The Standard Model

Gela Devidze
High Energy Physics Institute of Iv. Javakhishvili Tbilisi State University,
9, University str. 0186 Tbilisi, Georgia.
gela.devidze@tsu.ge gela.devidze@cern.ch gela_devidze@yahoo.co.uk

References:

1.David Griffiths, Introduction to elementary Particles, (2010)

2.Alessandro Bettini, Introduction to elementary particle physics, (2014).

3.T.P. Cheng, L.F. Li , Gauge theory of elementary particle physics (Oxford University Press, 2000)

4. T.P. Cheng, L.F. Li , Gauge theory of elementary particle physics , Problems and solutions (Oxford University Press, 2000)

5. John F. Donoghue, Eugene Golowich, Barry R. Holstein, Dynamics of the standard model, (Cambridge University Press, 1992).

6. L.B. Okun, "Leptons and quarks" (North-Holland)

7. M. E. Peskin, D.V. Schroeder, "An introduction to quantum field theory" (1995)

8. Francis Halzen, Alan D. Martin, Quarks and Leptons, (John Wiley & Sons, 1984)

9. I.I.Bigi, A.I. Sanda, “CP Violation” (Cambridge University Press, Cambridge & New York, second ed., 2009).

10. Abdelhak DJOUADI “The Anatomy of Electro–Weak Symmetry Breaking . Tome I: The Higgs boson in the Standard Model”, arXiv:hep-ph/0503172v2.

11. “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC”, The ATLAS Collaboration ,Phys. Lett. B 716, 1 (2012); “Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC” The CMS Collaboration, Phys. Lett. B 716, 30 (2012)

12. A. J. Buras, “Testing the CKM Picture of Flavour and CP Violation in Rare K and B Decays and Particle-Antiparticle Mixing,” arXiv:0904.4917 [hep-ph]

13. Steven Weinberg, The Making of the Standard Model, arXiv:040110v1 [hep-ph]

14. J.Iliopoulos, “Introduction to the STANDARD MODEL of the ElectroWeak Interactions, arXiv:1305.6779v1 [hep-ph]

15. Guido Altareli, “Collider Physics within the Standard Model: a Primer”, arXiv:1303.2842v2 [hep-ph]

16.P.A.Boyle, Standard Model 2014.

17.J. Ellis, Higgs physics, arXiv:1312.5672

Lecture I

Introduction (main properties of elementary particles)

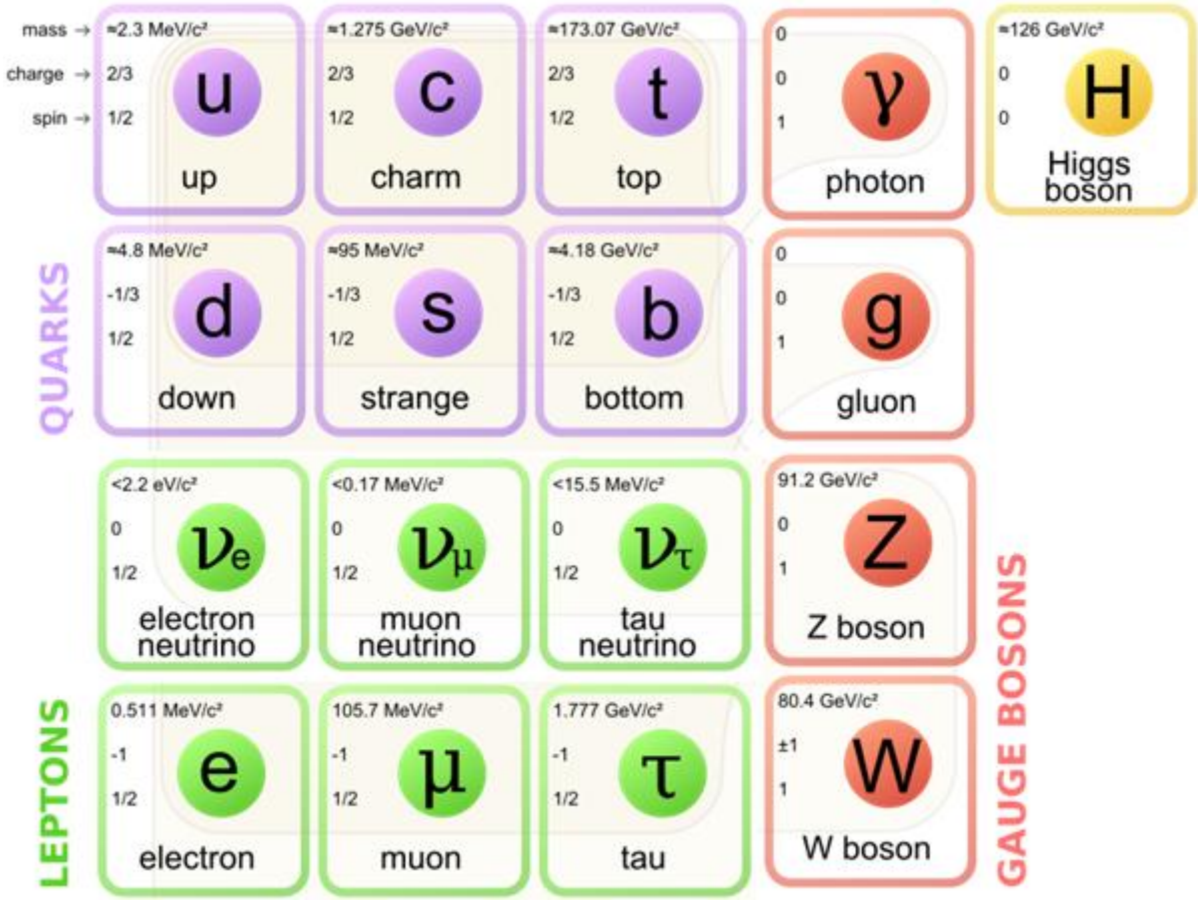
Fundamental Particles

At the end of the 1940's, only p, n, π, e, γ, ν_e were known. The Standard Model developed in an incredible period from 1955-1975 .
Up to now, all observed fundamental (not composite) particles in nature carry spin – 0, 1/2 or 1.

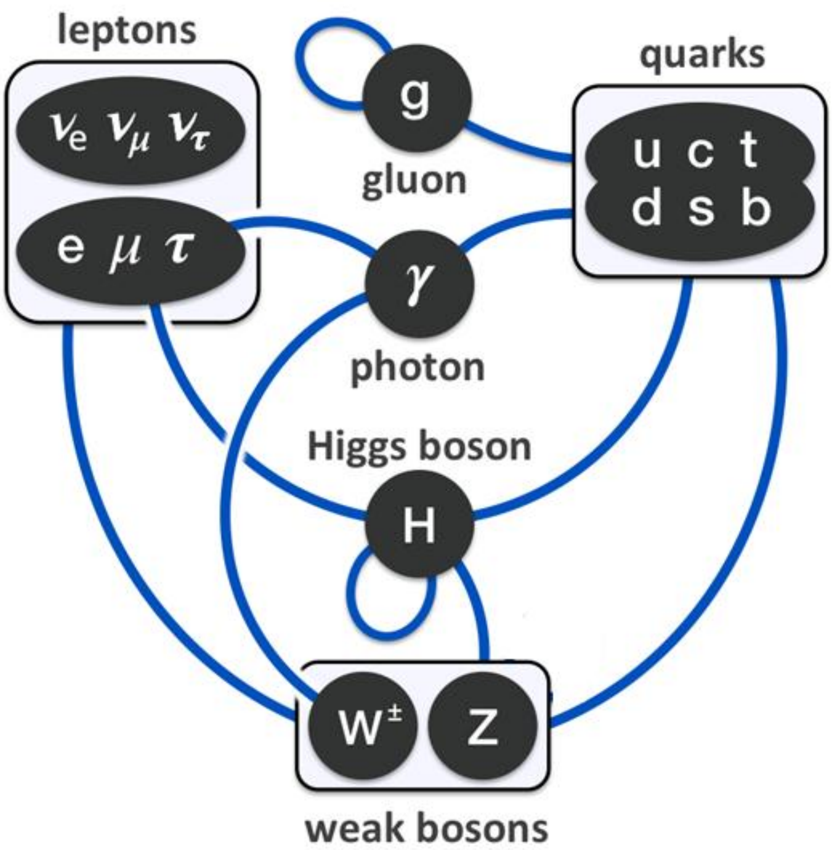
Fermions Matter fields (half-integer spin)		Bosons (integer spin)		
Leptons	Quarks	Higgs	Vector(or 'gauge')	'Graviton'
$\begin{pmatrix}\nu_e\\ e\end{pmatrix},\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix},\begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}$	$\begin{pmatrix}u\\ d\end{pmatrix},\begin{pmatrix}c\\ s\end{pmatrix},\begin{pmatrix}t\\ b\end{pmatrix}$	H	$\gamma, W^{\pm}, Z^0, g_{i=1,\ldots,8}$	G(?)

(The field quantum related to the gravitational field, the 'graviton' carries spin=2)

In the Standard Model, the fundamental fermionic constitutents of matter are the quarks and the leptons. Quarks, but not leptons, engage in the strong interactions as a consequence of their color-charge. Each quark and lepton has spin one-half. Collectively, they display conventional Fermi-Dirac statistics. No attempt is made in the Standard Model either to explain the variety and number of quarks and leptons or to compute any of their properties. That is, these particles are taken at this level as truly elementary. This is not unreasonable. There is no experimental evidence for quark or lepton compositeness, such as excited states or form factors associated with intrinsic structure.
In table we enumerate the quarks and leptons, and display each particle's mass and electric charge. Mass values are in units of GeV/c2 and electric charges are given as multiples of the proton charge e. There are three lepton types: electron (ν_e , e), muon (ν_μ, μ), and tau (ν_τ,τ)- The leptons fall into two classes according to electric charge, the neutral neutrinos ν_e , ν_μ , ν_τ and the negatively charged e, μ, τ
Like the leptons the quarks fall into two classes according to their electrical charge. The *u*, c, *t* quarks have charge 2e/3 and the *d, s, b* quarks have charge -e/3. Unlike the leptons, there are no neutral quarks and quark electrical charge is fractional.



mass →
charge →
spin →
≈2.3 MeV/c²
2/3
1/2
u
up
≈1.275 GeV/c²
2/3
1/2
c
charm
≈173.07 GeV/c²
2/3
1/2
t
top
0
0
1
γ
photon
≈126 GeV/c²
0
0
H
Higgs boson
QUARKS
≈4.8 MeV/c²
-1/3
1/2
d
down
≈95 MeV/c²
-1/3
1/2
s
strange
≈4.18 GeV/c²
-1/3
1/2
b
bottom
0
0
1
g
gluon
<2.2 eV/c²
0
1/2
νe
electron neutrino
<0.17 MeV/c²
0
1/2
νμ
muon neutrino
<15.5 MeV/c²
0
1/2
ντ
tau neutrino
91.2 GeV/c²
0
1
Z
Z boson
LEPTONS
0.511 MeV/c²
-1
1/2
e
electron
105.7 MeV/c²
-1
1/2
μ
muon
1.777 GeV/c²
-1
1/2
τ
tau
80.4 GeV/c²
±1
1
W
W boson
GAUGE BOSONS



leptons
νe νμ ντ
e μ τ
g
gluon
quarks
u c t
d s b
γ
photon
Higgs boson
H
W±
Z
weak bosons

Free quarks are not observed and are confined in bound states called Hadrons.
Baryons (bound state of 3 quarks), Mesons (quark-antiquark pairs: q¯q).

Gauge groups of Standard Model

The gauge fields of the Standard Model include matrix valued Maxwell fields. These gauge field structure is written as: SU(3) × SU(2) × U(1)

Group	Lagrangian fields	After EWSB
SU(3)	gluons	gluons
SU(2)	$W_\mu^{1,2,3}$	W_μ, Z_μ
U(1)	B_μ	A_μ

In the above the labels SU(N) refer to special unitary Lie groups. These are the groups consisting of N ×N complex matrices $g \in C_{N\times N}$ with det g = 1. These are the groups describing the force carrying bosons for a given fundamental force. It is therefore a necessary prerequisite to understand these groups.
The abelian U(1) group consists of the set of complex numbers $e^{i\theta}$ lying on the unit circle. Quantum electrodynamics is the U(1) gauge theory describing electromagnetism. The gauge bosons of QED are photons.
Will consider QED first, then generalise to SU(N) gauge theory.

Symmetries of Standard Model

The Standard Model has a number of important approximate and exact symmetries. Certain symmetries are almost held and these can lead to approximate relations or effective theories valid in certain limits.

Exact symmetries
The exact symmetries include invariance under Lorent transformations (momentum conservation, angular momentum conservation, CPT invariance), invariance under gauge transformations (charge conservation). Global U(1) invariance leads to charge conservation.

Approximate quark flavour symmetry
The masses of the up, down and strange quarks are almost identical (10-100 MeV).
SU(3)flavour matrix operations mixing the up, down and strange fields leave the action almost invariant.
If these quarks had identical masses this would become an exact symmetry. The representation theory of SU(3) describes the structure of the meson and baryon spectrum very well.
The up and down masses differ by only a few MeV and results in the near degeneracy of the proton and neutron, and of the three pions.
The SU(N) gauge group and flavour SU(N) symmetries are unrelated.

The Standard Model and our Universe

A long time ago in a Galaxy far far away... something went Bang.

Epoch	Time	Theory
Planck Epoch (Bang)	10^{-43} s	TOE (Gravi-Strong-Electro-Weak force (Strings???))
GUT Epoch	10^{-36} s	GUT (Strong-Electro-Weak force (SU(5)???))
Inflation + reheating	10^{-32} s	?? Poorly understood
Electroweak Epoch	10^{-12} s	SM (Electro-Weak force + Strong force)
Quark Epoch	10^{-6} s	SM (Electro-Weak force + Strong force)
Hadron Epoch	1 s	SM (Electro-Weak force + Strong force)
Lepton Epoch	10 s	SM (Electro-Weak force + Strong force)
Nucleosynthesis	20 m	QCD (nucleus formation)
Photon Epoch	380000 yr	QED(atoms)
Gravity Epoch	150000000yr	GR+QCD (Galaxy, star formation)
Now	5 billion yr	GR+QED (Homo-sapiens)

TOE hypothetical unified theory of everything:
strong, weak, electromagnetic and gravitational forces
GUT hypothetical grand unified theory:
strong+weak+electromagnetic forces)
SM Standard Model: unified theory of electroweak forces
separate theory of strong force
GR General Relativity
classical theory of relativistic graviation
QCD Quantum chromodynamics
QED Quantum electrodynamics

Lecture II

Free scalars, fermions, gauge bosons

Free fields

Minimising action means δS = 0 under *arbitrary* change δφvanishing at infinity ⇒equations of motion:

$$\delta S = \int d^4x \left[\left(\frac{\partial L}{\partial(\partial_\mu \phi)} \right) \partial_\mu \delta\phi + \frac{\partial L}{\partial \phi} \delta\phi \right] = 0 \Leftrightarrow \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \right) - \frac{\partial L}{\partial \phi} = 0$$

Where,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \vec{\nabla}), \quad a^\mu = g^{\mu\nu} a_\nu, \quad g^{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In classical mechanics the particle equations of motion can be obtained from Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

Where q_i are the generalized coordinates of the particles, t is the time variables, and $\dot{q}_i = dq_i / dt$. The Lagrangian is $L \equiv T - V$ where T and V are kinetic and potential energies of the system, respectively. It is straightforward to extend the formalism from a discrete system, with coordinate $q_i(t)$, to a continuous system, that is, a system with continuously varying coordinates $\phi(\vec{x}, t)$. The Lagrangian

$$L(q_i, \dot{q}_i, t) \to L\left(\phi, \frac{\partial \phi}{\partial x_\mu}, x_\mu \right),$$

Where the field ϕ itself is a function of continuous parameters x_μ, and the Lagrangian density L satisfy the Euler-Lagrange equation

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial(\partial \phi / \partial x_\mu)} \right) - \frac{\partial L}{\partial \phi} = 0$$

The Lagrangian $L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$ gives the Klein –Gordon equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

The Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$ follows from $L = i\overline{\psi}\gamma_\mu \partial^\mu \psi - m\overline{\psi}\psi$, where ψ and $\overline{\psi}$ is regarded as an independent field variable;

Where $\psi_\alpha \equiv \psi_\alpha(x^\mu)$, $\overline{\psi}_\alpha \equiv (\psi^+\gamma^0)_\alpha$, $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$; γ^μ is a 4×4 matrix satisfying the Clifford algebra: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. We will use Pauli-Dirac representation for the γ^μ:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And Pauli matrices are defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{21} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ldots.$$

The substitution of the Lagrangian $L = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$ into the Euler-Lagrang equation for A_μ gives the Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^\nu$$

With

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^1 \\ E^2 & B^3 & 0 & -B^2 \\ E^3 & -B^1 & B^2 & 0 \end{pmatrix}$$

1.Real scalar field (spin-0 particles: π^0, Higgs boson, …)

$$L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \quad \Rightarrow \quad \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

2.Complex scalar field ($\pi^\pm$, $K^\pm$,…, spin-0 charged particles)

$$L = (\partial_\mu \phi)^* (\partial^\mu \phi) - \frac{1}{2}m^2 \phi^* \phi \quad \Rightarrow \quad \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

3.Maxwell field (spin-1 particles: γ,…)

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow \qquad \partial_{\mu} F^{\mu\nu} = 0$$

4. Dirac field (spin-1/2 particles: e , μ , quarks,…)

$$L = i\overline{\psi}\gamma_{\mu}\partial^{\mu}\psi - m\overline{\psi}\psi \Rightarrow (i\partial\!\!\!/ - m)\psi = 0$$

The identification of the Feynman rules proceeds as follows:

1.We associate with the various terms in the Lagrangian a set of propagators and vertex factors.

2. The propagators are determined by the terms quadratic in the fields, that is, the terms in the Lagrangiancontaining ϕ^2 , $\overline{\psi}\psi$, and so on, such as $1/2(\partial_{\mu}\phi)^2 - 1/2m^2\phi^2$ and $\overline{\psi}(i\gamma_{\mu}\partial^{\mu} - m)\psi$.

3.The other terms in the Lagrangian are associated with interaction vertices. The Feynman vertex factor is just given by the coefficient of the corresponding term in iL containing the interacting fields.

Noether’s theorem: Symmetries and conservation laws

An electron is described by a complex field and the corresponding Lagrangian $L = i\overline{\psi}\gamma_{\mu}\partial^{\mu}\psi - m\overline{\psi}\psi$ is invariant under the phase transformation

$$\psi(x) \to e^{i\alpha}\psi(x) ,$$

where α is a real constant. This can be easely checked by noting

$$\partial_{\mu}\psi \to e^{i\alpha}\partial_{\mu}\psi ,$$

$$\overline{\psi} \to e^{-i\alpha}\overline{\psi} .$$

The family of phase transformations $U(\alpha) \equiv e^{i\alpha}$, where a single parameter α may run continuously over real numbers, forms a unitary Abelian group known as the $U(1)$ group

$$U(\alpha_1)U(\alpha_2) = U(\alpha_2)U(\alpha_1) .$$

The observation of $U(1)$ invariance is not trivial; through Neuther’s theorem it implies the existence of a conserved current. Consider the invariance of L under an infinitesimal $U(1)$ transformation,

$$\psi \to (1 + i\alpha)\psi ,$$

$$0=\delta L=\frac{\partial L}{\partial\psi}\delta\psi+\frac{\partial L}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi)+\frac{\partial L}{\partial\overline{\psi}}\delta\overline{\psi}+\frac{\partial L}{\partial(\partial_\mu\overline{\psi})}\delta(\partial_\mu\overline{\psi})$$

$$0=\frac{\partial L}{\partial\psi}(i\alpha\psi)+\frac{\partial L}{\partial(\partial_\mu\psi)}\delta(i\alpha\partial_\mu\psi)+...$$

$$=i\alpha\left[\frac{\partial L}{\partial\psi}-\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\right)\right]\psi+i\alpha\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\psi\right)+...$$

$$\partial_\mu j^\mu=0,\quad j^\mu=\frac{1}{2}\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\psi-\overline{\psi}\frac{\partial L}{\partial(\partial_\mu\overline{\psi}}\right)=-e\overline{\psi}\gamma^\mu\psi\ .$$

The proportionality factor is chosen so that matches up with the electromagnetic charge current density of an electron of charge –e. The charge $Q=\int d^3xj^0$ must be conserved $dQ/dt=0$.

$U(1)$ phase invariance of the Lagrangian $L=(\partial_\mu\phi)^*(\partial^\mu\phi)-m^2\phi^*\phi$ of a complex scalar field implies the existence of a conserved current $j^\mu=-ie(\phi^*\partial^\mu\phi-\phi\partial^\mu\phi^*)$

$U(1)$ local invariance and QED

The Lsgrangian $L=i\overline{\psi}\gamma_\mu\partial^\mu\psi-m\overline{\psi}\psi$ is invariant under the global phase transformation $\psi(x)\to e^{i\alpha}\psi(x)$, but is not invariant under local phase transformations:

$$\psi(x)\to e^{i\alpha(x)}\psi(x)$$

where $\alpha(x)$ now depends on space and time in a completely arbitrary way.

The derivative of $\psi(x)$

$$\partial_\mu\psi(x)\to e^{i\alpha(x)}\partial_\mu\psi(x)+ie^{i\alpha(x)}\psi\partial_\mu\alpha(x)$$

and the $\partial_\mu\alpha(x)$ term breaks the invariance of L. We must seek a modified derivative D_μ, that transforms covariantly under phase transformations, that is, like ψ itself

$$D_\mu\psi\to e^{i\alpha(x)}D_\mu\psi\ .$$

To form covariant derivative D_μ we must introduce a vector field A_μ with transformation properties such that the unwanted term is canceled. This can be accomplished by the construction

$$D_\mu \equiv \partial_\mu - ieA_\mu ,$$

where A_μ transforms as $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$.

The Lagrangian $L = i\overline{\psi}\gamma_\mu D^\mu \psi - m\overline{\psi}\psi$ is invariant under local phase transformation.

$$L = i\overline{\psi}\gamma_\mu D^\mu \psi - m\overline{\psi}\psi = \overline{\psi}(i\gamma_\mu \partial^\mu - m)\psi + e\overline{\psi}\gamma_\mu \psi A^\mu .$$

The Lagrangian of QED

$$L = \overline{\psi}(i\gamma_\mu \partial^\mu - m)\psi + e\overline{\psi}\gamma_\mu \psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} .$$

Problems:

1.Verify that the Dirac equation follows from the Lagrangian $L = i\overline{\psi}\gamma_\mu \partial^\mu \psi - m\overline{\psi}\psi$, where each of the four components of ψ and $\overline{\psi}$ is regarded as an independent field variable.

2.Show that the substitution of the Lagrangian $L = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$ into the Euler-Lagrang equation for A_μ gives the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$, where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. Show that the current is conserved $\partial_\nu j^\nu = 0$.

3.With the addition of a term $1/2m^2 A^\mu A_\mu$, show that the Lagrangian $L = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$ leads to an equation of motion $(\partial_\nu \partial^\nu + m^2)A^\mu = J^\mu$.

4. Show that $dQ/dt = 0$, where $Q = \int d^3x j^0$, $j^\mu = -e\overline{\psi}\gamma^\mu \psi$.

5.Show that U(1) phase invariance of the Lagrangian $L = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi$ of a complex scalar field implies the existence of a conserved current $j^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$.

6. Show that the $D_\mu \equiv \partial_\mu - ieA_\mu$ transforms covariantly $D_\mu \psi \to e^{i\alpha(x)} D_\mu \psi$ under $\psi(x) \to e^{i\alpha(x)}\psi(x)$, $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$.

Lecture III

Feynman rules for QED, Lorentz invariant phase space, cross-section phase space

i)Fermion propagator

$$L = \bar{\psi}(i\not{\partial} - m)\psi$$

$$\bar{\psi}(i\not{p} - m)\psi$$

$$\frac{i}{\not{p} - m}$$

$$S(p)_{\alpha\beta} = i\left(\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}\right)_{\alpha\beta} = \left(\frac{i}{\not{p} - m}\right)_{\alpha\beta}$$

ii)Gauge boson propagator

$$L_{Maxwell} + L_{gauge-fixi} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\lambda}\partial_\mu A^\mu \partial_\nu A^\nu$$

$\lambda = 1$ Feynman Gauge

$\lambda = 0$ Landau Gauge

$$\frac{1}{4}(q_{1\mu}A_\nu(q_1) - q_{1\nu}A_\mu(q_1))(q_2^\mu A^\nu(q_2) - q_2^\nu A^\mu) + \frac{1}{2\lambda}q_{1\mu}A^\mu(q_1)q_{2\nu}A^\nu(q_2)$$

$$\left[-q^2 g^{\sigma\nu} - q^\sigma q^\nu(1 - 1/\lambda)\right]D_{\nu\rho} = g^\sigma_\rho$$

$$D^{\mu\nu}(q) = \frac{i}{q^2 + i\varepsilon}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - i\varepsilon}(1 - \lambda)\right)$$

iii) $\bar{e}eA$-vertex

$$L = -e\bar{\psi}\gamma_\mu \psi A^\mu$$

$$-e\overline{\psi}(k_1)\gamma_\mu\psi(k_2)A^\mu(k_3)$$

$$-e(\gamma_\mu)_{\alpha\beta}$$

External particles and external lines

	Incoming	**Outgoing**
Electron	$[u_s(k)]_\alpha$	$[\overline{u}_s(k)]_\alpha$
Positron	$[\overline{v}_s(k)]_\alpha$	$[v_s(k)]_\alpha$
photon	$\varepsilon_\mu^\lambda(k)$	$\varepsilon_\mu^\lambda *(k)$

Dirac matrix manipulation

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$$

$$\gamma_0^+=\gamma_0,\qquad \vec{\gamma}^+=-\vec{\gamma},\qquad \gamma_5^+=\gamma_5,\qquad \gamma_0\gamma_\mu^+\gamma_0=\gamma_\mu$$

$$\gamma_\mu\gamma^\mu=4,\qquad \gamma_\mu \not{a}\gamma^\mu=-2\not{a},\qquad \gamma_\mu\not{a}\not{b}\gamma^\mu=4(a\cdot b),\qquad \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu=-2\overline{c}\,\overline{b}\,\overline{a}$$

$$tr(\not{a}_1\ldots\not{a}_{2n-1})=0,\qquad tr(\gamma_5)=0,\qquad tr(\gamma_5\not{a}\not{b})=0,\qquad tr(y_5\not{a}\not{b}\not{c}\not{d})=4i\varepsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma$$

$$tr(\not{a}\not{b})=4(a\cdot b),\qquad tr(\not{a}\not{b}\not{c}\not{d})=4[(a\cdot b)(c\cdot d)-(a\cdot c)(b\cdot d)+(a\cdot d)(b\cdot c)]$$

Lorentz invariant phase space

The Lorentz invariant phase space for n final state particles is

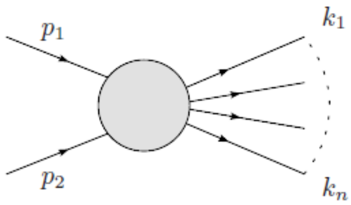
$$d[LIPS](p_1,...,p_n) = (2\pi)^4 \delta^{(4)}(p - \sum_{i=1}^{n} p_i) \prod_{j=i}^{n} \frac{d^3 p_j}{(2\pi)^3 2E_j}$$

where $E_i^2 = |\vec{p}_i|^2 + m_i^2$.

Cross-section phase space

The general differential cross-section is

$$d\sigma = \frac{\varsigma}{2\sqrt{\lambda(s, m_1^2, m_2^2)}} |M_{i\to f}|^2 d[LIPS](k_1,...,k_n)$$



where $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, $s = (p_1 + p_2)^2$ and ς is the symmetry factor. For nidentical particles in the final state it is ζ = 1/n!. For two sets(n1, n2) of identical particles it is ζ = 1/(n1!n2!);

$$\lambda(s, m_1^2, m_2^2) = s^2 + m_1^4 + m_2^4 - 2s(m_1^2 + m_2^2) - 2m_1^2 m_2^2$$

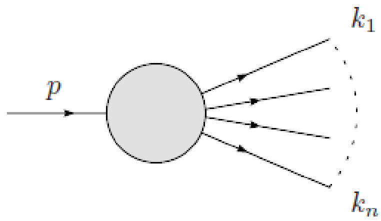
When $s >> m_1^2, m_2^2$ we have

$$d\sigma = \frac{\varsigma}{2s} |M_{i\to f}|^2 d[LIPS](k_1,...,k_n)$$

Decay phase space

$d\Gamma$ in the rest frame of the decaying particle where p = (M, 0)

$$d\Gamma = \frac{\varsigma}{2M} |M_{i\to f}|^2 d[LIPS](k_1,...,k_n)$$



Problems:

1.Verify that $\gamma_0^+ = \gamma_0$, $\vec{\gamma}^+ = -\vec{\gamma}$, $\gamma_5^+ = \gamma_5$, $\gamma_0 \gamma_\mu^+ \gamma_0 = \gamma_\mu$.

2.Verify that $\gamma_\mu \gamma^\mu = 4$, $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$, $\gamma_\mu \not{a}\not{b} \gamma^\mu = 4(a\cdot b)$, $\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$.

3.Verify that $tr(\not{a}\not{b}) = 4(a \cdot b)$, $\quad tr(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$.

4.Using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $tr(y_5\not{a}\not{b}\not{c}\not{d}) = 4i\varepsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma$ verify that

$$tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) = 4[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1 \cdot p_2)]$$

$$tr(\gamma^\mu (1-\gamma^5)\not{p}_1 \gamma^\nu (1-\gamma^5)\not{p}_2) = tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) + 8i\varepsilon^{\mu\alpha\nu\beta} p_{1\alpha} p_{2\beta}$$

$$tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) tr(\gamma_\mu \not{p}_3 \gamma_\nu \not{p}_4) = 32[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

$$tr(\gamma^\mu \not{p}_1 \gamma^\nu \gamma^5 \not{p}_2) tr(\gamma_\mu \not{p}_3 \gamma_\nu \gamma^5 \not{p}_4) = 32[(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

$$tr(\gamma^\mu (1-\gamma^5)\not{p}_1 \gamma^\nu (1-\gamma^5)\not{p}_2) tr(\gamma_\mu (1-\gamma^5)\not{p}_3 \gamma_\nu \not{p}_4) = 256(p_1 \cdot p_3)(p_2 \cdot p_4)$$

5.Particle A, at rest decays into particles B and C ($A \to B + C$)

(a)find the energy of the outgoing particles, in terms of the various masses
(answer $E_B = (m_A^2 + m_B^2 - m_C^2)/2m_A$)
(b)find the magnitude of the outgoing momenta
(answer $|\vec{p}_B| = |\vec{p}_C| = \sqrt{\lambda(m_A^2, m_B^2, m_C^2)}/2m_A$, $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$)

6.Find the CM energy of each decay product in the following reactions
(a) $\pi^- \to \mu^- + \bar{\nu}_\mu$
(b) $\pi^0 \to \gamma + \gamma$
(c) $K^+ \to \pi^+ + \pi^0$
(d) $\Lambda \to p + \pi^-$
(e) $\Omega^- \to \Lambda + K^-$

7.In two-body scattering event $A + B \to C + D$ it is convenient to introduce the Mandelstam variables $\quad s \equiv (p_A + p_B)^2$, $t \equiv (p_A - p_C)^2$, $u \equiv (p_A - p_D)^2$.
(a)Show that $\quad s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$
(b)Find the CM energy of A, in terms of s, t, u and the masses
(answer $E_A^{CM} = (s + m_A^2 - m_B^2)/2\sqrt{s}$)
(c) Find the Lab (B at rest) energy of A (answer $E_A^{Lab} = (s - m_A^2 - m_B^2)/2m_B$)

8.For elastic scattering of identical particles, $A + A \to A + A$, show that the Mandelstam variables become

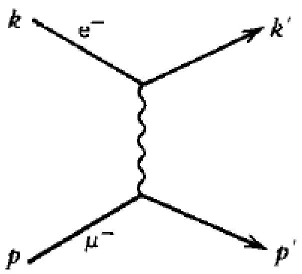
$$s = 4(|\vec{p}|^2 + m^2), \quad t = -2|\vec{p}|^2(1 - \cos\theta), \quad u = -2|\vec{p}|^2(1 + \cos\theta)$$

Where $\vec{p}$ is the CM momentum of identical particle, and θ is the scattering angle.

Lecture IV

Leading order processes

The process $e\mu \to e\mu$



The invariant amplitude: $A(e\mu \to e\mu) = -e^2 \bar{u}(k')\gamma^{\mu}u(k)\dfrac{1}{q^2}\bar{u}(p')\gamma_{\mu}u(p)$, where $q = k - k'$.

To obtain unpolarized cross section, we have to take the square of the modulus of amplitude and then carry out the spin sums

$$\overline{|A|^2} = \frac{e^4}{q^4} A_e^{\mu\nu} A_{\mu\nu}^{muon}$$

$$A_e^{\mu\nu} \equiv \frac{1}{2}\sum_{e\ \ spins}[\bar{u}(k')\gamma^{\mu}u(k)][\bar{u}(k')\gamma^{\nu}u(k)]^*$$

$$A_{\mu\nu}^{muon} \equiv \frac{1}{2}\sum_{e\ \ spins}[\bar{u}(p')\gamma_{\mu}u(p)][\bar{u}(p')\gamma_{\nu}u(p)]^*$$

note that

$$[\bar{u}(k')\gamma^{\nu}u(k)]^* = [\bar{u}(k')\gamma^{\nu}u(k)]^+ = [u^+(k)\gamma^{\nu+}\gamma^0 u(k')] = [\bar{u}(k)\gamma^{\nu}u(k')].$$

So,

$$A_e^{\mu\nu} \equiv \frac{1}{2}\sum_{e\ \ spins}[\bar{u}(k')\gamma^{\mu}u(k)][\bar{u}(k)\gamma^{\nu}u(k')] = \frac{1}{2}\sum_{s'}\bar{u}_{\alpha}^{(s')}(k')\gamma_{\alpha\beta}^{\mu}\sum_{s}u_{\beta}^{(s)}(k)\bar{u}_{\gamma}^{(s)}\gamma_{\gamma\delta}^{(s')}u_{\delta}^{(s')}(k') \Rightarrow$$

$$\Rightarrow \frac{1}{2}Tr\{(\not{k}'+m_e)\gamma^{\mu}(\not{k}+m)\gamma^{\nu}\}$$

$$A_e^{\mu\nu} = \frac{1}{2}Tr\{(\not{k}'+m_e)\gamma^{\mu}(\not{k}+m)\gamma^{\nu}\}$$

and, similarly

$$A^{muon}_{\mu\nu} = \frac{1}{2} Tr\{(\not p' + m_\mu)\gamma_\mu(\not p + m_\mu)\gamma_\nu\}$$

We have (using trace theorems)

$$A^{\mu\nu}_e = \frac{1}{2} Tr\{\not k'\gamma^\mu \not k \gamma^\nu\} + \frac{1}{2} m_e^2 Tr\{\gamma^\mu\gamma^\nu\} = 2\{k'^\mu k^\nu + k'^\nu k^\mu - (k'\cdot k)g^{\mu\nu} + m_e^2 g^{\mu\nu}\}$$

$$A^{muon}_{\mu\nu} = 2\{p'_\mu p_\nu + p'_\nu p_\mu - (p'\cdot p)g_{\mu\nu} + m_\mu^2 g_{\mu\nu}\}$$

The spin-averaged amplitude is

$$\overline{|A|^2} = \frac{8e^4}{q^4}\left\{(k'\cdot k)(k\cdot p) + (k'\cdot p)(k\cdot p') - m_e^2(p'\cdot p) - m_\mu^2(k'\cdot k) + 2m_e^2 m_\mu^2\right\}$$

In the massless limit (the extreme relativistic limit)

$$\overline{|A|^2} = \frac{8e^4}{q^4}\left\{(k'\cdot k)(k\cdot p) + (k'\cdot p)(k\cdot p')\right\}$$

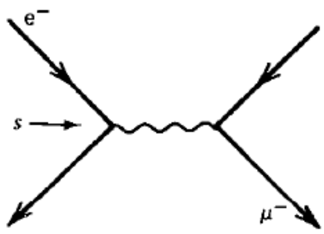
$$s = (k+p)^2 \cong 2k\cdot p \cong 2k'\cdot p, \quad t = (k-k')^2 \cong -2k'\cdot k \cong -2p'\cdot p, \quad u = (k-p')^2 \cong -2k\cdot p' \cong 2k'\cdot p$$

At high energies unpolarized $e\mu \to e\mu$ scattering is given by

$$\overline{|A|^2} = 2e^4 \frac{s^2+u^2}{t^2}$$

We may obtain amplitude for $e^- e^+ \to \mu^- \mu^+$ by crossing $k' \leftrightarrow -p, s \leftrightarrow t$

$$\overline{\left|A(e^- e^+ \to \mu^- \mu^+)\right|^2} = 2e^4 \frac{t^2+u^2}{s^2}$$



In the center of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}(1+\cos^2\theta)$$

$$\sigma\left(e^- e^+ \to \mu^- \mu^+\right) = \frac{4\pi\alpha_{em}^2}{3s}$$

Problems:

1.Proof that

$$\overline{|A|^2} = \frac{8e^4}{q^4}\left\{(k\cdot' k)(k\cdot p) + (k'\cdot p)(k\cdot p') - m_e^2(p'\cdot p) - m_\mu^2(k'\cdot k) + 2m_e^2 m_\mu^2\right\}$$

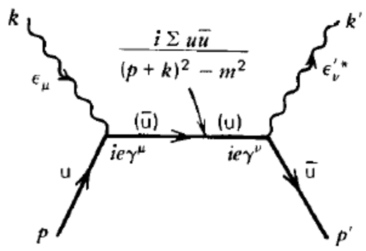
2.Verify that

$$\sigma\left(e^- e^+ \to \mu^- \mu^+\right) = \frac{4\pi\alpha_{em}^2}{3s}$$

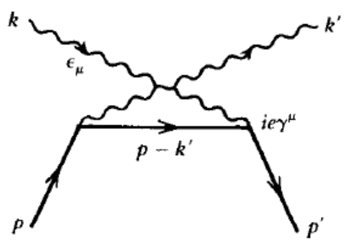
Compton scattering $\gamma e^- \to \gamma e^-$

Using the Feynman rules we obtain the following amplitudes for the two Feynman diagrams

$$-iA_1 = \overline{u}^{(s')}(p')\left[\varepsilon'^*_\nu(ie\gamma^\nu)\frac{i(\not p + \not k + m)}{(p+k)^2 m^2}(ie\gamma^\mu)\varepsilon_\mu\right]u^{(s)}(p)$$



$$-iA_2 = \overline{u}^{(s')}(p')\left[\varepsilon_\mu(ie\gamma^\mu)\frac{i(\not p \not k' + m)}{(pk')^2 m^2}(ie\gamma^\nu)\varepsilon'^*_\nu\right]u^{(s)}(p)$$



we neglect the mass of the electron**,**

$s = (k+p)^2 = 2k\cdot p = 2k'\cdot p$, $t = (k-k')^2 = -2k'\cdot k = -2p'\cdot p$, $u = (k-p')^2 = -2k\cdot p' = 2k'\cdot p$,

and we obtain

$$A_1 = \varepsilon'^*_\nu\,\varepsilon_\mu e^2 \overline{u}(p')\gamma^\nu(\not p + \not k)\gamma^\mu u^{(s)}(p)/s$$

$$A_1 = \varepsilon'^{*}_{\nu}\, \varepsilon_{\mu} e^2 \bar{u}(p')\gamma^{\mu}(\not p - \not k')\gamma^{\nu} u^{(s)}(p)/u$$

$$\overline{|A_1 + A_2|^2}$$

$$\overline{|A_1|^2} = \frac{e^4}{4s^2} Tr\{\not p'\gamma^{\nu}(\not p + \not k)\gamma^{\mu}\not p \gamma_{\mu}(\not p + \not k)\gamma_{\nu}\} = \frac{e^4}{4s^2} Tr\{(-2\not p')(\not p + \not k)(-2\not p)(\not p + \not k)\}$$

$$= \frac{e^4}{s^2} Tr\{\not p'\not k \not p \not k\} = \frac{8e^4}{s^2}(p'\cdot k)(p\cdot k) = 2e^4\left(-\frac{u}{s}\right)$$

Similarly,

$$\overline{|A_2|^2} = 2e^4\left(-\frac{s}{u}\right), \quad \overline{|A_1 A_2 *|^2} = 0$$

And,

$$\overline{|A\ |^2} = \overline{|A_1 + A_2|^2} = 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right)$$

In the center of mass frame

$$s = 4k^2, \quad t = -2k^2(1-\cos\theta), \quad u = -2k^2(1+\cos\theta),$$

Where θ is the center of mass scattering angle, and $k = |\vec{k}_i| = |\vec{k}_f|$.

$$\overline{|A\ |^2} = \overline{|A_1 + A_2|^2} = 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right) = \frac{4 + (1+\cos\theta)^2}{1+\cos\theta}$$

Problems:

1.Show that individually the amplitudes A_1 and A_2 are not gauge invariant but that their sum is.

2.Show that for $\gamma^* e^- \gamma e^-$ (where γ^* denotes a virtual photon of mass $k^2 \equiv -Q^2$)

$$\overline{|A\ |^2} = 2e^4\left(-\frac{u}{s} - \frac{s}{u} + \frac{2Q^2 t}{su}\right)$$

Lecture V

Weak interaction; V-A form of weak current.

The observed lifetimes of the pion and muon are considerably longer than those of particles which decay either through color (i.e. strong) or electromagnetic interactions. It is found that

$$\pi^- \to \mu^- \bar{\nu}_\mu \qquad \text{with } \tau = 2.6\times 10^{-8}\,\text{sec},$$

$$\mu^- \to e^- \bar{\nu}_e \nu_\mu \qquad \text{with } \tau = 2.2\times 10^{-8}\,\text{sec},$$

Whereas particles decay by olor interactions in about 10^{-23} sec and through electromagnetic interactions in about 10^{-16} sec (for example, $\pi^0 \to \gamma\gamma$). The lifetimes are inversely related to the coupling strength of these interactions, with the longer lifetime of the π^0 reflecting the fact that $\alpha << \alpha_s$. The pion and muon decays are evidence for another type of interaction with an even weaker coupling than electromagnetism.

Though all hadrons and leptons experience this weak interaction, and hence can udergo weak decays, they are often hidden by much more rapid color or electromagnetic decays. However, the $\pi^\pm$ and μ are special. They can not decay via the latter two interactions. The $\pi^\pm$ is the lightest hadron. Whereas the neutral π can decay into photons, the charged pions cannot. As a result, the weak decay $\pi^- \to \mu^- \bar{\nu}_\mu$ is dominant one. The reason why $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ is dominant decay of the μ is interesting. In principle, the μ could decay via $\mu \to e\gamma$. The fact that the decay mode $\mu \to e\gamma$ is not seen and that the particular decay modes $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ occur are evidence for additive conserved lepton numbers: the electron number (L_e) and the muon number (L_μ). For example, the electron number assignments are

$L_e = +1$: e^- and ν_e,

$L_e = -1$: e^+ and $\bar{\nu}_e$,

$L_e = 0$: all other particles.

Similar assignments are made for L_μ and L_τ. Clearly, $L_\mu = 1$ and $L_e = 0$ for both the initial and final states of $\mu^- \to e^- \bar{\nu}_e \nu_\mu$, so this decay is consistent with the conservation of these quantum numbers; but $\mu \to e\gamma$ is not.

The two examples of weak decays $\pi^- \to \mu^- \bar{\nu}_\mu$, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ involve neutrinos. Neutrinos are unique in that they can only interact by weak interactions. They are colorless and electrically neutral. Neutrinos are frequently found among the products of a weak decay, but not always. For example, a K^+ meson has the following decay modes

$$\left.\begin{array}{r} K^+ \to \mu^+\nu_\mu,\ \ e^+\nu_e \\ K^+ \to \pi^0\mu^+\nu_\mu,\ \ \pi^0 e^+\nu_e \end{array}\right\} \quad \text{semileptonic decays,}$$

$$K^+ \to \pi^+\pi^0,\ \ \pi^+\pi^+\pi^-,\ \ \pi^+\pi^0\pi^0 \qquad \text{nonleptonic decays.}$$

The weak interaction is also responsible for β-decay of atomic nuclei, which involves the transformation of a proton to neutron (or vice versa). Examples involving the emission of an lepton pair $e^+\nu_e$ are

$$^{10}C \to {}^{10}B^* + e^+ + \nu_e,$$

$$^{14}O \to {}^{14}N^* + e^+ + \nu_e.$$

Here, one of the protons in the nucleus transforms into a neutron via

$$p \to n + e^+ + \nu_e$$

For free protons, this is energetically impossible, but the crossed reaction, the β-decay process

$$n \to p + e^- + \bar{\nu}_e,$$

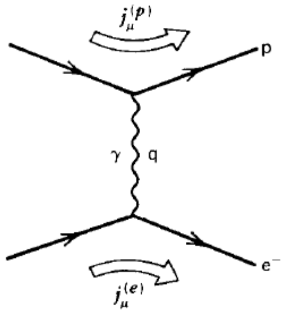
Is allowed and is the reason for neutron's instability (mean life 920 sec). Without the weak interaction, the neutron would be as stable as the proton.

Parity Violation and V-A Form of the Weak Current

Fermi's explanation of β-decay was inspired by the structure of the electromagnetic interaction. Recall that the invariant amplitude for electromagnetic electron-proton scattering is

$$A = \left(e\bar{u}_p\gamma^\mu u_p\right)\left(-\frac{1}{q^2}\right)\left(-e\bar{u}_e\gamma_\mu u_e\right),$$

Where we have treated the proton asc a srtuctureless Dirac particle.*A* is the product of the electron and proton electromagnetic currents, together with the propagator of the exchanged photon.



To facilitate the comparison with weak interactions, we define, for example, an electron electromagnetic current of the form

$$ej_\mu^{em} \equiv j_\mu^{fi}(0) = -e\bar{u}_f \gamma_\mu u_i \,,$$

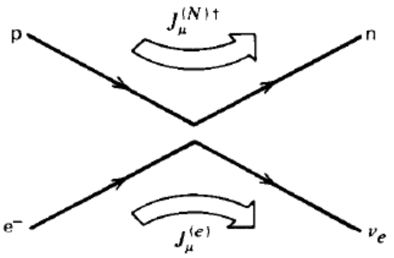
Thus, the invariant amplitude A becomes

$$A = -\frac{e^2}{q^2}\left(j_\mu^{em}\right)_p \left(j^{em,\mu}\right)_e$$

By analogy with the current-current form, Fermi proposed that the invariant amplitude for β-decay be given by

$$A = G\left(\bar{u}_n \gamma^\mu u_p\right)\left(\bar{u}_{\nu_e} \gamma_\mu u_e\right),$$

Where G is the weak coupling constant with remains to be determined by experiment; G is called the Fermi constant. Note the charge-raising or charge-lowering structure of the weak current. We speak of these as the "charged weak currents". (The existence of weak current that is electrically neutral, like the electromagnetic current, was not revealed until 1973; also note the absence of a propagator).



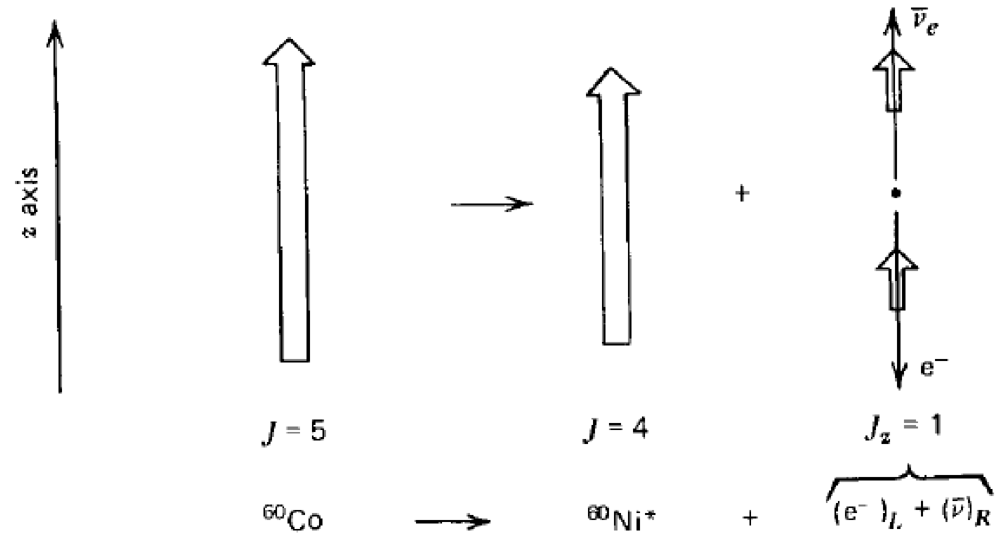
The diagram for β-decay, $p \to ne^+\nu_e$ showing the weak currents.

Fermi's inspired guess of a vector-vector form of the weak amplitude A is a very specific choice from among the various Lorentz invariant amplitudes that can in general be constructed using the biliniearcovariants. There is a priori no reason to use only vectors. The amplitude A explained the properties of some features of β-decay, but not others. Over the following 25 years or so, attempts to unravel the true form of the weak interaction led to a whole series of ingenious β-decay experiments, reaching a climax with the discovery of parity violation in 1956. Amazingly, the only essential change required in Fermi's original proposal was the replacement of γ^{μ} by $\gamma^{\mu}(1-\gamma^{5})$. Fermi had not foreseen parity violation and had not reason to include a $\gamma^{5}\gamma^{\mu}$ contribution; a mixture of γ^{μ} and $\gamma^{5}\gamma^{\mu}$ terms automatically violates parity conservation.

In 1956, Lee and Yang made a critical survey of all the weak interaction data. A particular concern at the time was the observed decay modes of the kaon, $K^{+}\rightarrow 2\pi$ and 3π, in which the two final states have opposite parities. (people, in fact, believed that two different particles were needed to explain the two final states). Lee and Yang argued persuasively that parity was not conserved in weak interactions. Experiments to check assertion followed immediately. The first of these historic experiments serves as a good illustration of the effects of parity violation. The experiment studied β-transitions of polarized cobalt nuclei:

$$^{60}Co\rightarrow {}^{60}Ni^{*}+e^{-}+\bar{\nu}_{e}$$

The nuclear spins in a sample ^{60}Co were aligned by an external magnetic field, and an asymmetry in the direction of emitted electrons was observed. The asymmetry was found to change sign upon reversal of the magnetic field such that electrons prefer to be emitted in a direction opposite to that of the nuclear spin. The essence of the argument is sketched in Fig.



The ^{60}Co experiment: the electron is emitted preferentially opposite the direction of spin of the sample ^{60}Co nucleus.

The observed correlation between the nuclear spin and the electron momentum is explained if the required $J_{z}=1$ is formed by right-handed antineutrino, $\bar{\nu}_{R}$, and a left-handed electron e_{L}.

The cumulative evidence of many experiments is that indeed only $\bar{\nu}_{R}$ (and ν_{L}) are involved in weak interactions. The absence of the "mirror image"staes $\bar{\nu}_{L}$ and ν_{R}, is a clear violation of parity invariance. Also, charge conjugation, C, invariance is violated, since C transforms a ??state into a ?? state. However, the $\gamma^{\mu}(1-\gamma^{5})$ form leaves the weak interaction invariant under the combined CP operation. For instance,

$$\Gamma(\pi^+ \to \mu^+ \nu_L) \neq \Gamma(\pi^+ \to \mu^+ \nu_R) = 0 \qquad \text{P violation}$$

$$\Gamma(\pi^+ \to \mu^+ \nu_L) \neq \Gamma(\pi^+ \to \mu^+ \nu_R) = 0 \qquad \text{C violation}$$

But

$$\Gamma(\pi^+ \to \mu^+ \nu_L) = \Gamma(\pi^- \to \mu^- \overline{\nu}_R) = 0 \qquad \text{CP invariance.}$$

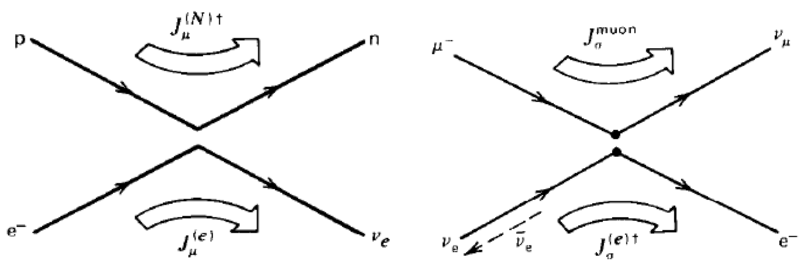
(we discuss CP invariance in next sections).

A charge-lowering weak current of the form $\overline{u}_e \gamma^\mu (1-\gamma^5)/2u_\nu$ involves only left-handed electrons (or right-handed positrons). The factor $(1-\gamma^5)/2$ automatically selects a left-handed neutrino (or a right-handed antineutrino). The V-A (vector-axial vector) structure of the weak current can be directly exposed by scattering ν_e's off electrons just as the γ^μ structure of electromagnetism was verified by measurements of the angular distribution of e^+e^- scattering.

It is natural to hope that all weak interaction phenomena are described by a V-A current-current interaction with a universal coupling G_F. For example

$$A(p \to ne^+\nu_e) = \frac{G_F}{\sqrt{2}} \left[\overline{u}_n \gamma^\mu (1-\gamma^5) u_p\right]\left[\overline{u}_{\nu_e} \gamma_\mu (1-\gamma^5) u_e\right]$$

$$A(\mu^- \to e^- \overline{\nu}_e \nu_\mu) = \frac{G_F}{\sqrt{2}} \left[\overline{u}_{\nu_\mu} \gamma^\mu (1-\gamma^5) u_\mu\right]\left[\overline{u}_e \gamma_\mu (1-\gamma^5) u_{\nu_e}\right]$$



The $1/\sqrt{2}$ is pure convention (to keep original definition of G_F which did not include γ^5).

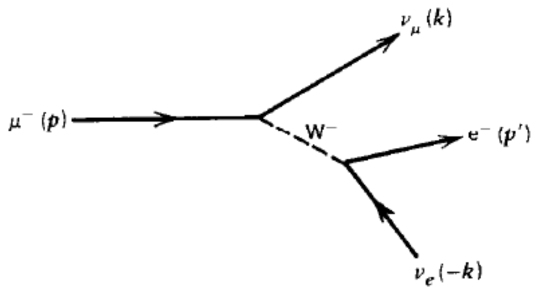
Weak interaction amplitudes are of the form

$$A = \frac{4G_F}{\sqrt{2}} J^\mu J^*_\mu$$

Interpretation of the coupling G_F

Examination of the electromagnetic an weak amplitudes shoes that in Fermi's model the analogy between the two interactions has not been fully developed. We see that G_F essentially replaces by e^2/q^2.Thus , G_F ,in contrast to the dimensionless coupling e, has dimensions GeV^{-2} . It is tempting to try and extand the analogy by postulating that the weak interactions are generated by the emission and adsorption of charged vector bosons, which we call weak bosons, $W^{\pm}$. The weak bosons are the analogous of the photon for the electromagnetic force and gluons for the color force.

For example, μ^- decay is mediated by a W^- boson



and the amplitude is of the form

$$A = \left(\frac{g}{\sqrt{2}} \bar{u}_{\nu_\mu} \gamma^\sigma \frac{1}{2}(1-\gamma^5) u_\mu \right) \frac{1}{M_W^2 - q^2} \left(\frac{g}{\sqrt{2}} \bar{u}_e \gamma_\sigma \frac{1}{2}(1-\gamma^5) u_{\nu_e} \right)$$

Where $g/\sqrt{2}$ is a dimensionless weak coupling and q is the momentum carried by the weak boson (the factor $1/\sqrt{2}$ and $1/2$ are inserted so that we have the conventional definition of g). In contrast to the photon, the weak boson is massive. We are interested in situations where $q^2 << M_W^2$, then the amplitude reverts to

$$A(\mu^- \to e^- \bar{\nu}_e \nu_\mu) = \frac{G_F}{\sqrt{2}} \left[\bar{u}_{\nu_\mu} \gamma^\mu (1-\gamma^5) u_\mu \right] \left[\bar{u}_e \gamma_\mu (1-\gamma^5) u_{\nu_e} \right]$$

And $$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}.$$

Problems:

1.Give the π^+ and μ^+ decay processes. List the possible decay modes of the τ^- lepton.

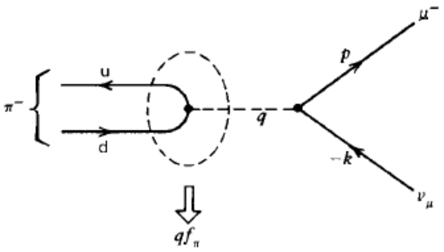
2.Show that a weak current of the form $\bar{u}_e \gamma^\mu (1-\gamma^5) u_\nu$ involves only left-handed electrons (or right-handed positrons)

Lecture VI

Pion decay, Muon decay

Pion decay

$$\pi^-(q) \rightarrow \mu^-(p) + \bar{\nu}(k)$$



The amplitude is of the form

$$A\left(\pi^-(q) \rightarrow \mu^-(p) + \bar{\nu}(k)\right) = \frac{G_F}{\sqrt{2}}(\ldots)^\mu \bar{u}(p)\gamma_\mu(1-\gamma_5)v(k)$$

Where $(\ldots)^\mu$ represents the weak quark current. It is tempting to write it as $\bar{u}_d \gamma^\mu (1-\gamma_5) v_{\bar{u}}$, but this is incorrect since quarks in Fig are not free quark states but are quarks bound into π^--meson. We know however, that

- The amplitude is Lorentz invariant, so $(\ldots)^\mu$ must be a vector or axial-vector,
- The π^- is spinless, so that q is the only four-vector available to construct $(\ldots)^\mu$.

We therefore have

$$(\ldots)^\mu = q^\mu f(q^2) \equiv q^\mu f_\pi$$

Where f_π is a function of q^2 since it is only Lorentz scalar that can be formed from q, but $q^2 = m_\pi^2$ and $f(m_\pi^2) \equiv f_\pi$ is a constant. So, the $\pi^-(q) \rightarrow \mu^-(p) + \bar{\nu}(k)$ decay amplitude is

$$A\left(\pi^-(q) \rightarrow \mu^-(p) + \bar{\nu}(k)\right) = \frac{G_F}{\sqrt{2}}(p+k)^\mu \bar{u}(p)\gamma_\mu(1-\gamma_5)v(k) = \frac{G_F}{\sqrt{2}}\bar{u}(p)(\not{p}+\not{k})(1-\gamma_5)v(k)$$

Using Dirac equations for the neutrino and muon ($\not{k}v(k) = 0, \bar{u}(p)\not{p} = m_\mu \bar{u}(p)$) one get

$$A\left(\pi^- \rightarrow \mu^- \bar{\nu}\right) = \frac{G_F}{\sqrt{2}} m_\mu \bar{u}(p)(1-\gamma_5)v(k)$$

In its rest frame, the π^- decay rate is

$$d\Gamma\left(\pi^- \to \mu^-\bar{\nu}\right) = \frac{1}{2m_\pi}\overline{|A|^2}\frac{d^3p}{(2\pi)^3 2E}\frac{d^3k}{(2\pi)^3 2\omega}(2\pi)^4\delta(q-p-k),$$

Where the sum over spins of outgoing lepton pair can be performed by familiar 'tracelogy'.

$$\overline{|A|^2} = \frac{G_F^2}{2} f_\pi^2 m_\mu^2 Tr\left((\not{p}+m_\mu)(1-\gamma_5)\not{k}(1+\gamma_5)\right) = 4G_F^2 f_\pi^2 m_\mu^2 (p\cdot k)$$

In the π^- rest frame ($\vec{k} = -\vec{p}$)

$$p\cdot k = E\omega - \vec{k}\cdot\vec{p} = E\omega + \vec{k}^2 = \omega(E+\omega)$$

Gatherinh these results together, we have

$$\Gamma\left(\pi^- \to \mu^-\bar{\nu}\right) = \frac{G_F^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi}\int\frac{d^3p d^3k}{E\omega}\omega(E+\omega)\delta(m_\pi - E - \omega)\delta^{(3)}(\vec{k}+\vec{p})$$

The integration d^3p is taken care of by the $\delta^{(3)}$ function and, since there is no angular dependence, we are left with only the integration over $d\omega$:

$$\Gamma\left(\pi^- \to \mu^-\bar{\nu}\right) = \frac{G_F^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi}4\pi\int d\omega\omega^2(1+\omega/E)\delta(m_\pi - E - \omega)$$

Where $E = (m_\mu^2 + \omega^2)^{1/2}$. The result of integration is ω_0^2 , where

$$\omega_0^2 \equiv \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

This can be seen by rewriting the δ –function as

$$\delta[f(\omega)] = \delta(\omega-\omega_0)/\left|\frac{\partial f}{\partial\omega}\right|_{\omega=\omega_0} = \delta(\omega-\omega_0)/(1+\frac{\omega_0}{E})$$

Therefore, finally we obtain

$$\Gamma\left(\pi^- \to \mu^-\bar{\nu}\right) = \frac{G_F^2 f_\pi^2 m_\mu^2}{8\pi} m_\pi\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

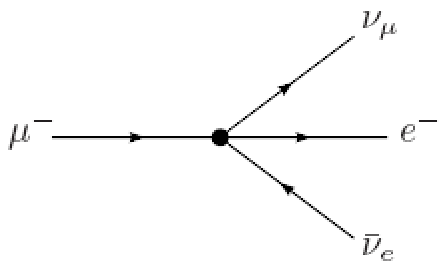
A quantitative test is possible. If, we repeat the calculation for the decay mode $\pi^- \to e^- + \bar{\nu}_e$, we obtain the same result with m_μ replaced by m_e. Therefore

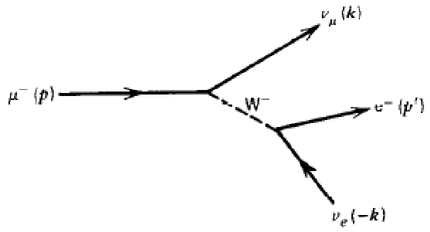
$$\frac{\Gamma(\pi^- \to e^-\bar{\nu}_e)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \sim 10^{-4}$$

The charged π-meson prefers to decay into a muon, which has a similar mass, rather than into the much lighter electron. This is quite contrary to what one would expect from phase-space considerations, so some dynamical mechanism must be at work

Problem: Predict the ratio of the $K^- \to e^- \bar{\nu}_e$ and $K^- \to \mu^- \bar{\nu}_\mu$ decay rates. Given that the lifetime of the K^- is $\tau = 1.2 \cdot 10^{-8}$ sec and the branching ratio is 64%, estimate the decay constant f_K.

Muon decay





The invariant amplitude for muon decay $\mu^-(p) \to e^-(p') + \bar{\nu}(k') + \nu(k)$ is

$$A(\mu^-(p) \to e^-(p') + \bar{\nu}(k') + \nu(k)) = \frac{G_F^2}{\sqrt{2}} \left[\bar{u}(k)\gamma^\mu (1-\gamma^5) u(p)\right]\left[\bar{u}(p')\gamma_\mu (1-\gamma^5) v(k')\right]$$

The muon decay rate can now be obtained

$$d\Gamma = \frac{1}{2E} \overline{|A|^2} d\Phi$$

Where the invariant phase space is

$$d\Phi = \frac{d^3 p'}{(2\pi)^3 2E'} \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 k'}{(2\pi)^3 2\omega'} (2\pi)^4 \delta^{(4)}(p - p' - k - k')$$

Using familiar 'tracelogy' we find the spin-averaged probability

$$\overline{|A|^2} \equiv \frac{1}{2}\sum_{spins}|A|^2 = 64G_F^2(k\cdot p')(k'\cdot p)$$

(since $m_\mu > 200m_e$, we can safely neglect the mass of the electron). Neutrinos being not observed, let us integrate over their momenta. This means that we must calculate the integral

$$I_{\alpha\beta} = \int k_\alpha k'_\beta \frac{d^3k}{\omega}\frac{d^3k'}{\omega'}\delta^{(4)}(q-k-k')$$

Where $q = p - p'$ is total momentum of two neutrinos. The anticipated result can be written as a sum of two mutually orthogonal terms:

$$I_{\alpha\beta} = a(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta) + b(q^2 g_{\alpha\beta} - 2q_\alpha q_\beta)$$

By multiplying both sides of the equality by $q^2 g_{\alpha\beta} - 2q_\alpha q_\beta$ we obtain

$$b4q^4 = \int k_\alpha k'_\beta (q^2 g_{\alpha\beta} - 2q_\alpha q_\beta)... = \int [q^2(kk') - 2(qk)(qk')]... = 0$$

since $q^2(k+k')^2 = 2kk'$, $qk = (k+k',k) = kk'$, $qk' = (k+k',k') = kk'$, $k^2 = k'^2 = 0$.

By multiplying both sides of tensor equality by $q^2 g_{\alpha\beta} + 2q_\alpha q_\beta$, we obtain

$$a12q^4 = q^4\int \frac{d^3k}{\omega}\frac{d^3k'}{\omega'}\delta^{(4)}(q-k-k') = q^4\int \frac{d^3k}{\omega\omega'}\delta(2\omega_1 - \omega) = q^4 4\pi\frac{1}{2}$$

So, $a = \pi/6$, $b = 0$. Finally,

$$I_{\alpha\beta} = \frac{\pi}{6}(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta)$$

Substitution of this result into the expression for the decay width yields

$$d\Gamma = \frac{G_F^2}{2\cdot 2m_\mu}\frac{128}{(2\pi)^5 2\cdot 2}\frac{\pi}{6}p^\alpha p'^\beta[q^2 g_{\alpha\beta} + q_\alpha q_\beta]\frac{d\vec{k}}{2E'}$$

$$= \frac{G_F^2}{48\pi^4 m_\mu}[q^2(pp') + (pq)(p'q)]\frac{d\vec{k}}{2E'}$$

Integration over the electron direction yields 4π , and we obtained

$$d\Gamma = \frac{G_F^2}{12\pi^3 m_\mu}(pp')[p^2 - 2pk + 2p^2 - 2pk]EdE = \frac{G_F^2}{12\pi^3 m_\mu}[3m_\mu^2 - 4m_\mu E]E^2 dE$$

Integration over the electron energy yields the total decay width

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Problems:

1.Verify that $\overline{|A|^2} \equiv \frac{1}{2}\sum_{spins}|A|^2 = 64G_F^2(k \cdot p')(k' \cdot p)$.

2.'Predict' the rate for the decay $\tau^- \to e^- \overline{\nu}_e \nu_\tau$. The observed branching ratio of this decay mode is approximately 20%. Calculate the lifetime of the τ -lepton

Lecture VII

Review of Lie Groups

In order to generalize abelian U(1) gauge theory to non-abelian gauge groups, we need to understand the properties of the SU(N) class of Lie groups and the corresponding su(N) Lie algebras.

Lie groups are a set of continuous groups that are also a differentiable manifold (surface). And in which the group multiplication and inverse are smooth functions.

The concept of generators and matrix exponentiation can be introduced in the simpler context of x-y plane rotation group SO(2) which is isomorphic to U(1).

Generator of translations

The derivative is the generator of translations. When exponentiated a finite translation is induced as follows:

$$e^{i\alpha\partial} f(x) = 1 + \alpha \frac{\partial f(x)}{1!} + \alpha^2 \frac{\partial^2 f(x)}{2!} + \ldots = f(x+\alpha)$$

Matrix exponentiation in SO(2)

Consider rotation of (r, θ) to $(r, \theta+\phi)$.

$$\begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = r\begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\cos\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\phi}{N} & -\sin\frac{\phi}{N} \\ \sin\frac{\phi}{N} & \cos\frac{\phi}{N} \end{bmatrix}^N \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

General rotation matrix is

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Infinitesimal rotation through angle ε is

$$1 + \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Direction one can take away from the unit matrix 1 while staying in the rotation group is

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This is the "tangent" matrix at 1 i.e. $dR/d\phi\big|_{\phi=0} = \tau$.

Define the exponential of a matrix via usual power series for exp

$$\exp\left(\begin{bmatrix} 0 & \phi \\ \phi & 0 \end{bmatrix}\right) = 1 + \frac{(\phi\tau)^1}{1!} + \frac{(\phi\tau)^2}{2!} + \ldots = \underset{N\to\infty}{Lim}\left(1 + \frac{\phi}{N}\tau\right)^N$$

This builds up a finite rotation as the composition of a large number of infinitesimal rotations. The infinitesimal rotations are built out of τ, and τ is called the generator of rotations.

Exponential of general Hermitian matrix

Consider a matrix $D = diag(\lambda_1,\ldots,\lambda_n)$. Then $D^N = diag(\lambda_1^N,\ldots,\lambda_n^N)$.

$$e^{iD} = \underset{N\to\infty}{Lim}\left(1 + i\frac{D}{N}\right)^N = \underset{N\to\infty}{Lim}\, diag\left((1 + i\frac{\lambda_1}{N})^N,\ldots,(1 + i\frac{\lambda_n}{N})^N\right)^N$$

$$= diag(e^{i\lambda_1},\ldots,e^{i\lambda_n})$$

Recall that any Hermitian (symmetric) matrix H is diagonalizable. That is

$$\exists P \text{ such that } P^{-1}HP = D = diag(\lambda_1,\ldots,\lambda_n).$$

Then

$$e^{iH} = \underset{N\to\infty}{Lim}\left(1 + i\frac{H}{N}\right)^N$$

$$= e^{iD} = \underset{N\to\infty}{Lim} P\left(1 + i\frac{D}{N}\right)^N p^{-1}$$

$$= Pdiag(e^{i\lambda_1},\ldots,e^{i\lambda_n})P^{-1}$$

And similarly

$$e^{H} = Pdiag(e^{\lambda_1},...,e^{\lambda_n})P^{-1}$$

Generators of SU(N)

SU(N) is the space of complex matrices $G \in C_{N\times N}$ for which $\det G = 1$ and $G^{+}G = 1_{N\times N}$.

For SU(N) we relate the group element G to a generator as

$$G = e^{i\Lambda\tau}.$$

In the vicinity of 1 the derivations of a matrix 1 are in some tangent space; we take the generators as a complete basis, lying in and spanning this sub-space.

Hermitian

Consider a matrix in the vicinity of 1, $G = 1 + i\varepsilon\tau$; this must be unitarity

$$(1+i\varepsilon\tau)^{+}(1+i\varepsilon\tau) = (1i\varepsilon\tau^{+})(1+i\varepsilon\tau)$$

$$= 1+i\varepsilon(\tau-\tau)^{+}...$$

$$= 1$$

Thus $\tau = \tau^{+}$ and is τ Hermitian.

Traceless

The determinant of a matrix in the vicinity of 1 must remain one. The determinant through $O(\varepsilon)$ is $$\det(1+i\varepsilon\tau) = 1 + i\varepsilon tr\,\tau + O(\varepsilon^2)... = 1$$

Thus the generators τ must be traceless.

Normalisation

Conventionally the generators satisfy a trace orthnormality condition

$$Tr\,\tau^{a}\tau^{b} = \frac{1}{2}\delta^{ab}.$$

Dimension. The number of linearly independent generators must equal the dimension of space.

For SU(N), hermiticity requires that diagonal element be real and tracelessness implies there are N-1 free parameters on the diagonal. The off-diagonal elements are constrained by Hermiticity: there are $(N^2 - N)/2$ off-diagonal elements, each of which has two parameters. The dimension of the traceless hermitian space of generators is thus

$$N^2 - 1 = 2\frac{N^2 - N}{2} + (N-1)$$

The idea that the group also be a differentiable manifold, or surface is connected to the concept of connecting the logarithm of the group element (define by its Taylor series) to a coordinate in the linear space spanned by the Lie algebra.

We might therefore ask what the action of group product looks like in terms of corresponding Lie algebra coordinate. That is, consider solving in the neighbourhood of the identity (small A,B,C) the following $e^A e^B = e^C$.

If the group is abelian and we Taylor expand, one find $AB = BA$ and our generators must commute. For non-abelian group A and B may be non-commutative ($C = A + B + \frac{1}{2}[A, B]..$)

A group is closed under product. As a Lie group should have a smooth product the product $e^A e^B$ should therefore also correspond to an element of Lie algebra C lying in the space spanned by the generators. Thus, order by order, the commutators involved in the Campbell-Baker Hausdorff formula should lie in Lie algebra. The commutator is often called the Lie product.

Structure Constants

If the group is non-abelian it will support non-zero structure constants f_{abc}

$$[\tau_a, \tau_b] = i f_{abc} \tau_c$$

The factor of i is related to our choice of $G = e^{i\Lambda\tau}$.

The structure constants are totally anti-symmetric: they are clearly anti-symmetric in the first two indices due to the commutator. For total anti-symmetry the trace orthonormality, and the cyclic property of the trace, gives cyclic symmetry for $f_{abc} = -2i tr\, \tau_c[\tau_a, \tau_b]$, and this leads to total anti-symmetry.

For SU(N) to be a Lie group, this must be true. But it is not a given that SU(N) is a Lie group, so; why is this the case for SU(N)?

The generators are traceless and Hermitian, and span the traceless Hermitian subspace of $C_{N\times N}$.

The commutator is necessarily traceless because the cyclic property gives $tr(AB - BA) = 0$

$(-i[\tau_a, \tau_b])^+ = (i[\tau_b^+, \tau_a^+]) = -i[\tau_a, \tau_b]$

So $-i[\tau_a, \tau_b]$ is traceless and Hermitian and can be written as a linear combination of the generators.

Thus for SU(N) we must be able to write

$$[\tau_a, \tau_b] = i f_{abc} \tau_c$$

SU(2)

SU(2) has 3 generators which are normalized versions of the Pauli matrices $\tau_j = \sigma_j / 2$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since $$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\varepsilon^{abc}\frac{\sigma^c}{2}$$

The SU(2) structure constants are ε^{abc}.

SU(3)

SU(3) has 8 generators which are normalised Gell-Mann matrices: $\tau_a = \lambda_a / 2$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 10 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ & 0 & 0 \\ i & 0 & 0 \end{pmatrix};$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The algebra is: $\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = if^{abc} \frac{\lambda^c}{2}$ with structure constants:

$$f^{123} = 1$$

$$f^{123} = f^{246} = f^{257} = f^{345} = f^{376} = f^{516} = \frac{1}{2}$$

$$f^{458} = f^{678} = \sqrt{\frac{3}{2}}$$

$$f^{other} = 0$$

Representations

A d-dimensional representation R of a group is a Homonorphism D^R mapping a group G to a space of matrices $C_{d\times d}$ (or $R_{d\times d}$)

$$D^R(g \in G \to C_{d\times d})$$

Such that the matrix product respects the group product:

$$D^R(g_1 g_2) = D^R(g_1) D^R(g_2)$$

This is sufficient to ensure that the subspace of matrices consisting of the image of G are a well defined group (identity/inverse/closed under multiplication).

The representation R contains generators $\tau^R \in C_{d\times d}$, and these are specific to each representation. The τ^R are easily defined as

$$\left. \frac{d}{dR} D^R(1 + i\lambda_a \tau_a) \right|_{\lambda_a = 0}$$

i.e. via $\quad D(1_{N\times N} + i\varepsilon\tau_a) = 1_{d\times d} + i\varepsilon(\tau_a^R)_{d\times d}$

Equivalence and reality

Two representations are equivalent if they are related by a unitary basis change

$$\tau_a^R = U^{-1} T_a^{R'} U$$

Remembering the factor of I in $e^{i\Lambda^a \tau^a}$, we categorise the reality of a representation R, if it is equivalent to one in which reality of the exponent is:

Real if R is the same as its complex conjugate $\bar{R}$:

$$i\tau_a^R = -i(\tau_a^R)^* = i\tau_a^{\bar{R}}$$

Pseudoreal if R is equivalent to its complex conjugate $\bar{R}$ under basis change:

$$i\tau_a^R = -iV^{-1}(\tau_a^R)^* V = iV^{-1}(\tau_a^{\bar{R}})^* V$$

Complex if R is inequivalent to its complex conjecture $\bar{R}$:

$$i\tau_a^{\bar{R}} = -i(\tau_a^R)^* \neq i\tau_a^R$$

Singlet Representation

There is always a very trivial and boring representation of any group. This is called a singlet representation.

$$D(g) = 1_{d\times d} \qquad \forall g \in G$$

Here, if d=1 the representation is a mapping to real or complex numbers. If d>1 the representation is to matrices, but the singlet image of G consists of only the identity 1 in all cases.

States (i.e. wavefunctions) transformed by this representation not really transformed at all – they are unaltered by the symmetry transformation as it simply multiplies by one. For example, spin-0 states are unaltered by rotations as they have no spin direction that needs to be rotated as the axes change.

Fundamental representation

$D^{fundamental}(g) \equiv SU(N)$ is the defining, or fundamental, representation.

States transformed by the fundamental $D(g)$ have an index j with N components which are acted upon by multiplication by this matrix, in the same way that a matrix multiplies a vector.

These states may, of course, have *other* components α in tensor product with j. The multiplication by D(g) is then of course done for each value of α as one expects of tensors.
For example, we might consider a flavor triplet f ∈ (u, d, s) of Dirac fields with spin index α. A SU(3) flavor basis rotation $\psi'_{f'\alpha} = g_{f'f}\psi_{f\alpha}$ is performed for each spin component α. The field ψ is a vector describing each of the up, down and strange quark spinors.

Adjoint representation

$D^{adjoint}(g) \in C_{(N^2-1)\times(N^2-1)}$

We introduced covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, transforming as $D'_\mu = gD_\mu g^+$. If the group for g is promoted to a Lie group such as SU(N), then $A_\mu = A^a_\mu \tau^a$ lies in the Lie Algebra and becomes a (heavily constrained) complex $N \times N$ matrix.

For now we take be a global transformation so that not terms involving derivatives appear, we see that

$$A'_\mu \to gA_\mu g^+$$

The field transforming like this is said to be in the Adjoint representation.

Viewed from the perspective of the $N^2 - 1$ real valued coefficents of the generators A_μ, we can ask how these components transform.

We may write for an infinitesimal transformation

$$(A^b_\mu)'\tau^b = (1 + i\varepsilon\tau^a)A^c_\mu\tau^c(1 - i\varepsilon\tau^a)$$

$$= A^b_\mu\tau^b + i\varepsilon[\tau^a, \tau^c]A^c_\mu = A^b_\mu\tau^b + if_{acb}A^b_\mu\varepsilon\tau^b = A^b_\mu\tau^b - if_{abc}A^b_\mu\varepsilon\tau^b$$

In terms of the adjoint field components A^a_μ the field has been multiplied by the matrix $1_{(N^2-1)\times(N^2-1)} + i(T^a)_{bc}$ where

$$(T^a_{adjoint})_{bc} = -if^{abc}$$

Gluon fields are in the adjoint representation of SU(3), and their transformation can equivalently be viewed in terms of complex 3x3 or real 8x8 operations on the Lie Algebra.

Complex conjugated representation

Antiparticles transform as

$$\overline{\psi}_j \to \overline{\psi}'_j = (\overline{\psi} U^+)_j = (U^* \overline{\psi})_j = U^*_{jl} \overline{\psi}_l$$

Here $U^* = (e^{i\Lambda_a \tau_a})^* = e^{-i\Lambda_a \tau_a^*}$ and $(U^*)^+ U^* = 1$.

This conjugated fundamental representation has generators $\tau^{conj} = -\tau_a^*$ satisfying

$$[\tau_a^{conj}, \tau_b^{conj}] = [-\tau_a^*, -\tau_b^*] = [\tau_a^*, \tau_b^*] = (if^{abc}\tau^c)^* = -if^{abc}\tau^{c*} = if^{abc}\tau_c^{conj}$$

For N>2 this is a new inequivalent representation of the Lie group. For SU(2), however, the fundamental representation is pseudoreal,

$$\exists \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \text{, with } \varepsilon^2 = 1_2, \quad \varepsilon = -\varepsilon^{-1}$$

And, $\varepsilon \sigma^j \varepsilon^{-1} = -(\sigma^j)^*$.

So that

$$U^* = e^{-i\Lambda_a T_a^*} = e^{i\varepsilon \Lambda_a T_a \varepsilon^{-1}} = \varepsilon e^{i\Lambda_a T_a} \varepsilon^{-1} = \varepsilon U \varepsilon^{-1}$$

$$\overline{\psi} \to \overline{\psi}' U^* \overline{\psi} = \varepsilon U \varepsilon^{-1} \overline{\psi} \Leftrightarrow \varepsilon^{-1} \overline{\psi}' = U \varepsilon^{-1} \overline{\psi}$$

But, e.g.

$$\varepsilon^{-1}\overline{\psi}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \overline{d} \end{pmatrix} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix}$$

Is just a basis change and hence the two representations are equivalent for N=2.

This fact implies tgat in SU(2) one can form two bilinear invariants:

(i) $\overline{\psi}\psi \xrightarrow{U} \overline{\psi} U^+ U \psi = \overline{\psi}\psi$

(ii) $\psi^T \varepsilon \psi \xrightarrow{U} (U\psi)^T \varepsilon U \psi = \psi^T U^T \varepsilon U \psi$

$$= \psi^T (U^+)^* \varepsilon U \psi$$

$$= \psi^T \varepsilon U^+ \varepsilon U \psi = \psi^T \varepsilon \psi$$

Tis is important for the Higgs couplings and hence Fermion mass terms in the standard model.

This subsection bellow may be insultingly simple and obvious. It will likely be not included or only skimmed in the lectures. On the other hand, if it is not obvious to you it may be very useful background reading.

Representations of SU(2) and spin

Recall systems of two or more spin 1/2 particles have state that is a *tensor* product.

$$\begin{array}{lcl} |\uparrow\rangle & \rightarrow & |\uparrow\downarrow\rangle \\ |\downarrow\rangle & & |\downarrow\uparrow\rangle \\ & & |\uparrow\downarrow\rangle \\ & & |\downarrow\downarrow\rangle \end{array}$$

Recall the spins can be coupled to form $\begin{array}{cc} \mathit{Spin} & S_Z \\ 1 & -1,0,1 \\ 0 & 0 \end{array}$

We will consider this coupling in a more group-theoretic fashion to (hopefully) make connection with the language of the Standard Model lectures in a familiar context.

For absolute clarity, please note the *tensor* product is important. The resulting four states is the product 2×2, not sum $2 + 2$. For example, combining three particles yields $2^3 = 8$ states (not 6!).

$$\begin{array}{lcl} |\uparrow\rangle & \rightarrow & |\uparrow\uparrow\uparrow\rangle \\ |\downarrow\rangle & & |\downarrow\uparrow\uparrow\rangle \\ & & |\uparrow\downarrow\uparrow\rangle \\ & & |\downarrow\downarrow\uparrow\rangle \\ & & |\uparrow\uparrow\downarrow\rangle \\ & & |\downarrow\uparrow\downarrow\rangle \\ & & |\uparrow\downarrow\downarrow\rangle \\ & & |\downarrow\downarrow\downarrow\rangle \end{array}$$

These can be comined by pairing the first two as above

$$\begin{pmatrix} s=1 & ; & s_z \in -1,0,1 \\ s=0 & ; & s_z \in 0 \end{pmatrix} \otimes \begin{pmatrix} s=\frac{1}{2} & ; & s_z \in -\frac{1}{2},\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} s=\frac{3}{2} & ; & s_z \in -\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2} \\ s=\frac{1}{2} & ; & s_z \in -\frac{1}{2},\frac{1}{2}\left(1\otimes\frac{1}{2}\right) \\ s=\frac{1}{2} & ; & s_z \in -\frac{1}{2},\frac{1}{2}\left(0\otimes\frac{1}{2}\right) \end{pmatrix}$$

Matrix notation for spin

In matrix form, and unit vector notation the single particle states are:

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_0 \qquad |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_1$$

We can enumerate the tensor product of two states as a four component vector

$$E_\alpha = e_j \otimes e_k \quad ; \quad \alpha = j + 2k$$

The states of the two particle system are then defined by four amplitudes

$$c_{jk} \equiv C_\alpha ; \alpha = j + 2k$$

As $\psi = c_{jk} e_j \otimes e_k$

Equivalently

$$\psi = C_\alpha E_\alpha = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} |\uparrow\uparrow\rangle \\ |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\downarrow\rangle \end{bmatrix}$$

Single particle operators

Define $S_x = \frac{1}{2}\sigma_1 \qquad S_y = \frac{1}{2}\sigma_2 \qquad S_z = \frac{1}{2}\sigma_3$

Then $S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}1 = \frac{1}{2}\left(\frac{1}{2}+1\right)1 = s(s+1)1$

and eigenvalues of S_z for eigenvectors e_j are $S_z e_j \pm \frac{1}{2} e_j$

The raising and lowering operators for S_z are $S_\pm = S_x \pm iS_y$. Explicitly,

$$S_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad S_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Note, however, that the eigenvalues of S_x, and S_y are also $\pm 1/2$, and that the commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

are symmetrical among x ↔ y ↔ z. The basis we have chosen has S_z diagonal, but a good change of basis would equally leave S_x or S_y diagonal. Physics doesn't care which axes are used. Rotation must be a good symmetry of the equations provided we know how to transform the states under rotation.

Problems:

1.Show that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$, ("1" here really means the 2x2 unit matrix; if no matrix is specified, the unit matrix is understood.)

2.Show that $\sigma_x \sigma_y = i\sigma_z$, $\sigma_y \sigma_z = i\sigma_x$, $\sigma_z \sigma_x = i\sigma_y$,

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k ,$$

where δ_{ij} is the Kronecker delta and ε_{ijk} is the Levi-Civita symbol.

3. Use the results of previous problems to show that

3.1.The commutator $[A,B] \equiv AB - BA$ of two Pauli matrices is $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$.

3.2.The anticommutator, $\{A,B\} \equiv AB + BA$ is $\{\sigma_i, \sigma_j\} = 2i\delta_{ij}$.

3.3.For any two vectors $\vec{a}$ and $\vec{b}$, $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$

4.Show that $e^{i\pi\sigma_z/2} = i\sigma_z$.

Lecture VIII

Quark model

The action of the QCD sector of the Standard Model becomes symmetric under SU(3) mixing of the up, down, strange quark fields in the mass degenerate limit. As this limit is (only) approximately the case in the real world, the composite states of the real world can be organised in the representations of this SU(3) mixing symmetry in multiplets that are *almost* mass degenerate and predicted by the irreducible representations of this SU(3) approximate symmetry. Hadrons are particles that feel the strong force; they are classified as:

• spin - 1/2 , 3/2 , . . . baryons: ~ qqq
• spin - 0, 1, . . . mesons: ~ $q\bar{q}$

Consider the up, down, strange quark fields as a vector of Dirac fields ψ_f : f ∈ {u, d, s}. The action is

$$L_{quark} = \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f$$

Ignoring the gauge group for now, we can see that transforming the field as

$$\psi_f \to \psi'_f = U_{ff'}\psi_{f'}$$

$$\bar{\psi}_f \to \bar{\psi}'_f = \bar{\psi}_{f'}(U^+)_{ff'}$$

leaves the action invariant if (and only if) $m_u = m_d = m_s$, and $U \in SU(3)$.

ψ_f is in the fundamental 3 representation of SU(3)

$\bar{\psi}_f$ is in the adjoint $\bar{3}$ representation of SU(3)

Bound states form SU(3) tensor product representations according to the number of quarks and anti-quarks States in the same multiplet will have similar masses.

Representations of SU(3)

The SU(3) generators {λi} can be put in a more useful basis.

$$I_1 = \frac{1}{2}\lambda_1 \quad ; \quad U_1 = \frac{1}{2}\lambda_4 \quad ; \quad V_1 = \frac{1}{2}\lambda_6$$
$$I_2 = \frac{1}{2}\lambda_2 \quad ; \quad U_2 = \frac{1}{2}\lambda_5 \quad ; \quad V_2 = \frac{1}{2}\lambda_7$$
$$I_3 = \frac{1}{2}\lambda_3 \quad ; \quad U_3 = \frac{1}{2}\left(-\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8\right) \quad ; \quad V_3 = \frac{1}{2}\left(\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8\right)$$

This basis has commutation relations:

$$\left.\begin{array}{ll} [I_i, I_j = i\varepsilon_{ijk} I_k & I-spin \\ [U_i, U_j = i\varepsilon_{ijk} U_k & U-spin \\ [V_i, V_j = i\varepsilon_{ijk} V_k & V-spin \end{array}\right\} SU(2) suba\lg ebras$$

The topology of SU(2) is simple. Pick a given three dimensional unit vector; this defines a linear combination of Pauli matrices. Traveling in any such direction through the group move simply along a line combining ident with this matrix with period 2π (and 4π in terms of a rotation angle).

Considering the su(2) sub-algebras of su(3) is more entertaining. $I_1, I_2, U_1, U_2, V_1, V_2$ look like three pairs of x − y planes, and are “toroidal”. However, the corresponding z-torii are lie at 60 degrees to each other as I_3, U_3, V_3 are *not* linearly independent.

The two diagonal generators λ_3 and λ_8 correspond to the space spanned by I_3, U_3, V_3. These may be simultaneously diagonalised. We can define raising and lowering operators as before and investigate the states using our knowledge of SU(2).

$$I_\pm = I_1 \pm iI_2, \qquad U_\pm = U_1 \pm iU_2, \qquad V_\pm = V_1 \pm iV_2$$

The ladder operator commutation relations

$$[I_3, I_\pm] = \pm I_\pm, \qquad [U_3, U_\pm] = \pm U_\pm, \qquad [V_3, V_\pm] = \pm V_\pm$$

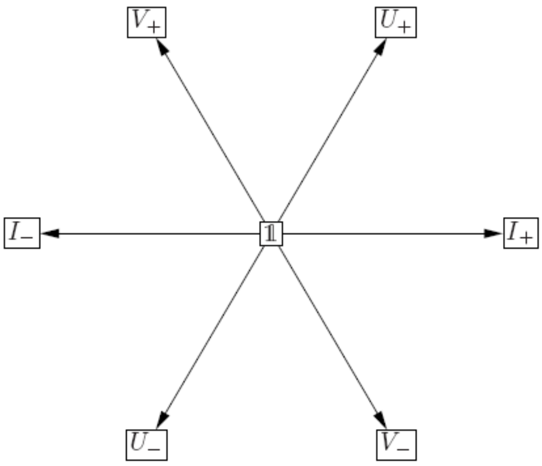
allow to raise and lower eigenvalues of ($\lambda_3/2$, $\lambda_8/2$) by amounts

$$I_\pm \quad \Rightarrow \quad \left(\delta\frac{\lambda_3}{2}, \delta\frac{\lambda_8}{2}\right) = \pm(1,0)$$

$$U_\pm \quad \Rightarrow \quad \left(\delta\frac{\lambda_3}{2}, \delta\frac{\lambda_8}{2}\right) = \pm(\frac{1}{2}, \frac{\sqrt{3}}{2})$$

$$V_\pm \quad \Rightarrow \quad \left(\delta\frac{\lambda_3}{2}, \delta\frac{\lambda_8}{2}\right) = \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$$

These are conceptually just like raising and lower operators for Sz, but now for SU(3) we have two simulta-neously diagonalisable “spin” directions.



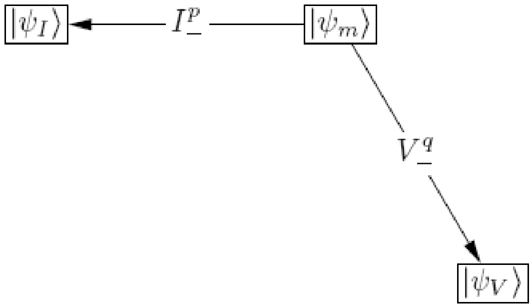
Construction of representations

Define a state of greatest weight $|\psi_m\rangle$ s.t. (analogous to $|\uparrow\uparrow\uparrow\rangle$)

$$I_+|\psi_m\rangle = U_m|\psi_m\rangle = V_m|\psi_m\rangle = 0$$

Find new states by acting on $|\psi_m\rangle$ with I_-, U_- until you get 0, obtaining

$$I_-^n|\psi_m\rangle \quad ; \quad n=0,\ldots,p \quad ; \quad |\psi_I\rangle = I_-^p|\psi_m\rangle$$
$$U_-^n|\psi_m\rangle \quad ; \quad n=0,\ldots,q \quad ; \quad |\psi_U\rangle = U_-^q|\psi_m\rangle$$



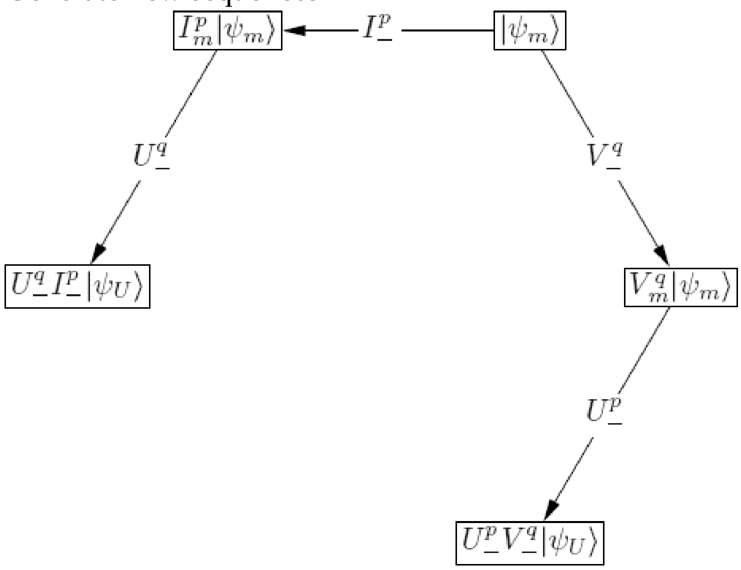
Note using additional easily derived commutation relations that we did not write down gives

$$U_+(I_-)^n|\psi_m\rangle = V_+(I_-)^n|\psi_m\rangle = 0$$

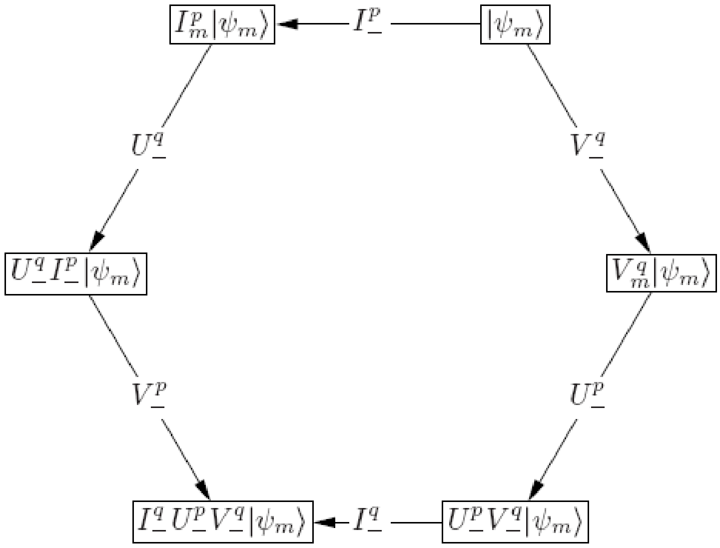
And

$$I_+(U_-)^n|\psi_m\rangle = V_+(U_-)^n|\psi_m\rangle = 0$$

so that all these states also lie on the upper boundary of the allowed quantum numbers of the representation. Generate new sequences



From these end points apply V_- and I_- and we have mapped out a boundary, constrained by I ↔ U ↔ V symmetry to have three faces of length p + 1 and three faces of length q + 1.



Thus we find irregular hexagonal shapes satisfying 120 degree rotation symmetry (p $\neq$ q $\neq$ 0). Special cases of a triangular representation occur when p$\neq$ 0, q = 0 (particles ▽) or p = 0, q $\neq$ 0 (anti-particles Δ). The case p = q = 0 is the singlet case.
States in the interior may be found by applying raising/lowering operators to states on the boundaries. When both p and q are non-zero, the interior can be shown to have degeneracy raised by one each time we step inwards until a triangular interior is attained. For example, an octet has *two* states in the central point (π^0, η).
The general expression for the number of states in the multiplet is

$$N = (p+1)(q+1)\frac{(p+q+2)}{2}$$

Meson flavour states

As discussed we consider the up, down, strange quark fields as a vector of Dirac fields ψ_f : f ∈ {u, d, s}. In the limit $m_u = m_d = m_s = m$ the Lagrangian density possesses a continuous SU($N_f = 3$) flavor symmetry, rotating the quark fields into each other.

$$\begin{aligned} L_{quark} &= \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f \\ &= \overline{\psi}_u (i\not{D} - m_u)\psi_u + \overline{\psi}_d (i\not{D} - m_u)\psi_d + \overline{\psi}_s (i\not{D} - m_u)\psi_s \\ &= \overline{\psi}^T (i\not{D} - m1)\psi \end{aligned}$$

Where

$$\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

transforms in the fundamental representation (3) and

$$\overline{\psi} \equiv \begin{pmatrix} \overline{u} \\ \overline{d} \\ \overline{s} \end{pmatrix}$$

transforms in the conjugate fundamental $\overline{3}$ representation.

We are firstly interested in the action of I_-, U_- , and V_- on these representations, and secondly forming their action on tensor product representation $3 \otimes \bar{3}$ relevant for the transformation of meson ($q\bar{q}$) bound states under this symmetry.

Fundamental

We can form the fundamental ladder operators taking their explicit representation in terms of Gell Mann matrices.

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$I_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad U_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$V_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad V_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We identify the quark flavour wavefunctions with basis column vectors as one of

$$u = (1,0,0)^T ; \quad d = (0,1,0)^T ; \quad s = (0,0,1)^T$$

The action of the raising and lowering operators is then

Operator	Action	Operator	Action
I_+	$d \to u$	I_-	$u \to d$
U_+	$s \to u$	U_-	$u \to s$
V_+	$s \to d$	V_-	$d \to s$

Conjugate Fundamental

We can form the conjugate fundamental ladder operators taking their explicit representation in terms of the conjugate representation generators, For example,

$$I_+^c = I_1^c + iI_2^c = -I_1^* - iI_2^* = -\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly we can collect the image of all the raising and lowering operators in this representation.

$$I_+^c = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I_-^c = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We identify the quark flavour wavefunctions with basis column vectors as one of

$$\bar{u} = (1,0,0)^T ; \quad \bar{d} = (0,1,0)^T ; \quad \bar{s} = (0,0,1)^T$$

The action of the raising and lowering operators is then to left multiply these column vectors and my be summarised as:

Operator	Action	Operator	Action
I_+^c	$\bar{u} \to -\bar{d}$	I_-	$d \to -\bar{u}$
U_+^c	$\bar{u} \to -\bar{s}$	U_-	$\bar{s} \to -\bar{u}$
V_+^c	$\bar{d} \to -\bar{s}$	V_-	$\bar{s} \to -\bar{d}$

Note that as anticipated these operators are indempotent. Applying them multiple times will eventually lead to a zero state. In the fundamental rep + moves towards u and − moves towards s. In the conjugate rep + moves towards $\bar{s}$ and − moves towards $\bar{u}$.

Meson states

We label the meson representation M = (conj) ⊗ (fund). We identify the ladder operators in the tensor product using the tensor product generator rule:

$$I_+^M = I_+^c \otimes 1 + 1 \otimes I_+$$

We first seek the state of greatest weight |W> such that

$$I_+|W\rangle = U_+|W\rangle = V_+|W\rangle = 0$$

This identifies the state as

$$|W\rangle = \bar{s}u$$

We can also find representations for the diagonalised quantum numbers

$$\lambda_3^M = -\lambda_3^* \otimes 1 + 1 \otimes \lambda_3$$

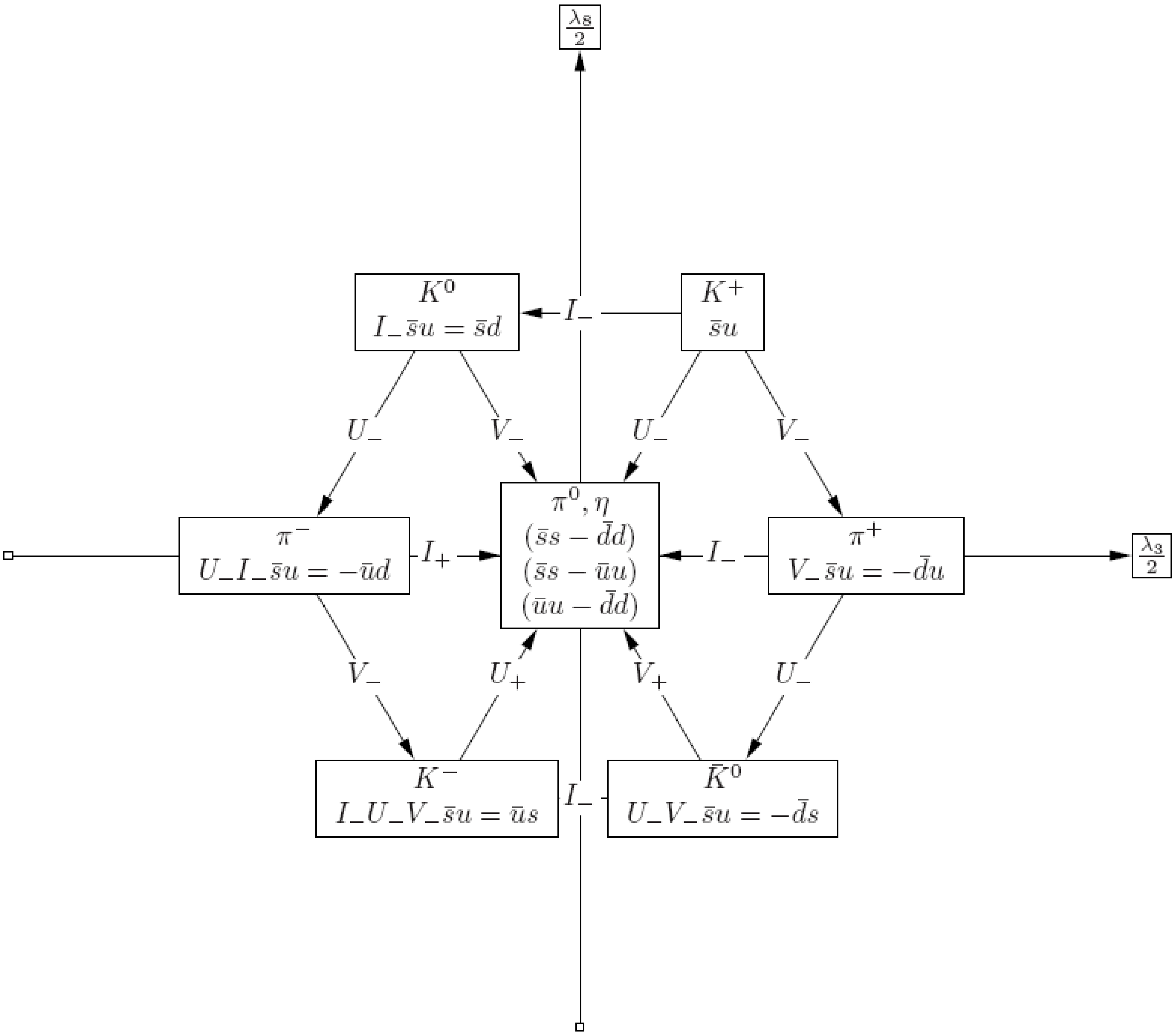
This expression tells us we simply add the flavour quantum numbers of the constituent particles to find the quantum number of the composite. We may note that

$$\lambda_3 u = 1; \qquad \lambda_3 d = -1; \qquad \lambda_3 s = 0$$
$$-\lambda_3^* \bar{u} = -1; \quad -\lambda_3^* \bar{d} = +1; \quad -\lambda_3^* \bar{s} = 0$$

while, similarly,

$$\lambda_8 u = \frac{1}{\sqrt{3}}; \qquad \lambda_8 d = \frac{1}{\sqrt{3}}; \qquad \lambda_8 s = -\frac{2}{\sqrt{3}}$$
$$-\lambda_8^* \bar{u} = -\frac{1}{\sqrt{3}}; \quad -\lambda_8^* \bar{d} = -\frac{1}{\sqrt{3}}; \quad -\lambda_8^* \bar{s} = \frac{2}{\sqrt{3}}$$

We can also identify *strangeness* as $S = \sqrt{3}\lambda_8 - 1$, and (yes!) for historical reasons the strangeness of a strange quark is strangely negative. λ_8 is also historically related to a quantity called *hypercharge* (Y), although this quantity is not particularly useful in modern physics.
Using this and the raising and lowering operators we may now map out the entire set of flavour wavefunctions in the multiplet, plotting in the ($\lambda_3/2$, $\lambda_8/2$) plane as follows.



This hexagonal multiplet has right hand vertex at $I_3 = \lambda_3/2 = +1$, and the state of greatest weight K+ lies at ($\lambda_3/2$, $\lambda_8/2$) = (1 /2 ,√3/2). The interior states at (0, 0) may be determined by applying

$$V_-\left|\bar{s}d\right\rangle = \left|\bar{s}s - \bar{d}d\right\rangle$$
$$U_-\left|\bar{s}u\right\rangle = \left|-\bar{u}u + \bar{s}s\right\rangle$$
$$I_-\left|-\bar{d}u\right\rangle = \left|-\bar{d}d + \bar{u}u\right\rangle$$

The linear dependence of the SU(3) generator basis results in these three states being linearly dependent: and there are only two independent states accessible with ladder operators at ($\lambda_3/2$, $\lambda_8/2$) = (0, 0).

The degeneracy $m_u \sim m_d \leq$ 10MeV is almost exact in nature, but m $m_s \sim$ 100 MeV. As a result the $I_3 = 0$, $\lambda_8/2$ = 0 eigenstates are identified as two normalised and orthogonal flavour states

$$\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)\,; \quad \eta = \frac{1}{\sqrt{6}}(2\bar{s}s - \bar{u}u - \bar{d}d)$$

The π^0 is a triplet I = 1 state, with the triplet consisting of π^+, π^0, π^-. Note that due to the conjugate representation the sign is reversed compared to the $S_z = 0$ wavefunction for the S = 1, $S_z = 0$ state when combining SU(2) ⊗ SU(2).

This completes the derivation of the wavefunctions for meson octet – and the Nobel prize winning the famous eightfold way (near) degeneracy in the meson spectrum.
There must of course be nine states in total and final remaining state lives in a different representation multiplet of $SU(N_f = 3)$. The irreducible representation decomposition is

$$\bar{3} \otimes 3 = 8 \oplus 1$$

The remaining orthogonal state is the flavour singlet $\eta' = \frac{1}{\sqrt{3}}(\bar{s}s + \bar{u}u + \bar{d}d)$. It is much heavier because of the coupling of flavour singlet states to topological structures in the gluon field such as instantons.

For clarity, note that if seeking a concrete vector notation, one can always enumerate the tensor product states as a single, nine component vector. For example

$$(\bar{s}u, \bar{d}u, \bar{u}u, \bar{s}d, \bar{d}d, \bar{u}d, \bar{s}s, \bar{d}s, \bar{u}s)$$

The orthonormality of the (λ_3, λ_8) = (0, 0) states is then clear:

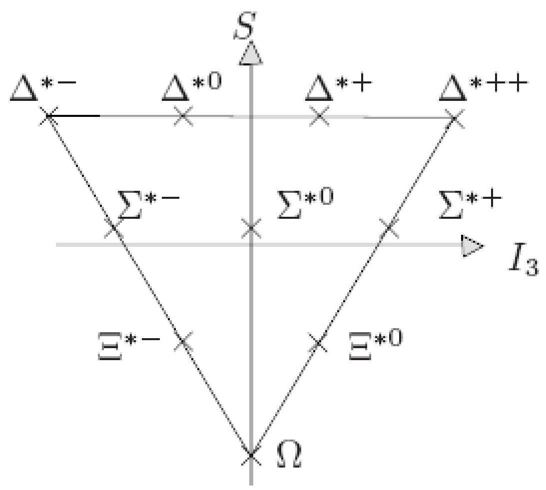
$$\eta' = \frac{1}{\sqrt{3}}(0,0,1,0,1,0,1,0,0)$$

$$\eta = \frac{1}{\sqrt{6}}(0,0,-1,0,-1,0,2,0,0)$$

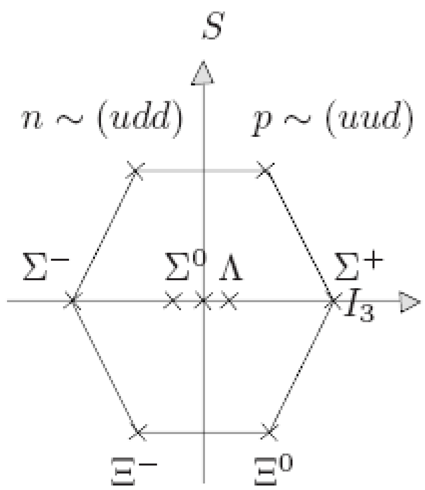
$$\pi^0 = \frac{1}{2}(0,0,1,0,-1,0,0,0,0)$$

Baryon decuplet/octet 3 ⊗ 3 ⊗ 3

A similar analysis predicts both the baryon decuplet and the octet. See tutorial questions.



Baryon Decuplet (spin-3/2)



Baryon Octet (spin-1/2)

Historical context

Prior to 1950's these consisted of p, n (spin – ½), $\pi^{\pm}, \pi^0$ (spin – 0) where $m_p \simeq m_n \simeq 1$GeV and $m_{\pi^{\pm}} \simeq m_{\pi^0} \simeq 140$ MeV.

We consider p (udu) and n (udd) in SU(2) an “isospin” doublet (fundamental representation 2)

$$\begin{pmatrix} |p\rangle & I_z = 1/2 \\ |n\rangle & I_z = -1/2 \end{pmatrix}$$

The three pions are an isospin triplet (3 representation from $\bar{2} \otimes 2 = 1 \oplus 3$)):

$$\begin{pmatrix} \pi^+ & I_z = 1 \\ \pi^0 & I_z = 0 \\ \pi^- & I_z = -1 \end{pmatrix}$$

Around 1950 new particles were observed $K^{\pm}, \Sigma^{\pm}, \ldots$, typically produced in pairs These were “strangely” long lived but heavy particles, and acquired their lifetime because the strange quark could only decay via weak interactions.

Rationalising the previously “bizarre” spectrum in terms of consituent quarks was a major triumph of group theory, and a triumph for which Murray Gell-Mann won the Nobel Prize in Physics.

Spin 0	Spin 1/2	Spin 3/2
$K^0 \sim \bar{s}d$	$p \sim uud$	$\Delta^{++} \sim uuu$
$K^0 \sim \bar{s}d$	$n \sim udd$	$\Xi^{+-} \sim dss$
	$\Sigma_0 \sim uds$	$K^0 \sim \bar{s}d\Omega^- \sim sss$
	$\Xi^- \sim dss$	

Observations in the strong interaction

Baryon number (B), Lepton number (L), charge are conserved: they are related to symmetry under global U(1) transformations:

$$\psi \to \psi' = e^{i\Lambda_B B} e^{i\Lambda_L L} e^{i\Lambda_Q Q} \psi$$

We also note

- Q is always measured in terms of electron charge
- anti-particles have opposite B, L, Q quantum numbers

	B	L	Q
e^-	0	1	-1
γ	0	0	0
ν_e	0	1	0
n	1	0	0
p	1	0	1
π^0	0	0	0
$\pi^{\pm}$	0	0	± 1

For example: quantum numbers for β-decay

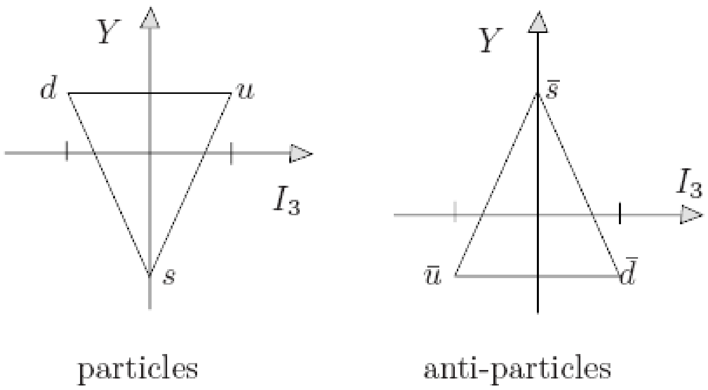
	n	$\to$	p	+	e^-	+	ν_e
B	1	=	1	+	0	+	0
L	0	=	0	+	1	+	-1
Q	0	=	1	+	1	+	0

|

These patterns were naturally explained by introducing 3 underlying objects:
'up' quark ~ $|u\rangle$, 'down' quark ~ $|d\rangle$, 'strange' quark ~ $|s\rangle$; quarks come in different 'flavours'

	B	I	I_3	S	Q
u	1/3	½	½	0	2/3
d	1/3	½	-1/2	0	-1/3
s	1/3	0	0	-1	-1/3
$p \sim uud$	1	½	½	0	1
$n \sim udd$	1	1/2	-1/2	0	0

Note: $Q = I_3 \frac{1}{2}(B+S)$, where B+S=Y is called 'Hypercharge'/



Third generation

$m_d \sim m_u \sim$ *few MeV* and $m_s \sim 100 MeV$ /
Three more (heavier) quarks were discovered along with the τ -lepton in 1976 and ν_τ in 2000.

'charm'	c	1974, with	$m_c \sim 1.2$ GeV
'botton/beaity'	b	1977,	$m_b \sim 5$ GeV
'top/truth'	t	1995,	$m_t \sim 175$ GeV

Color charge

The state of greatest weight in the baryon sector is the $\Delta^{++} = uuu$. There is a curious thing about the $\Delta^{++} = uuu$:
• spin – 3/2 = | ↑↑↑> – *symmetric* spin state
• charge ++ = |uuu> – *symmetric* flavor state

Pauli anti-symmetry? ⇒ SU(3) color degree of freedom
• Totally anti-symmetric color charge wavefunction $|\Delta^{++}\rangle = |\varepsilon_{ijk} u_i^{\uparrow} u_j^{\uparrow} u_k^{\uparrow}\rangle$
• This is a color single because for g ∈ SU(3), then u → gu implies

$$\varepsilon_{ijk}u_i u_j u_k \to \varepsilon_{ijk} g_{ii'}u_{i'} g_{jj'}u_{j'} g_{kk'}u_{k'} = \det \varepsilon_{i'j'k'}u_{i'}u_{j'}u_{k'}$$

We discover QCD is a SU(3) non-abelian gauge theory involving six quark flavors (three generations) and the bound states are color singlet mesons and baryons.

Problems:

1.Show that $Tr(\lambda^\alpha \lambda^\beta) = 2\delta^{\alpha\beta}$ (notice that all the λ are traceless).

2.Use isospin invariance to show that the reaction cross sections σ must satisfy

$$\frac{\sigma(pp \to \pi^+ d)}{\sigma(np \to \pi^o d)} = 2$$

Given that the deuteron d has isospin I=0 and the π has isospin

Lecture IX

SU(N) Yang-Mills theory, Quantum Chromodynamics

Non-abelian gauge theories Yang, Mills (1954)

Renormalizibility Fadeev, Popov (1969)

SU(N) gauge theory involves a non-abelian gauge transformation group. Non-abelian gauge fields support *self interactions* of the gauge bosons. Important realisations of such theories are:

$$SU(3)_C \to QCD$$

$$SU(2)_L \otimes U(1) \to \text{Weinberg-Salam Model}$$

The Dirac field transforms as a Fermion in the fundamental representation of SU(N). Consider the free Dirac Lagrangian density

$$L_D^0 = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$$

again with (in a condensed notation)

$$\psi \to \psi' = U\psi$$
$$\overline{\psi} \to \overline{\psi}' = \overline{\psi}U^+$$

In the fundamental representation the field, ψ, must be a N-vector, with each component being a spinor field:

$$\psi \sim \begin{pmatrix} \psi_{1\alpha} \\ \cdot \\ \cdot \\ \psi_{N\alpha} \end{pmatrix} \quad \text{where } \psi_{j\alpha} \text{ has j as SU(N) index, } \alpha \text{ as the spinor index}$$

Examples

$$SU(3)_C \qquad \psi = \begin{pmatrix} q^r \\ q^b \\ q^g \end{pmatrix}$$

with q ∈ {u, d, s, c, b, t} (the six quark 'flavours') and 'r' = red, 'b' = blue, 'g' = green (three 'colours').

$$SU(2)_L \qquad \psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} u \\ d \end{pmatrix}_L$$

SU(N) Lagrangian

Construct Lagrangian density to be invariant under *local* group transformations g(_(x)) ∈ SU(N).

$$\psi \to \psi' = g(\Lambda(x))\psi = \exp(ig_N \Lambda^a(x) T^a)\psi\,,$$

where

• g(Λ(x)) ∈ SU(N) is an N × N matrix in the Lie group
• $\Lambda^a(x)T^a$ is in the Lie algebra su(N) and is a linear combination of generators with a = 1, . . . ,$N^2 - 1$
• g_N a coupling constant

Construct gauge covariant derivative Dμ introducing $N^2 - 1$ gauge bosons A^μ via “minimal coupling”

$$\partial^\mu \to D^\mu = 1\partial^\mu + ig_N T^a A^a_\mu$$

Sometimes write $T^a A^a_\mu$ as a A_μ, and D^μ is matrix-valued in color indices.

$$[g, A_\mu] \neq 0 \quad !!!$$

For invariance under local transformations we have to require

$$D^\mu \to D^\mu{}' = g(x) D^\mu g^+(x)$$

As in the Abelian case local group transformations are placed in one-to-one correspondence to gauge trans-formations of gauge fields, and we define the gauge transformation property of A_μ via

$$A_\mu{}' = T^a A'^a_\mu = A^a_\mu g T^a g^+ - \frac{i}{g_N} g\partial_\mu g^+$$

This results in the required property

$$\partial_\mu{}' + ig_N A_\mu{}' = \partial_\mu + ig_N T^a A'^a_\mu$$

$$= g(\partial_\mu g^+) g g^{+\partial_\mu} + ig_N A^a_\mu g T^a g^+$$

$$= g(\partial_\mu + ig_N A_\mu) g^+$$

Using this derivative the Dirac Lagrangian is invariant under local group transformation and is now given by:

$$L_D = i\overline{\psi}\gamma_\mu(\partial^\mu + ig_N T^a A^{\mu a})\psi - m\overline{\psi}\psi = L^0_D + L_{Int}$$

Where

$$L_{Int} = g_N \overline{\psi}\gamma_\mu T^a \psi A^{\mu a} = -J^a_\mu A^{\mu a}$$

where the $N^2 - 1$ Noether currents

$$J^a_\mu = g_N \overline{\psi}\gamma_\mu T^a \psi$$

couple to the gauge bosons.

For the gauge field action we choose our ansatz

$$F_{\mu\nu} = -\frac{i}{g_N}[D^\mu, D^\nu] \quad \text{where} \quad F^{\mu\nu} \equiv T^a F^{a\mu\nu}$$

Evaluating this commutator (where a, b, c = 1, . . . ,$N^2 - 1$) we have

$$[D^\mu, D^\nu] = [\partial^\mu + ig_N T_a A^\mu_a, \partial^\nu + ig_N T_b A^\nu_b]$$

$$= [\partial^\mu, \partial^\nu] + [\partial^\mu, ig_N T_b A^\nu_b] + [ig_N T_a A^\mu_a, \partial^\nu] + [ig_N T_a A^\mu_a, ig_N T_b A^\nu_b]$$

$$= ig_N T_b(\partial^\mu A^\nu_b) - ig_N T_b A^\nu_b \partial^\mu +$$

$$+ig_N T_a A_a^\mu \partial^\nu - ig_N T_a(\partial^\nu A_a^\mu) - g_N^2 A_a^\mu A_b^\nu [T_a, T_b]$$

$$= ig_N T_a(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - g_N^2 f_{abc} A_a^\mu A_b^\nu T_c$$

$$= ig_N T_a(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_N f_{abc} A_b^\mu A_c^\nu)$$

Hence we read off that

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_N f_{abc} A_b^\mu A_c^\nu$$

This new term, $g_N f_{abc} A_b^\mu A_c^\nu$, leads to non-trivial interactions in the Yang-Mills theory and to a qualita-tively different behaviour of the corresponding quantum field theory compared to the Abelian case. Gauge invariance of tr $(F^{\mu\nu} F_{\mu\nu})$ follows from the transformation property of the covariant derivative:

$$D'^\mu = g D^\mu g^+$$

which we can readily see

$$F'^{\mu\nu} = -\frac{1}{g_N}[D'^\mu, D'^\nu] = -\frac{1}{g_N}[gD^\mu g^+, gD^\nu g^+] = -\frac{1}{g_N}(gD^\mu g^+ gD^\nu g^+ - gD^\nu g^+ gD^\mu g^+)$$

$$-\frac{1}{g_N}(gD^\mu D^\nu g^+ - gD^\nu D^\mu g^+) = -\frac{1}{g_N} g[D^\mu, D^\nu] g^+ = gF^{\mu\nu} g^+$$

AS a result

$$tr(F'^{\mu\nu} F'_{\mu\nu}) = tr(F^{\mu\nu} F_{\mu\nu})$$

This is the gauge invariant expression for the kinetic term.

The Yang-Mills Lagrangian is:

$$L_{YM} = -\frac{1}{2} tr(F^{\mu\nu} F_{\mu\nu}) \quad \text{where} \quad F^{\mu\nu} \equiv F^{a\mu\nu} T^a$$

$$= -\frac{1}{2} tr(T^a T^b) F^{a\mu\nu} F^b_{\mu\nu})$$

$$= -\frac{1}{2}(\frac{1}{2}\delta^{ab}) F^{a\mu\nu} F^b_{\mu\nu}) = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$$

Where $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_N f_{abc} A_b^\mu A_c^\nu$

Note that the gauge boson acquires self-interaction due to the extra term in the Yang-Mills theory. Since Fμν acquires a term quadratic in the gauge fields, the action acquires additional terms involving three and four gluon fields. The full Yang-Mills theory including interactions with Dirac Fermions has the Lagrangian:

$$L_D = L_{YM} + L_D$$

Remarks

• Gauge bosons are charged → *only true in the Non-Abelian case!*

→ self-interactions of the gauge fields and LYM is already an interacting theory

→ field eqns are nonlinear and difficult to solve (e.g. classical “monopole” and “instanton” solutions).

• No gauge boson mass is allowed because mass terms ~ $M^2 A_a^\mu A_{a\mu}$ are not gauge invariant.

• Classical Yang-Mills theory is qualitatively different to the quantum field theory.
For instance, $g_N^2(Q^2) \xrightarrow[Q^2 \to 0]{} 0$ where Q is some scale; this is known as *asymptotic freedom*

• Problem with long-range Yang-Mills interactions:
→ in principle, the massless boson has 1/R interaction
→ no long range gauge fields are observed apart from the photon (and graviton)
→ in nature, Yang-Mills gauge bosons are confined or massive
• gN is the coupling between the fermions and gauge bosons and for self interactions of the gauge bosons

→ this property is known as *universality* of the gauge coupling and can be checked experimentally.

Quantum Chromodynamics

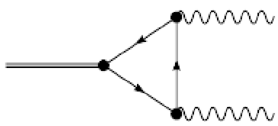
We can now define the QCD sector of the Standard Model as this is an SU(3) gauge theory coupled to six massive Dirac Fermions known as quarks.

Quark model and non-abelian g.t. Fritzsch, Gell-Mann, Leutwyler
(1972/73) Asymptotic freedom Gross, Politzer, Wilczek
(1973)

a.) The quark model has state: Δ^{++}, Ω^- → bound states of 3 identical fermions, $\Delta^{++} = u\uparrow u\uparrow u\uparrow$, which is not compatible with the exclusion principle.
Motivates a distinguishing, extra label organised in an anti-symmetric way called *colour*:

$$\Delta^{++} \sim \sum_{i,j,k\in\{1,2,3\}} u_i^\uparrow u_j^\uparrow u_k^\uparrow \varepsilon_{ijk}$$

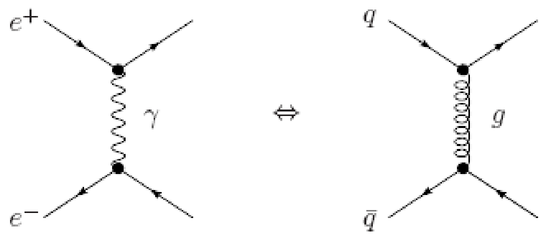
b) $\Gamma(\pi^0 \to \gamma\gamma)$

$\approx$  $\approx \left(\frac{N_C}{3}\right)^2$

If NC = 1 (1 colour), the theoretical prediction for decay rate is a factor of 10 too small (see tutorial).
c.) Quark dynamics must explain why no free quarks (confinement):
→ all hadrons are colour singlets (white)
→ non point-like substructure of hadrons observed in collider experiments ⇒ parton model

→ hadrons made from partons (Feynman, Bjorken 1972 – later indentified as quarks & gluons)

Following success of QED, it was attempted to describe quark dynamics with non-Abelian gauge theory.



QCD was formulated as an SU(3)C gauge theory with quarks (fermions) in the fundamental representation.

$$L = L_{gluon} + L_{quarks} + L_{gauge-fixing} + L_{ghost}$$

$$L_{gauge} = -\frac{1}{4}\sum_{a=1}^{N^2-1} F^a_{\mu\nu} F^{a\mu\nu} \quad \text{where} \quad F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_N f_{abc} A^\mu_b A^\nu_c$$

$$= -\frac{1}{4} tr(F_{\mu\nu} F^{\mu\nu}) \quad \text{where} \quad F^{\mu\nu} = F^{\mu\nu}_a T_a$$

$$L_{quarks} = \sum_{flavours} \sum_{i,j=1}^{N} \bar{q}^i_f (i\not{D} - m_f)_{ij} q^j_f$$

Where $(D_\mu)_{jk} = \partial_\mu \delta_{jk} + igA^a_\mu T^a_{jk}$. The gluon-quark interaction is induced by the covariant derivative. We need a gauge fixing term to arrive at an invertible gluon propagator.

$$L_{gauge-fixing} = -\frac{1}{2\lambda}(\partial_\nu A^{\mu c})(\partial_\mu A^{\nu c})$$

Note that is is not unique, there are many possible choices.

The ghost sector, Lghost, is needed for the perturbative expansion of non-abelian gauge theories.

$$L_{ghost} = \partial_\mu \bar{\eta}_a \partial^\mu \eta_a + g(\partial_\mu \bar{\eta}_c) f^{abc} A^{b\mu} \eta_a$$

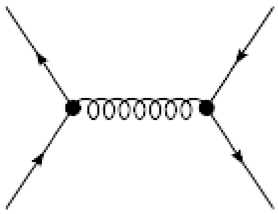
• η "Fadeev-Popov ghost" = complex scalar with "wrong" (i.e. fermionic) statistics. (i.e. closed loops get a minus sign).
• Ghost contribution compensates longitudinal degree of freedom in gluon loops in a covariant way.
Ghosts never appear as "external" particles in scattering amplitudes.
Ghost sector decouples in QED as fabc = 0.

• For a proper derivation path integral methods (original) or BRS methods necessary (→ MQFT).

Asymptotic Freedom in QCD

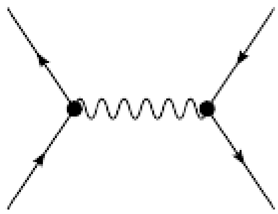
QCD is qualitatively different than classical field theory.

QCD (classical)



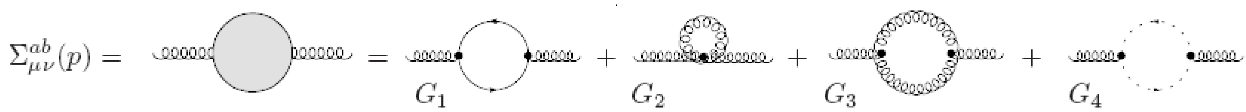
$\alpha_s/Q^2 \leftrightarrow \alpha s/r$ potential

Electrodynamics:



$\alpha/Q^2 \leftrightarrow \alpha/r$ potential

However, the quantum effects change qualitatively the low energy behaviour. Let's consider the loop correc- tions to the gluon propagator. These corrections are given to order $\alpha_s = g_s^2/4\pi$ by



These diagrams correspond to divergent integrals. We must apply a regularisation and renormalisation pro-
cedure to deal with the divergencies. Observables will not depend on the chosen regulator. One approach is *dimensional regularisation* where we evaluate these integrals in dimension = 4 − 2ε. For the Feynman rule this would amount to (MS-scheme):

$$\alpha_s \int \frac{d^4k}{(2\pi)^4} \to \alpha_s^{\overline{MS}} \frac{\mu^{2\varepsilon}}{(4\pi)^{2\varepsilon}} \frac{1}{\Gamma(1+\varepsilon)} \int \frac{d^nk}{(2\pi)^n}$$

As $\varepsilon \to 0$, divergences occur as $1/\varepsilon$ poles.

The physics is unchanged under a reparameterisation of the coupling constant and fields

$$\left.\begin{array}{r} A^\mu \to \sqrt{Z_3} A^\mu \\ g_s \to Z_g g_s \end{array}\right\} rescaling\ \ doesn't change\ \ physics$$

where the constants Z_3 , Z_g are defined perturbatively

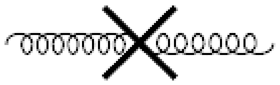
$$Z_3 = 1 + Z_3^{(1)} + Z_3^{(2)} + \ldots$$

$$Z_3^{(1)} = \alpha_s (Z_3^{1,1)} \frac{1}{\varepsilon} + Z_3^{1,0)})$$

$$Z_3^{(2)} = \alpha_s^2 (Z_3^{(2,2)} \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} Z_3^{(2,1)} + Z_3^{(2,0)}), \dots$$

$Z_3^{(k)}$ are constants which can be chosen freely.

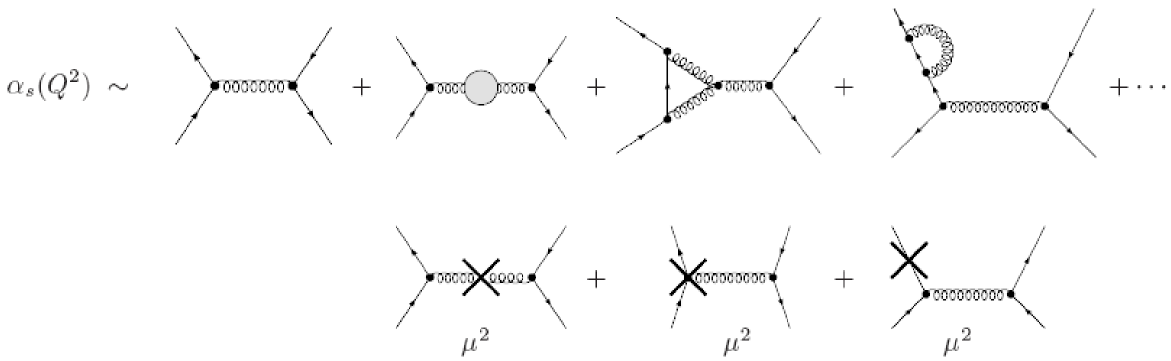
These renormalisation constants lead to additional Feynman graphs like

$G_5 =$ 

Evaluating the divergent part of $G_1 + \ldots + G_4$ gives (beyond the scope of this course)

$$\Sigma_{\mu\nu}^{ab}(p)\Big|_{div} = \frac{\alpha_s}{4\pi} \frac{r_3}{\varepsilon} \delta^{ab} (T_R N_f \frac{4}{3} - C_A \frac{13}{6})(-p^2 g^{\mu\nu} - p^\mu p^\nu)$$

As it is a constant, we choose $Z_3^{(1)}$ to cancel this (unphysical) contribution at a certain scale $\mu^2 = -p^2 > 0$. For the coupling at one loop we find, schematically,



The latter are divergences which come from the renormalisation constants. The are adjusted so to cancel the divergences and with an experimental value at a certain scale.

$$\alpha_s(Q^2) = \alpha_s + \alpha_s^2 \frac{b}{\varepsilon} (Q^2)^{-\varepsilon} - \alpha_s^2 (\mu^2)^{-\varepsilon} = \alpha_s (1 - \alpha_s b \log(\frac{Q^2}{\mu^2}) + O(\alpha^2))$$

Quantum corrections in this way lead to a scale-dependent coupling

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(\mu^2)} + b \log(\frac{Q^2}{\mu^2}); \quad b = \frac{11 C_A - 4 T_R N_f}{12\pi}$$

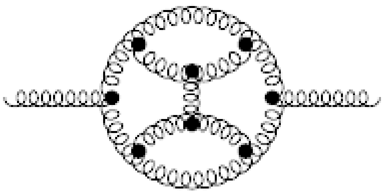
This is the *renormalised coupling* and it depends logarithmically on the scale of the process. The same is true for QED but b has a different sign.

The logarithmic scale dependence of the strong coupling

The so-called β-function measures the scale dependence of the coupling:

$$\beta(\alpha_s(Q^2)) = \frac{\partial}{\partial \log(Q^2)} \alpha_s = -\alpha_s^2 (Q^2) b$$

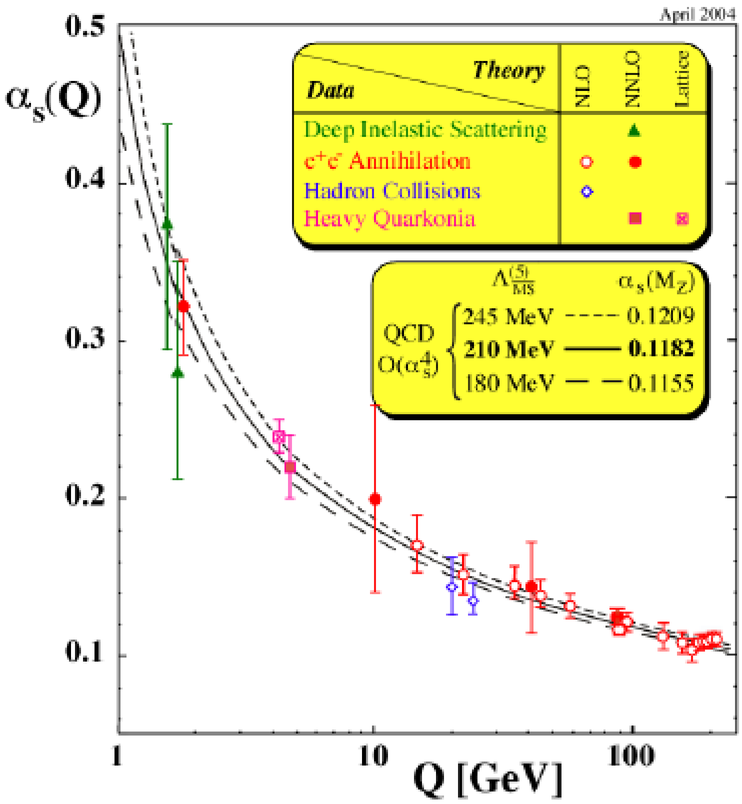
to the 4-loop level



$$\beta = -\alpha_s^2 (b + \alpha_s b' + \alpha_s^2 b'' + \alpha_s^3 b''' + \ldots)$$

At one loop $\quad b = \dfrac{11C_A - 4T_R N_f}{12\pi}$

The sign of the β-function depends on the particle content of the respective theory. If the number of flavours in the loop is Nf < 33/2 ⇒ b > 0. Thus, in the Standard Model, β < 0 (or b > 0). The sign is crucial for the high energy (ultraviolet) and low energy (infrared) behaviour of the theory.



The running of αs is experimentally confirmed.

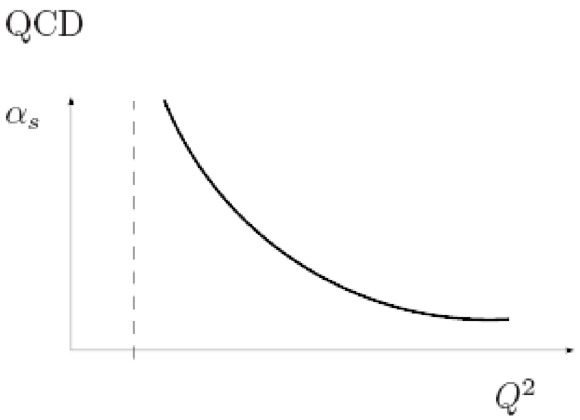
Ultraviolet behaviour in QCD For b > 0 (β < 0) ⇒ $\alpha_s \xrightarrow{Q^2 \to \infty} 0$. Hence, the quarks feel no gluon exchange and can be considered as quasi-free particles. This is known as *asymptotic freedom* (2004 Nobel prize: Gross, Politzer, Wilczek).

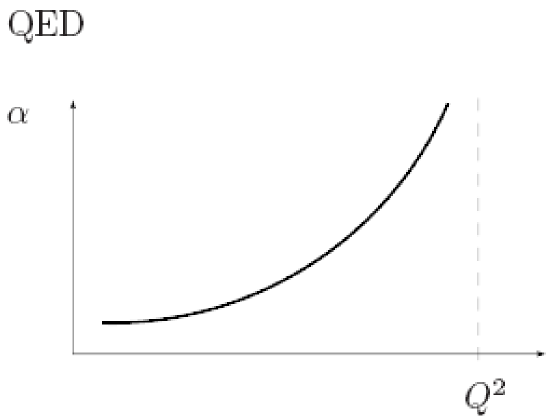
Infrared behaviour in QCD For b > 0 implies that there exists a scale, Λ such that $\alpha_s \xrightarrow[Q^2 \to \Lambda^2]{} 0$. The point where the coupling goes to infinity is called the *Landau pole*, it indicates the *strong coupling* regime. Of course, perturbation theory itself breaks down long before the Landau pole is reached.

As α_s is large at hadronic scale, $\sim$ 1 GeV , perturbation theory is not applicable. QCD is a strongly interacting theory at low energy and most of our calculational tools fail in this regime.

This is unfortunate, but also a necessary condition for the theory to have a chance of explaining confinement.
In QED, we have no photon self interaction, and the running is reversed:

$$b_{QED} = b_{QCD}\Big|_{\substack{C_A=0\\ T_R=1\\ N_f=1}} = -\frac{1}{3\pi} < 0$$

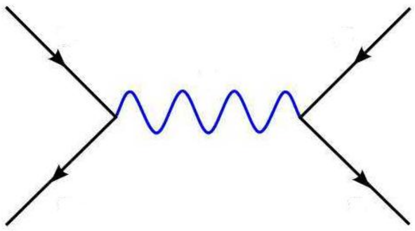




We see that QCD and QED are qualitatively different from one another with opposite asymptotic behaviour. Quantitative solutions in low-energy QCD are intrinsically difficult. The dynamics is non-perturbative.
Our lack of knowledge of the internal dynamics of the low energy bound states of QCD can be parametrised using general functions with the correct allowable Lorentz structure.
These can in some cases be measured in one process and reapplied in another. We shall see an example of this with the pion decay constant.
Better yet we can calculated such quantities using non-perturbative numerical methods → *Lattice gauge theory*.

Problems:

1.Find the amplitude for the diagram



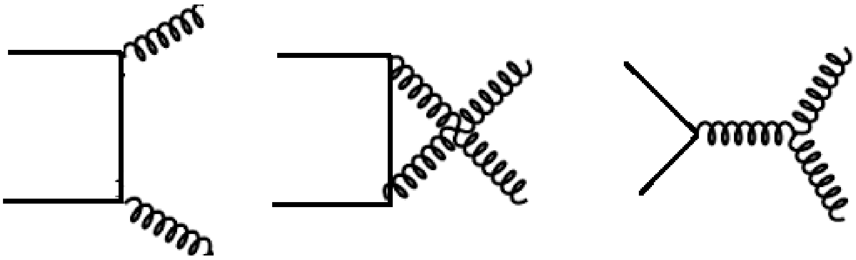
What is the color factor.

2. Color factors always involve expressions of the form $\lambda_{ij}^{\alpha}\lambda_{kl}^{\alpha}$ (summed over α). There is a formula for this quantity, which shortens the arithmetic:

$$\lambda_{ij}^{\alpha}\lambda_{kl}^{\alpha} = 2\delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}$$

Chek this theorem for
(a)i=j=k=l=1
(b)i=j=1, k=l=2
(c)i=l=1, j=k=2.

3.There is a simple test for the gauge invariance of an amplitude in QCD (or QED): Replace any gluon (or photon) polarization vector by its momentum and you must get zero. Show that form $A = A_1 + A_2 + A_3$ is gauge invariant, but $A = A_1 + A_2$ alone is not.



Lecture X

Goldstone's theorem, Higgs Mechanism

We now study gauge invariant models with a nontrivial vaccuum structure. Such models display *Spontaneous Symmetry Breaking*: this means the action is gauge invariant under some symmetry but the ground state (or vaccuum) is not and this breaks the symmetry.

$$S[\phi,\psi,F^{\mu\nu}]=S[U\phi,U\psi,UF^{\mu\nu}U^{+}]$$

We use <…> to denote the *vacuum expectation value* (or vev).
A vev for fields with angular momentum would be incompatible with the observed isotropy of space. For example, a vev for fermions (→ spin) or gauge fields (→$\langle\vec{E}\rangle,\langle\vec{B}\rangle$)

$$\Rightarrow\langle\psi\rangle,\langle F^{\mu\nu}\rangle=0$$

However a vev for a scalar is allowed $\langle\phi\rangle\neq 0$.

We will see below that the Goldstone Theorem implies that there exists a massless mode for each generator T_a which does not leave the vacuum invariant, $T_a\langle\phi\rangle\neq\langle\phi\rangle$. These massless fields are called Nambu-Goldstone bosons.
Recall that an explicit gauge boson mass term in the Lagrangian is not invariant. In combination with a gauge theory, the massless *Nambu-Goldstone boson* will lead to a massive vector boson (→*Higgs mechanism*). A massless gauge boson together with a Nambu-Goldstone boson combines to a massive gauge field. One may say that the gauge field acquires a longitudinal component through interaction with the nontrivial vacuum.
Spontaneous Symmetry Breaking (SSB) is not specific to particle physics. Ferromagnets and Superconduc-tors are two other examples.

SSB in an U(1) scalar field theory

We will see: If the vacuum breaks the symmetry of the theory then a massless mode is created.
Note: As U(1) is isomorphic to SO(2) one can view the model as a so-called SO(2) symmetric σ-model.

$$\phi=\frac{1}{\sqrt{2}}(\phi_1+\phi_2)\qquad(\text{or}\quad\tilde{\phi}=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\quad\text{for SO(2))}$$

$$L=\partial_\mu\phi^{+}\partial^\mu\phi-\mu^2\phi^{+}\phi-\lambda(\phi^{+}\phi)^2\qquad\qquad\text{with }\lambda>0$$
$$=\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1+\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2-\frac{\mu^2}{2}(\phi_1^2+\phi_2^2)-\frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$$

L is invariant under a global transformation

$$\phi\to e^{i\theta}\phi\qquad(\text{or}\quad\tilde{\phi}\to\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\tilde{\phi}\quad)$$

The ground state is defined by minimising the energy:

$$\mathrm{H}=\vec{\pi}\dot{\vec{\phi}}-L=\frac{\partial L}{\partial\dot{\phi}_1}\dot{\phi}_1+\frac{\partial L}{\partial\dot{\phi}_2}\dot{\phi}_2-L$$

$$= \frac{1}{2}(\pi_1^2 + \pi_2^2) + \frac{1}{2}(\nabla\phi_1 \cdot \nabla\phi_1 + \nabla\phi_2 \cdot \nabla\phi_2) + V(\phi_1, \phi_2)$$

Hence, we see that to find the minimum of H we need to find the minimum of the potential

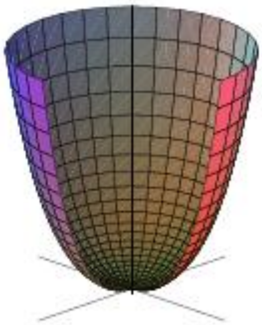
$$V(\phi_1, \phi_2) = \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2$$

The minimum condition reads

$$\frac{\partial V}{\partial \phi_1} = \frac{\partial V}{\partial \phi_2} = 0 \quad \Leftrightarrow \quad \left.\begin{array}{l} \phi_1(\mu^2 + \lambda(\phi_1^2 + \phi_2^2)) = 0 \\ \phi_2(\mu^2 + \lambda(\phi_1^2 + \phi_2^2)) = 0 \end{array}\right\} (*)$$

case 1. $\mu^2 > 0$: $\phi_1 = \phi_2 = 0$ is the ground state or vaccuum solution.

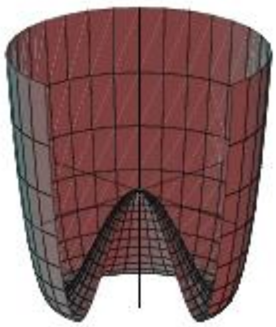
ϕ_1, ϕ_2 are real scalar fields with mass μ



The vaccuum state $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$ is trivially invariant under rotations in the ϕ_1, ϕ_2-plane.

case 2. $\mu^2 < 0$: $(*)$ has a nontrivial solution

$$2\phi^*\phi = \phi_1^2 + \phi_2^2 = -\frac{\mu^2}{\lambda} = v^2 > 0$$



The minimum of the potential is along a circle → infinite degeneracy. This is called the "champagne bottle" or "Mexican hat" potential. The ground state has to pick one point of this circle, i.e. it breaks the symmetry of the system.
As the theory is U(1) (or SO(2)) invariant we may choose

$$\langle \phi_1 \rangle = 0, \quad \langle \phi_2 \rangle = v \qquad (\text{or} \quad \tilde{\phi} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix})$$

Applying a phase transform to the vaccuum we find

$$e^{i\Lambda\theta}\langle \phi \rangle \neq \langle \phi \rangle \qquad \Rightarrow U(1) \text{ symmetry "broken"}$$

The physical spectrum is obtained after expanding around the vev of the theory

$$\phi_1 = \pi\,, \qquad \phi_2 = \sigma = H + \langle \phi_2 \rangle = H + v$$

where the new fields have

$$\langle \pi = 0 \rangle, \qquad \langle H = 0 \rangle$$

$$L = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{\mu^2}{2}[\pi^2 + (v+H)^2] - \frac{\lambda}{4}[\pi^2 + (v+H)^2]^2$$
$$= \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{\mu^2}{2}[\pi^2 + v^2 + 2vH + H^2] - \frac{\lambda}{4}[\pi^2 + v^2 + 2vH + H^2]^2$$

Collect the terms with different powers of π, H:

~ π^0, H^0: $\quad -\frac{\mu^2}{2}v^2 - \frac{\lambda}{4}v^4$ irrelevant constant;

~ π^1, H^1: $\quad -\mu^2 v - \lambda v^3 = 0$ linear terms give rise to tadpoles, nonexistent in the theory

~ π^2, H^0: $\quad -\frac{\mu^2}{2} - \frac{\lambda}{2}v^2 = 0$ π's are massless

~ π^0, H^2: $\quad \mu^2$ Gives mass: $\frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}m_H^2 H^2$

where $m_H^2 = -2\mu^2 > 0$

The other terms define the interactions between π,H.
We conclude that

- π is a massless spin-zero boson, the *Nambu-Goldstone boson*
- H is a massive spin-zero boson, the *Higgs boson*

Generalisation to SO(N)

$$\vec{\phi} = \begin{pmatrix} \pi_1 \\ . \\ . \\ \pi_{N-1} \\ \sigma \end{pmatrix} \in R^N$$

This is in the fundamental representation of SO(N) where the generators U ∈ SO(N) are such that $UU^{-1} = 1$ and detU = 1. There are N(N − 1)/2 generators, all of which are antisymmetric matrices.

$$U = \exp(i\sum_{i<j}^{N} \Lambda_{ij} T^{(ij)}) \qquad \in SO(N)$$

Where

$$(T^{(ij)})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

The Lagrangian

$$L = \frac{1}{2}\partial_\mu \vec{\phi}^T \partial^\mu \vec{\phi} - \frac{\mu^2}{2}(\vec{\phi}^T\vec{\phi}) - \frac{\lambda}{4}(\vec{\phi}^T\vec{\phi})^2$$

is invariant under global SO(N) transformations. For $\mu^2 < 0$, $V(\vec{\phi})$ is minimal if $\phi_i\phi_i = \frac{-\mu^2}{\lambda} = v^2 > 0$, and hence we choose

$$\langle\vec{\phi}\rangle = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ v \end{pmatrix}$$

$\langle\vec{\phi}\rangle$ is invariant under SO(N − 1) transformations (generators $T^{(ij)}$ with i < j <k which are defined by the $T^{(ij)}\langle\vec{\phi}\rangle = \langle\vec{\phi}\rangle$). The remaining N − 1 generators $T^{(ik)}$ *break* the vacuum as $T^{(ik)}\langle\vec{\phi}\rangle \neq \langle\vec{\phi}\rangle$. There are (1/2) (N(N − 1)) – (1/2) (N − 1)(N − 2) = N − 1 broken generators. We see that the vev breaks SO(N) spontaneously to SO(N − 1).

Looking at the spectrum, we find

- $\pi_{j=1,\ldots,N-1}$ massless Goldstone bosons
- σ = H + v → H is a massive state with mass $M_H^2 = 2\lambda v^2$ known as the Higgs boson mass

Classical Goldstone Theorem

To each generator which breaks the vacuum, i.e. $T^a\langle\vec{\phi}\rangle \neq 0$, corresponds a massless field (Nambu-Goldstone boson).

Proof: We need the definition of the mass matrix in the following:

$$M_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\vec{\phi}=\langle\vec{\phi}\rangle}$$

The symmetry of the action implies

$$V(\vec{\phi}) = V(\vec{\phi} + i\varepsilon^a T^a \vec{\phi}) = V(\vec{\phi}) + \frac{\partial V}{\partial\phi_j} i\varepsilon^a T^a \vec{\phi} + O(|\varepsilon^2|)$$

$$\Rightarrow \frac{\partial V}{\partial\phi_j} T^a_{jl}\phi_l = 0$$

Applying another derivative on the last line gives

$$0 = \frac{\partial}{\partial\phi_k}\left.\left(\frac{\partial V}{\partial\phi_j}T^a_{jl}\right)\right|_{\vec{\phi}=\langle\vec{\phi}\rangle} = \left.\left(\frac{\partial^2 V}{\partial\phi_k\partial\phi_j}T^a_{jl}\phi_l + \frac{\partial V}{\partial\phi_j}T^a_{jk}\right)\right|_{\vec{\phi}=\langle\vec{\phi}\rangle}$$

and finally one obtaines

$$0 = M_{kj}T^a_{jl}\langle\phi_l\rangle + 0$$

If T^a is a "broken" generator one has $T^a\langle\vec{\phi}\rangle \neq \vec{0} \Rightarrow M_{kj}$ has a null eigenvector ⇒ null eigenvalues ⇒ massless particle for each such generator. (Note that the eigenvalues of the mass matrix are the particle masses, as the particle are defined as their mass eigenstates.)

This completes the proof.

We now combine the concept of a spontaneously broken symmetry with a gauge theory.

Higgs mechanism

The Higgs Mechanism for U(1) gauge theory

Consider

$$L = (D^{\mu}\phi^*)(D_{\mu}\phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

With $D_{\mu} = \partial_{\mu} + iQA_{\mu}$ and $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$. Gauge symmetry here means invariance under $A_{\mu} \to A_{\mu} - \partial_{\mu}\Lambda$.

case a) $\mu^2 > 0$: $V(\phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$ unbroken case, with a minimum at φ = 0.

The ground state or vaccuum is U(1) symmetric. The corresponding theory is known as *Scalar Electrody-namics* of a massive spin-0 boson with mass μ and charge Q.

case b) $\mu^2 > 0$ nontrivial vaccuum case

V (φ) has a minimum for $2(\phi^*\phi) = -\mu^2/\lambda = v^2$ which gives $\langle\phi\rangle = \frac{v}{\sqrt{2}}e^{i\alpha}$.

We may choose α = 0 as α is arbitrary but fixed.

$$e^{i\Lambda Q}\langle\phi\rangle \neq \langle\phi\rangle \quad \to \quad \text{SSB}$$

We may parameterise the field in polar coordinates

$$\phi = \frac{1}{\sqrt{2}}\mathrm{Re}^{i\theta} = \frac{1}{\sqrt{2}}(v + H + i\pi)$$

The kinetic term is

$$D_{\mu}\phi = (\partial_{\mu} + iQA_{\mu})\frac{1}{\sqrt{2}}\mathrm{Re}^{i\theta}$$

$$= \frac{1}{\sqrt{2}}(\partial_{\mu}R + iR\partial_{\mu}\theta + iQA_{\mu}R)e^{i\theta}$$

$$D_{\mu}\phi^* = \frac{1}{\sqrt{2}}(\partial_{\mu}R - iR\partial_{\mu}\theta - iQA_{\mu}R)e^{-i\theta}$$

The potential term is

$$V(\phi^*\phi) = \frac{1}{2}\mu^2R^2 + \frac{\lambda}{4}R^4$$

$$L=\frac{1}{2}(\partial^\mu R\partial_\mu R)+\frac{1}{2}R^2\partial^\mu\theta\partial_\mu\theta-\frac{1}{2}Q^2R^2A^\mu A_\mu+QR^2A_\mu\partial^\mu\theta-V(R^2)-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$=\frac{1}{2}(\partial^\mu R\partial_\mu R)-V(R^2)-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}Q^2R^2(A_\mu+\frac{1}{Q}\partial_\mu\theta)^2$$

Now we note that the term $A_\mu+\frac{1}{Q}\partial_\mu\theta$ looks like a gauge field transformation. In fact, it can be gauged away1: $A_\mu\to A'_\mu+\frac{1}{Q}\partial_\mu\theta$. Now, we see that $\langle R=v\rangle$ as R=v+H.

$$L=\frac{1}{2}\partial^\mu H\partial_\mu H)-V((v+H)^2)-\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu}-\frac{1}{2}Q^2(v+H)^2A'^\mu A'_\mu$$
$$=\frac{1}{2}\partial^\mu H\partial_\mu H)-V((v+H)^2)-\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu}-\frac{1}{2}Q^2v^2A'^\mu A'_\mu-\frac{1}{2}Q^2(2vH+H^2)A'^\mu A'_\mu$$

In the unbroken case the particle content contained A_μ, a massless gauge boson (2 d.o.f's) and two massive scalar fields ϕ_1,ϕ_2 (2 d.o.f's) all in all 4 d.o.f's.

SSB: After mapping H, π ↔ R, θ it is manifest that θ is a massless Nambu-Goldstone boson that can be gauged away; and A_μ is massive with 3 d.o.f's: 2 transverse + 1 longitudinal. We say that "θ is eaten" by the gauge field. H is massive: 1 degree of freedom. The number of d.o.f's (=4) is unchanged.
We see, that the Goldstone boson is absorbed by the gauge boson leading to a longitudinal degree of freedom for the gauge field. This is called the *Higgs mechanism* after P. W. Higgs (1964). Ideas along the same line were developed by Brout, Englert, Hagen and Kibble around the same time.

Problem:

The Lagrangian for three real fields is

$$L=\frac{1}{2}(\partial^\mu\phi_i)^2-\frac{1}{2}\mu^2\phi_i^2-\frac{1}{4}\lambda(\phi_i^2)^2$$

Show that ($\mu^2<0$, $\lambda>0$; a summation over is is implied) it describes a massive field of mass $\sqrt{-2\mu^2}$ and two massless Goldstone bosons.

Lecture XI

Electroweak unification

The Standard Model is a gauge theory with the gauge group:

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$$

It defines the fundamental interaction of Fermions (leptons and quarks), gauge bosons and the Higgs boson.
The electroweak sector $SU(2)_L \otimes U(1)_Y$ is spontaneously broken via the Higgs mechanism. It displays other odd features.

Weirdness in the Weak sector

Up to around 1956 it was expected that charge-conjugation (C), parity (P), and time reversal (T) were eachsymmetries of nature.
Why?
• A breakage of joint CPT would break Lorentz invariance
• Dirac equation: suggests C should be a good symmetry
• US drives on left, UK drives on right: suggests P should be a good symmetry
• Classical mechanics is reversible: suggests T should be a good symmetry
In 1956 T. D. Lee and C. N. Yang observed while C and P invariance had been rigorously checked experimentally for the strong interactions, no such test had been made for weak decays.

Parity and CP violation history
1956 T. D. Lee & C. N. Yang suggest Parity, Charge and CP violation experiments
1957 C.S. Wu et al discover Co60 beta decay has strong parity asymmetry
1957 T.D. Lee & C. N. Yang : two component neutrino theory
1958 Marshak & Sudarshan,Feynman & Gell-Mann : V-A four fermi coupling
1957 Lee and Yang Nobel Prize
1964 Cronin and Fitch discover CP violation in neutral kaon system
1963 Cabbibo quark flavour mixing
c1970 Glashow-Salam-Weinberg Theory unifies electro-weak sector
1972 Kobayashi Maskawa
1980 Cronin and Fitch Nobel Prize
2008 Kobayashi-Maskawa-Nambu Nobel Prize

Wu experiment
The Wu experiment placed Co^{60} in a magnetic field aligning nuclear spin
Electrons preferentially emitted in direction opposite to magnetic field
→ Parity breaking angular distribution of beta (e^-) particles
T. D. Lee and C. N. Yang concluded that the reclusive neutrino *always* spin aligns with the direction of propagation. Breaks parity (defines the sense of a left-handed screw).
The anti-neutrino always spin anti-aligns with the direction of propagation.
Helicity and chirality coincide for massless neutrinos ⇒ V-A current

$$\bar{\nu}\gamma_\mu(1-\gamma_5)e$$

enters the four fermi interaction (Marshak & Sudarshan, Feynman & Gell-Mann).
Model does *not* violate joint CP:
neutrino↔ anti-neutrino *and* left handed spin ↔ right handed spin

Parity breaking

We have mentioned the V − A weak coupling vertex several times.
Consider an $e^- + \bar{\nu}_e \to W^-$ transition, where the electron momentum is large and in the $\hat{z}$ direction. The W-boson couples to the V-A current

$$J_\mu = \bar{u}_\nu(p')\gamma_\mu(1-\gamma_5)u_e(p) = \frac{1}{2}\bar{u}_\nu(p')(1+\gamma_5)\gamma_\mu(1-\gamma_5)u_e(p)$$

We take the case of $p_\mu(p, p\hat{z})$ and $p'_\mu(p', p'\hat{z})$

The we take $\chi \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ as up/down two spinors and our external four spinors are

$$u_e\begin{pmatrix} \chi_e \\ \sigma_z\chi_e \end{pmatrix}, \qquad \bar{u}_\nu = (\chi_\nu^+, -\chi_\nu^+\sigma_z)$$

We then have

$$(1-\gamma_5)u_e = \begin{pmatrix} (1-\sigma_z)\chi_e \\ -(1-\sigma_z)\chi_e \end{pmatrix} \qquad \bar{u}_\nu(1+\gamma_5) = (\chi_\nu^+(1-\sigma_z), \quad -\chi_\nu^+(1-\sigma_z))$$

Observe that $\frac{1}{2}(1-\sigma_z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projects out *only* left handed components. The corresponding W-boson processes for the parity flipped right handed spins *do not exist* and this mechanism lead to the Wu experiment asymmetry.

Glashow-Salam-Weinberg Theory *SU(2)$_L$ ⊗ U(1)$_Y$*

Both Parity violation and CP violation are explained by the unified electro-weak sector, also referred to as Glashow-Salam-Weinberg Theory (c. 1970), Electro-Weak Theory, or Quantum Flavour Dynamics.

Electroweak theory describes electromagnetic and weak interactions. The gauge group is spontaneously broken via the Higgs mechanism $SU(2)_L \otimes U(1)_Y \xrightarrow[SSBreakdown]{} U(1)_{EM}$

Theory is chirally coupled gauge theory:

left handed fields ψ_L, ϕ_L trandform as $SU(2)_L$ doublets

right handed fields ψ_R, ϕ_R transforms as $SU(2)_L$ singlets

$$\psi_{L,R}\frac{1}{2}(1\mp\gamma_5) \text{ such that } P\psi_{L,R} = \psi_{R,L}$$

Left-handed doublets are in the fundamental representation of $SU(2)_L$ (weak isospin):

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \rightarrow \begin{pmatrix} \nu'_e \\ e' \end{pmatrix}_L = e^{i\Lambda^a T^a} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$

$$e_R \rightarrow e'_R = e_R \qquad \text{(singlet)}$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \rightarrow \begin{pmatrix} u' \\ d' \end{pmatrix}_L = e^{i\Lambda^a T^a} \begin{pmatrix} u \\ d \end{pmatrix}_L$$

$$u_R \rightarrow u'_R = u_R$$

$$d_R \rightarrow d'_R = d_R$$

Theory does not distinguish between quarks, *colour blind.*

Lagrangian

We can write the theory as the sum of Lagrangian densities:

$$L = L_{gauge} + L_{Higgs} + L_{fermion} + L_{Yukawa}[+L_{ghost}]$$

Note that the Yukawa sector, L_{Yukawa}, allows for boson and fermion interactions; the "ghost" sector is needed to maintain covariance whilst quantising a non-Abelian gauge theory.

The gauge sector: L_{gauge}

Electromagnetic interactions are related to an unbroken gauge theory as it contains a massless boson; electric charge is conserved leaving the photon massless.
Weak interactions are short range hence we need SSB to create exchange particle mass

$$massles \;\; \sim \frac{1}{r} \rightarrow \;\; massive \;\; \sim \frac{e^{-mr}}{r}$$

Glashow, Salam, and Weinberg constructed a Lagrangian which explained all existing data so far.

$$\begin{array}{lccc} gauge\;\; group: & SU(2)_L & \otimes & U(1)_Y \\ generators: & \underbrace{T^1, T^2, T^3}_{weak\; isospin} & & Y \\ gauge\;\; couplings: & g & & g' \end{array}$$

The Lagrangian may be written as

$$L_{gauge} = -\frac{1}{4} W^{j\mu\nu} W^{j}_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}$$

Where

$$W^{j}_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu - g\varepsilon^{jlm} W^l_\mu W^m_\nu , \qquad j \in [1,2,3]$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

This theory describes four massless vector bosons but we need to break this to only one using the Higg's mechanism.

The Higgs sector: L_{Higgs}

We generate gauge boson masses by spontaneous symmetry breakdown:
need 1 massless gauge boson (photon) and 3 massive gauge bosons (weak interactions)
Minimal choice: introduce a scalar field as an SU(2) doublet

$$\vec{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1 + i\pi_2 \\ \sigma + i\pi_3 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

Where $T(\vec{\phi}) = \frac{1}{2}$ transforms under fundamental representation of $SU(2)_L$.

Assign hypercharge: $Y(\vec{\phi}) = \frac{1}{2}$.

To make the theory invariant under local transformations, we need

$$D_\mu = \partial_\mu + \underbrace{+ig\frac{1}{2}\vec{\sigma}\cdot\vec{W}_\mu}_{SU(2)\ \ coupling} + \underbrace{ig' T B_\mu}_{U(1)\ \ coupling}$$

The Lagrangian is then

$$L_{Higgs} = (D_\mu \vec{\phi})^+ (D^\mu \vec{\phi}) - V(\vec{\phi}^+ \vec{\phi})$$

.where $V(\vec{\phi}^+\vec{\phi}) = \mu^2 \vec{\phi}^+ \vec{\phi} + \lambda(\vec{\phi}^+\vec{\phi})^2$.

Note that $\mu^2 < 0$ and $\lambda > 0$ leads to SSBreakdown.

$$\langle \vec{\phi} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with} \quad v = \sqrt{\frac{-\mu^2}{\lambda}}$$

Consider

$$\underbrace{(T^3 + Y)}_{Q} \langle \vec{\phi} \rangle = \underbrace{\left[\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]}_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \langle \vec{\phi} \rangle \Rightarrow e^{i\Lambda Q} \langle \vec{\phi} \rangle = \langle \vec{\phi} \rangle$$

$$T^1\langle\vec{\phi}\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \neq \langle\vec{0}\rangle$$

Hence, T^1 is a broken generator. The same is true for T^2 and $T^3 - Y$ meaning that we have three Goldstone bosons which we may gauge away. The SU(2)L doublet can be written as

$$\vec{\phi} = \exp(-i\frac{1}{v}T^j\theta^j)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H+v \end{pmatrix} = U^+\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H+v \end{pmatrix}$$

The term $U^+ = \exp(-i\frac{1}{v}T^j\theta^j)$ looks like a $SU(2)_L$ local group element. The Goldstone bosons, θ^j , play here the role of the space-time dependent parameters.

Applying the gauge transformation U leads to the so-called unitary gauge:

$$\vec{\phi} \to \vec{\phi}' = U\vec{\phi} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H+v \end{pmatrix}$$

We obtain

$$L_{Higgs} = \frac{1}{2}\left[D_\mu\begin{pmatrix} 0 \\ H+v \end{pmatrix}\right]^+ D^\mu\begin{pmatrix} 0 \\ H+v \end{pmatrix} + \underbrace{V(\frac{1}{2}(H+v)^2)}_{L_{quad}}$$

We can write

$$\vec{\sigma}\cdot\vec{W}_\mu = \sigma^1 W_\mu^1 + \sigma^2 W_\mu^2 + \sigma^3 W_\mu^3 = \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}$$

Thus in this gauge:

$$D^\mu\begin{pmatrix} 0 \\ H+v \end{pmatrix} = (\partial^\mu + i\frac{g}{2}\vec{\sigma}\cdot\vec{W}^\mu + i\frac{g'}{2}B^\mu)$$

$$= \begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} + i(\frac{g'}{2}B^\mu - \frac{g}{2}W^{3\mu})\begin{pmatrix} 0 \\ v+H \end{pmatrix} + i\frac{g}{2}(W^{1\mu} - iW^{2\mu})\begin{pmatrix} v+H \\ 0 \end{pmatrix}$$

And similarly

$$\left(D_\mu\begin{pmatrix} 0 \\ H+v \end{pmatrix}\right)^+ = (0,\partial_\mu H) - i\frac{g}{2}(W_\mu^1 + iW_\mu^2)(v+H,0) - i(\frac{g'}{2}B_\mu - \frac{g}{2}W_\mu^3)(0,v+H)$$

$$L_{Higgs} = \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{4}(v+H)^4 +$$
$$+\frac{g^2}{2}(v+H)^2(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}) + \frac{1}{8}(g'B^\mu - gW^{3\mu})(g'B^\mu - gW^{3\mu})(v+H)^2$$

This Lagrangian defines interaction and mass terms
The charged vector boson masses can be read off directly from

$$\frac{M_W^2}{2}(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}) = \frac{g^2 v^2}{8}((W_\mu^1)^2 + (W_\mu^2)^2)$$

leading to

$$M_{W^{\mp}} = M_{W^{1,2}} = \frac{gv}{2}$$

The interactions of $W^{1,2}$ arise through the combinations $W_\mu^\pm = W_\mu^1 \pm iW_\mu^2$, and these linear combinations are the W^+ and W^- gauge bosons. For the quadratic term in the W_μ^3, B_μ bosons we find

$$L_{quad} = \frac{v^2}{8}(W_\mu^3, B_\mu)\begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

The eigenvalues of the mass matrix are: $\lambda = 0$, $\lambda = g^2 + g'^2$

The mass term is a diagonal quadratic form of the field $g'B^\mu - gW^{3\mu}$, and we find the normalised eigenvectors are parallel and orthogonal to this

$$\frac{1}{\sqrt{g^2 + g'^2}}\begin{pmatrix} -g \\ g' \end{pmatrix} \qquad \frac{1}{\sqrt{g^2 + g'^2}}\begin{pmatrix} g' \\ g \end{pmatrix}$$

Making the field redefinition,

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}$$

We get

$$L_{quad} = \frac{v^2}{8}(g^2 + g'^2)\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

We see that A^μ is massless and that A^Z is a massive vector boson with mass $M_Z^2 = (g^2 g'^2)v^2/4$, and the other massive state in Higgs sector is the Higgs boson with mass $M_H^2 = -2\mu^2 = 2\lambda v^2$.

The fermion sector: L$_{\text{fermion}}$

Massive Dirac-fermions can be split into left and right-handed chiral components by using projectors $P_{L,R}$:

$$\psi = \psi_L + \psi_R = P_L\psi + P_R\psi \quad , \quad P_{L,R} = \frac{1 \mp \gamma_5}{2}$$

In the original formulation of the Standard Model the massless left handed neutrinos had no right-handed partners. Recently observed neutrino oscillations suggest right-handed neutrinos but we shall stick to the original formulation in the following.

The SU(2)$_L$ $\otimes$ U(1)$_Y$ quantum numbers for the fermions are

Leptons	T (isospin)	Y (hypercharge)	T_3	$Q = T_3 + Y$

$\begin{pmatrix}\nu_e \\ e\end{pmatrix}, \begin{pmatrix}\nu_\mu \\ \mu\end{pmatrix}, \begin{pmatrix}\nu_\tau \\ \tau\end{pmatrix}$	$\frac{1}{2}$	$-\frac{1}{2}$	$1/2$ $-1/2$	0 -1
e_R , μ_R , τ_R	0	-1	0	-1

Quarks	T (isospin)	Y (hypercharge)	T_3	$Q = T_3 + Y$
$\begin{pmatrix}u \\ d\end{pmatrix}, \begin{pmatrix}c \\ s\end{pmatrix}, \begin{pmatrix}t \\ b\end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{6}$	$1/2$ $-1/2$	0 -1
u_R , c_R , t_R	0	2/3	0	2/3
d'_R , s'_R , b'_R	0	-1/3	0	-1/3

To give an example for the notation:

$$\psi_L^{c=red,\, f\in\{1,2\}} = \begin{pmatrix} c_{red} \\ s'_{red} \end{pmatrix}_L \;, \quad \psi_R^{c=blue,\, f=2} = (d'_{blue})_R$$

The reason for introducing the primed quark fields q′ in the list will become clear below.
We can now construct the fermion Lagrangian

$$L_{fermion} = \sum_{\substack{colors \\ flavors}} \{ \bar{\psi}_L^{cf} i\not{D} \psi_L^{cf} + \bar{\psi}_R^{cf} i\not{D} \psi_L^{cf} \}$$

where we have to distinguish the covariant derivative acting on the left fields

$$D_\mu = \partial_\mu + ig\frac{\vec{\sigma}}{2}\vec{W}_\mu + ig'YB_\mu$$

from the one acting on the right fields, as the latter do not couple to $SU(2)_L$ gauge bosons.

$$\tilde{D}_\mu = \partial_\mu + ig'YB_\mu$$

$L_{fermion}$ is not sufficient to provide mass terms; we need to couple the fermions to the Higgs sector to achieve that.
There are no mass terms because gauge invariance does not allow them as L/R fields cannot be adequately combined. For example:

$$m\bar{e}e = m(\bar{e}P_R e + \bar{e}P_L e) = m(\bar{e}_L e_R + \bar{e}_R e_L) = m\bar{e}_L e_R + Hermitian \quad conjugate$$

But this is not a gauge invariant term under $SU(2)_L$ and hence it is forbidden.
Note that we have used $\bar{e}P_R = e^+\gamma_0 P_R = e^+ P_L \gamma_0 = e^+ P_L^+ \gamma_0 = (P_L e)^+ \gamma_0 = \bar{e}_L$

The Yukawa sector: L_{Yukawa}

L_{Yukawa} is defined by $SU(2)_L \otimes U(1)_Y$ invariants composed out of the Higgs doublet and the fermion multiplets where $\vec{\phi} = \begin{pmatrix} \phi^+ \\ \phi \end{pmatrix}$ is the Higgs doublet.

SU(2) invariants are

$$\vec{\phi}^+ \psi_L = (\phi^-, \phi^0) \cdot \begin{pmatrix} \psi_{1,L} \\ \psi_{2L} \end{pmatrix}, \quad \vec{\phi}^T \cdot \varepsilon \cdot \psi_L = (\phi^+, \phi^0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{1,L} \\ \psi_{2L} \end{pmatrix}$$

For the first generation:

$$Y(\vec{\phi}) = \frac{1}{2} \quad Y(e_R) = -1 \quad Y\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = -\frac{1}{2} \quad Y(u_R) = \frac{2}{3} \quad Y\begin{pmatrix} u \\ d \end{pmatrix}_L = \frac{1}{6}$$

Electron mass:

$$\lambda_e \bar{e}_R \vec{\phi}^+ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L + Herm.conj. \xrightarrow{\vec{\phi} \to \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \frac{\lambda_e v}{\sqrt{2}} (\bar{e}_R e_L + Herm/conj.) = m_e \bar{e} e$$

Y: 1-1/2-1/2=0 as it must be for an $U(1)_Y$ singlet.

Up quark mass:

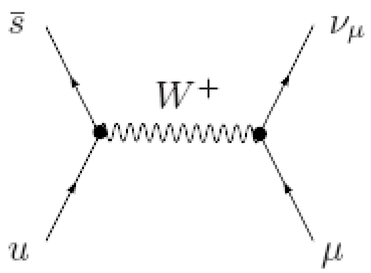
$$\lambda_u \bar{u}_R \vec{\phi}^T \cdot \varepsilon \cdot \begin{pmatrix} u \\ d' \end{pmatrix}_L + Herm.conj. \to m_u \bar{u} u$$

Y: -2/3+1/2+1/6=0

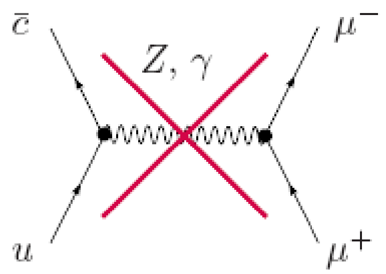
Down quark mass:

$$\lambda_d \bar{d'}_R \vec{\phi}^+ \begin{pmatrix} u \\ d' \end{pmatrix}_L + Herm.conj. \to m_{d'} \bar{d'} d'$$

Flavour mixing: seen in experiments, e.g. $K^+(\sim u\bar{s}) \to \mu^+ + \nu_\mu$ implies



Flavour changing neutral currents do not arise in the Feynman rules.
(complex loop processes can have this effect and are involved in searches for new physics)



The most general Yukawa interaction for 3 generation is

$$L_{Yukawa} = -(\bar{e}_R, \bar{\mu}_R, \bar{\tau}_R) C_l \begin{pmatrix} \vec{\phi}^+ \cdot \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ \vec{\phi}^+ \cdot \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \\ \vec{\phi}^+ \cdot \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \end{pmatrix} + (\bar{d}'_R, \bar{s}'_R, \bar{b}'_R) C_q \begin{pmatrix} \vec{\phi}^+ \cdot \begin{pmatrix} u \\ d' \end{pmatrix}_L \\ \vec{\phi}^+ \cdot \begin{pmatrix} c \\ s' \end{pmatrix}_L \\ \vec{\phi}^+ \cdot \begin{pmatrix} t \\ b' \end{pmatrix}_L \end{pmatrix}$$

$$(\bar{u}_R, \bar{c}_R, \bar{t}_R) C'_q \begin{pmatrix} \vec{\phi}^T \cdot \varepsilon \cdot \begin{pmatrix} u \\ d' \end{pmatrix}_L \\ \vec{\phi}^T \cdot \varepsilon \cdot \begin{pmatrix} c \\ s' \end{pmatrix}_L \\ \vec{\phi}^T \cdot \varepsilon \cdot \begin{pmatrix} t \\ b' \end{pmatrix}_L \end{pmatrix} \text{ +Herm. Conj.}$$

Where $C_l, C_q, C'_q \in C^{3\times 3}$

.

$SU(2)_L \otimes U(1)_Y$ invariantce seems to allow for generation mixing.
We now have to ask the question how many entries in the matrices are actually physical, means cannot be absorbed by a redefinition of fields and complex phases.
Consider making a basis change (this does not affect the physics):

$$\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_R \to U_1 \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_R, \quad \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R \to U_2 \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R, \quad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_R \to U_3 \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_R$$

$$\begin{pmatrix}\psi_e\\ \psi_\mu\\ \psi_\tau\end{pmatrix}_R \to V_1\begin{pmatrix}\psi_e\\ \psi_\mu\\ \psi_\tau\end{pmatrix}_R, \quad \begin{pmatrix}\psi_u\\ \psi_c\\ \psi_t\end{pmatrix}_R \to V_2\begin{pmatrix}\psi_u\\ \psi_c\\ \psi_t\end{pmatrix}_R$$

We can now use this transformation to reduce the degrees of freedom in C_l, C_q, C'_q.

$$C_l \to U_1^+ C_l V_1, \quad C'_q \to U_2^+ C'_q V_2, \quad C_q \to U_{31}^+ C_q V_{21}$$

Simplifications

Recall Hermitian ⇒ diagonalisable: C^+C is automaticall Hermitian positive semi-definite with eigenvalues λ^2.
Our matrices C are merely complex and not diagonalisable. However, complex matrix has a singular value decomposition

$$C = UDV^+$$

where U, V are unitary, D is diagonal with elements $\lambda \geq 0$ and λ^2 are the eigenvalues of C^+C.
Thus we can write

$$C_l = U_l D_l V_l^+ \quad C_q = U_q D_q V_q^+ \quad C_{q'} = U_{q'} D_{q'} V_{q'}^+$$

Lepton sector

Choose $U_1 = U_l$, $V_1 = V_l \Rightarrow C_l \to D_l = diag(\lambda_e, \lambda_\mu, \lambda_\tau)$
No lepton flavour mixing
Lepton-gauge couplings diagonal in same basis as that with a diagonal mass matrix.

The quark sector

Choose $U_2 = U_{q'}$, $V_2 = V_{lq'} \Rightarrow C_{q'} = diag(\lambda_u, \lambda_c, \lambda_t)$
But $C_q \to U_3 U_q D_q V_q^+ V_2^+$ is *not* diagonalisable. We have already chosen V_2.
Greatest simplification obtained by taking

$$U_3 = V_2 V_q U_q^+ \Rightarrow C_q = VD_q V_+$$

where $V = V_2 V_q$ is the unitary Cabibbo-Kobyashi-Maskawa matrix (1973).

By using our free choice for U_2 and V_2 to diagonalise C'_q we see that C_q can in general not be diagonalised at the same time. We are left with a unitary rotation which acts on the d, s, b quarks.

There is still some freedom that allows us to restrict V ∈ U(3) further. We are free to apply a phase transformation on $\psi_{L,R}$ since the wave functions are defined up to a global phase.

$$V \to \begin{pmatrix} e^{-i\varphi_1} & 0 & 0 \\ 0 & e^{-i\varphi_2} & 0 \\ 0 & 0 & e^{-i\varphi_3} \end{pmatrix} V \begin{pmatrix} e^{i\chi_1} & 0 & 0 \\ 0 & e^{i\chi_2} & 0 \\ 0 & 0 & e^{i\chi_3} \end{pmatrix}$$

Consider first the case of two generations:

$$V_{2\ gens.} = \begin{pmatrix} e^{-i(\varphi_1-\chi_1)}V_{11} & e^{-i(\varphi_1-\chi_{21})}V_{12} \\ e^{-i(\varphi_2-\chi_1)}V_{21} & e^{-i(\varphi_2-\chi_2)}V_{22} \end{pmatrix}$$

The three phase differences can be chosen such that: $V_{11} \geq 0$, $V_{12} \geq 0$, $V_{21} \leq 0$.
The 4th phase is fixed as $\varphi_2 - \chi_2 = (\varphi_2 - \chi_1) + (\varphi_1 - \chi_2) - (\varphi_1 - \chi_1)$. This gives

$$V = \begin{pmatrix} V_{11} & V_{12} \\ -|V_{21}| & e^{i\rho}V_{22} \end{pmatrix}, \qquad V^+ = \begin{pmatrix} V_{11} & -|V_{21}| \\ V_{12} & e^{-i\rho}V_{22} \end{pmatrix}$$

As $V \in U(2) \Rightarrow VV^+ = 1 \Rightarrow$

$$\begin{aligned} V_{11}^2 + V_{21}^2 &= 1 \\ -V_{11}|V_1| + V_{12}V_{22}e^{-i\rho} &= 0 \\ |V_{21}|^2 + V_{22}^2 &= 1 \end{aligned}$$

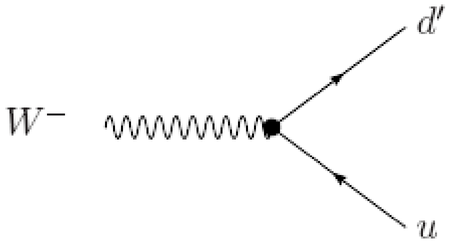
To fulfill these three conditions we need ρ = 0, an imaginary phase is not allowed. Hence

$$\left.\begin{aligned} V_{11} = V_{22} = \cos\theta_c \\ V_{12} = |V_{21} = \sin\theta_c| \end{aligned}\right\} \theta_C \in [0, \pi/2] \quad \textit{is called the Cabibbo angle}$$

This defines the original Cabibbo matrix for two generations. It describes the mixing between the electroweak eigenstates d′, s′ and the mass eigenstates d and s.

$$V = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}$$

It describes the mixing between the electroweak eigenstates d′, s′ and the mass eigenstates d and s.



W couples to the electroweak doublet $\begin{pmatrix} u \\ d' \end{pmatrix}_L = \begin{pmatrix} u \\ d\cos\theta_C + s\sin\theta_C \end{pmatrix}$. We see that the "strangeness" quantum number is not conserved in electroweak interaction. Mesons and baryons containing strange quarks decay through charged currents, i.e. the exchange of charged vector bosons. For example it allows K^+ to decay dominantly to leptons via a vector boson $\mu^+\nu_\mu$ (~ 64%), $\pi^0\mu^+ + \nu_\mu$ (~ 3%), $\pi^0 e^+\nu_e$ (~ 5%).

In the 3-generation case we start with

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

which can be written as a matrix defined by 3 angles and 1 phase if we follow the same reasoning as in the 2-generation case. We employ a reduced notation to express this more compactly: $\cos\theta_C \to c_i$, $\sin\theta_C \to s_i$. The mixing matrix is then

$$\begin{pmatrix} c_1 & s_1c_2 & s_1s_2 \\ -s_1c_2 & (c_1c_2c_3 - s_2s_3e^{i\delta}) & (c_1c_2s_3 + s_2c_3e^{i\delta}) \\ -s_1s_2 & (c_1s_2c_3 + c_2s_3e^{i\delta}) & (c_1s_2s_3 - c_2c_3e^{i\delta}) \end{pmatrix}$$

Where $\theta_i \in [0, \pi/2]$, $\delta \in [0, 2\pi]$.

Remark: Of all parameters in the Standard Model, $e^{i\delta}$ is the only complex one. Such terms are not invari-ant under CP transformations. This has two important applications:
i.) mixing in kaon and anti-kaon → first experimental indication of CP violation in nature.
ii.) for baryon asymmetry → need CP violation.

Theoretical prediction of CP violation: Nobel-Prize for Kobayashi and Maskawa 2008.

Summary

Collecting things together, the Yukawa sector of the Standard Model leads to mass terms for fermions, $m_f = \lambda v/\sqrt{2}$, and Higgs-fermion interactions

$$L_{Yukawa} = \left\{(\bar{e},\bar{\mu},\bar{\tau})\begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} + \right.$$

$$\left. + (\bar{u},\bar{c},\bar{t})\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}\begin{pmatrix} u \\ c \\ t \end{pmatrix} + (\bar{d},\bar{s},\bar{b})\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_{b\tau} \end{pmatrix}\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}\right\}\left(1 + \frac{H}{v}\right)$$

Recall that dsb is a linear combination of electroweak eigenstates:

$$\underbrace{\begin{pmatrix} d \\ s \\ b \end{pmatrix}}_{mass\ eigenstates} = V \underbrace{\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}}_{e.w.\ eigenstates}$$

We see that the Yukawa sector contains 9 masses + 3 angles + 1 phase = 13 parameters. The parametersof the flavour sector have to be fixed by experiment.

The Standard Model – Final Lagrangian

$$L = -\frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \qquad \begin{cases} W^{\pm}, Z, \gamma \quad \textit{kinetic} \quad \textit{energies} \\ \textit{and} \quad \textit{self}-\text{int}\,\textit{eractions} \end{cases}$$

$$+\bar{L}\gamma^{\mu}(i\partial_{\mu} - \frac{1}{2} g \tau_i W^i_{\mu} - \frac{Y}{2} g' B_{\mu}) L \qquad \begin{cases} \textit{lepton} \quad \textit{and} \quad \textit{kinetic} \quad \textit{energies} \quad \textit{and} \quad \textit{their} \\ \qquad \text{int}\,\textit{eractions} \quad \textit{with} \quad W^{\pm}, Z, \gamma \end{cases}$$

$$\left| (i\partial_{\mu} - \frac{1}{2} g \tau_i W^i_{\mu} - \frac{Y}{2} g' B_{\mu}) \phi \right|^2 - V(\phi) \qquad \begin{cases} \quad W^{\pm}, Z, \quad \textit{and} \quad \textit{Higgs} \\ \textit{massesand} \quad \textit{and} \quad \textit{couplings} \end{cases}$$

$$-(y_1 \bar{L} \phi R + y_2 L \phi_c R) + h.c. \qquad \begin{cases} \textit{lepton} \quad \textit{and} \quad \textit{quark} \quad \textit{masses} \\ \textit{and} \quad \textit{coupling} \quad \textit{to} \quad \textit{Higgs} \end{cases}$$

Problems:

1.The Lagrangian for the scalar field contains trilinear $hW^{+}W^{-}$, and quadrilineaer $hhW^{+}W^{-}$ Higgs boson couplings. Use $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$

(1)show that the vertex factor for $hW^{+}W^{-}$ and igM_W are $i/4g^2$ respectively.

(2)determine the hZZ and hhZZ vertex factors.

2.Show that

$$\frac{1}{2v^2} = \frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}$$

3.Show that $m_h^2 = 2v^2\lambda$.

Lecture XII

Neutrino oscillation

Oscilations

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_1 & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} + c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$

$$U = \begin{pmatrix} |U_{11}| & |U_{12}| & |U_{13}| \\ |U_{21}| & |U_{22}| & |U_{23}| \\ |U_{31}| & |U_{32}| & |U_{33}| \end{pmatrix} \cong \begin{pmatrix} 0.82 & 0.55 & 0.16 \\ 0.44 & 0.65 & 0.62 \\ 0.36 & 0.52 & 0.77 \end{pmatrix}$$

Consider two generations

$\nu_1 = \nu_\mu \cos\theta - \nu_e \sin\theta$, $\nu_2 = \nu_\mu \sin\theta + \nu_e \cos\theta$

$$\nu_1(t)=\nu_1(0)e^{-iE_1t/\hbar}\text{,}\quad \nu_2(t)=\nu_2(0)e^{-iE_2t/\hbar}$$

Suppose,

$$\nu_e(0)=1\text{,}\quad \nu_\mu(0)=0$$

So, $\nu_1(0)=-\sin\theta$**,** $\nu_2(0)=\nu\cos\theta$

$$\nu_1(t)=-\sin\theta e^{-iE_1t/\hbar}\text{,}\quad \nu_{21}(t)=\cos e^{-iE_2t/\hbar}$$

$$\nu_\mu(t)=\nu_1(t)\cos+\nu_2(t)\sin\theta=\sin\theta\cos\theta(-e^{-iE_1t/\hbar}+e^{-iE_2t/\hbar})$$

The probability that the electron neutrino has convert into a muon neutrino is

$$\left|\nu_\mu(t)\right|^2=\sin^2\theta\cos^2\theta(-e^{-iE_1t/\hbar}+e^{-iE_2t/\hbar})(-e^{iE_1t/\hbar}+e^{iE_2t/\hbar})$$

$$=\frac{1}{4}\sin^2(2\theta)(1-e^{i(E_2-E_1)t/\hbar}-e^{-i(E_2-E_1)t/\hbar}+1)=\frac{1}{4}\sin^2(2\theta)\left(2-2\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right)$$

$$=\frac{1}{4}\sin^2(2\theta)(1-e^{i(E_2-E_1)t/\hbar}-e^{-i(E_2-E_1)t/\hbar}+1)=\frac{1}{4}\sin^2(2\theta)\left(2-2\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right)$$

$$=\sin^2(2\theta)\sin^2\left(\frac{(E_2-E_1)t}{\hbar}\right)$$

Or $P(\nu_e\to\nu_\mu)=\left[\sin(2\theta)\sin^2\left(\frac{(E_2-E_1)t}{\hbar}\right)\right]^2$

$$E^2=|\vec{p}|^2c^2+m^2c^4=|\vec{p}|^2c^2\left(1+\frac{m^2c^2}{|\vec{p}|^2}\right)$$

$$E\approx|\vec{p}|c\left(1+\frac{1}{2}\frac{m^2c^2}{|\vec{p}|^2}\right)=|\vec{p}|c+\frac{m^2c^3}{2|\vec{p}|}$$

$$E_2 - E_1 \approx \frac{m_2^2c^3 - m_1^2c^3}{2|\vec{p}|} \approx \frac{m_2^2 - m_1^2}{2E}c^4$$

And hence

$$P(\nu_e \to \nu_\mu) = \left[\sin(2\theta)\sin^2\left(\frac{(m_2^2 - m_1^2)c^4}{4\hbar E}t\right)\right]^2$$

$$z \approx ct\text{,} \quad P(\nu_e \to \nu_\mu) = \left[\sin(2\theta)\sin^2\left(\frac{(m_2^2 - m_1^2)c^3}{4\hbar E}z\right)\right]^2$$

After a distance $L = \dfrac{2\pi\hbar E}{(m_2^2 - m_1^2)c^3}$ the probability hits a maximum $\sin^2(2\theta)$ and at *2L* they are all back to electron neutrinos.

The Super-Kamiokande detector

The pp chain:

Two protons make a deuteron

$$p + p \to d + e^+ + \nu_e\text{,} \quad p + p + e^- \to d + \nu_e$$

Deuteron and proton make ^3He

$$d + p \to {}^3He + \gamma$$

Helim-3 makes alpha particle or ^7Be

$$^3He + p \to \alpha + e^+ + \nu_e$$

$$^3He + {}^3He \to \alpha + p + p$$

$$^3He + \alpha \to {}^7Be + \gamma$$

Beryllium makes alpha particle

$$^7Be + e^- \to {}^7Li + \nu_e$$

$$^7Li + p \to \alpha + \alpha$$

$$^7Be + p \to {}^8B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be^{*} + e^{+} + \nu_e$$

$$^{8}Be^{*} \rightarrow \alpha + \alpha$$

Problems:

1.An important reaction producing electron neutrinos in the Sun is $e^{-} + {}^{7}Be \rightarrow \nu_e + {}^{7}Li$. In the vast majority of the cases (90%) th Li nucleus is produced in its ground state. Consequently, the 'Be neutrinos' energy spectrum is monoenergetic with $E_\nu = 0.862$ MeV. The corresponding total neutrino flux at the Earth is $\Phi_\nu = 4.6 \times 10^{13} \;\; m^{-2}$. The BOREXINO experiment at LNGS detects neutrinos via the reaction $\nu_e + e^{-} \rightarrow \nu_e + e^{-}$. Its fiducial volume contains 100 t of liquid scintilliator. The liquid is pseudocumene C_9H_{12}. The light produced by the final electron is detected by an array of photomultipliers covering the surface surrounding the volume. Assume $\sigma(\nu_e e) = 0.6 \times 10^{-48} m^2$, $\sigma(\nu_{\mu,\tau} e) \approx 1/6\sigma(\nu_e e)$, $\theta_{12} = 34^0$, $\delta m^2 = 80 meV^2$.

(1)If electron neutrinos did not change flavor, how many events would be expected per day in the 100 t target mass?

(2)Which is the principal mechanism of flavor conversion for Be neutrinos from the Sun?

(3)Under these conditions, calculate the expected number of events per day in BOREXIMO of Be neutrinos.

2.Consider the muon neutrinos generated by the decays of the mesons produced by the collisions of cosmic rays in the atmosphere ('atmospheric neutrinos'). Their energy spectrum at the surface of the Earth extends over several orders of magnitudes, decreasing with energy roughly as E_ν^{-3} and with important dependence on the angle to the zenith. At $E_\nu \approx 1 GeV$ their flux around the zenith is approximately $\Phi_{\nu_\mu} \approx 130 m^{-2} s^{-1} sr^{-1} GeV^{-1}$.

The Super-Kamiokande detector is a 22.5 kt fiducial Cherenkov detector in the Kamioka underground observatory. Muon neutrinos (and antineutrinos, but we will not consider them) are mainly detected via their CC interactions on ^{16}O nuclei $\nu_\mu + {}^{16}O \rightarrow \mu^{-} + X$. Assume $\sigma(\nu_\mu {}^{16}O) \approx 10^{-42} m^2$, $\theta_{23} = 45^0$, $\theta_{13} = 0^0$, $\Delta m^2 2500 meV^2$.

(1)How many interactions per year will happen induced by muon neutrinos arriving with directions within $\Delta\Omega = 1$sr around the zenith in 1 GeV energy interval? (for the purpose of thisproblem, assume, unrealistically, all quantities to be constant in these intervals).

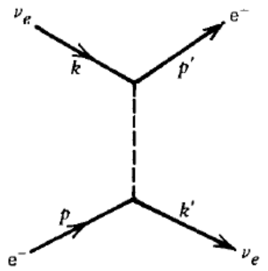
(2)What is the fraction of surviving neutrinos coming vertically upwards?

Lecture XIII

Discovery of the Hggs Boson

Let us consider charged current neutrino-electron scattering

(on purely dimensional grounds one can conclude that the cross section (for a point interaction) must behave $\sim G_F^2 s$ at high energies)



The invariant amplitude has the form

$$A = \frac{G_F}{\sqrt{2}} (\bar{u}(k')\gamma^{\mu}(1-\gamma^5)u(p))(\bar{u}(p')\gamma_{\mu}(1-\gamma^5)u(k))$$

And

$$\overline{|A|^2} = \frac{1}{2}\frac{G_F^2}{2} Tr\{\gamma^{\mu}(1-\gamma^5)\not{p}\gamma^{\nu}(1-\gamma^5)\not{k}'\} Tr\{\gamma_{\mu}(1-\gamma^5)\not{k}\gamma_{\nu}(1-\gamma^5)\not{p}'\}$$

$$= 64 G_F^2 (k\cdot p)(k'\cdot p') = 16 G_F^2 s^2$$

(the electron mass was neglected; so $s = (k+p)^2 = 2k\cdot p = 2k'\cdot p'$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|A|^2} = \frac{G_F^2 s}{4\pi^2}$$

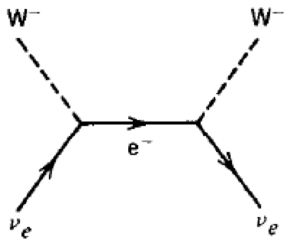
Integration over angular distribution gives

$$\sigma = \frac{G_F^2 s}{\pi}$$

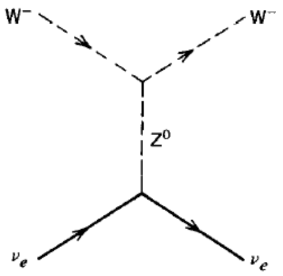
It becomes infinite as $s \to \infty$. The introduction of finite-mass W boson removes the divergence, and for large s it can be shown that

$$\sigma(\nu_e e \xrightarrow{W} \nu_e e) = \frac{G_F^2 M_W^2}{\pi}$$

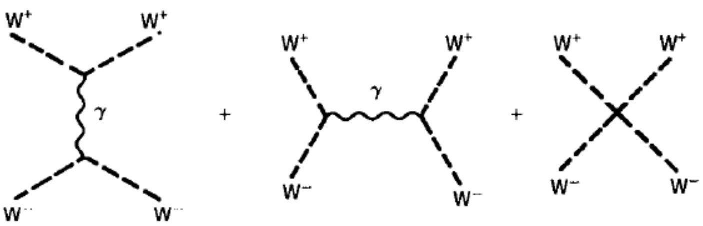
The introduction of W boson causes its own problems; consider neutrino W boson scattering.



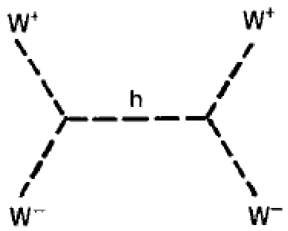
The corresponding cross section $\sigma = G_F^2 s / 3\pi$ similarly behaves (diverges) at large s. We may tempted to conclude that we are required to introduce the neutral current.



The process $e^- e^+ \rightarrow W^- W^+$ is another example where the self-coupling of gauge bosons ensures a finite answer.



Direct computation reveals that the individual diagrams at high energies behaves as s^2 / M_w^4 , but the sum is more gentle $\sim s / M_w^2$. Even after introducing similar diagrams with Z exchange the sum of all diagrams behaves as $\sim s / M_w^2$. Heavy leptons cannot help us, so the only solution is to introduce a scalar particle which cancels these residual divergences through diagrams of the tipe

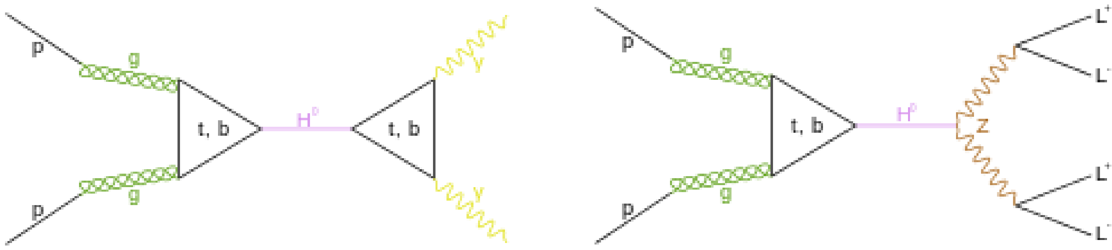


If we had not previously introduced it to generate the heavy boson masses, we would have been forced to invent it now to guarantee renormalizibility. A detailed investigation of this point would reveal that the higgs couplings are proportional to masses, a result we are also familiar with.

Discovery of the H boson

The LHC was designed to be able to reach $M_H = 1000\,GeV$. Sensitive searches for the H boson were carried out in the LEP experiments. The final limit on the mass at 95% confidence level was (hep-ex/0612034)

$$M_H = 114.4\ GeV$$

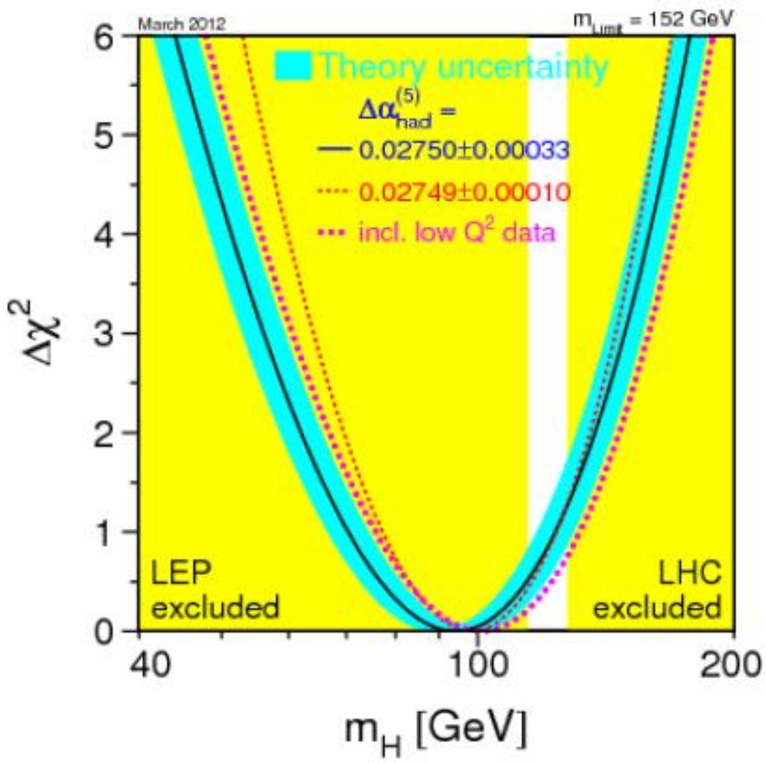


The mass of observed boson was calculated with a best-fit procedure in the four-lepton and two-photon channels (arXiv:1408.5191 arXiv:1412.8662)

$$M_H = 125.7 \pm 0.4\ GeV$$

The Standard Model of particle physics codifies the properties and interactions of the fundamental constituents of all the visible matter in the Universe. It describes successfully the results of myriads of accelerator experiments, some of them to a very high degree of precision. However, for quite some time the Standard Model resembled a jigsaw puzzle with one piece missing: the Higgs boson. It, or something capable of replacing it, was essential for the calculability of the Standard Model and its consistency with experimental data. The last piece of the puzzle, at times (somewhat dubiously) termed the “Holy Grail” of particle physics, or even the “God Particle”, was finally put into place with the July 4, 2012, announcement of the discovery [1] at the CERN Large Hadron Collider (LHC) of a “Higgs-like” particle at a mass of approximately 125 GeV. Subsequently, measurements of its properties by the ATLAS and CMS collaborations have shown more detailed consistency with predictions for the Higgs boson of the Standard Model, but searches for possible discrepancies indicative of new physics beyond the Standard Model are continuing.

The existence of the Higgs boson was first postulated in 1964 [2: P. W. Higgs, Broken symmetries and the masses of gauge bosons, Phys. Rev. Lett. 13 (1964) 508], following earlier theoretical work that introduced spontaneous symmetry breaking into condensed-matter [3: P. W. Anderson, Plasmons, gauge invariance, and mass, Phys. Rev. 130 (1963) 439; see also Y. Nambu, Quasi-particles and gauge invariance in the theory of superconductivity, Phys. Rev. 117 (1960) 648.] and particle physics [4–6: Y. Nambu, Axial vector current conservation in weak interactions, Phys. Rev. Lett. 4 (1960) 380; F. Englert and R. Brout, Broken symmetry and the mass of gauge vector mesons, Phys. Rev. Lett. 13 (1964) 321; P. W. Higgs, Broken symmetries, massless particles and gauge fields, Phys. Lett. 12 (1964) 132; G. S. Guralnik; C. R. Hagen; T. W. B. Kibble (1964). "Global Conservation Laws and Massless Particles". *Physical Review Letters*. **13** (20): 585–587.]. It was incorporated into the Standard Model in 1967 [7, 8: S. Weinberg, A model of leptons, Phys. Rev. Lett. 19, 1264 (1967). This seems to be the first paper in which it is shown that matter particles can also acquire their masses from 'spontaneous' symmetry breaking.; A. Salam, Weak and electromagnetic interactions, in the Proceedings of 8th Nobel Symposium, Lerum, Sweden, 19-25 May 1968, pp 367-377.], and shown in 1971 [9: G. 't Hooft, Renormalizable lagrangians for massive Yang-Mills fields, Nucl. Phys. B 35, 167 (1971); G. 't Hooft and M. J. G. Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. B 44 (1972) 189.] to lead to a calculable and predictive unified theory of the weak and electromagnetic interactions. With the discovery of neutral currents in 1973 [10: F. J. Hasert et al. [Gargamelle Neutrino Collaboration], Search for elastic muon neutrino electron scattering, Phys. Lett. B 46 (1973) 121 and Observation of neutrinolike interactions without muon or electron in the Gargamelle neutrino experiment, Phys. Lett. B 46 (1973) 138.], the discovery of charmonium in 1974 [11: J. J. Aubert et al. [E598 Collaboration], Experimental Observation Of A Heavy Particle J, Phys. Rev. Lett. 33 (1974) 1404; J. E. Augustin et al. [SLAC-SP-017 Collaboration], Discovery of a narrow resonance in e +e − annihilation, Phys. Rev. Lett. 33 (1974) 1406.], the discoveries of the W± and Z 0 particles in 1983 [12: G. Arnison et al. [UA1 Collaboration], Experimental observation of isolated large transverse energy electrons with associated missing energy at s 1/2 = 540 GeV, Phys. Lett. B 122 (1983) 103 and Experimental observation of lepton pairs of invariant mass around 95-GeV/c2 at the CERN SPS collider, Phys. Lett. B 126 (1983) 398] and subsequent detailed measurements, the predictions of the Standard Model have been crowned with a series of successes



*The status of the Higgs search in March 2012 [*LEP Electroweak Working Group, http://lepewwg.web.cern.ch/LEPEWWG/ *]. The left-hand yellow-shaded region is the LEP exclusion, and the right-hand yellow-shaded region is the Tevatron exclusion at that time [*Tevatron New Phenomena and Higgs Working Group, http://tevnphwg.fnal.gov/ *].*

Problems:

1.Proof that charged current neutrino-electron scattering evaraged amplitude

$$\overline{|A|^2} = 16 G_F^2 s^2$$

2.Proof that the neutrino W boson scattering cross section at large s

$$\sigma = G_F^2 s / 3\pi$$

Lecture 14

Beyond the standard model

Grand unification

With the success of electroweak unification (in the 1960s) the logical next step was to include the strong interaction in a 'Grand Unified Theory' (GUT) that would identify all three forces as different manifestation of a single underlying interaction.

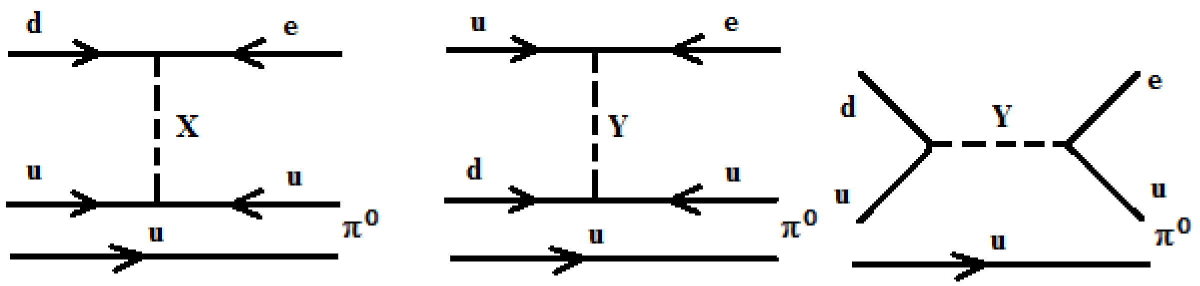
GUT [PRL, 32, 438 (1974)]

It led to a spectacular prediction: the proton is unstable, decaying (for example) into a positron and a pion

$$p \to e^{+} + \pi^{0}$$

The lifetime is reassuringly long – at least 10^{30} years. In 30 years of increasingly precise experiments, however, proton decay has never been observed ($\tau(proton) > 10^{33\div 34}$ years). More elaborate GUTs have been proposed, but almost all of them require proton decay at some level.

Grand unification contemplates an overarching symmetry group (SU(5)) that contains as subgroups the (color) SU(3) and $SU(2)\otimes U(1)$ symmetry of the Standard Model. The fundamental fermions (quarks and leptons) are assigned to representations of this group. The first generation comprises 15 particle states. There are 24 mediators: the 8 gluons, the photon, W^{+}, W^{-}, and Z, and 12 new ones – the X (charge $\pm 4/3$, 3 colors, hence 6 in all), and the Y (charge $\pm 1/3$, 3 colors, for another 6). The couple leptons to (anti)quarks, and hence are known as leptoquarks ($\bar{d} \to e + X, \bar{u} \to e + Y$). The also couple quarks to antiquarks ($u \to \bar{u} + X, d \to \bar{d} + Y$)



Proton decay in the SU(54) GUT

	Charge	Mass
8 gluons	0	0
1 photon	0	0
3 $W^{\pm}$, Z	1, -1, 0	$\sim 10^{2} GeV/c^{2}$
6 X	4/3, -4/3	$\sim 10^{16} GeV/c^{2}$
6 Y	1/3, -1/3	$\sim 10^{16} GeV/c^{2}$

Grand unification purports to 'explain' the relation between quark and lepton charges. The sum of the charges in a multiplet must be zero, and putting quarks and leptons into the same multiplet forces (in case of SU(5)) $q_e - 3q_d = 0$.

Supersymmetry, Strings, Extra Dimensions, …

Supersymmetry

Over the past 40 years an enormous amount of work has been done on supersymmetry. Supersymmetry carries the stupendous implication that every fermion has a bosonic partner (identified by putting an ‘s’ in front of the name – thus ‘squark’, selectron’ sneutrino’, etc) and every boson has a fermionic partner (identified by putting ‘ino’ after the name – thus ‘photino’, ‘gluino’, ‘wino’, higgsino’, etc). If supersymmetry were unbroken, the particles would share the masses of their twins – the photino would be massless particle of spin ½, and the selectron a spin-0 particle with a mass of 0.5 MeV; no such particle exist. So the symmetry must be badly broken. Presumably the supersymmetric particles are much heavier – too heavy to be produced by any existing machine ,though there are strong indications that at least some of them should be accessible to the LHC.

Supersymmetry has the potential to solve several theory problems:

1.By introducing a number of new particles, it modifies the energy dependence of the three running coupling constant, making possible their perfect convergence at the GUT scale.

2.It offers a ‘natural’ solution to the so-called hierarchy problem. The higgs mass is renormalized by various loop diagrams, which drive it way out acceptable range unless there are magical cancellations (‘fine tunning’). But loop corrections are opposite sign for bosons and fermions, so supersymmetry, by pairing particles with ‘sparticles’ makes the cancellation exact and automatic.

3.In most models, the lightest supersymmetric particle is colorless, neutral and stable, making it an attractive candidate for Dark Matter.

Strings

In string theory the basic units of matter are not (zero-dimensional)particles, but rather one-dimensional ‘strings’ (or higher dimensional ‘branes’), of which ‘particles’are various vibrational modes. The theory underwent an extraordinary evolution between the 1970s, when a few lonely visionaries took up the cause, and 2000, by which time it was wellestablished as the dominant paradigm.Early versions contained only bosons; fermions were later incorporated via supersymmetry (hence ‘superstrings’) and the number of space dimensions dropped to 9 or 10. Meanwhile, it was realized that the theory automatically includes the gravitation, making it a natural candidate for quantum gravity.

Dark Matter/Dark Energy

Persuasive astronomical evidence now indicates that the matter we know about – described by the Standard Model – represents a measly 5% of the mass/energy content of the universe. The rest is Dark Matter (about 20%) and Dark Energy (75%). The implications for particle physics are humbling: we see only the tip of the iceberg. What is all this other stuff, and how it managed to elude us?

In 1933 Fritz Zwicky measured the velocities of galaxies in the Coma cluster (from the Doppler shift of their atomic spectra), and used this information to determine the mass of the cluster. The result was surprising: 400 times larger than the visible stars in the cluster. Evidently the galaxies contain a lot of matter that does not radiate (and is called therefore dark matter). More recently rotation curves have been measured for a number of galaxies (including our own). These plot the (tangential) velocity v as a function of distance r from the galactic center. Newton's law of universal gravitation says that for stars well away from the core v should decrease as $1/\sqrt{r}$; instead, it typically increases. This suggests that the dark matter permeates a spherical 'halo' extending well outside the galactic nucleus.

So far, though, our only evidence for dark matter comes from its large-scale gravitational effects, and it is natural to wonder whether perhaps Newton's laws (and also general relativity) are incorrect on some scale, and there is actually no dark matter out there. Short of such a radical alternative, the question remains: what is this stuff? Could it be ordinary could matter – sand and gravel, perhaps, the remnants of extinct stars or dead planets. Almost certainly not. Cosmological models that are convincingly corroborated by the observed abundances of light elements do not allow for anywhere near enough baryons to account for dark matter. What about neutrinos? Probably not – even though there are enormous numbers of them, they are much too light to contribute more than a small fraction of the observed dark matter. Evidently we are looking for something much more massive than neutrinos, but (like neutrinos) weakly interacting; WIMPs (Weakly Interacting Massive Particles). Their mass is tentatively estimated to lie in the range 100 – 200 GeV; they are certainly neutral (otherwise they would radiate) and stable (left over from the Big Bang). No such particle is known in the Standard Model. But supersymmetry does suggest a candidate: the lightest supersymmetric particle (probably a mixture of the photino and higgsino – or possibly the Zino – called the 'neutralino'). Large numbers might be left over from the Big Bang. Another possibility is the axion – the hypothetical particle introduced to account for the absence of strong CP violation. Since the late 1980s a number of WIMP searches have been under way and convincing evidence may well come in the next few years.

Before 1998 it was taken for granted that the expansion of the universe is slowing down, due to the gravitational attraction of all matter; the only question was whether the energy density of the universe is great enough to reverse the expansion completely, leading to a 'big crunch'. Visible matter and dark matter together amount to about a third of 'critical density', so for those who believed the expansion 'should' reverse there was a second 'missing mass' paradox, unrelated to the dark matter problem: where is all that 'extra' energy?

The problem was turned inside out by the astonishing discovery that the expansion of the universe is not slowing down at all, but rather accelerating. Evidently Newtonian gravity (universal attraction) is not right on the largest scale – either that or there is some new force that is repulsive in nature and overwhelms gravity in this case. In general relativity, there is a (sort of) natural place for an extra term that could account for the phenomenon: the cosmological; constant Λ. Einstein's original theory (with no cosmological constant) implied that the universe expands – something he regarded as absurd. He was able to rescue the theory by introducing an ad hoc source term, whose strength (Λ) could be adjusted to stabilize the universe. (Mathematically, the cosmological constant introduces a kind of primordial repulsion, or negative pressure, that balances the universe attraction on a cosmic scale.) Later, when Hubble discovered that the universe in fact expanding, a chagrined Einstein disowned the cosmological constant, calling it 'my greatest blunder'. But, when the accelerated expansion was discovered, the obvious remedy was to resurrect the cosmological constant. There is however a subtle distinction between the original notation of a cosmological constant and its contemporary reincarnation. Einstein conceived Λ as an unexplained fundamental constant of nature – analogous to Planck's constant or Boltzmann's constant; there were two distinct sources of gravitation: matter (actually the stress tensor, incorporating energy, momentum, and stress of all forms) and Λ. In the modern version Λ is taken to have a dynamical origin, in the form of dark energy associated with the vacuum expectation value of some quantum field.

(Large) Extra Dimensions

The modern models with extra space-time dimensions could be built in several ways. Among them the following major approaches are most remarkable: 1) ADD model of Arcani-Hammed, Dimopoulos and Dvali (1998) [N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys.Lett. B429 (1998) 263; hep-ph/9803315. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B436 (1998) 257; hepph/9804398]. In this approach all elementary particles except graviton are localized on the Brane, while the graviton propagates in the whole Bulk. There are at least two extra space dimensions in the above model. 2) RS model of Randal and Sundrum with warped 5-dimensional space-time and nonfactorized geometry [L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370; hep-ph/9905221. L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 46090; hep-th/9906064]. There are two such models with compactified and noncompactified dimensions (RS1 and RS2 models). 3) ACD models of Appelquist, Cheng and Dobrecu (so called Universal Extra Dimensional Model), where all the particles move in the whole Bulk [T. Appelquist, H.C. Cheng, B.A. Dobrescu, Phys. Rev. D64 (2001) 035002; hep-ph/0012100]. The idea of extra dimensions is as old as almost century age. It was G. Nordstrom, who in 1912-1914 has formulated relativistic theory in 5 dimensions, which simultaneously described gravity and electromagnetism [Nordstrom G, Phys. Ztsch., 1914, Bd. 15, S.504]. Of course Nordstrom unification looked rather formal. Nevertheless the possibility of this 5dimensional unification hinted on deep relation between these two fundamental interactions, known in that time. In 1915 Einstein created the General Theory of Relativity [A. Einstein, Published in Sitzungsber. Preuss.Akad.Wiss.Berlin (Math.Phys.) 1915:778- 786,1915, Addendum-ibid.1915:799-801,1915], where gravity was considered as geometrical deformation of the space-time. In 1919 mathematician Th. Kaluza has shown that five dimensional relativistic gravity (with Einstein-Hilbert action) manifests itself in the four dimensional space-time as both electromagnetism and gravity [Th. Kaluza. Sitzungsber. Preuss.Akad.Wiss.Berlin(Math.Phys) 1921 (1921) 966]. In other words, Nordstrom formulated first five dimensional electrodynamics, while Kaluza created first 5D gravity. These two were just first nave attempts towards first unification of interactions (gravity and electromagnetism) established that time. In 1926 O. Klein rediscovered Kaluza's theory [O. Klein. Z.Phys.37 (1926) 895]. Moreover, there were also numerous efforts to unify electromagnetism, gravity and quantum mechanics in 5D by several authors. Later, the discovery of new types of interactions (strong and weak ones) complicated above efforts. The use of one additional dimension seemed to be unnatural and insufficient for these goals. One of the difficulties of multidimensional theories is the mechanism, due to which extra dimensions are hidden. Thus during study of ordinary physical phenomena the space-time looks like effectively four-dimensional. Until recently mostly Kaluza-Klein type theories were considered. In these types of theories extra dimensions are assumed as essentially compact and in essence homogenous. Just compactness of extra dimensions provides effective 4D character of the space-time dimension at the distances above the compactification scale (size of extra dimensions). At that the extra dimensions must be of microscopic size. Following to widespread opinion, the compactification scale should be of the Planck scale size (though electroweak scale have been discussed in this role). On the other hand, direct observation of extra dimensions at the Planck scale ($l_{Pl} \sim 10^{-33} cm$, $M_{Pl} \sim 10^{19} GeV$) seems to be hopeless. However "Brane World" conception permitted to change situation on this direction: we mean just the localization of ordinary matter (with the exception maybe gravitons and other hypothetical particles which interact very weakly with the matter) on the three dimensional manifold which is cold the Brane [V.A. Rubakov, M.E. Shaposhnikov, Phys. Lett. B125 (1983) 136. K.Akama, in Gauge Theory and Gravitation: Proc. of the Intern. Symp., Nara, Japan, 1982 (Lecture Notes in Physics, Vol. 176, Eds. K. Kikkawa, N. Nakanishi, H. Hariai)(Berlin: SpringerVerlag, 1983), p.267. I. Antoniadis, Phys. Lett. B246 (1990) 377]. The Brane is embedded into ambient higher dimensional manifold (Bulk). Extra dimensions in the Brane World approach may have large and even infinitely large size, leading to experimentally observable effects. Recent development of multidimensional models is encouraged mainly due to superstring theories and their generalization M-theory, which is only consistent quantum theory containing (at least in principle) all interactions including gravity for today. Both superstring and M-theory most naturally are formulated in the d=10 and d=11 dimensions correspondingly. Just this latter circumstances indicate the possibility of the existence of extra dimensions. There are no experimental evidences in favor extra dimensions yet. From the phenomenological point of view the driving forces for modern extra dimensional approaches are connected with the existence of hierarchy problem ($M_Z << M_{Pl}$) and with that of non vanishing cosmological Λ term ($\Lambda \sim 10^{-48}$ 1GeV). It is very hard to explain such a small but nonzero Λ value in the framework of 4D theory. However, it would be mentioned that none of above phenomenological motivations could be considered as a direct indication of the ultimate prove in favor of extra spatial dimensions. For example: hierarchy ($M_Z << M_{Pl}$) has beautiful explanation in frame of 4D GUTs, and convincing solution of Λ-problem is not found in multidimensional theories yet, though there are very interesting new approaches in this direction. Let us discuss the model, which illustrates the new understanding of hierarchy problem [N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys.Lett. B429 (1998) 263; hep-ph/9803315. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B436 (1998) 257; hepph/9804398]. Let us consider the question, how gravity for the particles on the Brane becomes four dimensional. There are some answers on this question. Simple possibility to answer the question is that extra dimensions are compact and are characterized by size R. Gravity in this model is four dimensional at r ≫ R, but stops to be such at r ∼ R. One has N-dimensional Newton law at r ≪ R: $V(r) = G_N m_1 m_2 / r^{1+n}$ (G_N being fundamental gravitational constant in N+1 dimensional space-time, n=N-3 being the number of extra dimensions). For r ≫ R the four dimensional Newton law works: $V(r) = G\ m_1 m_2 / r^1$. Lacing potentials at r ∼ R leads to the conditions $GR^n = G_N$. Introducing some fundamental mass parameter M, which is connected with G_N as $G_N = 1/M^{2+n}$, we have $M_{Pl} = M(RM)^{n/2}$. So, 4D gravity coupling G and M_{Pl} are effective values and M_{Pl} could be different from the new mass parameter M. This allows us to solve the hierarchy problem in the new unexpected way. One can assume that fundamental scale M coincides with electroweak scale by order of magnitude. In this case lacing condition will define the size R of extra dimensions. So, putting M ∼ 1T eV , we have $r = 10^{30/n-17} cm$.

Remarkable feature of this new insight to the hierarchy problem is that gravitational interactions become sizable not at the Planck scale, but on the new scale of M ∼ O(T eV), which in fact must be considered as only fundamental scale of the nature. Such alternative permits us to explore the TeV scale region at the forthcoming accelerators, including from the point of viability of extra dimension approaches.