sigma(z) - \frac{1}{L^2}g_{\lambda\rho}\nabla_\mu\sigma(z)\ , \tag{9.67}$$

$$\nabla^\mu\nabla^2_{(\mu\lambda\rho)}\sigma(z) = \frac{28}{3L^2}\nabla^2_{(\lambda\rho)}\sigma(z) - \frac{8}{3l^4}g_{\lambda\rho}\sigma(z)\ , \tag{9.68}$$

$$\left[\nabla_\mu, \nabla^3_{(\nu\lambda\rho)}\right]\nabla^\mu\sigma(z) = \frac{12}{L^2}\nabla^3_{(\nu\lambda\rho)}\sigma(z) - \frac{9}{L^4}g_{(\nu\lambda}\nabla_{\rho)}\sigma(z)\ , \tag{9.69}$$

$$g^{\lambda\rho}\nabla^3_{(\mu\lambda\rho)}\sigma(z) = \frac{4}{L^2}\nabla_\rho\sigma(z)\ , \tag{9.70}$$

$$g^{\lambda\rho}\nabla^4_{(\mu\nu\lambda\rho)}\sigma(z) = \frac{20}{3L^2}\nabla^2_{(\mu\nu)}\sigma(z) - \frac{4}{3L^4}\sigma(z)\ . \tag{9.71}$$

Now we can construct directly the conserved spin 4 traceless current. First of all we note that from four derivatives we can construct only three bilinear combinations

$$T^{0,4}_{\mu\nu\lambda\rho} = \sigma\nabla_{(\mu}\nabla_\nu\nabla_\lambda\nabla_{\rho)}\sigma\ , \tag{9.72}$$

$$T^{1,3}_{\mu\nu\lambda\rho} = \nabla_{(\mu}\sigma\nabla_\nu\nabla_\lambda\nabla_{\rho)}\sigma\ , \tag{9.73}$$

$$T^{2,2}_{\mu\nu\lambda\rho} = \nabla_{(\mu}\nabla_\nu\sigma\nabla_\lambda\nabla_{\rho)}\sigma\ . \tag{9.74}$$

For constructing the conserved (on-shell) combination of the traceless parts of these tensors we need first of all the on-shell value of their first and second traces

$$T^{0,4}_{\lambda\rho} = \frac{20}{3l^2}\sigma(z)\nabla^2_{(\mu\nu)}\sigma(z) - \frac{4}{3L^4}\sigma^2(z) \quad , \quad T^{0,4} = \frac{8}{L^4}\sigma^2(z)\ , \tag{9.75}$$

$$T^{1,3}_{\lambda\rho} = \frac{1}{2}\nabla^\mu\sigma\nabla^2_{(\lambda\rho)}\nabla_\mu\sigma + \frac{13}{6L^2}\nabla_\lambda\sigma\nabla_\rho\sigma - \frac{1}{6L^2}g_{\lambda\rho}\sigma\nabla_\mu\sigma, \quad T^{1,3} = \frac{4}{L^2}\nabla^\mu\sigma\nabla_\mu\sigma\ , \tag{9.76}$$

$$T^{2,2}_{\lambda\rho} = \frac{2}{3}\nabla_\lambda\nabla^\mu\sigma\nabla_\rho\nabla_\mu\sigma + \frac{2}{3L^2}\sigma\nabla^2_{(\lambda\rho)}\sigma, \quad T^{2,2} = \frac{2}{3}\nabla^{2(\mu\nu)}\sigma\nabla^2_{(\mu\nu)}\sigma + \frac{4}{3L^4}\sigma^2\ . \tag{9.77}$$

Then introducing the following third rank symmetric tensor bilinear terms

$$\mathbf{A} = \nabla_{(\nu}\nabla^\mu\sigma\nabla^2_{\lambda\rho)}\nabla_\mu\sigma, \quad \mathbf{a} = \nabla_{(\nu}\sigma\nabla^2_{\lambda\rho)}\sigma\ , \tag{9.78}$$

$$\mathbf{B} = g_{(\nu\lambda}\nabla_{\rho)}\left(\nabla^{2(\mu\nu)}\sigma\nabla^2_{(\mu\nu)}\sigma\right), \quad \mathbf{b} = g_{(\nu\lambda}\nabla_{\rho)}\left(\nabla^\mu\sigma\nabla_\mu\sigma\right)\ , \tag{9.79}$$

$$\mathbf{C} = \nabla^\mu\sigma\nabla^3_{(\nu\lambda\rho)}\nabla_\mu\sigma, \quad \mathbf{c} = \sigma\nabla^3_{(\nu\lambda\rho)}\sigma, \quad \mathbf{d} = g_{(\nu\lambda}\nabla_{\rho)}\left(\sigma^2\right) \tag{9.80}$$

and using (9.63)-(9.71), we obtain the following on-shell relations

$$\nabla^\mu T^{2,2\text{traceless}}_{\mu\nu\lambda\rho} = \frac{1}{2}\mathbf{A} - \frac{1}{12}\mathbf{B} + \frac{23}{4L^2}\mathbf{a} - \frac{9}{8L^2}\mathbf{b} - \frac{1}{4L^2}\mathbf{c} - \frac{19}{24L^4}\mathbf{d}\ , \tag{9.81}$$

$$\nabla^\mu T^{1,3\text{traceless}}_{\mu\nu\lambda\rho} = \frac{9}{16}\mathbf{A} - \frac{3}{32}\mathbf{B} + \frac{1}{16}\mathbf{C} + \frac{51}{8L^2}\mathbf{a} - \frac{11}{8L^2}\mathbf{b} + \frac{1}{2L^2}\mathbf{c} - \frac{13}{8L^4}\mathbf{d}\ , \tag{9.82}$$

$$\nabla^\mu T^{0,4\text{traceless}}_{\mu\nu\lambda\rho} = \mathbf{C} - \frac{3}{2L^2}\mathbf{a} - \frac{7}{4L^2}\mathbf{b} + \frac{25}{2L^2}\mathbf{c} - \frac{47}{4L^4}\mathbf{d}\ . \tag{9.83}$$

Now we can see that the following unique combination of (9.72)-(9.74) is conserved

$$T^{s=4,\text{traceless}}_{\mu\nu\lambda\rho} = T^{2,2\text{traceless}}_{\mu\nu\lambda\rho} - \frac{8}{9}T^{1,3\text{traceless}}_{\mu\nu\lambda\rho} + \frac{1}{18}T^{0,4\text{traceless}}_{\mu\nu\lambda\rho}\ , \tag{9.84}$$

$$\nabla^\mu T^{s=4,\text{traceless}}_{\mu\nu\lambda\rho} = 0\ . \tag{9.85}$$

The expression (9.84) for the current is again in agreement with the flat space case general formula after a replacement of ordinary derivatives by covariant ones (compare the coefficients in (9.84) with the solution (9.38) and overall factor $\frac{1}{36}$).

10 Spin two and four currents interaction with gauge field

The action for the conformally coupled scalar field in D dimensions in external gravity is

$$S = \frac{1}{2}\int d^D z\sqrt{-G}\left[G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{(D-2)}{4(D-1)}R(G)\phi^2\right]. \tag{10.1}$$

In this section we restore the linearized form of this action in fixed AdS background using a gauging procedure both for the gauge and Weyl symmetry on the linearized level. We do this derivation just for methodical reasons because the final nonlinear answer is known (10.1). But we would like to extend this consideration to the higher spin case and try to elaborate a linearized construction which works in the case $\ell = 4$ where the final answer is unknown.

We start from the massive free scalar action in the fixed AdS external metric

$$S_0(\phi) = \frac{1}{2}\int d^D z\sqrt{-g}\left[\nabla_\mu\phi\nabla^\mu\phi + \lambda\phi^2\right]. \tag{10.2}$$

For getting an interaction with linearized gravity using the gauging procedure we have to variate S_0 with respect to $\delta^1_\varepsilon\phi = \varepsilon^\mu(z)\nabla_\mu\phi$

$$\delta^1_\varepsilon S_0 \;=\; \int d^D z\sqrt{-g}\nabla^{(\mu}\varepsilon^{\nu)}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{g_{\mu\nu}}{2}\left(\nabla_\alpha\phi\nabla^\alpha\phi + \lambda\phi^2\right)\right] \tag{10.3}$$

and solving (we assume that ε^μ and $h^{\mu\nu}$ have the same infinitesimal order) the equation

$$\delta^1_\epsilon S_0(\phi) + \delta^0_\varepsilon S_1(\phi, h^{(2)}) = 0, \quad \delta^0_\varepsilon h^{(2)}_{\mu\nu} = 2\nabla_{(\mu}\varepsilon_{\nu)}, \tag{10.4}$$

we immediately find the following cubic interaction linear in the gauge field

$$S_1(\phi, h^{(2)}) = \frac{1}{2}\int d^D z\sqrt{-g}h^{(2)\mu\nu}\left[-\nabla_\mu\phi\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}\left(\nabla_\mu\phi\nabla^\mu\phi + \lambda\phi^2\right)\right]. \tag{10.5}$$

Note that here we used many times partial integration which means that we admit that all fields or at least parameters of symmetry are zero on the boundary, otherwise we would have to check all symmetries taking into account some boundary terms and their variations also. It is clear that for constructing the local interaction on the bulk we can use partial integrations without watching the boundary effects.

So we see that gauge invariance

$$\delta^1_\varepsilon \phi(z) = \varepsilon^\mu(z)\nabla_\mu\phi(z), \quad \delta^0_\varepsilon h^{(2)}_{\mu\nu}(z) = 2\nabla_{(\mu}\varepsilon_{\nu)}(z) \tag{10.6}$$

in this linear approach does not fix the free parameter λ and the corresponding spin two Noether current (energy-momentum tensor)

$$\Psi^{(2)}_{\mu\nu}(\phi,\lambda) = -\nabla_\mu\phi\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}\left(\nabla_\mu\phi\nabla^\mu\phi + \lambda\phi^2\right) \tag{10.7}$$

is conserved but not traceless. But we can fix this problem having noted that there is one more gauge invariant combination of two derivatives and one $h_{\mu\nu}$ field

$$r^{(2)}(h^{(2)}(z)) = \nabla_\mu\nabla_\nu h^{(2)\mu\nu} - \nabla^2 h^{(2)\mu}_\mu - \frac{D-1}{L^2}h^{(2)\mu}_\mu, \quad \delta^1_\varepsilon r^{(2)}(h^{(2)}) = 0. \tag{10.8}$$

It is of course the linearized Ricci scalar-but at this moment it is important for us that there is only one gauge invariant combination of $h^{(2)}_{\mu\nu}(z)$, two scalars $\phi(z)$ and two derivatives

$$\int d^D z\sqrt{g}r^{(2)}(h^{(2)})\phi^2, \tag{10.9}$$

which we can add to our linearized action with one more free parameter. So finally we can write the most general gauge invariant action in this approximation of the first order in the gauge field

$$\begin{aligned} S^{GI}(\lambda,\xi,\phi,h^{(2)}) &= \frac{1}{2}\int d^D z\sqrt{-g}\left[\nabla_\mu\phi\nabla^\mu\phi + \lambda\phi^2\right] \\ &+ \frac{1}{2}\int d^D z\sqrt{-g}h^{(2)\mu\nu}\left[-\nabla_\mu\phi\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}\left(\nabla_\mu\phi\nabla^\mu\phi + \lambda\phi^2\right)\right] \\ &+ \xi\int d^D z\sqrt{-g}\left[\nabla_\mu\nabla_\nu h^{(2)\mu\nu} - \nabla^2 h^{(2)\mu}_\mu - \frac{D-1}{L^2}h^{(2)\mu}_\mu\right]\phi^2. \end{aligned} \tag{10.10}$$

Then we search for the additional local symmetry permitting to remove the trace of the gauge field $h_{\mu\nu}$ and therefore leading to the traceless conformal spin two current.

The natural choice here is of course Weyl invariance and we will define local Weyl transformation in linear approximation in the following way

$$\delta^1_\sigma \phi(z) = \Delta\sigma(z)\phi(z), \quad \delta^0_\sigma h^{(2)}_{\mu\nu}(z) = 2\sigma(z) g_{\mu\nu}, \tag{10.11}$$

where Δ is the conformal weight (one more additional free parameter to fit) of the scalar field. The important point here is that when we impose on the gauge invariant action (10.10) conformal (Weyl) invariance (10.11) we obtain the condition

$$\frac{\delta}{\delta\sigma(z)} S^{GI}(\lambda,\xi,\phi,h^{(2)}) = \left[\Delta\lambda + \frac{\lambda D}{2} - \frac{2\xi D(D-1)}{L^2}\right]\sigma\phi^2 \tag{10.12}$$

$$+\left[\Delta - 1 + \frac{D}{2}\right]\sigma\nabla_\mu\phi\nabla^\mu\phi + \left[2\xi(1-D) - \frac{\Delta}{2}\right]\nabla^2\sigma\phi^2 = 0 \tag{10.13}$$

with the unique solution for all free constants

$$\Delta = 1 - \frac{D}{2}, \quad \xi = \frac{1}{8}\frac{D-2}{D-1}, \quad \lambda = \frac{D(D-2)}{4L^2}. \tag{10.14}$$

So finally we come to the gauge and conformal invariant action

$$S^{WI}(\phi, h_{\mu\nu}) = S_0(\phi) + S_1^{\Psi^{(2)}}(\phi, h^{(2)}) + S_1^{r^{(2)}}(\phi, h^{(2)}) \tag{10.15}$$

where

$$S_0(\phi) = \frac{1}{2}\int d^D z\sqrt{-g}\left[\nabla_\mu\phi\nabla^\mu\phi + \frac{D(D-2)}{4L^2}\phi^2\right], \tag{10.16}$$

$$S_1^{\Psi^{(2)}}(\phi, h^{(2)}) = \frac{1}{2}\int d^D z\sqrt{-g} h^{(2)\mu\nu}\left[-\nabla_\mu\phi\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}\left(\nabla_\mu\phi\nabla^\mu\phi + \frac{D(D-2)}{4L^2}\phi^2\right)\right], \tag{10.17}$$

$$S_1^{r^{(2)}}(\phi, h^{(2)}) = \frac{1}{8}\frac{D-2}{D-1}\int d^D z\sqrt{-g}\left[\nabla_\mu\nabla_\nu h^{(2)\mu\nu} - \nabla^2 h^{(2)\mu}_\mu - \frac{D-1}{L^2} h^{(2)\mu}_\mu\right]\phi^2, \tag{10.18}$$

which is of course the linearized action (10.1) and can be obtained from that after expansion near to the AdS_D background $G_{\mu\nu}(z) = g_{\mu\nu} + h^{(2)}_{\mu\nu}(z)$ in the first order on $h^{(\ell)}_{\mu\nu}$.

Now we turn to the spin four case.

We start from action (10.16) to apply Noether's method for the following transformation of the scalar field with a traceless third rank symmetric tensor parameter

$$\delta^1_\epsilon\phi = \epsilon^{\mu\nu\lambda}\nabla_\mu\nabla_\nu\nabla_\lambda\phi \quad , \quad \epsilon^\alpha_{\alpha\mu} = 0 \tag{10.19}$$

First of all we have to calculate $\delta_1 S_0$. For brevity we introduce the notation (and in a similar way for any other tensor)

$$\tilde{\epsilon}^{\mu\nu} = \nabla_\lambda \epsilon^{\lambda\mu\nu}, \quad \tilde{\tilde{\epsilon}}^{\mu} = \nabla_\nu \nabla_\lambda \epsilon^{\nu\lambda\mu} \tag{10.20}$$

Then after variation of (10.16) we obtain

$$\begin{aligned}\delta^1_\epsilon S_0(\phi) = \int dx^4 \sqrt{-g} \Big\{ &-\nabla^{(\alpha}\epsilon^{\mu\nu\lambda)}\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla_\lambda\phi + \frac{3}{2}\tilde{\epsilon}^{\nu\lambda}\nabla_\nu\nabla_\alpha\phi\nabla_\lambda\nabla^\alpha\phi \\ -\frac{1}{2}\tilde{\epsilon}^{\nu\lambda}\nabla^2\left(\nabla_\nu\phi\nabla_\lambda\phi\right) &+ \frac{1}{8L^2}\left[3D(D+2)-8\right]\tilde{\epsilon}^{\nu\lambda}\nabla_\nu\phi\nabla_\lambda\phi \\ -\nabla^{(\alpha}\tilde{\tilde{\epsilon}}^{\lambda)}\Big[-\nabla_\mu\phi\nabla_\nu\phi &+ \frac{g_{\mu\nu}}{2}\left(\nabla_\mu\phi\nabla^\mu\phi + \frac{D(D-2)}{4L^2}\phi^2\right)\Big]\Big\}\end{aligned} \tag{10.21}$$

We see that we can introduce an interaction with the spin four gauge field $h^{(4)}_{\mu\nu\alpha\beta}$ in the minimal way if we will deform the transformation law for the spin two field. The solution for the equation

$$\delta^1_\epsilon S_0(\phi) + \delta^0_\epsilon\left[S_1^{\Psi^{(2)}}(\phi, h^{(2)}) + S_1^{\Psi^{(4)}}(\phi, h^{(4)})\right] = 0 \tag{10.22}$$

is

$$\begin{aligned}S_1^{\Psi^{(4)}}(\phi, h^{(4)}) = \frac{1}{4}\int dx^4\sqrt{-g}\,\Big[&h^{(4)\mu\nu\alpha\beta}\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi - 3h^{(4)\alpha\mu\nu}_{\alpha}\nabla_\mu\nabla_\beta\phi\nabla_\nu\nabla^\beta\phi \\ &+ h^{(4)\alpha\mu\nu}_{\alpha}\nabla^2\left(\nabla_\mu\phi\nabla_\nu\phi\right) - \frac{3D(D+2)-8}{4L^2}h^{(4)\alpha\mu\nu}_{\alpha}\nabla_\mu\phi\nabla_\nu\phi\Big],\end{aligned} \tag{10.23}$$

$$\delta^0_\epsilon h^{(4)\mu\nu\alpha\beta} = 4\nabla^{(\mu}\epsilon^{\nu\alpha\beta)}, \quad \delta^1_\epsilon\phi = \epsilon^{\mu\nu\alpha}\nabla_\mu\nabla_\nu\nabla_\alpha\phi, \tag{10.24}$$

$$\delta^0_\epsilon h^{(4)\alpha\mu\nu}_{\alpha} = 2\tilde{\epsilon}^{\mu\nu}, \quad \delta^0_\epsilon h^{(2)\mu\nu} = 2\nabla^{(\mu}\tilde{\tilde{\epsilon}}^{\nu)}. \tag{10.25}$$

So we obtain the following gauged action with linearized interaction with both spin two and spin four gauge fields and linearized usual Weyl invariance

$$S^{GI}(\phi, h^{(2)}, h^{(4)}) = S^{WI}(\phi, h^{(2)}) + S_1^{\Psi^{(4)}}(\phi, h^{(4)}) \tag{10.26}$$

$$\delta^0 h^{(4)\mu\nu\lambda\alpha} = 4\nabla^{(\mu}\epsilon^{\nu\lambda\alpha)}, \quad \delta^0 h^{(2)\mu\nu} = 2\nabla^{(\mu}\varepsilon^{\nu)} + 2\nabla^{(\mu}\tilde{\tilde{\epsilon}}^{\nu)} + 2\sigma g_{\mu\nu} \tag{10.27}$$

$$\delta^1\phi = \varepsilon^\mu\nabla_\mu\phi + \epsilon^{\mu\nu\lambda}\nabla_\mu\nabla_\nu\nabla_\lambda\phi + (1-\frac{D}{2})\sigma\phi \tag{10.28}$$

where $S^{WI}(\phi, h^{(2)})$ can be read from (10.15)-(10.18) and we note that on this linearized level usual Weyl transformation does not affect the spin four part of the action but the spin four gauge transformation deforms the gauge transformation for spin two gauge field.

Now we turn to the construction of the conformal invariant coupling of the scalar field with the spin four gauge field in a similar way as in the case of spin two. For this we note first that here we can construct also the gauge invariant combination of two derivatives and $h^{(4)\mu\nu\alpha\beta}$. This is the following traceless symmetric second rank tensor

$$r^{(4)\alpha\beta} = \nabla_\mu \nabla_\nu h^{(4)\mu\nu\alpha\beta} - \nabla^2 h^{(4)\mu\alpha\beta}_{\mu} - \nabla^{(\alpha}\nabla_\nu h^{(4)\beta)\mu\nu}_{\mu} - \frac{3(D+1)}{L^2} h^{(4)\alpha\beta\mu}_{\mu}, \tag{10.29}$$

$$\delta^1_\epsilon r^{(4)\alpha\beta} = 0, \qquad r^{(4)\alpha}_{\alpha} = 0 \tag{10.30}$$

This is the analogue of the Ricci scalar in the spin four case and we can construct using this tensor *two* additional gauge invariant combinations of the same order.

$$S^{r^{(4)}}_1(\xi_1, \xi_2, \phi, h^{(4)}) = \xi_1 \int d^D z \sqrt{-g} r^{(4)\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \xi_2 \int d^D z \sqrt{-g} \nabla_\mu \nabla_\nu r^{(4)\mu\nu} \phi^2. \tag{10.31}$$

Then we can define the *generalized* "Weyl" transformation for the scalar and spin four gauge field with the second rank symmetric traceless parameter $\chi^{\mu\nu}(z)$

$$\delta^0_\chi h^{(4)\mu\nu\alpha\beta}(z) = 12\chi^{(\mu\nu}(z) g^{\alpha\beta)}, \quad \delta^1_\chi \phi(z) = \tilde{\Delta} \chi^{\alpha\beta}(z) \nabla_\alpha \nabla_\beta \phi(z), \tag{10.32}$$

where we introduced the "conformal" weight $\tilde{\Delta}$ for the scalar field. Computing the following χ variations

$$\delta^1_\chi S_0(\phi) + \delta^0_\chi S_1^{\Psi^{(4)}}(\phi, h^{(4)}) = \int \Big\{ (\tilde{\Delta} - 1)\nabla^{(\alpha}\tilde{\chi}^{\beta)}\Psi^{(2)}_{\alpha\beta}(\phi, \frac{D(D-2)}{4L^2})$$
$$-(\tilde{\Delta} + \frac{3D}{2} + 3)\chi^{\alpha\beta}\nabla_\alpha\nabla_\mu\phi\nabla_\beta\nabla^\mu\phi + \frac{\tilde{\Delta} + D + 3}{2}\nabla^2\chi^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$$
$$-\frac{1}{L^2}C(\tilde{\Delta}, D)\chi^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \frac{D(D-2)}{8L^2}\tilde{\tilde{\chi}}\phi^2 \Big\} \sqrt{-g}d^D z, \tag{10.33}$$

$$C(\tilde{\Delta}, D) = (\tilde{\Delta} - 1)(D-1) + \frac{\tilde{\Delta}}{4}D(D-2) + (D+4)(\frac{3D(D+2)}{8} - 1), \tag{10.34}$$

$$\delta^0_\chi S_1^{r^{(4)}}(\phi, h^{(4)}) = \xi_1 \int \Big[2D\nabla^{(\alpha}\tilde{\chi}^{\beta)}\Psi^{(2)}_{\alpha\beta}(\phi, \frac{D(D-2)}{4L^2}) - (D-2)\tilde{\tilde{\chi}}\nabla_\alpha\phi\nabla^\alpha\phi$$
$$-2(D+3)\nabla^2\chi^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - \frac{2}{L^2}(D+3)(3D+4)\chi^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi \Big] \sqrt{-g}d^D z$$
$$-\Big[\xi_1\frac{D^2(D-2)}{4L^2} + \xi_2\frac{12(D+1)(D+2)}{L^2}\Big]\int d^D z\sqrt{-g}\tilde{\tilde{\chi}}\phi^2$$
$$-\xi_2 4(D+1)\int d^D z\sqrt{-g}\nabla^2\tilde{\tilde{\chi}}\phi^2 \tag{10.35}$$

we see again that for obtaining a "Weyl" invariant interaction we have to deform the gauge and usual Weyl transformation of the spin two gauge field $h^{(2)}_{\mu\nu}$

$$\delta^0_\chi h^{(2)}_{\mu\nu} = 2(1 - \tilde{\Delta} - 2D\xi_1)\nabla^{(\mu}\tilde{\chi}^{\nu)} + 2\xi_1\tilde{\tilde{\chi}}g_{\mu\nu} \tag{10.36}$$

Then solving the symmetry condition

$$\delta^1_\chi S_0(\phi) + \delta^0_\chi \left(S_1^{\Psi^{(2)}}(\phi, h^{(2)}) + S_1^{r^{(2)}}(\phi, h^{(2)}) + S_1^{\Psi^{(4)}}(\phi, h^{(4)}) + S_1^{r^{(4)}}(\phi, h^{(4)}) \right) = 0 \tag{10.37}$$

we obtain again a unique solution for all three free parameters

$$\tilde{\Delta} = -3 - \frac{3}{2}D, \tag{10.38}$$
$$\xi_1 = -\frac{1}{8}\frac{D}{D+3}, \tag{10.39}$$
$$\xi_2 = \frac{1}{64}\frac{D(D-2)}{(D+1)(D+3)}. \tag{10.40}$$

Thus we constructed the linearized action for a scalar field interacting with the spin two and four field in a conformally invariant way

$$S^{WI}(\phi, h^{(2)}, h^{(4)}) = S^{WI}(\phi, h^{(2)}) + S_1^{\Psi^{(4)}}(\phi, h^{(4)}) + S_1^{r^{(4)}}(\phi, h^{(4)}), \tag{10.41}$$

$$\delta^1\phi = \varepsilon^\mu\nabla_\mu\phi + \epsilon^{\mu\nu\lambda}\nabla_\mu\nabla_\nu\nabla_\lambda\phi + \Delta\sigma\phi + \tilde{\Delta}\chi^{\mu\nu}\nabla_\mu\nabla_\nu\phi, \tag{10.42}$$

$$\delta^0 h^{(2)\mu\nu} = 2\nabla^{(\mu}\varepsilon^{\nu)} + 2\nabla^{(\mu}\tilde{\tilde{\epsilon}}^{\nu)} + 2(1 - \tilde{\Delta} - 2D\xi_1)\nabla^{(\mu}\tilde{\chi}^{\nu)} + 2\sigma g_{\mu\nu} + 2\xi_1\tilde{\tilde{\chi}}g_{\mu\nu}, \tag{10.43}$$

$$\delta^0 h^{(4)\mu\nu\alpha\beta} = 4\nabla^{(\mu}\epsilon^{\nu\lambda\alpha)} + 12\chi^{(\mu\nu}g^{\alpha\beta)}. \tag{10.44}$$

This interaction has an additional local symmetry permitting to gauge away the trace of spin two and four gauge fields. So we can say that this is a linearized interaction for *conformal higher spin theory* of the type discussed in [27],[28].

11 De Donder gauge and Goldstone mode

We will use Euclidian AdS_{d+1} (the same as in (1.15)-(1.20)) with conformal flat metric, curvature and covariant derivatives satisfying

$$
\begin{aligned}
& ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{L^2}{(z^0)^2}\delta_{\mu\nu}dz^\mu dz^\nu, \quad \sqrt{g} = \frac{L^{d+1}}{(z^0)^{d+1}}\,, \\
& [\nabla_\mu, \nabla_\nu]\, V^\rho_\lambda = R_{\mu\nu\lambda}{}^\sigma V^\rho_\sigma - R_{\mu\nu\sigma}{}^\rho V^\sigma_\lambda\,, \\
& R_{\mu\nu\lambda}{}^\rho = -\frac{1}{(z^0)^2}\left(\delta_{\mu\lambda}\delta^\rho_\nu - \delta_{\nu\lambda}\delta^\rho_\mu\right) = -\frac{1}{L^2}\left(g_{\mu\lambda}(z)\delta^\rho_\nu - g_{\nu\lambda}(z)\delta^\rho_\mu\right)\,, \\
& R_{\mu\nu} = -\frac{d}{(z^0)^2}\delta_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu}(z) \quad , \quad R = -\frac{d(d+1)}{L^2}\,.
\end{aligned}
$$

As before in spin ℓ case, for shortening the notation and calculation, we contract all rank ℓ symmetric tensors with the ℓ-fold tensor product of a vector a^μ. In this notation Fronsdal's equation of motion [24] for the double traceless spin ℓ field is (from now on we put $L = 1$)

$$
\begin{aligned}
& \mathcal{F}(h^{(\ell)}(z;a)) = \Box h^{(\ell)}(z;a) - (a\nabla)\nabla^\mu \frac{\partial}{\partial a^\mu}h^{(\ell)} + \frac{1}{2}(a\nabla)^2\Box_a h^{(\ell)}(z;a) \\
& - \left(\ell^2 + \ell(d-5) - 2(d-2)\right) h^{(\ell)} - a^2\Box_a h^{(\ell-2)}(z;a) = 0, \qquad (11.1)
\end{aligned}
$$

$$
\Box_a\Box_a h^{(\ell)} = 0, \qquad (11.2)
$$

$$
\Box = \nabla^\mu\nabla_\mu \quad , \Box_a = g^{\mu\nu}\frac{\partial^2}{\partial a^\mu \partial a^\nu}, \quad (a\nabla) = a^\mu\nabla_\mu, \quad a^2 = g_{\mu\nu}(z)a^\mu a^\nu. \qquad (11.3)
$$

The basic property of this equation is higher spin gauge invariance with the traceless parameter $\epsilon^{(\ell-1)}(z;a)$,

$$
\delta h^{(\ell)}(z;a) = (a\nabla)\epsilon^{(\ell-1)}(z;a), \quad \Box_a\epsilon^{(\ell-1)}(z;a) = 0, \quad \delta\mathcal{F}(h^{(\ell)}(z;a)) = 0. \qquad (11.4)
$$

The equation (11.1) can be simplified by gauge fixing. It is easy to see that in the so called de Donder gauge

$$
\mathcal{D}^{(\ell-1)}(h^{(\ell)}) = \nabla^\mu\frac{\partial}{\partial a^\mu}h^{(\ell)} - \frac{1}{2}(a\nabla)\Box_a h^{(\ell)} = 0, \qquad (11.5)
$$

Fronsdal's equation simplifies to

$$
\mathcal{F}^{dD}(h^{(\ell)}) = \Box h^{(\ell)} - \left(\ell^2 + \ell(d-5) - 2(d-2)\right) h^{(\ell)} - a^2\Box_a h^{(\ell-2)} = 0. \qquad (11.6)
$$

It was shown (see for example [34]) that in the de Donder gauge the residual gauge symmetry leads to the tracelessness of the *on-shell* fields. So we can define our massless *physical* spin ℓ modes as traceless and transverse symmetric tensor fields satisfying the equation (11.6)

$$[\Box + \ell]h^{(\ell)} = \Delta_\ell(\Delta_\ell - d)h^{(\ell)}, \tag{11.7}$$

$$\Box_a h^{(\ell)} = \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} = 0, \tag{11.8}$$

$$\Delta_\ell = \ell + d - 2. \tag{11.9}$$

Note that equation (11.7) for $\ell = 0$ coincides with the equation for the massless conformal coupled scalar only for $d = 3$.

In a similar way we can describe the massive higher spin modes using the same set of constraints on the general symmetric tensor field $\phi^{(\ell)}(z, a)$ [35] but with the independent conformal weight Δ (dimension) of the corresponding massive (in means of AdS field) representation of the $SO(d + 1, 1)$ isometry group. This general representation with two independent quantum numbers $[\Delta, \ell]$ under the maximal compact subgroup goes, after imposing a shortening condition $\Delta = \Delta_\ell = \ell + d - 2$, to the massless higher spin case (11.7)-(11.9) with the following decomposition [11, 29, 30]

$$\lim_{\Delta \to \ell + d - 2} [\Delta, \ell] = [\ell + d - 2, \ell] \oplus [\ell + d - 1, \ell - 1]. \tag{11.10}$$

The additional massive representation $[\ell + d - 1, \ell - 1]$ is the Goldstone field. Reading this decomposition from the opposite side, we can interpret it as swallowing of the massive spin $\ell - 1$ Goldstone field by the massless spin ℓ field with generation of a mass for the latter one [13]. For better understanding of this phenomenon we need a more careful investigation of the gauge invariant equation (11.1) in more general gauges.

First of all note that the gauge parameter $\epsilon^{(\ell-1)}$ is a traceless rank $\ell - 1$ tensor and therefore in any off-shell consideration (quantization, propagator and perturbation theory) we can use only gauge conditions with the same number of degrees of freedom.

The de-Donder gauge (11.5) is just such a type of the gauge due to the tracelessness of the $\mathcal{D}^{(\ell-1)}(h^{(\ell)})$. Nevertheless for on-shell states we can impose more restrictive gauges. Here we consider a one-parameter family of gauge fixing conditions

$$\mathcal{G}_{\alpha}^{(\ell-1)}(h^{(\ell)}) = \nabla^{\mu}\frac{\partial}{\partial a^{\mu}}h^{(\ell)} - \frac{1}{\alpha}(a\nabla)\Box_{a}h^{(\ell)} = 0 \tag{11.11}$$

This gauge condition coincides with the traceless de Donder gauge if $\alpha = 2$ ($\Box_{a}\mathcal{G}_{2}^{(\ell-1)} = \Box_{a}\mathcal{D}^{(\ell-1)} = 0$). Then we can write our double traceless field $h^{(\ell)}(z;a)$ as a sum of the two traceless spin ℓ and $\ell-2$ fields $\psi^{(\ell)}(z;a)$ and $\theta^{(\ell-2)}(z;a)$

$$h^{(\ell)}(z;a) = \psi^{(\ell)} + \frac{a^2}{2\alpha_0}\theta^{(\ell-2)}(z;a) \quad , \tag{11.12}$$

$$\Box_{a}h^{(\ell)} = \theta^{(\ell-2)} \quad , \quad \Box_{a}\psi^{(\ell)} = \Box_{a}\theta^{(\ell-2)} = 0, \tag{11.13}$$

$$\alpha_0 = d + 2\ell - 3. \tag{11.14}$$

In this parametrization Fronsdal's equation of motion with the gauge condition (11.11) can be written in the form of the following system of equations for the two independent traceless fields $\psi^{(\ell)}$ and $\theta^{(\ell-2)}$

$$\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}\psi^{(\ell)} + \frac{a^2}{2\alpha_0}\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}\theta^{(\ell-2)} = \frac{\alpha_0-\alpha}{\alpha\alpha_0}(a\nabla)\theta^{(\ell-2)}, \tag{11.15}$$

$$(\Box+\ell)\,\psi^{(\ell)} + \frac{\alpha-2}{2\alpha}\left[(a\nabla)^2\theta^{(\ell-2)} - a^2\frac{\alpha(\alpha_0-1)}{\alpha_0(\alpha-1)}\theta^{(\ell-2)}\right] = \Delta_{\ell}(\Delta_{\ell}-d)\psi^{(\ell)}, \tag{11.16}$$

$$(\Box+\ell-2)\,\theta^{(\ell-2)} = \left[\Delta_{\theta}(\Delta_{\theta}-d) + \frac{\alpha_0-\alpha}{\alpha-1}\right]\theta^{(\ell-2)}, \tag{11.17}$$

$$\Delta_{\ell} = d+\ell-2 \quad , \quad \Delta_{\theta} = d+\ell-1. \tag{11.18}$$

Now we are ready to discuss different gauge conditions. First of all we see that the de Donder gauge ($\alpha = 2$) leads to the complete separation of the equations of motion for $\psi^{(\ell)}$ and $\theta^{(\ell-2)}$ fields. On the other hand the gauge condition (11.15) becomes just traceless for $\alpha = 2$ and keeps on to connect the divergence of $\psi^{(\ell)}$ and the traceless part of the gradient of $\theta^{(\ell-2)}$

$$\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}\psi^{(\ell)}(z;a) = \frac{\alpha_0-2}{2\alpha_0}G^{(\ell-1)}(z;a), \tag{11.19}$$

$$G^{(\ell-1)}(z;a) = (a\nabla)\theta^{(\ell-2)}(z;a) - \frac{a^2}{\alpha_0-2}\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}\theta^{(\ell-2)}(z;a). \tag{11.20}$$

Here $G^{(\ell-1)}$ corresponds to the Goldstone representation. Indeed using the equations of motion (11.16) and (11.17) with $\alpha = 2$ one can derive that the $G^{(\ell-1)}$ field obeys the following on-shell equation

$$(\Box + \ell - 1)\, G^{(\ell-1)}(z;a) = \Delta_\theta(\Delta_\theta - d) G^{(\ell-1)}(z;a) \tag{11.21}$$

corresponding to the Goldstone representation $[\Delta_\theta = \ell + d - 1, \ell - 1]$ arising in (11.10). This mode can be gauged away on the classical level together with the trace $\theta^{(\ell-2)}$ but only *on-shell*. Therefore on the quantum level this mode can arise in loop diagrams and will play the crucial role in the mechanism of mass generation for the higher spin gauge fields [13].

Now we return to (11.15)-(11.18) and consider the next interesting gauge $\alpha = d + 2\ell - 3$. This is a generalization for the higher spin case of the so-called "Landau" gauge considered in [44] for the case of the graviton in AdS_{d+1}. But the difference between the higher spin and graviton ($\ell = 2$) cases is essential. For the graviton we can apply this "Landau" gauge

$$\nabla^\mu h_{\mu\nu} = \frac{1}{d+1}\partial_\nu h^\mu_\mu \tag{11.22}$$

off-shell also because the trace is scalar here and this gauge fixes the same number of degrees of freedom as the de Donder gauge. For $\ell > 2, \alpha \neq 2$ it is easy to see that condition (11.15) after taking the trace forces the trace components $\theta^{(\ell-2)}$ of our double traceless field $h^{(\ell)}$ to be transverse

$$\nabla^\mu \frac{\partial}{\partial a^\mu}\theta^{(\ell-2)} = 0, \tag{11.23}$$

$$\nabla^\mu \frac{\partial}{\partial a^\mu}\psi^{(\ell)} = \frac{\alpha_0 - \alpha}{\alpha\alpha_0}(a\nabla)\theta^{(\ell-2)}. \tag{11.24}$$

Moreover in the "Landau" gauge ($\alpha = \alpha_0$) the $\psi^{(\ell)}$ component is also transverse but it's equation of motion is not diagonal like in the de Donder gauge. On the other hand the equation of motion for the field $\theta^{(\ell-2)}$ is simplified and we have for this field the realization of the representation $[\Delta_\theta = \ell + d - 1, \ell - 2]$

$$(\Box + \ell - 2)\,\theta^{(\ell-2)} = \Delta_\theta(\Delta_\theta - d)\theta^{(\ell-2)}. \tag{11.25}$$

So we see that only in the de Donder gauge we have a diagonal equation of motion for the physical $\psi^{(\ell)}$ components but this component is not transversal due to the presence of the $[\ell + d - 1, \ell - 1]$ Goldstone mode $G^{(\ell-1)}$. This gauge is most suitable for the quantization and construction of the bulk-to-bulk propagator and for the investigation of the AdS_4/CFT_3 correspondence in the case of the critical conformal $O(N)$ boundary sigma model.

12 Bulk to Bulk Propagators

Propagators in de Donder's gauge

On AdS space which is a constant curvature space the geodesic distance η enters all invariant expressions of the relative distance of two points. The standard variable $\zeta = \cosh\eta$ can be expressed by Poincaré coordinates as

$$\zeta(z_1, z_2) = \frac{(z_1^0)^2 + (z_2^0)^2 + (\vec{z}_1 - \vec{z}_2)^2}{2z_1^0 z_2^0} = 1 + \frac{(z_1 - z_2)^\mu (z_1 - z_2)^\nu \delta_{\mu\nu}}{2z_1^0 z_2^0}. \tag{12.1}$$

The propagators are bitensorial quantities which are presented in the algebraic basis of homogeneous functions of I_1, I_2, I_3, I_4

$$I_1(a,c) := (a\partial_1)(c\partial_2)\zeta(z_1, z_2), \tag{12.2}$$

$$I_2(a,c) := (a\partial_1)\zeta(z_1, z_2)(c\partial_2)\zeta(z_1, z_2), \tag{12.3}$$

$$I_3(a,c) := a_1^2 I_{2c}^2 + c_2^2 I_{1a}^2, \tag{12.4}$$

$$I_4 := a_1^2 c_2^2, \tag{12.5}$$

$$I_{1a} := (a\partial_1)\zeta(z_1, z_2) \quad , \quad I_{2c} := (c\partial_2)\zeta(z_1, z_2), \tag{12.6}$$

$$(a\partial_1) = a^\mu \frac{\partial}{\partial z_1^\mu}, \quad (c\partial_2) = c^\mu \frac{\partial}{\partial z_2^\mu}, \tag{12.7}$$

$$a_1^2 = g_{\mu\nu}(z_1) a^\mu a^\nu, \quad c_2^2 = g_{\mu\nu}(z_2) c^\mu c^\nu. \tag{12.8}$$

of degree ℓ, the spin of the field. All important formulas for this "advanced technology" of working with higher spin field theory in AdS space one can find in Appendix A. We are interested only in that part of the propagator expansion which neglects traces. So it is a map from a space of $\ell+1$ functions $\{F_k(\zeta)\}_{k=0}^{\ell}$ to a space of bitensors parameterized by I_1 and I_2 only, namely

$$\Psi^{(\ell)}[F_k] = \sum_{k=0}^{\ell} F_k(\zeta) I_1^{\ell-k} I_2^k, \tag{12.9}$$

$$(\Box + \ell)\,\Psi^{(\ell)}[F_k] = \Delta_\ell(\Delta_\ell - d)\Psi^{(\ell)}[F_k] + O(a_1^2; c_2^2). \tag{12.10}$$

In the variable ζ the analytic properties of QFT n-point functions are conveniently described. In particular the two-point functions or propagators are analytic in the ζ

plane with singularities at $\zeta = \pm 1$ and at $\zeta = \infty$, which in most cases are logarithmic branch points. Analyticity is therefore meant in general on infinite covering planes. All *AdS* field theories are symmetric under the exchange ζ against $-\zeta$.

Another variable used often is $u = \zeta - 1$, the "chordal distance", more precisely one half the square of the chordal distance. The series expansions for two-point functions in u converge in a radius 2, whereas the series expansions in powers of ζ^{-1} converge for $|\ \zeta\ | > 1$. These analytic properties remind us of Legendre functions. Indeed if propagator functions can be identified as Gaussian hypergeometric functions, these are Legendre functions and the "quadratic transformations" can be applied. Using formulas from Appendix A we can show that in de Donder's gauge the propagator satisfy the following set of differential equations for the functions $F_k(\zeta)$ following from equation (12.10)

$$(\zeta^2 - 1)F_k'' + (d + 1 + 4k)\zeta F_k' + X_k F_k + 2\zeta(k + 1)^2 F_{k+1} + 2(\ell - k + 1)F_{k-1}' = 0, \quad (12.11)$$

$$X_k = k(d + 2\ell - k) + 2l - (\ell - 2)(\ell + d - 2). \quad (12.12)$$

The "dimension" of the higher spin field $\Delta_\ell = \ell + d - 2$ has been inserted. Moreover we use $F_{-1} = F_{\ell+1} = 0$. The dimension of the AdS space is $d + 1$, we interpolate analytically in d if this is technically required. Our issue is to solve these equations by expansion in powers of ζ^{-1} or u. This leads to matrix recursion equations which necessitate some linear algebra operations.

As an ansatz for the series expansion of $F_k(\zeta)$ at $\zeta = \infty$ we use

$$F_k(\zeta) = \zeta^{-\alpha-k} \sum_{n=0}^{\infty} c_{kn} \zeta^{-2n}. \quad (12.13)$$

Denote $\xi = \alpha + 2n$. Then a two term recursion of the form

$$D_n \begin{pmatrix} c_{0n} \\ c_{1n} \\ \vdots \\ c_{\ell,n} \end{pmatrix} = C_{n-1} \begin{pmatrix} c_{0,n-1} \\ c_{1,n-1} \\ \vdots \\ c_{\ell,n-1} \end{pmatrix}, \quad (12.14)$$

results with the two matrices

$$C_{n-1} = diag\{(\xi-1)(\xi-2), \xi(\xi-1), \dots (\xi+\ell-1)(\xi+\ell-2)\}, \tag{12.15}$$

and the entries of the matrix D_n

$$\begin{aligned}
(D_n)_{k,k-1} &= -2(\ell-k+1)(\xi+k-1), &(12.16)\\
(D_n)_{k,k} &= \xi^2 - \xi(d+2k) - 4k^2 + 2\ell(k+1) - (l-2)(\ell+d-2), &(12.17)\\
(D_n)_{k,k+1} &= 2(k+1)^2. &(12.18)
\end{aligned}$$

The determinant of D_0 is a polynomial of degree $2(\ell+1)$ of the variable α with roots which we identify with the "roots" of the differential equation system. For arbitrary ℓ we have

$$\begin{aligned}
detD_0 = [(\alpha+\ell-2)(\alpha+2-\ell-d)][(\alpha+\ell-2)(\alpha-\ell-d)]\\
\times \prod_{n=0}^{\ell-2}[\alpha^2 - (d+4+2n)\alpha - ((\ell-2)d + (\ell+n)^2 - (n+2)(3n+4))].
\end{aligned} \tag{12.19}$$

Each square bracket represents one eigenvalue of D_0 and contributes two roots. The quadratic factors lead in almost all cases to two irrational roots that are neither degenerate among themselves nor with the other roots, but there are exceptions which have two integer roots e.g. for $d = 3 : (\ell, n) \in \{(4,1), (6,4), (9,2), (9,5), (11,8), (15,8)...\}$. Two roots are said to be degenerate, if their difference is an integer. For the case of expansions in powers of ζ^2 as in (12.13), this integer must be even. In such case the solution with the bigger root enters the other one with a $log\zeta$ factor.

The following roots are of particular (physical) importance

$$\begin{aligned}
\alpha_p &= \ell+d-2, &(12.20)\\
\alpha_c &= \ell+d. &(12.21)
\end{aligned}$$

We call the first root α_p "principal" because it has the value of the dimension Δ of the field which enters the field equation in the form $\Delta(\Delta-d)$. The second root is a "companion" of it, since they appear for all ℓ as such pair (see (12.19)). It is degenerate with the principal root and the solution of it enters the principal solution with a $log\zeta$ factor on the next to leading power in the expansion. The bigger ones of the two roots in the exceptional cases quoted above are also bigger than the principal root $\ell+1$ (for the same ℓ) but their distance to it are odd numbers except for the case $(\ell, n) = (15, 8)$, where the distance to $\ell+1$ is sixteen and the $log\zeta$ term appears at a very high power.

For the principal root the equation for the eigenvector of D_0

$$D_0(\alpha_p) \begin{pmatrix} c_{00}^{(\alpha_p)} \\ c_{10}^{(\alpha_p)} \\ \vdots \\ c_{l0}^{(\alpha_p)} \end{pmatrix} = 0, \tag{12.22}$$

can be solved for each ℓ. We find

$$c_{k,0}^{(\alpha_p)} = (-1)^k \binom{\ell}{k}, \tag{12.23}$$

which is easy to prove by using the general expression for the rows of the matrix D_n as given in (12.16) - (12.18). The consequence of this result is that the leading term of $\Psi^{(\ell)}[F_k(\alpha_p)]$ at $\zeta = \infty$ is the well known expression $\zeta^{-\Delta}(I_1 - \zeta^{-1} I_2)^{\ell}$. Already at next order in ζ^{-2} log-terms appear.

For the companion root α_c the eigenvector for D_0 can be derived by a little bit more algebra for any ℓ

$$c_{k,0}^{(\alpha_c)} = (-1)^k \left(\binom{\ell}{k} + (d + 2\ell - 2) \binom{\ell - 1}{k - 1} \right). \tag{12.24}$$

The actual construction of a solution for the pair of roots starts with the bigger one, α_c. Its solution takes the form

$$F_k(\zeta; \alpha_c) = \zeta^{-\Delta-2} \sum_{n=0}^{\infty} \zeta^{-2n} \sum_{s=0}^{\ell} \Pi_n(\alpha_c)_{k,s} c_{s,0}^{(\alpha_c)}, \tag{12.25}$$

where we used

$$
\begin{aligned}
H_n(\alpha_c) &= D_n(\alpha_c)^{-1}C_{n-1}(\alpha_c) \\
&= H_1(\alpha_c+2(n-1)), && (12.26) \\
\Pi_n(\alpha_c) &= \Pi_{r=0}^{n-1,\leftarrow} H_1(\alpha_c+2r). && (12.27)
\end{aligned}
$$

and the left arrow denotes ordering of the product with increasing r from right to left. In this context we note that if a nonsingular matrix $S(\alpha)$ would exist, so that H_1 could be diagonalized by

$$H_1(\alpha) = S^{-1}(\alpha+2)\Delta(\alpha)S(\alpha), \tag{12.28}$$

then $F_k(\zeta;\alpha)$ would be a generalized hypergeometric function.

Having constructed the solution for the companion root we turn to the principal root. We recognize that $D_n(\alpha_p)$ can be spectrally decomposed in the following fashion

$$
\begin{aligned}
D_n\chi_i &= \lambda_i\chi_i, && (12.29) \\
D_n^T\psi_i &= \lambda_i\psi_i, && (12.30) \\
D_n &= \sum_{i=0}^{l} \lambda_i\chi_i \otimes \psi_i^T, && (12.31) \\
\psi_i^T\chi_j &= \delta_{ij} && (12.32)
\end{aligned}
$$

Denote further

$$\rho^T = \psi^T C_{n-1}. \tag{12.33}$$

All these quantities can be determined as functions of ξ, and it is easily verified that (12.28) is not fulfilled.

One of the eigenvalues of $D_1(\alpha_p)$ vanishes, we denote it λ_0, so that $D_1(\alpha_p)$ cannot be inverted. We perform a deformation of our differential equation system replacing α_p only in λ_0 and in the prefactor $\zeta^{-\alpha_p}$ by $\alpha_p+\epsilon$. All other eigenvalues and the eigenvectors remain unchanged. Then we continue the whole procedure known from the companion root, all H_n will remain singularity free. At the end we subtract a certain multiple γ of $(\epsilon^{-1} + \mu)\Psi^{(\ell)}[F_k(\alpha_c)]$ so that the limit $\epsilon \to 0$ can be performed and the log-terms appearing are $-\gamma log\zeta\Psi^{(l)}[F_k(\alpha_c)]$. The additional parameter μ is in principle arbitrary showing that the principal solution containing a log factor is a coset with respect to adding the companion solution. This parameter can, however, be normalized in a standard fashion by requiring that the $(l+1)$-tupel of coefficients $c_{k,n}^{(\alpha_p)}$ where at level n the log term appears first, is orthogonal to the eigenvector ψ_0 belonging to the deformed eigenvalue. We close this discussion with the remark that on the boundary of AdS space i.e. $\zeta = \infty$ any linear combination

$$\Psi^{(\ell)}[F_k(\alpha_p)] + A\Psi^{(\ell)}[F_k(\alpha_c)] \tag{12.34}$$

is indistinguishable from the pure principal solution. Thus the boundary constraint fixes only the whole coset and not any representative of it.

In order to render the expansions of F_k around $\zeta = 1 (u = 0)$ a visually different expression, we shall denote them Φ_k. The expansions are

$$\Phi_k(u) = u^\alpha \sum_{n=0}^{\infty} a_{k,n} u^n. \tag{12.35}$$

Again we obtain matrix recursion relations

$$A_n \begin{pmatrix} a_{0,n} \\ a_{1,n} \\ \vdots \\ a_{\ell,n} \end{pmatrix} + B_{n-1} \begin{pmatrix} a_{0,n-1} \\ a_{1,n-1} \\ \vdots \\ a_{\ell,n-1} \end{pmatrix} + E \begin{pmatrix} a_{0,n-2} \\ a_{1,n-2} \\ \vdots \\ a_{\ell,n-2} \end{pmatrix} = 0. \tag{12.36}$$

We define

$$\xi = \alpha + n, \tag{12.37}$$

and obtain the matrices

$$
\begin{aligned}
(A_n)_{k,k} &= \xi(2\xi + d + 4k - 1), && (12.38)\\
(A_n)_{k,k-1} &= 2\xi(\ell - k + 1), && (12.39)\\
(B_{n-1})_{k,k} &= (\xi - 1)(\xi + d + 4k - 1) + X_k, label 3.6 && (12.40)\\
(B_{n-1})_{k,k+1} &= 2(k+1)^2 = (E)_{k,k+1}. && (12.41)
\end{aligned}
$$

Here we used the shorthand (see (12.12))

$$X_k(\lambda) = k(2\lambda + 2\ell - k + 1) + 2\ell - (\ell - 2)(2\lambda + \ell - 1), \tag{12.42}$$

and $d = 2\lambda + 1$ has been introduced. Therefore A_n is of lower triangular shape with eigenvalues $\xi(2\xi + d + 4k - 1)$. The root system is therefore

- $\ell + 1$ times the root zero;
- the $\ell + 1$ roots $\alpha_m = -\lambda - 2m, \qquad 0 \leq m \leq \ell.$

Both sets are degenerate among themselves, and if d is odd, the second set is degenerate with respect to the first one. The first set produces regular solutions, the second set produces poles if d is odd, which it is in the case of present interest. Nevertheless we will regard d as a free real parameter in order to handle the degeneracy with the regular cases. The solution for α_0 in combination with any regular solution has the appropriate singular behaviour at $u = 0$ needed for a propagator, namely applying Fronsdal's differential operator the correct delta function is created.

Any solution is obtained by requiring

$$A_0 \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{\ell,0} \end{pmatrix} = 0. \tag{12.43}$$

This requirement is solved for the regular solutions $\Phi_k^{(r)}(u)$ (for which $A_0 = 0$ and the solution is trivial) by

$$a_{k,0}^{(r)} = \delta_{k,r}. \tag{12.44}$$

For any such solution r we obtain next

$$\begin{aligned} a_{k,1}^{(r)} &= -(A_1^{-1}B_0)_{k,r} \\ &= -(A_1^{-1})_{k,r}(B_0)_{rr} - (A_1^{-1})_{k,r-1}(B_0)_{r-1,r}, \end{aligned} \tag{12.45}$$

where we insert

$$(A_1)_{r,r} = d + 4r + 1, \tag{12.46}$$

$$(A_1)_{r,r-1} = 2(\ell - r + 1), \tag{12.47}$$

$$(B_0)_{r,r} = X_r, \tag{12.48}$$

$$(B_0)_{r-1,r} = 2r^2, \tag{12.49}$$

and obtain

$$(A_1^{-1})_{k,r} = (-2)^{k-r} \prod_{s=r+1}^{k} (\ell - s + 1) \quad [\prod_{s=r}^{k}(d + 4s + 1)]^{-1} \quad (\text{for} \quad k > r), \tag{12.50}$$

$$(A_1^{-1})_{r,r} = (d + 4r + 1)^{-1}, \tag{12.51}$$

$$(A_1^{-1})_{k,r} = 0 \quad \text{for} \quad k < r. \tag{12.52}$$

There is no sign of any singularity caused by the degeneracy. Finally we get

$$a_{k,1}^{(r)} = -X_r(A_1^{-1})_{k,r} - 2r^2(A_1^{-1})_{k,r-1}, \tag{12.53}$$

which vanishes for $r > k + 1$.

We turn now to the nonanalytic solutions $\Phi_k(u, \alpha_m)$ with roots $\alpha_m = -\lambda - 2m$ and concentrate on the case $m = 0$ because this is the perturbative Green function for the Fronsdal differential operator. At the beginning we assume $\lambda \notin \mathbf{Z}$ in order to avoid the degeneracy with the regular solutions. In this case we have

$$(A_0)_{k,k} = -4\lambda k, \tag{12.54}$$

$$(A_0)_{k,k-1} = -2\lambda(\ell - k + 1), \tag{12.55}$$

and the equation

$$\sum_r (A_0)_{k,r} c_{r,0}^{(\alpha_0)} = 0 \tag{12.56}$$

is solved by

$$c_{k,0}^{(\alpha_0)} = \left(-\frac{1}{2}\right)^k \binom{\ell}{k}. \tag{12.57}$$

Next we treat the A_1 matrix

$$\begin{aligned}
(A_1)_{k,k} &= 2(1-\lambda)N_k, \quad N_k = 2k+1, &(12.58)\\
(A_1)_{k,k-1} &= 2(1-\lambda)(\ell-k+1), &(12.59)\\
(A_1^{-1})_{k,r} &= [2(1-\lambda)]^{-1}\beta_{k,r}, \text{for} \quad k \geq r \quad \text{and zero else}, &(12.60)\\
\beta_{k,r} &= (-\ell)_{k-r}[\prod_{s=r}^{k} N_s]^{-1}. &(12.61)
\end{aligned}$$

The B_0 matrix is

$$\begin{aligned}
(B_0)_{k,k} &= -\lambda(\lambda+4k+1) + X_k := Z_k(\lambda), &(12.62)\\
(B_0)_{k,k+1} &= 2(k+1)^2. &(12.63)
\end{aligned}$$

The matrix E is still not needed for $n = 1$.

We define the matrix

$$(H_1)_{k,r} = -(A_1^{-1}B_0)_{k,r} = [2(\lambda-1)]^{-1}\{\beta_{k,r}(B_0)_{r,r} + \beta_{k,r-1}(B_0)_{r-1,r}\}, \tag{12.64}$$

and obtain for the coefficients $c_{k,1}^{(\alpha_0)}$

$$c_{k,1}^{(\alpha_0)} = \sum_{r=0}^{k+1} (H_1)_{k,r} \left(-\frac{1}{2}\right)^r \binom{\ell}{r}. \tag{12.65}$$

All these coefficients inherit a pole in λ at the value 1.

This pole does not appear in one eigenvalue only as in the $\zeta = \infty$ case. This is due to the fact that for $\lambda = 1$ there exist $\ell+1$ degenerate regular solutions and therefore the pole appears in all $\ell+1$ eigenvalues simultaneously. It is straightforward to calculate the residues of all matrix elements of H_1 and to derive the expressions

$$\rho_k = \sum_{r=0}^{k+1} res(H_1)_{k,r} \left(-\frac{1}{2}\right)^r \binom{\ell}{r}. \tag{12.66}$$

Then we subtract from this solution at $n = 1$ the regular solution

$$(\lambda - 1)^{-1}[\sum_{r=0}^{\ell} \rho_r \Phi^{(r)}(u)], \tag{12.67}$$

obtaining in the limit the log term of $\Psi^{(\ell)}[\Phi_k(u, \alpha_0)]$

$$-\log u \quad [\sum_{r=0}^{\ell} \rho_r \Phi^{(r)}(u)]. \tag{12.68}$$

We mention that the leading term of $\Psi^{(\ell)}[\Phi_k(u, \alpha_0)]$ is

$$u^{-1}(I_1 - \frac{1}{2} I_2)^{\ell}. \tag{12.69}$$

The situation with the Green function type solution is the same as with the solution which is constrained by the AdS boundary condition: The UV constraint is satisfied by a coset, namely any linear combination of regular solutions can be added to the solution $\Psi^{(\ell)}[\Phi(\alpha_0)]$. In turn this may also be used to normalize the solutions $\Phi_k(\alpha_0)$. We can namely require that on the level $n = 1$ on which $\log u$ appears first, all the coefficients $c_{k,1}^{(\alpha_0)}$ are made to vanish by appropriate subtraction of regular solutions.

Propagators in Feynman's gauge

In this section we consider the higher spin gauge propagators analyzed in the previous section and in [12], [36], [37] in an approach developed originally for the spin $\ell = 0, 1, 2$ in [43], [39], [44] only, but now generalized for all ℓ with a slight modification of arguments. Namely we consider our propagator working directly in the space of conserved currents

$$h^{(\ell)}(z_1; a) = \int \sqrt{g} d^4 z_2 K^{(\ell)}(z_1, a; z_2, c) *_c J^{(\ell)}(z_2, c), \tag{12.70}$$

where

$$K^{(\ell)}(z_1, a; z_2, c) = \Psi^{(\ell)}[F_k(u(z_1; z_2))] + \text{traces}. \tag{12.71}$$

Taking into account the conservation properties of the current $J^{(\ell)}(z_2, c)$ we can formulate the ansatz following from (12.10)

$$\begin{aligned}[\Box_1 + \ell - \Delta_\ell(\Delta_\ell - d)]\Psi^{(\ell)}[F_k(u(z_1; z_2))] &= -I_1^\ell \delta_{d+1}(z_1; z_2) + \text{traces} \\ &+ (c\nabla_2)\left(I_{1a}\Psi^{(\ell-1)}[\Lambda_k(u(z_1; z_2))]\right). \quad (12.72)\end{aligned}$$

This means that applying the gauge fixed equation of motion at the first argument of the bilocal propagator we get zero (or more precisely a delta function in the coincident points) due to a gauge transformation at the second argument.

Here we should make some comments on the delta function in curved AdS space. Our notation in (12.72) means

$$\delta_{(d+1)}(z_1; z_2) = \frac{\delta_{(d+1)}(z_1 - z_2)}{\sqrt{g(z)}}, \qquad \int \delta_{(d+1)}(z_1 - z_2) d^{d+1} z_1 = 1. \qquad (12.73)$$

In the polar coordinate system defined in Appendix A the invariant measure (for $d = 3$) is

$$\sqrt{g} d^4 z = u(u + 2) du d\Omega_3. \qquad (12.74)$$

Therefore we can define

$$\begin{aligned}&\frac{\delta_{(4)}(z - z_{pole})}{\sqrt{g(z)}} = \frac{\delta(u)}{u(u + 2)\Omega_3} = -\frac{\delta^{(1)}(u)}{(u + 2)\Omega_3}, \qquad (12.75)\\ &u\delta^{(1)}(u) = -\delta(u).\end{aligned}$$

This u- dependence of the measure leads to the idea that short distance singularities in $D = d + 1 = 4$ dimensional AdS space should start from $\frac{1}{u^2}$ not from $\frac{1}{u}$.

Then using the gradient map (A.34), (A.35) we can derive

$$(c\nabla_2)\left(I_{1a}\Psi^{(\ell-1)}[\Lambda_k(u)]\right) = \Psi^{(\ell)}[\Lambda'_{k-1}(u) + (k + 1)\Lambda_k(u)], \quad \Lambda_\ell = 0 \qquad (12.76)$$

Combining this with the Laplacian map (A.31)-(A.33) and (12.70) we obtain the following set of $\ell + 1$ equations for $z_1 \neq z_2$ (unlike the case (12.11) we do not insert the value of Δ_ℓ here)

$$\begin{aligned}&u(u + 2)F''_k + (d + 1 + 4k)(u + 1)F'_k + 2(\ell - k + 1)F'_{k-1} + 2(u + 1)(k + 1)^2 F_{k+1} \\ &+[2\ell + k(d + 2\ell - k)]F_k - \Delta_\ell(\Delta_\ell - d)F_k = \Lambda'_{k-1} + (k + 1)\Lambda_k. \qquad (12.77)\end{aligned}$$

To analyze this system we write the $k = 0, 1$ and $\ell - 1, \ell$ cases explicitly

$$u(u+2)F_0'' + (d+1)(u+1)F_0' - \Delta_\ell(\Delta_\ell - d)F_0 + 2(u+1)F_1 + 2\ell F_0 = \Lambda_0, \quad (12.78)$$

$$O(F_1'', F_1', F_1, F_2) + 2\ell F_0' = \Lambda_0' + 2\Lambda_1, \quad (12.79)$$

$$\vdots$$

$$O(F_{\ell-1}'', F_{\ell-1}', F_{\ell-1}, F_\ell, F_{\ell-2}') = \Lambda_{\ell-2}' + \ell\Lambda_{\ell-1}, \quad (12.80)$$

$$u(u+2)F_\ell'' + (d+1+4\ell)(u+1)F_\ell' + [\ell^2 + \ell(d+2) - \Delta_\ell(\Delta_\ell - d)]F_\ell + 2F_{\ell-1}' = \Lambda_{\ell-1}', \quad (12.81)$$

and we see that this system for $2\ell + 1$ functions is separable. One solution is obtained if we put

$$F_k = 0, \quad k = 1, 2, \dots \ell, \quad (12.82)$$

$$\Lambda_k = 0, \quad k = 1, 2, \dots \ell - 1, \quad (12.83)$$

and submit $F_0(u)$ to the Gaussian hypergeometric equation

$$u(u+2)F_0''(u) + (d+1)(u+1)F_0'(u) - \Delta_\ell(\Delta_\ell - d)F_0(u) = 0, \quad (12.84)$$

supplemented with a noncontradictory solution for the remaining gauge parameter $\Lambda_0(u)$

$$\Lambda_0(u) = 2\ell F_0(u). \quad (12.85)$$

So we prove that with an appropriate choice of the gauge freedom we can obtain the propagator in Feynman's gauge in the form

$$K^{(\ell)}(z_1, a; z_2, c) = I_1^\ell F_0(u) + \text{traces}, \quad (12.86)$$

where $F_0(u)$ is the solution of the equation for the scalar field with dimension Δ_ℓ (12.84) [43]. The solution of this equation is well known and can be written in two different forms [44, 45]. The first form is ($\zeta = u + 1$)

$$F_0(\zeta) = C(\ell, d)2^{\Delta_\ell}\zeta^{-\Delta_\ell}{}_2F_1\left(\frac{\Delta_\ell}{2}, \frac{\Delta_\ell + 1}{2}, \Delta_\ell - \frac{d}{2} + 1; \frac{1}{\zeta^2}\right). \quad (12.87)$$

This form is suitable for an investigation of the infrared behaviour. We see immediately that near the boundary limit we have

$$F_0(\zeta) \sim \zeta^{-\Delta_\ell}|_{d=3} = \zeta^{-(\ell+1)}, \quad \text{if} \quad \zeta \to \infty, \tag{12.88}$$

which is just wanted for AdS/CFT correspondence. Indeed comparing Δ_ℓ and Δ_θ in (11.16)-(11.18) we see that the propagator of the nonphysical mode θ falls off in the boundary limit faster than the propagator for the physical mode ψ, as it should be.

But for us the second form of this expression obtained after a quadratic transformation of the hypergeometric function listed in the Appendix B (B.18) is more interesting

$$F_0(u) = C(\ell, d)\left(\frac{2}{u}\right)^{\Delta_\ell} {}_2F_1\left(\Delta_\ell, \Delta_\ell - \frac{d}{2} + \frac{1}{2}, 2\Delta_\ell - d + 1; -\frac{2}{u}\right). \tag{12.89}$$

The normalization constant $C(\ell, d)$ is chosen to obtain the δ function on the right hand side of (12.72)

$$C(\ell, d) = \frac{\Gamma(\Delta_\ell)\Gamma(\Delta_\ell - \frac{d}{2} + \frac{1}{2})}{(4\pi)^{\frac{(d+1)}{2}}\Gamma(2\Delta_\ell - d + 1)}|_{d=3} = \frac{\ell!(\ell-1)!}{16\pi^2(2\ell-1)!}. \tag{12.90}$$

To investigate the ultraviolet limit of (12.89) we have to use the second formula (B.19) of Appendix B and take carefully the limit $d \to 3$ to obtain

$$\left(\frac{2}{u}\right)^{\Delta_\ell} {}_2F_1\left(\Delta_\ell, \Delta_\ell - \frac{d}{2} + \frac{1}{2}, 2\Delta_\ell - d + 1; -\frac{2}{u}\right)|_{d\to 3} = \frac{(2\ell-1)!}{(\ell-1)!}\left\{\frac{2}{\ell! u}\right.$$
$$\left. + \frac{1}{(\ell-2)!}\sum_{n=0}^{\ell-2} \frac{(\ell+1)_n(2-\ell)_n}{n!(n+1)!}\left[\Upsilon_{\ell,n} + \log\frac{u}{2}\right]\left(-\frac{u}{2}\right)^n\right\}, \tag{12.91}$$

where the rational number $\Upsilon_{\ell,n}$ is expressed by the ψ functions

$$\Upsilon_{\ell,n} = \psi(\ell+n+1) + \psi(\ell-n-1) - \psi(n+1) - \psi(n+2). \tag{12.92}$$

So we see now that in the ultraviolet limit we get

$$F_0(u)|_{d=3} \cong \frac{1}{8\pi^2}\frac{1}{u} + O(1, u, \log u, u\log u, \dots). \tag{12.93}$$

This main singular term in the propagator of the scalar field with dimension Δ_ℓ does not depend on the field dimension and behaves always like $\frac{1}{8\pi^2 u}$. For example we have the same singularity in the propagator of the conformally coupled scalar in AdS_4 (see [14])

$$\Sigma[u(z_1,z_2)] = \frac{1}{8\pi^2}\left(\frac{1}{u} \pm \frac{1}{u+2}\right), \tag{12.94}$$

$$(\Box + 2)\Sigma[u(z_1,z_2)] = -\delta_{(4)}(z_1;z_2). \tag{12.95}$$

So we observe some universality in the UV behaviour of higher spin propagators in Feynman's gauge:

For any spin ℓ the propagator starts from $I_1^\ell \frac{1}{8\pi^2 u}$.

Comparing with (12.69) we deduce that in de Donder gauge we have the same picture because

- $I_1(a,c;u) \to a^\mu c_\nu$ if $u \to 0$.
- $I_2(a,c;u) = I_3(a,c;u) \to 0$ if $u \to 0$.
- $I_4(a,c;u) \to a^2c^2$ if $u \to 0$.

So finally we can formulate the following statement:

The higher spin propagator in Feynman's gauge is simplest and most convenient for the calculation of any Feynman diagram. Just we have to couple it with conserved currents to make sure that we preserve gauge invariance. The UV-behaviour of the propagator is universal and described by (12.93).

13 Bulk-to-boundary limit

Now we can take the boundary limit and obtain the spin ℓ bulk-to-boundary propagator from the bulk-to-bulk propagator directly. For this purpose we mention that the boundary of AdS space is approached in the limit

$$z^0 \to 0, \tag{13.1}$$

which is connected with the limit $\zeta \to \infty$ due to

$$\lim_{z_2^0 \to 0} 2 z_1^0 z_2^0 \zeta(z_1, z_2) = (z_1^0)^2 + (\vec{z}_1 - \vec{z}_2)^2. \tag{13.2}$$

Then following the explanation of the previous section we see that at the boundary only the main term (12.69) survives and we get

$$\lim_{\substack{z_2^0 \to 0 \\ c_\mu = (0, \vec{c})}} (z_2^0)^{\ell - \Delta} \left(I_1 - \frac{1}{\zeta} I_2 \right)^\ell F_0(\zeta) = 2^\Delta C \frac{(z_1^0)^{d-2}}{[(z_1^0)^2 + (\vec{z}_1 - \vec{z}_2)^2]^\Delta} \left[R(a, \vec{c}; z_1 - \vec{z}_2) \right]^\ell \tag{13.3}$$

$$R(a, \vec{c}; z_1 - \vec{z}_2) = < \vec{a}, \vec{c} > -2 \frac{(a, z_1) < \vec{z}_1 - \vec{z}_2, \vec{c} >}{(z_1^0)^2 + (\vec{z}_1 - \vec{z}_2)^2}. \tag{13.4}$$

Here we introduced the $d+1$ and d dimensional Euclidian scalar products

$$(a, z) = \sum_{\mu=0}^{d} a^\mu z_\mu, \quad < \vec{c}, \vec{z} > = \sum_{i=1}^{d} c^i z_i \tag{13.5}$$

and the Jacobian tensor

$$R_{\mu\nu}(z) = \delta_{\mu\nu} - 2 \frac{z_\mu z_\nu}{(z, z)}. \tag{13.6}$$

We see that the limit (13.3) really produces Dobrev's [38] boundary-to-bulk propagator without trace terms.

Actually we need only this leading term because all other trace terms depend on the gauge condition (11.5) applied to the bulk dependent side of the right hand side of (13.3). On the other hand we can fix the trace terms by requiring the tensor fields to approach a certain tensor type on the boundary. In the case of irreducible d dimensional CFT currents we have to claim tracelessness with respect to the indices contracted with

$\vec{c}$

$$\Box_{\vec{c}} G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = \frac{\partial^2}{\partial\vec{c}\partial\vec{c}}\left(G^{(\ell)}_m(a,\vec{c};z) + \text{trace terms}\right) = 0, \tag{13.7}$$

$$G^{(\ell)}_m(a,\vec{c};z) = \frac{(z^0)^{d-2}}{(z,z)^\Delta}\left[R(a,\vec{c};z)\right]^\ell. \tag{13.8}$$

Here we omit the normalization factor $2^\Delta C$ and put for simplicity $\vec{z_2} = 0$ and $z_1^\mu = z^\mu$ (we can always restore the right dependence on the boundary coordinate $\vec{z_2}$ using translation invariance in the flat boundary space).

Then considering the boundary limit of the I_3 and I_4 dependent terms we can easily render the propagator (13.7) traceless on the boundary by the projection[3]

$$G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = G^{(\ell)}_m(a,\vec{c};z) - \frac{(a,a) - [R^0(a;z)]^2}{2(\alpha_0-1)(z^0)^2}\Box_a G^{(\ell)}_m(a,\vec{c};z)$$
$$+O(a^4) + O(c^4). \tag{13.9}$$

The complete polynomial expression for $G^{(\ell)}_{AdS/CFT}(a,\vec{c};z)$ is presented in the Appendix B, Eqn. (B.10). But here we consider only the first order trace term

$$\Box_a G^{(\ell)}_m(a,\vec{c};z) = \ell(\ell-1)\frac{(z^0)^d}{(z,z)^\Delta} < \vec{c},\vec{c} > \left[R(a,\vec{c};z)\right]^{\ell-2}, \tag{13.10}$$

$$R^0(a;z) = a^\mu R^0_\mu(z) = a^0 - 2\frac{z^0(a,z)}{(z,z)}, \quad \alpha_0 = d + 2\ell - 3. \tag{13.11}$$

The important point of this consideration is the following: The expression (13.9) is automatically traceless on the AdS side.

$$\Box_a G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = 0, \tag{13.12}$$

due to the relations

$$\delta^{\mu\nu}R^0_\mu(z)R^0_\nu(z) = 1 \quad , \quad \delta^{\mu\nu}R^0_\mu(z)R_\nu(\vec{c};z) = 0 \tag{13.13}$$

This is natural because the original bulk-to-bulk basis $\{I_i(a,c;\zeta)\}_{i=1}^4$ was symmetric with respect to the $a \leftrightarrow c$ exchange . Then we see that this projection in agreement

[3] In this section we used the exact expression for Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{z^0}\left(\delta^\lambda_0\delta_{\mu\nu} - 2\delta^0_{(\mu}\delta^\lambda_{\nu)}\right)$ and the AdS trace rule $\Box_a = (z^0)^2\delta^{\mu\nu}\frac{\partial^2}{\partial a^\mu \partial a^\nu}$

with de Donder gauge condition (11.5) (for traceless case) leads to the transverse-traceless bulk-to-boundary propagator (13.9) for all higher spin fields on AdS side. For proving this we have to calculate several relations in first order of (a,a) and $<\vec{c},\vec{c}>$ (see details in Appendix B)

$$\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}G^{(\ell)}_{m}(a,\vec{c};z)=\ell(\ell-1)\frac{(z^0)^{d-1}}{(z,z)^{\Delta}}<\vec{c},\vec{c}>[R(a,\vec{c};z)]^{\ell-2}R^0(a;z), \tag{13.14}$$

$$a^{\mu}\nabla_{\mu}\Box_a G^{(\ell)}_{m}(a,\vec{c};z)=\ell(\ell-1)\frac{(z^0)^{d-1}}{(z,z)^{\Delta}}<\vec{c},\vec{c}>[R(a,\vec{c};z)]^{\ell-2}R^0(a;z)(\alpha_0-1), \tag{13.15}$$

$$\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}\frac{[R^0(a;z)]^2}{(z^0)^2}\Box_a G^{(\ell)}_{m}(a,\vec{c};z)=0. \tag{13.16}$$

Putting all together we obtain

$$\nabla^{\mu}\frac{\partial}{\partial a^{\mu}}G^{(\ell)}_{AdS/CFT}(a,\vec{c};z)=0. \tag{13.17}$$

So we see that the Goldstone mode (non transverseness of the propagator) can not be visible after trace projection on the boundary side corresponding to the case of the traceless currents in the large N limit of the $O(N)$ sigma model.

The next interesting question which we can ask is the transversal property of the bulk-to-boundary propagator on the boundary side. The answer is negative. The divergence on the boundary side of the traceless bulk-to-boundary propagator is not zero and equals a gauge term (gradient) with respect to the bulk gauge invariance. Using the formulas from Appendix B one can check that

$$\frac{\partial}{\partial\vec{z}}\cdot\frac{\partial}{\partial\vec{c}}G^{(\ell)}_{AdS/CFT}(a,\vec{c};z)=a^{\mu}\nabla_{\mu}\Lambda^{(\ell-1)}(a,\vec{c};z), \tag{13.18}$$

$$\Lambda^{(\ell-1)}(a,\vec{c};z)=2\ell\frac{(\alpha_0-1)(\ell+d-1)-2(\ell-1)}{\alpha_0^2-1}\frac{(z^0)^d}{(z,z)^{\Delta_\ell+1}}[R(a,\vec{c};z)]^{\ell-1}. \tag{13.19}$$

We see that the boundary trace projection generates the bulk gauge term on the boundary side and is equivalent to the residual on-shell gauge fixing preserving the bulk side de Donder off-shell gauge (this property of the bulk-to-boundary propagator was mentioned in fr[34] and in [44] for the vector field case).

Finalizing our consideration we can define now the CFT propagator from (13.9) by $a^0 = 0$ and the limit $z^0 \to 0$. Due to the vanishing of $R^0(a; z)$ in this limit we get

$$G^{(\ell)}_{CFT}(\vec{a}, \vec{c}; \vec{z}) = \lim_{z^0 \to 0} (z^0)^{2-d} G^{(\ell)}_{AdS/CFT}(\vec{a}, \vec{c}; z)$$
$$= G^{(\ell)}_m(\vec{a}, \vec{c}; \vec{z}) - \frac{< \vec{a}, \vec{a} >}{2(\alpha_0 - 1)} \Box_{\vec{a}} G^{(\ell)}_m(\vec{a}, \vec{c}; \vec{z}) + O(< \vec{a}, \vec{a} >^2) \tag{13.20}$$

(13.21)

Thus the limit (13.20) defines the correct CFT two point function for traceless conserved[4] currents.

So we prove that the boundary limit of our bulk-to-bulk propagator in the de Donder gauge is in agreement with the bulk-to-boundary propagator obtained from the AdS isometry group representation theory.

[4]Note that the gradient of the gauge term also vanishes on the boundary because $a^\mu \nabla_\mu \Lambda^{(\ell-1)}(a, \vec{c}; z) \sim R^0(a; z)$.

14 Exercises on spin one field couplings with the higher spin gauge fields

We start this section constructing the well known interaction of the electromagnetic field A_μ in flat D dimensional space-time with the linearized spin two field. Hereby we illustrate how Noether's procedure regulates the relation between gauge symmetries of different spin fields. The standard free Lagrangian of the electromagnetic field is

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial A)^2, \tag{14.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial A = \partial_\mu A^\mu. \tag{14.2}$$

To construct the interaction we propose a possible form for the action of the spin two linearized gauge symmetry

$$\delta^0_\varepsilon h^{(2)\mu\nu}(x) = 2\partial^{(\mu}\varepsilon^{\nu)}(x) = \partial^\mu \varepsilon^\nu(x) + \partial^\nu \varepsilon^\mu(x), \tag{14.3}$$

on the spin one gauge field $A_\mu(x)$. Then Noether's procedure fixes this coupling (1-1-2 interaction) of the electromagnetic field with linearized gravity correcting when necessary the proposed transformation.

We start from the following general ansatz for a gauge variation of A_μ with respect to a spin 2 gauge transformation with vector parameter ε^ρ

$$\delta^1_\varepsilon A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + C\varepsilon^\rho \partial_\mu A_\rho. \tag{14.4}$$

Then we apply this variation (14.4) to (14.1) and after some algebra neglecting total derivatives we obtain [5]

$$\begin{aligned}\delta^1_\varepsilon \mathcal{L}_0 &= \partial^{(\mu}\varepsilon^{\nu)}\partial_\mu A_\rho \partial_\nu A^\rho - \frac{1}{2}\varepsilon_{(1)}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\varepsilon_{(1)}(\partial A)^2 + C\partial^{(\mu}\varepsilon^{\nu)}\partial_\rho A_\mu \partial^\rho A_\nu \\ &- 2C\partial^{(\mu}\varepsilon^{\nu)}\partial_\rho A_{(\mu}\partial_{\nu)}A^\rho + \frac{C}{2}\varepsilon_{(1)}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{C}{2}\varepsilon_{(1)}(\partial A)^2 \\ &+ (C-1)(\partial A)\partial^\mu \varepsilon^\nu \partial_\nu A_\mu. \end{aligned} \tag{14.6}$$

Then we have to compensate (or integrate) this variation using the gauge variation of the spin 2 field (14.3) and its trace $\delta^0_\varepsilon h^{(2)\mu}_\mu = 2\varepsilon_{(1)}$. We see immediately that the last line in (14.6) is irrelevant but can be dropped by choice of the free constant $C = 1$. With this choice we have instead of (14.4)

$$\delta^1_\varepsilon A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + \varepsilon^\rho \partial_\mu A_\rho = \varepsilon^\rho F_{\mu\rho}, \tag{14.7}$$

so that our spin two transformation now is manifestly gauge invariant with respect to the spin one gauge invariance

$$\delta^0_\sigma A_\mu = \partial_\mu \sigma, \tag{14.8}$$

and our spin one gauge invariant free action (14.1) keeps this property also after spin two gauge variation. Namely (14.6) now can be written as

$$\delta^1_\varepsilon \mathcal{L}_0 = \partial^{(\mu}\varepsilon^{\nu)} F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4}\varepsilon_{(1)} F_{\mu\nu} F^{\mu\nu}. \tag{14.9}$$

This variation can be compensated introducing the following 2-1-1 interaction

$$\mathcal{L}_1(A_\mu, h^{(2)}_{\mu\nu}) = \frac{1}{2} h^{(2)\mu\nu}\Psi^{(2)}_{\mu\nu}, \tag{14.10}$$

[5] From now on we will never make a difference between a variation of the Lagrangians or the actions discarding all total derivative terms and admitting partial integration if necessary. For compactness we introduce also shortened notations for divergences of the tensorial symmetry parameters

$$\epsilon^{\mu\nu\ldots}_{(1)} = \nabla_\lambda \epsilon^{\lambda\mu\nu\ldots}, \quad \epsilon^{\mu\ldots}_{(2)} = \nabla_\nu \nabla_\lambda \epsilon^{\nu\lambda\mu\ldots}, \quad \ldots \tag{14.5}$$

where

$$\Psi^{(2)}_{\mu\nu} = -F_{\mu\rho}F_{\nu}{}^{\rho} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{14.11}$$

is the well known energy-momentum tensor for the electromagnetic field.

Thus we solved Noether's equation

$$\delta^1_\varepsilon \mathcal{L}_0(A_\mu) + \delta^0_\varepsilon \mathcal{L}_1(A_\mu, h^{(2)}_{\mu\nu}) = 0 \tag{14.12}$$

in this approximation completely, defining a first order transformation and interaction term at the same time. Finally note that the corrected Noether's procedure spin two transformation of the spin one field (14.7) can be written as a combination of the usual reparametrization for the contravariant vector $A_\mu(x)$ (non invariant with respect to (14.8)) and spin one gauge transformation with the special field dependent choice of the parameter $\sigma(x) = \varepsilon^\rho(x)A_\rho(x)$

$$\delta^1_\varepsilon A_\mu = \varepsilon^\rho F_{\mu\rho} = -\varepsilon^\rho\partial_\rho A_\mu - \partial_\mu\varepsilon^\rho A_\rho + \partial_\mu\left(\varepsilon^\rho(x)A_\rho(x)\right), \tag{14.13}$$

A symmetry algebra of these transformations can be understood from the commutator

$$[\delta^1_\eta, \delta^1_\varepsilon]A_\mu(x) = \delta^1_{[\eta,\varepsilon]}A_\mu(x) + \partial_\mu\left(\varepsilon^\rho\eta^\lambda F_{\rho\lambda}(x)\right) \tag{14.14}$$

$$[\eta, \varepsilon]^\lambda = \eta^\rho\partial_\rho\varepsilon^\lambda - \varepsilon^\rho\partial_\rho\eta^\lambda \tag{14.15}$$

So we see that the algebra of transformations (14.13) close on the field dependent gauge transformation (14.8) with parameter $\sigma(x) = \varepsilon^\rho\eta^\lambda F_{\rho\lambda}(x)$.

Now we turn to the first nontrivial case of the vector field interaction with a spin four gauge field with the following zero order spin four gauge variation

$$\delta^0_\epsilon h^{\mu\rho\lambda\sigma} = 4\partial^{(\mu}\epsilon^{\rho\lambda\sigma)}, \quad \delta^0_\epsilon h_\rho{}^{\rho\lambda\sigma} = 2\epsilon^{\lambda\sigma}_{(1)}. \tag{14.16}$$

where we have a symmetric and traceless gauge parameter $\epsilon^{\mu\nu\lambda}$ to construct a gauge variation for A_μ. In this case we first present final result and then explain details of the derivation.

The solution of the corresponding Noether's equation

$$\delta_\varepsilon^1 \mathcal{L}_0(A_\mu) + \delta_\varepsilon^0 \mathcal{L}_1(A_\mu, h_{\mu\nu}^{(2)}, h_{\mu\nu\lambda\rho}^{(4)}) = 0, \tag{14.17}$$

after field redefinitions is the linearized Lagrangian for the coupling of the electromagnetic field to the spin four and spin two fields

$$\mathcal{L}_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\nu\alpha\beta}) = \frac{1}{4} h^{(4)\mu\nu\alpha\beta} \Psi_{\mu\nu\alpha\beta}^{(4)} + \frac{1}{2} h^{(2)\mu\nu} \Psi_{\mu\nu}^{(2)}, \tag{14.18}$$

where the current $\Psi_{\mu\nu}^{(2)}$ is the same energy-momentum tensor (14.10) and

$$\Psi_{\mu\nu\alpha\beta}^{(4)} = \partial_{(\alpha} F_\mu{}^\rho \partial_\beta F_{\nu)\rho} - \frac{1}{2} g_{(\mu\nu} \partial^\lambda F_{\alpha\sigma} \partial^\sigma F_{\beta)\lambda} - \frac{1}{2} g_{(\mu\nu} \partial_\alpha F^{\sigma\rho} \partial_{\beta)} F_{\sigma\rho}. \tag{14.19}$$

The whole action

$$\mathcal{L}_0(A_\mu) + \mathcal{L}_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\nu\alpha\beta}), \tag{14.20}$$

is invariant with respect to the spin one gauge transformations and the following higher spin transformations

$$\delta^1 A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma} + \frac{1}{2} \partial_\rho \epsilon_{\mu\lambda\sigma} \partial^\lambda F^{\sigma\rho}, \tag{14.21}$$

$$\delta^0 h^{(4)\mu\nu\alpha\beta} = 4\partial^{(\mu} \epsilon^{\nu\alpha\beta)}, \;\; \delta_\epsilon^0 h_\mu{}^{\mu\alpha\beta} = 2\epsilon_{(1)}^{\alpha\beta}, \tag{14.22}$$

$$\delta^0 h^{(2)\mu\nu} = 2\partial^{(\mu} \epsilon_{(2)}^{\nu)}, \;\; \delta^0 h_\mu^{(2)\mu} = 2\epsilon_{(3)}. \tag{14.23}$$

Therefore we have to prove that like the previously investigated scalar–higher spin coupling case [33], the interaction with the spin four gauge field leads to the additional interaction with the lower even spin two field. To do that according to the previous lesson we start from a spin one gauge invariant ansatz for the spin four transformation of A_μ field

$$\delta_\epsilon^1 A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma}. \tag{14.24}$$

Thus we have now the following variation of $\mathcal{L}_0$

$$\delta_\epsilon^1 \mathcal{L}_0 = \delta_\epsilon^1 (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}) = (\delta_\epsilon^1 A_\nu) \partial_\mu F^{\mu\nu} = -\partial_\mu (\epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\nu\sigma}) F^{\mu\nu}. \tag{14.25}$$

After some algebra, again neglecting total derivatives and using the Bianchi identity for $F_{\mu\nu}$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{14.26}$$

and taking into account the important relation

$$\begin{aligned} -\partial^\mu \epsilon^{\rho\lambda\sigma} \partial_\rho F_\mu{}^\nu \partial_\lambda F_{\sigma\nu} = {} & -\partial^{(\mu} \epsilon^{\rho\lambda\sigma)} \partial_{(\rho} F_\mu{}^\nu \partial_\lambda F_{\sigma)\nu} + \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial^\nu F_{\mu\lambda} \partial^\mu F_{\nu\sigma} \\ & -\frac{1}{2} \partial^\nu \epsilon^{\rho\lambda\sigma} \partial_\lambda F_{\sigma\nu} \partial^\mu F_{\mu\rho} - \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial^\mu F_{\mu\rho} \partial^\nu F_{\nu\sigma}, \end{aligned} \tag{14.27}$$

we arrive at the following form of the variation convenient for our analysis

$$\begin{aligned} \delta^1_\epsilon \mathcal{L}_0 \; = \; & -\partial^{(\mu} \epsilon^{\rho\lambda\sigma)} \partial_{(\rho} F_\mu{}^\nu \partial_\lambda F_{\sigma)\nu} + \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial^\nu F_{\mu\lambda} \partial^\mu F_{\nu\sigma} + \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial_\lambda F_{\mu\nu} \partial_\sigma F^{\mu\nu} \\ - \; & \partial_\lambda (\epsilon^{\lambda\sigma}_{(1)} F_{\mu\sigma}) \partial_\nu F^{\nu\mu} - \frac{1}{4} \epsilon^{\lambda\sigma}_{(1)} \partial^\mu F_{\mu\lambda} \partial^\nu F_{\nu\sigma} - \frac{1}{2} \partial^\rho \epsilon^{\nu\lambda\sigma} \partial_\lambda F_{\sigma\rho} \partial^\mu F_{\mu\nu} \\ + \; & \partial^{(\mu} \epsilon^{\nu)}_{(2)} F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{4} \epsilon_{(3)} F_{\mu\nu} F^{\mu\nu}. \end{aligned} \tag{14.28}$$

Returning to the gauge variation of the spin four field (14.16) we notice that all terms in the first line of (14.28) and the first two terms in the second line can be integrated to the interaction terms. The last term in the second line is proportional to the free field equations but is not integrable, so we can cancel this term only by changing the initial variation of A_μ (14.24). The modified form of (14.24) is

$$\delta^1_\epsilon A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma} + \frac{1}{2} \partial_\rho \epsilon_{\mu\lambda\sigma} \partial^\lambda F^{\sigma\rho}. \tag{14.29}$$

Therefore

$$\begin{aligned} \mathcal{L}_1 \; = \; & \frac{1}{4} h^{(4)\mu\rho\lambda\sigma} \partial_{(\rho} F_\mu{}^\nu \partial_\lambda F_{\sigma)\nu} - \frac{1}{8} h^{(4)\rho\lambda\sigma}_\rho \partial^\nu F_{\mu\lambda} \partial^\mu F_{\nu\sigma} - \frac{1}{8} h^{(4)\rho\lambda\sigma}_\rho \partial_\lambda F_{\mu\nu} \partial_\sigma F^{\mu\nu} \\ + \; & \partial_\lambda (\frac{1}{2} h^{(4)\rho\lambda\sigma}_\rho F_{\mu\sigma}) \partial_\nu F^{\nu\mu} + \frac{1}{8} h^{(4)\rho\lambda\sigma}_\rho \partial^\mu F_{\mu\lambda} \partial^\nu F_{\nu\sigma} \\ - \; & \frac{1}{2} h^{(2)\mu\nu} F_{\mu\sigma} F_\nu{}^\sigma + \frac{1}{8} h^{(2)\rho}_\rho F_{\mu\nu} F^{\mu\nu}. \end{aligned} \tag{14.30}$$

But the two terms in the second line are proportional to the equation of motion for the initial Lagrangian (14.1), hence they are not physical and can be removed by the following field redefinition

$$A_{\mu} \to A_{\mu} - \partial_{\lambda}(\frac{1}{2}h_{\alpha}^{\ \alpha\lambda\sigma}F_{\mu\sigma}) - \frac{1}{8}h^{\alpha}_{\ \alpha\mu\sigma}\partial_{\beta}F^{\beta\sigma}. \tag{14.31}$$

So we can drop the second line of (14.30).

Another novelty in (14.30) in comparison with the previous case is the third line of (14.28). Comparing with (14.9) we see that we can integrate these two terms introducing an additional spin two field coupling and compensate the first and third line introducing the linearized Lagrangian (14.18) for the coupling of the electromagnetic field to the spin four and spin two fields with the set of higher spin field transformations (14.21)-(14.23).

Therefore we proved that the interaction with the spin four gauge field leads to the additional interaction with the lower even spin two field.

15 Generalization to the 2-2-4 and 2-2-6 interactions

In this section we turn to the spin two field as a lower spin field in the construction of the higher spin gauge invariant interactions with spin 4 and spin 6 gauge potentials. And again we want to keep manifest the lower spin two gauge invariance.

So proceeding similarly as in the previous section we start from the free spin two Pauli-Fierz Lagrangian [17]

$$\mathcal{L}_0(h^{(2)}_{\mu\nu}) = \frac{1}{2}\partial_\mu h^{(2)}_{\alpha\beta}\partial^\mu h^{(2)\alpha\beta} - \partial_\alpha h^{(2)\alpha\beta}\partial_\mu h^{(2)\mu}_{\beta} + \partial_\mu h^{(2)\alpha}_{\alpha}\partial_\beta h^{(2)\beta\mu} - \frac{1}{2}\partial_\mu h^{(2)\alpha}_{\alpha}\partial^\mu h^{(2)\beta}_{\beta}, \quad (15.1)$$

and try to solve the following Noether's equations, either

$$\delta^1_\varepsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) + \delta^0_\varepsilon \mathcal{L}_1(h^{(2)}_{\mu\nu}, h^{(4)\alpha\beta\lambda\rho}) = 0, \quad (15.2)$$

or

$$\delta^1_\varepsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) + \delta^0_\varepsilon \mathcal{L}_1(h^{(2)}_{\mu\nu}, h^{(4)\alpha\beta\lambda\rho}, h^{(6)\mu\nu\alpha\beta\lambda\rho}) = 0. \quad (15.3)$$

Again we present first the final result for the 2-2-4 gauge invariant interaction

$$\begin{aligned}
&\mathcal{L}_1(h^{(2)}_{\mu\nu}, h^{(4)}_{\alpha\beta\mu\nu}) = \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}(h^{(2)}_{\mu\nu}) \\
&= \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Gamma_{\alpha\beta,\rho\sigma}\Gamma_{\mu\nu,}{}^{\rho\sigma} - \frac{1}{6}h^{(4)\alpha\mu\nu}_{\alpha}\Gamma_{\mu}{}^{\rho,\sigma\lambda}\Gamma_{\nu\rho,\sigma\lambda}, \quad (15.4)
\end{aligned}$$

with the following gauge transformations

$$\delta_\epsilon h^{(2)}_{\mu\nu} = \epsilon^{\rho\lambda\sigma}\partial_\rho\Gamma_{\lambda\sigma,\mu\nu} - \partial_\rho\epsilon_{\lambda\sigma(\mu}\Gamma_{\nu)}{}^{\rho,\lambda\sigma}, \quad (15.5)$$

$$\delta^0_\epsilon h^{(4)\mu\rho\lambda\sigma} = 4\partial^{(\mu}\epsilon^{\rho\lambda\sigma)}, \quad \delta^0_\epsilon h^{(4)\rho\lambda\sigma}_{\rho} = 2\epsilon^{\lambda\sigma}_{(1)}. \quad (15.6)$$

The final result for the 2-2-6 case correspondingly looks like

$$\begin{aligned}
&\mathcal{L}_1(h^{(2)}, h^{(4)}, h^{(6)}) = -\frac{1}{6}h^{(6)\alpha\beta\mu\nu\lambda\rho}\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} + \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} \\
&= -\frac{1}{6}h^{(6)\alpha\beta\mu\nu\lambda\rho}\partial_\alpha\Gamma_{\beta\mu,}{}^{\sigma\delta}\partial_\nu\Gamma_{\lambda\rho,\sigma\delta} + \frac{1}{6}h^{(6)\alpha\mu\nu\lambda\rho}_{\alpha}\partial_\mu\Gamma_{\nu}{}^{\kappa,\sigma\delta}\partial_\lambda\Gamma_{\rho\kappa,\sigma\delta} \\
&+\frac{1}{12}h^{(6)\alpha\mu\nu\lambda\rho}_{\alpha}\partial^\kappa\Gamma_{\mu\nu,}{}^{\sigma\delta}\partial_\sigma\Gamma_{\lambda\rho),\kappa\delta} + \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Gamma_{\alpha\beta,\rho\sigma}\Gamma_{\mu\nu,}{}^{\rho\sigma} - \frac{1}{6}h^{(4)\alpha\mu\nu}_{\alpha}\Gamma_{\mu}{}^{\rho,\sigma\lambda}\Gamma_{\nu\rho,\sigma\lambda}. \quad (15.7)
\end{aligned}$$

This formula together with the corrected gauge transformation

$$\delta^1_\epsilon h^{(2)}_{\alpha\beta} = \epsilon^{\mu\nu\rho\lambda\sigma}\partial_\mu\partial_\nu\partial_\rho\Gamma_{\lambda\sigma,\alpha\beta} - \frac{4}{3}\partial^\rho\epsilon_\alpha{}^{\mu\nu\lambda\sigma}\partial_\lambda\partial_\sigma\Gamma_{\beta\rho,\mu\nu} + \frac{1}{3}\partial^\rho\partial^\lambda\epsilon_{\alpha\beta}{}^{\mu\nu\sigma}\partial_\sigma\Gamma_{\rho\lambda,\mu\nu}, \tag{15.8}$$

$$\delta^0_\epsilon h^{(6)\mu\nu\alpha\beta\sigma\rho} = 6\partial^{(\mu}\epsilon^{\nu\alpha\beta\sigma\rho)}(x), \quad \delta^0_\epsilon h^{(6)\mu\alpha\beta\sigma\rho}_\mu = 2\epsilon^{\alpha\beta\sigma\rho}_{(1)}. \tag{15.9}$$

$$\delta^0_\epsilon h^{(4)\mu\rho\lambda\sigma} = 4\partial^{(\mu}\epsilon^{\rho\lambda\sigma)}_{(2)}, \quad \delta^0_\epsilon h^{(4)\rho\lambda\sigma}_\rho = 2\epsilon^{\lambda\sigma}_{(3)} \tag{15.10}$$

solves completely Noether's equation (15.3).

$\Gamma_{\lambda\sigma,\mu\nu}$ here is the spin two gauge invariant symmetrized linearized Riemann curvature

$$\Gamma_{\alpha\beta,\mu\nu} = \frac{1}{2}(R_{\alpha\mu,\beta\nu} + R_{\beta\mu,\alpha\nu}), \tag{15.11}$$

$$\Gamma_{(\alpha\beta,\mu)\nu} = 0, \tag{15.12}$$

introduced by de Witt and Freedman for higher spin gauge fields together with the higher spin generalization of the Christoffel symbols [16]. This symmetrized curvature is more convenient for the construction of an interaction with symmetric tensors. The corresponding Ricci tensor (Fronsdal operator for higher spin generalization) and scalar can be defined in the usual manner using traces

$$\mathcal{F}_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} = \Box h^{(2)}_{\mu\nu} - 2\partial_{(\mu}\partial^\alpha h^{(2)}_{\nu)\alpha} + \partial_\mu\partial_\nu h^{(2)\alpha}_\alpha, \tag{15.13}$$

$$\mathcal{F} = \mathcal{F}^\mu_\mu = 2(\Box h^{(2)\mu}_\mu - \partial_\mu\partial_\nu h^{(2)\mu\nu}). \tag{15.14}$$

In terms of these objects the Bianchi identities can be written as

$$\partial_\lambda\Gamma_{\mu\nu,\alpha\beta} = \partial_{(\mu}\Gamma_{\nu)\lambda,\alpha\beta} + \partial_{(\alpha}\Gamma_{\beta)\lambda,\mu\nu}, \tag{15.15}$$

$$\partial_\lambda\mathcal{F}_{\alpha\beta} = \partial^\mu\Gamma_{\mu\lambda,\alpha\beta} + \partial_{(\alpha}\mathcal{F}_{\beta)\lambda}, \tag{15.16}$$

$$\partial^\lambda\mathcal{F}_{\lambda\mu} = \frac{1}{2}\partial_\mu\mathcal{F}^\alpha_\alpha. \tag{15.17}$$

So to prove (15.4)-(15.6) we introduce the following starting ansatz for the spin four transformation of the spin two field

$$\delta^1_\epsilon h^{(2)}_{\mu\nu} = \epsilon^{\rho\lambda\sigma}\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}, \tag{15.18}$$

Then a variation of (15.1) with respect to (15.4) is

$$\delta^1_\epsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) = \frac{\delta \mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}} \delta^1_\epsilon h^{(2)}_{\mu\nu} = -(\mathcal{F}^{\mu\nu} - \frac{1}{2} g^{\mu\nu}\mathcal{F})\epsilon^{\rho\lambda\sigma}\partial_\rho \Gamma_{\lambda\sigma,\mu\nu}. \tag{15.19}$$

To integrate it and solve the equation (15.2) we submit to the following strategy:

1) First we perform a partial integration and use the Bianchi identity (15.16) to lift the variation to a curvature square term.

2) Then we make a partial integration again and rearrange indices using (15.12) and (15.15) to extract an integrable part.

3) Symmetrizing expressions in this way we classify terms as

- integrable
- integrable and subjected to field redefinition (proportional to the free field equation of motion)
- non integrable but reducible by deformation of the initial ansatz for the gauge transformation (again proportional to the free field equation of motion)

Then if no other terms remain we can construct our interaction together with the corrected first order transformation. Following this strategy after some fight with formulas we win the battle obtaining the following expression

$$\begin{aligned} \delta^1_\epsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) &= -\partial^{(\alpha}\epsilon^{\beta\mu\nu)}(\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} - \Psi^{(4)}_{(\mathcal{F})\alpha\beta\mu\nu}) \\ &- \epsilon^{\mu\nu}_{(1)}\Gamma_{\mu\nu,\alpha\beta}\frac{\delta \mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}} + \partial^\rho \epsilon_\alpha{}^{\mu\nu}\Gamma_{\beta\rho,\mu\nu}\frac{\delta \mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}}, \end{aligned} \tag{15.20}$$

where

$$\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} = \Gamma_{(\alpha\beta,}{}^{\rho\sigma}\Gamma_{\mu\nu),\rho\sigma} - \frac{2}{3} g_{(\alpha\beta}\Gamma_\mu{}^{\rho,\sigma\lambda}\Gamma_{\nu)\rho,\sigma\lambda}, \tag{15.21}$$

$$\Psi^{(4)}_{(\mathcal{F})\alpha\beta\mu\nu} = \mathcal{F}_{(\alpha\beta}\mathcal{F}_{\mu\nu)} - g_{(\alpha\beta}\mathcal{F}^\sigma_\mu \mathcal{F}_{\nu)\sigma} = -\frac{\delta \mathcal{L}_0}{\delta h^{(2)(\alpha\beta}}\mathcal{F}_{\mu\nu)} + g_{(\alpha\beta}\frac{\delta \mathcal{L}_0}{\delta h^{(2)\mu}_\sigma}\mathcal{F}_{\nu)\sigma}, \tag{15.22}$$

$$\frac{\delta \mathcal{L}_0}{\delta h^{(2)\alpha\beta}} = -\mathcal{F}_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta}\mathcal{F}. \tag{15.23}$$

So we see immediately that in (15.20) only the last term of the second line is not integrable but proportional to the equation of motion and can be dropped by the correction (15.5) to the initial gauge transformation (15.18). Other terms of (15.20) can be integrated to

$$\mathcal{L}_1(h^{(2)}_{\mu\nu}, h^{(4)}_{\alpha\beta\mu\nu}) = \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\left(\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}(h^{(2)}_{\mu\nu}) - \Psi^{(4)}_{(\mathcal{F})\alpha\beta\mu\nu}\right) + \frac{1}{2}h^{(4)\alpha\mu\nu}_{\alpha}\Gamma_{\mu\nu,\alpha\beta}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}}. \tag{15.24}$$

On the other hand taking into account (15.22) and (15.23) we can compensate $\Psi^{(4)}_{(\mathcal{F})}$ and the last term in (15.24) by the following field redefinition

$$h^{(2)}_{\mu\nu} \to h^{(2)}_{\mu\nu} - \frac{1}{2}h^{(4)\alpha\lambda\sigma}_{\alpha}\Gamma_{\lambda\sigma,\mu\nu} - \frac{1}{4}h^{(4)\alpha\lambda}_{\mu\nu}\mathcal{F}_{\alpha\lambda} + \frac{1}{4}h^{(4)\alpha\lambda}{}_{\alpha(\mu}\mathcal{F}_{\nu)\lambda}. \tag{15.25}$$

Thus after field redefinition we arrive at the 2-2-4 gauge invariant interaction (15.4) with the gauge transformations (15.5), (15.6).

Now in possession of knowledge about the 2-2-4 interaction we start to construct the most nontrivial interaction in this article between spin 2 and spin 6 gauge fields (15.7)-(15.10). We would like to check the appearance of the 2-2-4 coupling during the construction of 2-2-6 which we expect from the analogy with the scalar case considered in [47, 33] and the 1-1-4 case considered in the previous section.

To proceed we have to solve the following initial Noether's equation

$$\delta^1_\varepsilon\mathcal{L}_0(h^{(2)}_{\mu\nu}) + \delta^0_\varepsilon\mathcal{L}_1(h^{(2)}_{\mu\nu}, h^{(6)}_{\alpha\beta\lambda\rho\sigma\delta}) = 0, \tag{15.26}$$

with a starting ansatz for the spin 6 first order gauge transformation for the spin 2 field:

$$\delta^1_\epsilon h^{(2)}_{\mu\nu}(x) = \epsilon^{\alpha\beta\rho\lambda\sigma}(x)\partial_\alpha\partial_\beta\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}(x), \tag{15.27}$$

and the standard zero order gauge transformation for the spin 6 gauge field

$$\delta^0_\epsilon h^{(6)\mu\nu\alpha\beta\sigma\rho} = 6\partial^{(\mu}\epsilon^{\nu\alpha\beta\sigma\rho)}(x), \tag{15.28}$$

$$\delta^0_\epsilon h^{(6)\mu\alpha\beta\sigma\rho}_{\mu} = 2\epsilon^{\alpha\beta\sigma\rho}_{(1)}. \tag{15.29}$$

First of all we have to transform the variation

$$\delta^1_\varepsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) = -(\mathcal{F}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{F})\epsilon^{\alpha\beta\rho\lambda\sigma}\partial_\alpha\partial_\beta\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}, \tag{15.30}$$

into a form convenient for integration. Following the same strategy as before in the 2-2-4 case, using many times partial integration and Bianchi identities (15.12), (15.15)-(15.17), we obtain after tedious but straightforward calculations

$$\begin{aligned}\delta^1_\varepsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) &= \partial^{(\alpha}\epsilon^{\beta\mu\nu\lambda\rho)}\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} - \partial^{(\alpha}\epsilon^{\beta\mu\nu)}_{(2)}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}\\ &+\frac{4}{3}\partial^\rho\epsilon_\alpha{}^{\mu\nu\lambda\sigma}\partial_\lambda\partial_\sigma\Gamma_{\beta\rho,\mu\nu}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}} - \frac{1}{3}\partial^\rho\partial^\lambda\epsilon_{\alpha\beta}{}^{\mu\nu\sigma}\partial_\sigma\Gamma_{\rho\lambda,\mu\nu}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}}\\ &-R^{\mu\nu}_{int}(\Gamma,\mathcal{F},\epsilon)\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}},\end{aligned} \tag{15.31}$$

where

$$\begin{aligned}\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} &= \partial_{(\alpha}\Gamma_{\beta\mu,}{}^{\sigma\delta}\partial_\nu\Gamma_{\lambda\rho),\sigma\delta} - g_{(\alpha\beta}\partial_\mu\Gamma_\nu{}^{\kappa,\sigma\delta}\partial_\lambda\Gamma_{\rho)\kappa,\sigma\delta}\\ &\quad -\frac{1}{2}g_{(\alpha\beta}\partial^\kappa\Gamma_{\mu\nu,}{}^{\sigma\delta}\partial_\sigma\Gamma_{\lambda\rho),\kappa\delta},\end{aligned} \tag{15.32}$$

$$\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} = \Gamma_{(\alpha\beta,}{}^{\rho\sigma}\Gamma_{\mu\nu),\rho\sigma} - \frac{2}{3}g_{(\alpha\beta}\Gamma_\mu{}^{\rho,\sigma\lambda}\Gamma_{\nu)\rho,\sigma\lambda}, \tag{15.33}$$

and $R^{\mu\nu}_{int}(\Gamma,\mathcal{F},\epsilon)\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}}$ are remaining integrable terms proportional to the equation of motion. Indeed the symmetric tensor $R^{\mu\nu}_{int}(\Gamma,\mathcal{F})$ is expressed through the only integrable combinations of derivatives of the gauge parameter

$$\begin{aligned}R^{\mu\nu}_{int}(\Gamma,\mathcal{F},\epsilon) &= \epsilon^{\alpha\beta\lambda\delta}_{(1)}\partial_\alpha\partial_\beta\Gamma_{\lambda\delta,}{}^{\mu\nu} - \frac{1}{3}\partial^\lambda\epsilon^{\alpha\beta\delta(\mu}_{(1)}\partial_\alpha\Gamma^{\nu)}{}_{\lambda,\beta\delta} + \partial_\lambda\left[\partial^{(\lambda}\epsilon^{\alpha\beta\delta\mu\nu)}\partial_\alpha\mathcal{F}_{\beta\delta}\right]\\ &- \frac{2}{3}\partial_\lambda\left[\epsilon^{\lambda\alpha\mu\nu}_{(1)}\partial_\alpha\mathcal{F}\right] + \frac{1}{6}\epsilon^{\alpha\beta\mu\nu}_{(1)}\partial_\alpha\partial_\beta\mathcal{F} + \partial^{(\alpha}\epsilon^{\beta\mu\nu)}_{(2)}\mathcal{F}_{\alpha\beta} + \frac{5}{3}\partial^\alpha\epsilon^{\beta\lambda\mu\nu}_{(1)}\partial_\lambda\mathcal{F}_{\alpha\beta}\\ &- \frac{5}{3}\partial_\lambda\left[\epsilon^{\lambda\alpha\beta(\mu}_{(1)}\partial_\alpha\mathcal{F}^{\nu)}_\beta\right] + \frac{1}{6}\Box\epsilon^{\alpha\beta\mu\nu}_{(1)}\mathcal{F}_{\alpha\beta} - \frac{1}{6}\partial^\lambda\epsilon^{\alpha\beta\mu\nu}_{(1)}\partial_\lambda\mathcal{F}_{\alpha\beta} - \frac{1}{2}\epsilon^{\alpha(\mu}_{(3)}\mathcal{F}^{\nu)}_\alpha.\end{aligned} \tag{15.34}$$

The second line in (15.31) is not integrable and therefore can be cancelled by the following deformation of the initial ansatz for the transformation (15.27)

$$\delta^1_\epsilon h^{(2)}_{\alpha\beta} = \epsilon^{\mu\nu\rho\lambda\sigma}\partial_\mu\partial_\nu\partial_\rho\Gamma_{\lambda\sigma,\alpha\beta} - \frac{4}{3}\partial^\rho\epsilon_\alpha{}^{\mu\nu\lambda\sigma}\partial_\lambda\partial_\sigma\Gamma_{\beta\rho,\mu\nu} + \frac{1}{3}\partial^\rho\partial^\lambda\epsilon_{\alpha\beta}{}^{\mu\nu\sigma}\partial_\sigma\Gamma_{\rho\lambda,\mu\nu}. \tag{15.35}$$

Then substituting into (15.34) $\partial^{(\lambda}\epsilon^{\alpha\beta\delta\mu\nu)}$ with $\frac{1}{6}h^{(6)\lambda\alpha\beta\delta\mu\nu}$, $\partial^{(\alpha}\epsilon^{\beta\mu\nu)}_{(2)}$ with $\frac{1}{4}h^{(4)\alpha\beta\mu\nu}$, and correspondingly $2\epsilon^{\alpha\beta\mu\nu}_{(1)}$ and $2\epsilon^{\alpha\beta}_{(3)}$ with their traces, we can integrate the first and third line of (15.31) to

$$\begin{aligned}
&\mathcal{L}_1(h^{(2)},h^{(4)},h^{(6)}) = -\frac{1}{6}h^{(6)\alpha\beta\mu\nu\lambda\rho}\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} + \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} \\
&+R^{\mu\nu}_{int}(\Gamma,\mathcal{F},h^{(6)},h^{(4)})\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}}
\end{aligned} \tag{15.36}$$

where

$$\begin{aligned}
&R^{\mu\nu}_{int}(\Gamma,\mathcal{F},h^{(6)},h^{(4)}) = \frac{1}{2}h^{(6)\rho\alpha\beta\lambda\delta}_{\rho}\partial_\alpha\partial_\beta\Gamma_{\lambda\delta,}{}^{\mu\nu} - \frac{1}{6}\partial^\lambda h^{(6)\rho\alpha\beta\delta(\mu}_{\rho}\partial_\alpha\Gamma^{\nu)}_{\lambda,\beta\delta} + \partial_\lambda\left[\frac{1}{6}h^{(6)\lambda\alpha\beta\delta\mu\nu}\partial_\alpha\mathcal{F}_{\beta\delta}\right] \\
&-\frac{2}{6}\partial_\lambda\left[h^{(6)\rho\lambda\alpha\mu\nu}_{\rho}\partial_\alpha\mathcal{F}\right] + \frac{1}{12}h^{(6)\rho\alpha\beta\mu\nu}_{\rho}\partial_\alpha\partial_\beta\mathcal{F} + \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\mathcal{F}_{\alpha\beta} + \frac{5}{6}\partial^\alpha h^{(6)\rho\beta\lambda\mu\nu}_{\rho}\partial_\lambda\mathcal{F}_{\alpha\beta} \\
&-\frac{5}{6}\partial_\lambda\left[h^{(6)\rho\lambda\alpha\beta(\mu}_{\rho}\partial_\alpha\mathcal{F}^{\nu)}_{\beta}\right] + \frac{1}{12}\Box h^{(6)\rho\alpha\beta\mu\nu}_{\rho}\mathcal{F}_{\alpha\beta} - \frac{1}{12}\partial^\lambda h^{(6)\rho\alpha\beta\mu\nu}_{\rho}\partial_\lambda\mathcal{F}_{\alpha\beta} - \frac{1}{4}h^{(4)\rho\alpha(\mu}_{\rho}\mathcal{F}^{\nu)}_{\alpha}.
\end{aligned} \tag{15.37}$$

Now we define a field redefinition for $h^{(2)\mu\nu}$

$$h^{(2)\mu\nu} \to h^{(2)\mu\nu} - R^{\mu\nu}_{int}(\Gamma,\mathcal{F},h^{(6)},h^{(4)}), \tag{15.38}$$

using which we can drop the last term in (15.36).

Thus we arrive at the promised result that the 2-2-6 interaction automatically includes also the 2-2-4 interaction constructed above, and the corresponding trilinear interaction Lagrangian is (15.7). This formula together with the corrected gauge transformations (15.8)-(15.10) solves completely Noether's equation (15.3).

Finally note that these interactions should reproduce the flat space limit of the Fradkin-Vasiliev type nonlinear interactions [2] constructed in an AdS background. For some other vertices i.e. 2-s-s and 1-s-s with additional nonabelian symmetry such construction and connection with Fradkin-Vasiliev formalism can be found in [7], where authors used BRST-cohomological approach.

16 2s-s-s interaction Lagrangian

The most elegant and convenient way of handling symmetric tensors such as $h^{(s)}_{\mu_1\mu_2...\mu_s}(z)$ is by contracting it with the s'th tensorial power of a vector a^μ of the tangential space at the base point z [12]-[15]

$$h^{(s)}(z;a) = \sum_{\mu_i}(\prod_{i=1}^{s} a^{\mu_i})h^{(s)}_{\mu_1\mu_2...\mu_s}(z). \tag{16.1}$$

In this way we obtain a homogeneous polynomial in the vector a^μ of degree s. In this formalism the symmetrized gradient, trace and divergence are[6]

$$Grad : h^{(s)}(z;a) \Rightarrow Gradh^{(s+1)}(z;a) = (a\nabla)h^{(s)}(z;a), \tag{16.2}$$

$$Tr : h^{(s)}(z;a) \Rightarrow Trh^{(s-2)}(z;a) = \frac{1}{s(s-1)}\Box_a h^{(s)}(z;a), \tag{16.3}$$

$$Div : h^{(s)}(z;a) \Rightarrow Divh^{(s-1)}(z;a) = \frac{1}{s}(\nabla\partial_a)h^{(s)}(z;a). \tag{16.4}$$

The gauge variation of a spin s field is

$$\delta h^{(s)}(z;a) = s(a\nabla)\epsilon^{(s-1)}(z;a), \tag{16.5}$$

with traceless gauge parameter

$$\Box_a \epsilon^{(s-1)}(z;a) = 0, \tag{16.6}$$

for the double traceless gauge field

$$\Box_a^2 h^{(s)}(z;a) = 0. \tag{16.7}$$

We will use the deWit-Freedman curvature and Cristoffel symbols [16]. We contract them with the degree s tensorial power of one tangential vector a^μ in the first set of s

[6]To distinguish easily between "a" and "z" spaces we introduce for space-time derivatives $\frac{\partial}{\partial z^\mu}$ the notation ∇_μ and as before we will admit integration everywhere where it is necessary (we work with a Lagrangian as with an action) and therefore we will neglect all space-time total derivatives when making a partial integration

indices and with a similar tensorial power of another tangential vector b^ν in its second set. The deWit-Freedman curvature and n-th Cristoffel symbol are then written as

$$\Gamma^{(s)}(z;b,a): \qquad \Gamma^{(s)}(z;b,\lambda a) = \Gamma^{(s)}(z;\lambda b,a) = \lambda^s \Gamma^{(s)}(z;b,a), \tag{16.8}$$

$$\Gamma^{(s)}_{(n)}(z;b,a): \qquad \Gamma^{(s)}_{(n)}(z;b,\lambda a) = \lambda^s \Gamma^{(s)}_{(n)}(z;b,a), \tag{16.9}$$

$$\Gamma^{(s)}_{(n)}(z;\lambda b,a) = \lambda^n \Gamma^{(s)}_{(n)}(z;b,a), \tag{16.10}$$

$$\Gamma^{(s)}(z;b,a) = \Gamma^{(s)}_{(n)}(z;b,a)|_{n=s}. \tag{16.11}$$

Next we introduce the notation $*_a, *_b$ for a contraction in the symmetric spaces of indices a or b

$$*_a \quad = \quad \frac{1}{(s!)^2}\prod_{i=1}^{s} \overleftarrow{\partial}^{\mu_i}_a \overrightarrow{\partial}^a_{\mu_i}. \tag{16.12}$$

All required manipulations in the framework of this formalism are discussed in the Appendix of this paper. Here we will only present Fronsdal's Lagrangian in terms of these conventions:

$$\mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2}h^{(s)}(a) *_a \mathcal{F}^{(s)}(a) + \frac{1}{8s(s-1)}\Box_a h^{(s)}(a) *_a \Box_a \mathcal{F}^{(s)}(a). \tag{16.13}$$

where $\mathcal{F}^{(s)}(z;a)$ is so called Fronsdal tensor

$$\mathcal{F}^{(s)}(z;a) = \Box h^{(s)}(z;a) - (a\nabla)(\nabla\partial_a)h^{(s)}(z;a) + \frac{1}{2}(a\nabla)^2\Box_a h^{(s)}(z;a) \tag{16.14}$$

To obtain the equation of motion we vary (16.13) and obtain

$$\delta\mathcal{L}_0(h^{(s)}(a)) = -(\mathcal{F}^{(s)}(a) - \frac{a^2}{4}\Box_a\mathcal{F}^{(s)}(a)) *_a \delta h^{(s)}(a). \tag{16.15}$$

Zero order gauge invariance can be checked easily by substitution of (16.5) into this variation and use of the duality relation (C.5) and identity (C.29) taking into account tracelessness of the gauge parameter (16.6). Now we turn to the generalization of Noether's procedure of the 2-2-4 case to the general s-s-2s interaction construction. Noether's equation in this case looks like

$$\delta_{(1)}\mathcal{L}_0(h^{(s)}(a)) + \delta_0\mathcal{L}_1(h^{(s)}(a), h^{(2s)}(b)) = 0. \tag{16.16}$$

And we would like to show that the solution of the latter is (with generalized Bell-Robinson current [1])

$$\mathcal{L}_1(h^{(s)}(a), h^{(2s)}(b)) = \frac{1}{2s} h^{(2s)}(z;b) *_b \Psi^{(2s)}_{(\Gamma)}(z;b), \tag{16.17}$$
$$\Psi^{(2s)}_{(\Gamma)}(z;b) = \Gamma^{(s)}(b,a) *_a \Gamma^{(s)}(b,a) - \frac{b^2}{2(s+1)} \partial^b_\mu \Gamma^{(s)}(b,a) *_a \partial^\mu_b \Gamma^{(s)}(b,a). \tag{16.18}$$

To prove this we must propose a first order variation of the spin s field with respect to a spin 2s gauge transformation. Remembering that Fronsdal's higher spin gauge potential is double traceless, we must make sure that the same holds for the variation. Expanding the general variation in powers of a^2

$$\delta h^{(s)}(a) = \delta h^{(s)}_{(1)}(a) + a^2 \delta h^{(s-2)}(a) + (a^2)^2 \delta h^{(s-4)}(a) + \ldots, \tag{16.19}$$

we see that the double tracelessness condition $\Box^2_a \delta h^{(s)}(a) = 0$ expresses the third and higher terms of the expansion (16.19) through the first two free parameters $\delta h^{(s)}_{(1)}(a)$ and $\delta h^{(s-2)}(a)$[7]. From the other hand Fronsdal's tensor is double traceless by definition and therefore all these $O(a^4)$ terms are unimportant because they do not contribute to (16.15). This leaves us freedom in the choice of $\delta h^{(s-2)}(a)$. Substituting (16.19) in (16.15) we discover that the following choice of $\delta h^{(s-2)}(a)$

$$\delta h^{(s-2)}(a) = \frac{1}{2(D+2s-2)} \Box_a \delta h^{(s)}_{(1)}(a), \tag{16.20}$$

reduces our variation (16.15) to

$$\delta_{(1)} \mathcal{L}_0(h^{(s)}(a)) = -\mathcal{F}^{(s)}(a) *_a \delta h^{(s)}_{(1)}(a). \tag{16.21}$$

[7] For completeness we present here the solution for $\delta h^{(s-4)}(a)$ following from the double tracelessness condition

$$\delta h^{(s-4)}(a) = -\frac{1}{8\alpha_1\alpha_2} \left[\Box^2_a \delta h^{(s)}_{(1)}(a) + 4\alpha_1 \Box_a \delta h^{(s-2)}(a) \right],$$
$$\alpha_k = D + 2s - (4+2k), \quad k \in \{1,2\}.$$

Then we propose the following spin 2s transformation of the spin s potential

$$\delta h^{(s)}_{(1)}(a) = \tilde{\mathcal{U}}(b,a,2,s)\epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a), \tag{16.22}$$

where

$$\tilde{\mathcal{U}}(b,a,2,s) = \frac{(-1)^s}{(s-1)!}\prod_{k=2}^{s}\left[(\nabla\partial_b) - \frac{1}{k}A_b(\nabla\partial_a)\right], \tag{16.23}$$

is operator dual to

$$[(b\nabla) - \frac{1}{2}(a\nabla)B_a]\mathcal{U}(b,a,3,s) = \prod_{k=2}^{s}[(b\nabla) - \frac{1}{k}(a\nabla)B_a], \tag{16.24}$$

with respect to the $*_{a,b}$ contraction product. Taking into account (C.22) and Bianchi identities (C.28) we get

$$\begin{aligned}
&\delta_{(1)}\mathcal{L}_0(h^{(s)}(a)) = \epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a) *_a [(b\nabla) - \frac{1}{2}(a\nabla)B_a]\mathcal{U}(b,a,3,s)\mathcal{F}^{(s)}(z;a) \\
&= \epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a) *_a \frac{1}{s(s-1)}[(b\nabla) - \frac{1}{2}(a\nabla)B_a]\Box_b\Gamma^{(s)}(z;b,a) \\
&= \epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a) *_a \frac{1}{s}(\nabla\partial_b)\Gamma^{(s)}(z;b,a) \\
&= -(b\nabla)\epsilon^{2s-1}(b) *_b \Gamma^{(s)}(b,a) *_a \Gamma^{(s)}(b,a) - \epsilon^{2s-1}(b) *_b \nabla_\mu\Gamma^{(s)}(b,a) *_a \frac{1}{s}\partial^\mu_b\Gamma^{(s)}(b,a).
\end{aligned} \tag{16.25}$$

Then using a secondary Bianchi identity (C.27) and a primary one (C.6) one can show that

$$\begin{aligned}
&-\epsilon^{2s-1}(b) *_b \nabla_\mu\Gamma^{(s)}(b,a) *_a \frac{1}{s}\partial^\mu_b\Gamma^{(s)}(b,a) \\
&= \frac{1}{2s(s+1)(2s-1)}(\nabla\partial_b)\epsilon^{2s-1}(b) *_b \partial^b_\mu\Gamma^{(s)}(b,a) *_a \partial^\mu_b\Gamma^{(s)}(b,a).
\end{aligned} \tag{16.26}$$

Putting all together we see that the integrated first order interaction Lagrangian (16.17) supplemented with transformation (16.22) for $h^{(s)}(a)$ and the standard zero order transformations for $h^{(2s)}(a)$

$$\delta_0 h^{(2s)}(z;b) = 2s(b\nabla)\epsilon^{(2s-1)}(z;b), \tag{16.27}$$

$$\delta_0\Box_b h^{(2s)}(z;b) = 4s(\nabla\partial_b)\epsilon^{(2s-1)}(z;b), \tag{16.28}$$

completely solves Noether's equation (16.16). Note that here just as in the 2-2-4 case we did not obtain an interaction with lower spins because all derivatives included in the ansatz were used for the lifting to the second curvature.

Appendix A

The Euclidian AdS_{d+1} metric

$$ds^2 = g_{\mu\nu}(z) dz^\mu dz^\nu = \frac{1}{(z^0)^2} \delta_{\mu\nu} dz^\mu dz^\nu \tag{A.1}$$

can be realized as an induced metric for the hypersphere defined by the following embedding procedure in $d+2$ dimensional Minkowski space

$$X^A X^B \eta_{AB} = -X_{-1}^2 + X_0^2 + \sum_{i=1}^{d} X_i^2 = -1, \tag{A.2}$$

$$X_{-1}(z) = \frac{1}{2}\left(\frac{1}{z_0} + \frac{z_0^2 + \sum_{i=1}^d z_i^2}{z_0}\right), \tag{A.3}$$

$$X_0(z) = \frac{1}{2}\left(\frac{1}{z_0} - \frac{z_0^2 + \sum_{i=1}^d z_i^2}{z_0}\right), \tag{A.4}$$

$$X_i(z) = \frac{z_i}{z_0}. \tag{A.5}$$

Using these embedding rules we can identify the variable $\zeta(z, w)$ as an $SO(1, d+1)$ invariant scalar product

$$-X^A(z) Y^B(w) \eta_{AB} = \frac{1}{2 z_0 w_0}\left(2 z_0 w_0 + \sum_{\mu=0}^{d} (z - w)_\mu^2\right) = \zeta = u + 1, \tag{A.6}$$

and therefore can be realized by *cosh* of a hyperbolic angle. Indeed we can introduce another embedding

$$X_{-1}(\eta, \omega_\mu) = \cosh\eta, \tag{A.7}$$

$$X_\mu(\eta, \omega_\mu) = \sinh\eta\, \omega_\mu \quad , \qquad \sum_{\mu=0}^{d} \omega_\mu^2 = 1, \tag{A.8}$$

$$ds^2 = d\eta^2 + \sinh^2\eta\, d\Omega_d. \tag{A.9}$$

In these coordinates the chordal distance u between an arbitrary point $X^A(\eta, \Omega_\mu)$ and the pole of the hypersphere $Y^A(\eta = 0, \omega_\mu)$ is simply

$$\zeta = -X^A Y^B \eta_{AB} = \cosh\eta. \tag{A.10}$$

Therefore the invariant measure is expressed as

$$\sqrt{g} d\eta d\Omega_d = (\sinh\eta)^d d\eta d\Omega_d = [u(u+2)]^{\frac{d-1}{2}} du d\Omega_d. \tag{A.11}$$

So we see that the integration measure for $d = 3$ ($D = d + 1 = 4$) will cancel one order of u^{-n} in short distance singularities and we have to count the singularities starting from u^{-2} which is "logarithmically divergent" in standard QFT terminology.

In this manuscript we use the following rules and relations for $u(z, z')$, I_{1a}, I_{2c} and the bitensorial basis $\{I_i\}_{i=1}^{4}$

$$\Box u = (d+1)(u+1), \quad \nabla_\mu \partial_\nu u = g_{\mu\nu}(u+1), \quad g^{\mu\nu}\partial_\mu u \partial_\nu u = u(u+2), \tag{A.12}$$

$$\partial_\mu \partial_{\nu'} u \nabla^\mu u = (u+1)\partial_{\nu'} u, \quad \partial_\mu \partial_{\nu'} u \nabla^\mu \partial_{\mu'} u = g_{\mu'\nu'} + \partial_{\mu'} u \partial_{\nu'} u, \tag{A.13}$$

$$\nabla_\mu \partial_\nu \partial_{\nu'} u \nabla^\mu u = \partial_\nu u \partial_{\nu'} u, \quad \nabla_\mu \partial_\nu \partial_{\nu'} u = g_{\mu\nu} \partial_{\nu'} u, \tag{A.14}$$

$$\frac{\partial}{\partial a^\mu} I_{1a} \frac{\partial}{\partial a_\mu} I_{1a} = u(u+2), \quad \frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_{1a} = (u+1) I_{2c}, \tag{A.15}$$

$$\frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = c_2^2 + I_{2c}^2, \ \frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_2 = (u+1) I_{2c}^2, \ \Box_a I_4 = 2(d+1)c_2^2, \tag{A.16}$$

$$\frac{\partial}{\partial a^\mu} I_2 \frac{\partial}{\partial a_\mu} I_2 = u(u+2) I_{2c}^2, \quad \Box_a I_3 = 2(d+1) I_{2c}^2 + 2c_2^2 u(u+2), \tag{A.17}$$

$$\nabla^\mu \frac{\partial}{\partial a^\mu} I_1 = (d+1) I_{2c}, \ \nabla^\mu \frac{\partial}{\partial a^\mu} I_2 = (d+2)(u+1) I_{2c}, \quad \nabla^\mu I_1 \partial_\mu u = I_2, \tag{A.18}$$

$$\nabla^\mu \frac{\partial}{\partial a^\mu} I_3 = 4 I_1 I_{2c} + 2(d+2)(u+1) c_2^2 I_{1a}, \quad \nabla^\mu I_2 \partial_\mu u = 2(u+1) I_2, \tag{A.19}$$

$$\frac{\partial}{\partial a_\mu} I_1 \partial_\mu u = (u+1) I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_2 \partial_\mu u = u(u+2) I_{2c}, \ \frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_1 = I_1 I_{2c}, \tag{A.20}$$

$$\frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_2 = I_{2c}\left((u+1) I_1 + I_2\right) + c_2^2 I_{1a}, \ \frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_1 = I_{2c} I_2, \tag{A.21}$$

$$\frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_2 = 2(u+1) I_{2c} I_2, \quad \nabla^\mu I_1 \nabla_\mu I_1 = a_1^2 I_{2c}^2, \quad \Box I_1 = I_1, \tag{A.22}$$

$$\nabla^\mu I_1 \nabla_\mu I_2 = I_2 I_1 + a_1^2 (u+1) I_{2c}^2, \quad \Box I_2 = (d+2) I_2 + 2(u+1) I_1, \tag{A.23}$$

$$\nabla^\mu I_2 \nabla_\mu I_2 = I_2^2 + 2(u+1) I_1 I_2 + a_1^2 I_{2c}^2 (u+1)^2 + c_2^2 I_{1a}^2, \tag{A.24}$$

$$a^\mu \nabla_\mu I_{1a} = a^2 (u+1), \quad a^\mu \nabla_\mu I_{2c} = I_1, \quad a^\mu \nabla_\mu I_1 = a^2 I_{2c}, \tag{A.25}$$

$$a^\mu \nabla_\mu I_2 = a^2 (u+1) I_{2c} + I_{1a} I_1, . \tag{A.26}$$

Using these relations we can derive ($F_k' := \frac{\partial}{\partial u} F_k(u)$)

- Divergence map

$$\nabla_1^{\mu}\frac{\partial}{\partial a^{\mu}}\Psi^{\ell}[F] = I_{2c}\Psi^{\ell-1}[Div_{\ell}F] + O(c_2^2), \tag{A.27}$$

$$(Div_{\ell}F)_k = (\ell-k)(u+1)F'_k + (k+1)u(u+2)F'_{k+1}$$

$$+(\ell-k)(\ell+d+k)F_k + (k+1)(\ell+d+k+1)(u+1)F_{k+1}. \tag{A.28}$$

- Trace map

$$\Box_a\Psi^{\ell}[F] = I_{2c}^2\Psi^{\ell-2}[Tr_{\ell}F] + O(c_2^2), \tag{A.29}$$

$$(Tr_{\ell}F)_k = (\ell-k)(\ell-k-1)F_k + 2(k+1)(\ell-k-1)(u+1)F_{k+1}$$

$$+(k+2)(k+1)u(u+2)F_{k+2}. \tag{A.30}$$

- Laplacian map

$$\Box_1\Psi^{\ell}[F] = \Psi^{\ell}[Lap_{\ell}F] + O(a_1^2, c_2^2), \tag{A.31}$$

$$(Lap_{\ell}F)_k = u(u+2)F''_k + (d+1+4k)(u+1)F'_k + [\ell + k(d+2\ell-k)]F_k$$

$$+2(u+1)(k+1)^2F_{k+1} + 2(\ell-k+1)F'_{k-1}, \tag{A.32}$$

$$\Box F_k(u) = u(u+2)F''_k + (d+1)(u+1)F'_k. \tag{A.33}$$

- Gradient map

$$(a\cdot\nabla)_1\Psi^{\ell}[F] = I_{1a}\Psi^{\ell}[Grad_{\ell}F] + O(a_1^2), \tag{A.34}$$

$$(Grad_{\ell}F)_k = F'_k + (k+1)F_{k+1}. \tag{A.35}$$

At the end we present all important commutation relations working in the space of symmetric rank n tensors

$$[(\nabla\partial_a), \Box]f^{(n)}(z,a) = [2(a\nabla)\Box_a - (d+2n-2)(\nabla\partial_a)]\, f^{(n)}(z,a); \tag{A.36}$$

$$[(\nabla\partial_a), (a\nabla)]f^{(n)}(z,a) = \Box f^{(n)}(z,a) + [\nabla_{\mu}, (a\nabla)]\partial_a^{\mu} f^{(n)}(z,a); \tag{A.37}$$

$$[\nabla_{\mu}, (a\nabla)]\partial_a^{\mu} f^{(n)}(z,a) = \left[a^2\Box_a - n(d+n-1)\right] f^{(n)}(z,a); \tag{A.38}$$

$$[\Box, (a\nabla)]f^{(n)}(z,a) = \left[2a^2(\nabla\partial_a) - (d+2n)(a\nabla)\right] f^{(n)}(z,a); \tag{A.39}$$

$$\Box_a\left[a^2 f^{(n)}(z,a)\right] = 2(d+2n+1)f^{(n)}(z,a) + a^2\Box_a f^{(n)}(z,a). \tag{A.40}$$

Appendix B

Here we prove the relations (13.14)-(13.17). The more transparent way of working with the boundary-to-bulk propagator for higher spins is to introduce two additional objects

$$\phi^0(z) = \frac{z^0}{(z,z)}, \tag{B.1}$$

$$\psi(\vec{c},z) = \frac{<\vec{c},\vec{z}>}{(z,z)}, \tag{B.2}$$

satisfying the following relations

$$a^\mu \partial_\mu \phi^0(z) = \frac{R^0(a;z)}{(z,z)}, \quad a^\mu \partial_\mu \psi(\vec{c},z) = \frac{R(a,\vec{c};z)}{(z,z)}, \tag{B.3}$$

$$\Box \phi^0(z) = -(d-1)\phi^0(z), \quad \Box \psi(\vec{c},z) = 0, \tag{B.4}$$

$$a^\mu a^\nu \nabla_\mu \partial_\nu \psi(\vec{c},z) = 2[\phi^0(z)]^{-1} a^\mu \partial_\mu \phi^0(z) a^\nu \partial_\nu \psi(\vec{c},z), \tag{B.5}$$

$$\nabla^\mu \phi^0(z) \partial_\mu \phi^0(z) = (\phi^0)^2, \quad \nabla^\mu \phi^0(z) \partial_\mu \psi(\vec{c},z) = 0 \tag{B.6}$$

$$\nabla^\mu \psi(\vec{c},z) \partial_\mu \psi(\vec{c},z) = (\phi^0)^2 <\vec{c},\vec{c}>, \tag{B.7}$$

$$\Box = \nabla^\mu \partial_\mu, \quad \nabla^\mu \begin{Bmatrix} \phi^0(z) \\ \psi(\vec{c},z) \end{Bmatrix} = g^{\mu\nu} \partial_\nu \begin{Bmatrix} \phi^0(z) \\ \psi(\vec{c},z) \end{Bmatrix}, \tag{B.8}$$

$$\nabla_\mu \partial_\nu = g^{\mu\nu} \left(\partial_\mu \delta^\lambda_\nu - \Gamma^\lambda_{\mu\nu} \right) \partial_\lambda, \quad \Gamma^\lambda_{\mu\nu} = \frac{1}{z^0} \left(\delta^\lambda_0 \delta_{\mu\nu} - \delta^\lambda_\mu \delta_{\nu 0} - \delta^\lambda_\nu \delta_{\mu 0} \right). \tag{B.9}$$

Then using (B.1)-(B.3) we can rewrite the AdS/CFT bulk-to-boundary propagator (13.9) in the following complete form

$$G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = (\phi^0(z))^{d-2} \sum_{k=0}^{[\ell/2]} \frac{(-\ell)_{2k}}{2^{2k} k! (\frac{1-\alpha_0}{2})_k} [a^\mu \partial_\mu \psi(\vec{c},z)]^{\ell-2k}$$
$$\times \left[<\vec{c},\vec{c}> \left(a^\mu a_\mu (\phi^0(z))^2 - [a^\mu \partial_\mu \phi^0(z)]^2 \right) \right]^k. \tag{B.10}$$

After that the proof of the condition

$$\nabla^\mu \frac{\partial}{\partial a^\mu} G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = 0 \tag{B.11}$$

reduces to the differentiation of the right hand side of with the covariant Leibniz rules and use of the relations (B.4)-(B.6).

For taking the divergence on the boundary side of (13.9) or (B.10) we need the following identities for $\psi(\vec{c},z)$ and $\phi^0(z)$

$$\vec{\psi}(z) = \frac{\partial}{\partial \vec{c}}\psi(\vec{c},z) = \frac{\vec{z}}{(z,z)}, \quad \vec{\psi}(z)\cdot\vec{\psi}(z) = \frac{1}{(z,z)} - [\phi^0(z)]^2, \tag{B.12}$$

$$\frac{\partial}{\partial \vec{z}}\phi^0(z) = -2\phi^0(z)\vec{\psi}(z), \quad a^\mu\partial_\mu\frac{\partial}{\partial \vec{z}}\cdot\vec{\psi}(z) = a^\mu\partial_\mu\frac{d-2}{(z,z)} + 4\phi^0(z)a^\mu\partial_\mu\phi^0(z), \tag{B.13}$$

$$\frac{\partial}{\partial \vec{z}}\phi^0(z)\cdot a^\mu\partial_\mu\vec{\psi}(z) = 2[\phi^0(z)]^2a^\mu\partial_\mu\phi^0(z) - \phi^0(z)a^\mu\partial_\mu\frac{1}{(z,z)}, \tag{B.14}$$

$$\begin{aligned} a^\mu\partial_\mu\frac{\partial}{\partial \vec{z}}\psi(\vec{c},z)\cdot a^\nu\partial_\nu\vec{\psi}(z) &= 2\phi^0(z)a^\mu\partial_\mu\phi^0(z)a^\mu\partial_\mu\psi(\vec{c},z) \\ &\quad -2\psi(\vec{c},z)\left([\phi^0(z)]^2a^\mu a_\mu - [a^\mu\partial_\mu\phi^0(z)]^2\right) \end{aligned} \tag{B.15}$$

Then performing boundary differentiation of (B.10) and using (B.12)-(B.15) we obtain

$$\frac{\partial}{\partial \vec{z}}\cdot\frac{\partial}{\partial \vec{c}}G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = a^\mu\nabla_\mu\Lambda^{(\ell-1)}(a,\vec{c};z) + O(<\vec{c},\vec{c}>), \tag{B.16}$$

$$\Lambda^{(\ell-1)}(a,\vec{c};z) = 2\ell\frac{(\alpha_0-1)(\ell+d-1)-2(\ell-1)}{\alpha_0^2-1}[\phi^0(z)]^d[a^\mu\partial_\mu\psi(\vec{c},z)]^{\ell-1}. \tag{B.17}$$

At the end of this Appendix we present two useful hypergeometric identities we learned from the book of H. Bateman and A. Erdelyi “Higher transcendental functions” V.1, McGraw-Hill Book company Inc. 1953.

$${}_2F_1(a,b,2b;z) = \left(1-\frac{z}{2}\right)^{-a}{}_2F_1\left(\frac{a}{2},\frac{a+1}{2},b+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right), \tag{B.18}$$

$$\begin{aligned} {}_2F_1(a,b,c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,1-c+a,1-b+a;z^{-1}) \\ &+ \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-b}{}_2F_1(a,1-c+b,1-a+b;z^{-1}). \end{aligned} \tag{B.19}$$

Appendix C

To manipulate reshuffling of different sets of indices we employ two differentials with respect to a and b, e.g.

$$A_b = (a\partial_b), \tag{C.1}$$

$$B_a = (b\partial_a). \tag{C.2}$$

Then we see that operators A_b, a^2, b^2 are dual (or adjoint) to $B_a, \Box_a, \Box_b$ with respect to the "star" product of tensors with two sets of symmetrized indices (16.12)

$$\frac{1}{n} A_b f^{(m-1,n)}(a,b) *_{a,b} g^{(m,n-1)}(a,b) = f^{(m-1,n)}(a,b) *_{a,b} \frac{1}{m} B_a g^{(m,n-1)}(a,b), \tag{C.3}$$

$$a^2 f^{(m-2,n)}(a,b) *_{a,b} g^{(m,n)}(a,b) = f^{(m-2,n)}(a,b) *_{a,b} \frac{1}{m(m-1)} \Box_a g^{(m,n)}(a,b). \tag{C.4}$$

In the same fashion gradients and divergences are dual with respect to the full scalar product in the space (z, a, b)

$$(a\nabla) f^{(m-1,n)}(z;a,b) *_{a,b} g^{(m,n)}(z;a,b) = -f^{(m-1,n)}(z;a,b) *_{a,b} \frac{1}{m} (\nabla\partial_a) g^{(m,n)}(z;a,b). \tag{C.5}$$

Analogous equations can be formulated for the operators b^2 or $b\nabla$.

Now one can prove that [16, 15]:

$$A_b \Gamma^{(s)}(z;a,b) = B_a \Gamma^{(s)}(z;a,b) = 0. \tag{C.6}$$

These "primary Bianchi identities" are manifestations of the hidden antisymmetry. The n-th deWit-Freedman-Cristoffel symbol is

$$\begin{aligned} \Gamma^{(s)}_{(n)}(z;b,a) &\equiv \Gamma^{(s)}_{(n)\rho_1...\rho_n,\mu_1...\mu_\ell} b^{\rho_1}...b^{\rho_n} a^{\mu_1}...a^{\mu_\ell} \\ &= [(b\nabla) - \frac{1}{n}(a\nabla)B_a]\Gamma^{(s)}_{(n-1)}(z;b,a), \end{aligned} \tag{C.7}$$

or in another way

$$\Gamma^{(s)}_{(n)}(z;b,a) = (\prod_{k=1}^{s} [(b\nabla) - \frac{1}{k}(a\nabla)B_a]) h^{(s)}(z;a). \tag{C.8}$$

Using the following commutation relations

$$[B_a, (a\nabla)] = (b\nabla), \tag{C.9}$$

$$[B_a^k, (a\nabla)] = kB_a^{k-1}(b\nabla), \tag{C.10}$$

$$[B_a, (a\nabla)^k] = k(b\nabla)(a\nabla)^{k-1}, \tag{C.11}$$

$$\Box_b(b\nabla)^i = i(i-1)(b\nabla)^{i-2}\Box, \tag{C.12}$$

$$\partial_\mu^b(b\nabla)^i\partial_b^\mu B_a^j = ij(b\nabla)^{i-1}B_a^{j-1}(\nabla\partial_a), \tag{C.13}$$

$$\Box_b B_a^j = j(j-1)B_a^{j-2}\Box_a, \tag{C.14}$$

and mathematical induction we can prove that

$$\Gamma^{(s)}_{(n)}(z;b,a) = \sum_{k=0}^{n}\frac{(-1)^k}{k!}(b\nabla)^{n-k}(a\nabla)^k B_a^k h^{(s)}(z;a). \tag{C.15}$$

The gauge variation of the n-th Cristoffel symbol is

$$\delta\Gamma^{(s)}_{(n)}(z;b,a) = \frac{(-1)^n}{n!}(a\nabla)^{n+1}B_a^n\epsilon^{(s-1)}(z;a), \tag{C.16}$$

putting here $n = s$ we obtain gauge invariance for the curvature

$$\delta\Gamma^{(s)}_{(s)}(z;b,a) = 0. \tag{C.17}$$

Tracelessness of the gauge parameter (16.6) implies that b-traces of all Cristoffel symbols are gauge invariant

$$\Box_b\delta\Gamma^{(s)}_{(n)}(z;b,a) = \frac{(-1)^n}{(n-2)!}(a\nabla)^{n+1}B_a^{n-2}\Box_a\epsilon^{(s-1)}(z;a) = 0. \tag{C.18}$$

Thus for the second order gauge invariant field equation we can use the trace of the second Cristoffel symbol, the so called Fronsdal tensor:

$$\begin{aligned}\mathcal{F}^{(s)}(z;a) &= \frac{1}{2}\Box_b\Gamma^{(s)}_{(2)}(z;b,a)\\ &= \Box h^{(s)}(z;a) - (a\nabla)(\nabla\partial_a)h^{(s)}(z;a) + \frac{1}{2}(a\nabla)^2\Box_a h^{(s)}(z;a).\end{aligned} \tag{C.19}$$

Using equation (C.15) for Cristoffel symbols and after long calculations we obtain the following expression

$$\begin{aligned}&\Box_b\Gamma^{(s)}_{(n)}(z;b,a)\\&=\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}(n-k)(n-k-1)(b\nabla)^{n-k-2}(a\nabla)^k B^k_a\mathcal{F}^{(s)}(z;a).\end{aligned}\tag{C.20}$$

We have expressed the b-trace of any $\Gamma^{(s)}_{(n)}$ through the Fronsdal tensor or the b-trace of the second Cristoffel symbol, but this is not the whole story. Using mathematical induction and (C.9)-(C.14) again we can show that

$$\begin{aligned}&\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}(n-k)(n-k-1)(b\nabla)^{n-k-2}(a\nabla)^k B^k_a\mathcal{F}^{(s)}(z;a)\\&\qquad=n(n-1)(\prod_{k=3}^{n}[(b\nabla)-\frac{1}{k}(a\nabla)B_a])\mathcal{F}^{(s)}(z;a).\end{aligned}\tag{C.21}$$

In particular for the trace of the curvature we can write

$$\Box_b\Gamma^{(s)}(z;b,a)=s(s-1)\mathcal{U}(a,b,3,s)\mathcal{F}^{(s)}(z;a),\tag{C.22}$$

where we introduced an operator mapping the Fronsdal tensor on the trace of the curvature

$$\mathcal{U}(a,b,3,s)=\prod_{k=3}^{s}[(b\nabla)-\frac{1}{k}(a\nabla)B_a].\tag{C.23}$$

Now let us consider this curvature in more detail. First we have the symmetry under exchange of a and b

$$\Gamma^{(s)}(z;a,b)=\Gamma^{(s)}(z;b,a).\tag{C.24}$$

Therefore the operation "a-trace" can be defined by (C.22) with exchange of a and b at the end. The mixed trace of the curvature can be expressed through the a or b traces using "primary Bianchi identities" (C.6)

$$(\partial_a\partial_b)\Gamma^{(s)}(z;b,a)=-\frac{1}{2}B_a\Box_b\Gamma^{(s)}(z;b,a)=-\frac{1}{2}A_b\Box_a\Gamma^{(s)}(z;b,a).\tag{C.25}$$

The next interesting properties of the higher spin curvature and corresponding Ricci tensors are so called generalized secondary or differential Bianchi identities. We can formulate these identities in our notation in the following compressed form ([...] denotes antisymmetrization)

$$\frac{\partial}{\partial a^{[\mu}}\frac{\partial}{\partial b^{\nu}}\nabla_{\lambda]}\Gamma^{(s)}(z;a,b)=0. \tag{C.26}$$

This relation can be checked directly from representation (C.15). Then contracting with a^{μ} and b^{ν} we get a symmetrized form of (C.26)

$$s\nabla_{\mu}\Gamma^{(s)}(z;a,b)=(a\nabla)\partial^{a}_{\mu}\Gamma^{(s)}(z;a,b)+(b\nabla)\partial^{b}_{\mu}\Gamma^{(s)}(z;a,b). \tag{C.27}$$

Now we can contract (C.27) with a ∂^{μ}_{b} and using (C.25) obtain a connection between the divergence and the trace of the curvature

$$(s-1)(\nabla\partial_{b})\Gamma^{(s)}(z;a,b)=[(b\nabla)-\frac{1}{2}(a\nabla)B_{a}]\Box_{b}\Gamma^{(s)}(z;a,b). \tag{C.28}$$

These two identities with a similar identity for the Fronsdal tensor

$$(\nabla\partial_{a})\mathcal{F}^{(s)}(z;a)=\frac{1}{2}(a\nabla)\Box_{a}\mathcal{F}^{(s)}(z;a), \tag{C.29}$$

play an important role for the construction of the interaction Lagrangian.

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