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Two-dimensional conformal field theories with defects and boundaries

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Contents

0.1 Introduction

The study of two-dimensional conformal field theories passed long way. Their applications to the various topic of physics are so numerous that conformal field theory became one of the most powerful techniques in modern physics. The first great success was the precise computation of the critical exponents for the second-order phase transitions in two-dimensional statistical systems.

The tremendous branch of applications of conformal field theories is String theory. Today string theory offers the most well developed candidate for a fundamental theory of quantum gravity and an approach to the unification of all known interactions. Conformal field theories appear as solutions of string theoretic equations of motion.

The study of boundary conditions is very important problem in physics. Realistic systems possess boundaries and therefore their full understanding obviously requires control of boundary conditions. For two-dimensional conformal field theories, the study of boundaries was started by John Cardy in sequence of papers, in particular [41, 42]. The presence of powerful infinite-dimensional symmetries resulted to numerous exact results on boundary correlation functions.

Boundary conformal field theories are more directly applicable to real physical situations than conformal field theories on closed surfaces. Many processes in three space dimensions possess rotational symmetry and hence all the relevant quantities depend on time and radial coordinates. Therefore, conformal field theories on the half-plane appear naturally. Quantum impurity scattering, the Kondo effect, is the most well known example [1].

In string theory, we need two-dimensional conformal field theories with boundaries to describe open strings.

At low energy limit, p-branes appear as supergravity solitonic solution , describing stable objects whose mass and charge are distributed along $(p + 1)$-dimensional hypersurfaces in the spacetime. Beyond the low-energy regime, supergravity should to be replaced by full-fledged string theory, and we need to understand how to describe branes in string theory. For a class of

branes, those that called D-branes, the answer was found by Polchinski in [142]: D-branes are objects on which open strings can end. The "D" in D-branes stands for the Dirichlet boundary conditions, which constrain the open-string endpoints to live within the brane worldvolume. String theory contains many kinds of branes, which are characterized by their dimension and some additional data. All the data in fact are encoded in the choice of boundary states.

The importance of D-branes for our understanding of string theory, and perhaps many other branches of modern theoretical physics is enormous. Non-complete list of applications includes: Brane modelling of gauge theories [98], Braneworld scenario [11, 149], Braneworld cosmology and inflation [50], Counting of states on black holes by superstring theory [177], Holographic principle: gravitational description of quarks (also known as AdS/CFT correspondence) [2].

The boundary CFT can be generalized to consider a situation in which two (or more) non-trivial CFT are glued together along a common interface.

Interfaces in two-dimensional theories are oriented lines separating two different quantum field theories. In this thesis we consider special class of interfaces, for which the energy-momentum tensor is continuous across the defect. These interfaces are called topological defects [13].

During the last years topological defects in two-dimensional quantum field theories have appeared in the various topics. Let us mention some of them. Topological defects appear in quantum Hall problem [65], quantum wires problem [189], in the consideration of impurities [135, 154, 155]. Topological defects played an important role in the topologically twisted $N = 4$ SYM approach to the geometric Langland program [111]. Defects provide us with examples of 2-category in physics [45, 69, 166, 174]. Defects in Liouville and Toda field theories [48, 164, 168] appear as holographic counterpart of the Wilson lines in the AGT correspondence [5, 48, 49, 140]. Defects appear as domain wall in the Ads/CFT correspondence in the presence of D-branes [12]. Recently they were found to be useful also in study of the renormgroup flow [81, 116].

The topological defects have proved to be very useful in study of the boundary state transformation. Since the topological defect can be moved to the boundary without changing the correlator, it can be fused with the boundary producing new boundary condition. Remember-

ing that in String theory boundary states correspond to D-branes, one arrives to the conclusion that topological defects induce D-brane transformation. This property was crucial for example in the topologically twisted $N = 4$ SYM approach to the Langland problem [111]. On other side D-branes are classified by their Ramond-Ramond or K-theory charges. Therefore topological defects should induce also transformations in cohomolog or K-theory groups. It is expected that this transform should be of the Fourier-Mukai type [15, 35, 57, 78, 93, 163, 166].

In this dissertation we study D-branes and defects in the following CFTs: WZW models, Product of WZW models, Gauged WZW models, Liouville and Toda field theories, Gepner model. We also study duality defects implementing T-duality, Non-abelian T-duality. Fermionic T-duality.

Let us briefly review our findings and contributions in the mentioned topics.

Non-maximally symmetric D-branes in WZW models and D-branes on cosets

Based on papers [54–56, 156–161].

Given a Conformal Field Theory (CFT) on a world-sheet with boundary one inevitably encounters with problem of specifying of boundary conditions. Hence in Boundary Conformal Field Theory (BCFT) one of the most important problems is classification of the boundary conditions. One of the clues to this problem is the amount of the preserved symmetries. In typical situation one has some extended symmetries algebra, which contains conformal algebra as its subalgebra. Cardy in his seminal paper [42] has shown that for diagonal models so called maximally symmetric boundary states, preserving full diagonal subalgebra always exist, and labelled by primaries. On the other side, in CFT's admitting Lagrangian approach as 2D Sigma models, boundary conditions can be specified by constraints imposed on the boundary values of the fields. Amongst most important models are WZW model, providing Lagrangian description of affine algebras, and gauged WZW model, providing Lagrangian description of coset models. The geometrical description of the Cardy states in the WZW model is given by

conjugacy classes [8]. This set-up immediately raised the following problems:

1. to find geometrical realization of the Cardy states in coset models,

2. to find non-maximally symmetric boundary conditions breaking full diagonal symmetry to some subalgebra always containing conformal symmetry.

In [56], the first problem using the Lagrangian of the gauged WZW model has been solved. It was shown that the Cardy states in cosets realized by pointwise product of conjugacy classes. In this paper geometrical realization of selection rule and field identification in cosets was addressed as well. The methods developed in this paper, in particular the use of the Polyakov-Wiegmann identities, turned out to be very fruitful and enabled in the next publications [156–158, 160, 161] to address the second problem and build many new examples of non-maximally symmetric boundary conditions and branes in various cosets. In particular we would like to mention the following findings:

- Geometrical realization of the Maldacena-Moore-Seiberg parafermionic D-branes [156, 157]

- D-branes in asymmetric cosets [158]

- D-branes in cosmological Nappi-Witten model and in Guadagnini-Martellini-Mintchev model [158]

- Non-maximally symmetric non-factorizable D-branes on product of WZW models [159, 160]

- Geometrical realization of permutation D-branes and defects in coset models [161]

D-branes in Gepner model

Based on papers [10, 162].

The Gepner model is one of the most interesting exact-solvable compactification schemes. We have studied Cardy states in $(2)^4$ Gepner model. This model is still rather simple to handle,

and yet enough complicated to capture typical difficulties one encounters using Gepner model. In particular this model possesses a simple current extension fixed point, and requires to use corresponding machinery for fixed point resolution. The full list of 88 characters of this model has been presented, and using equivalence of this model with toroidal orbifold T^4/Z_4, partial geometric realization of Cardy states has been obtained.

Topological defects

Defects in two-dimensional quantum field theory are oriented lines separating different quantum field theories. The notion of the defects is very rich and defects appear in the numerous different topics, like condensed matter, string theory, algebraic topology, Langland theory, boundary conformal field theory, D-branes.

I have published numerous papers on topological defects. In these papers the following aspects of the defects have been discussed: applications of defects to string dualities, defects in Liouville and Toda field theories, defects in WZW and gauged WZW models.

Defects and dualities

Based on papers [57, 93, 163].

In [57, 93, 163], defects separating two bulk systems, each described by its own Lagrangian, where the two descriptions are related by a discrete symmetry, were considered, and defect equations of motion (defect analogue of boundary equation of motion) have been elaborated. In particular the descriptions related by T-duality, fermionic T-duality, and non-abelian T-duality were considered. This analysis implies that to each kind of dualities a bundle on a defect world-volume can be associated. A defect corresponding to a duality, sometimes called defect performing or implementing the duality, since it can be also considered as an operator implementing the duality. We have found that the defect equations of motion encode the duality relations. We observed that bundles on world-volumes of defects performing various T-dualities, are in fact different cousins of the Poincaré bundle. It was shown that the duality

action on D-branes and the Ramond-Ramond fields is identical to the Fourier-Mukai transform with a kernel given by the corresponding Poincaré bundles or the exponential of the gauge invariant flux on a defect respectively. This enabled us to develop new method of calculation of the Ramond-Ramond field transformation under the non-abelain T-duality. We also studied in detail T-duality between $SU(2)$ WZW model and the lens space, and axially and vectorially gauged WZW models.

Defects in WZW and gauged WZW models

Based on papers [165–167].

In [165], the famous relation between WZW model and Chern-Simons gauge theory [52, 186] has been elaborated in the presence of defects and permutation branes. Using the Lagrangian formulation of WZW model with defects [78] and boundaries [87] the following three symplectomorphisms have been established:

1. The symplectic phase space of the WZW model with N defects on a cylinder is symplectomorphic to that of Chern-Simons gauge theory on an annulus $\mathcal{A}$ times the time-line R with N time-like Wilson lines.

2. The symplectic phase space of the WZW model with N defects on a strip is symplectomorphic to that of Chern-Simons gauge theory on a disc D times the time-line R with $N + 2$ time-like Wilson lines.

3. The symplectic phase space of N-fold product of WZW models on a strip with boundary conditions given by the permutation branes is symplectomorphic to that of Chern-Simons gauge theory on a sphere with N holes times the time-line R and with two time-like Wilson lines.

In [166, 167], the relation between gauged WZW model G/H model and double Chern-Simons gauge theory [131] has been studied in the presence of defects and permutation branes and the following isomorphisms have been established:

1. The symplectic phase space of the gauged WZW G/H model on a cylinder with N defects is symplectomorphic to the symplectic phase space of the double Chern-Simons theory on an annulus $\mathcal{A}$ times the time-line R with G and H gauge fields both coupled to N Wilson lines.

2. The symplectic phase space of the gauged WZW G/H model on a strip with N defects is symplectomorphic to the symplectic phase space of the double Chern-Simons theory on a disc D times the time-line R with G and H gauge fields both coupled to $N + 2$ time-like Wilson lines.

3. The symplectic phase space of the N-fold product of the gauged WZW models on a strip with boundary conditions given by permutation branes is symplectomorphic to the symplectic phase space of the double Chern-Simons theory on a sphere with N holes times the time-line R with G and H gauge fields both coupled to two Wilson lines.

   In the special case of topological coset G/G these isomorphisms take the form:

4. The symplectic phase space of the gauged WZW G/G model on a cylinder with N defects is symplectomorphic to the symplectic phase space of the Chern-Simons theory on $T^2 \times R$ with $2N$ Wilson lines.

5. The symplectic phase space of the gauged WZW G/G model on a strip with N defects is symplectomorphic to the symplectic phase space of the Chern-Simons theory on $S^2 \times R$ with $2N + 4$ time-like Wilson lines.

6. The symplectic phase space of the N-fold product of the topological coset G/G on a strip with boundary conditions given by permutation branes is symplectomorphic to the symplectic phase space of the Chern-Simons theory on a Riemann surface of the genus $N - 1$ times the time-line with four Wilson lines.

Defects in the Liouville and Toda field theories

Based on papers [144, 164, 168]

In [164,168], using the Cardy-Lewellen cluster equation for defects, derived in [139], defects in the Liouville and Toda field theories have been constructed. It was shown that defects labelled by the physical and degenerate primaries of Liouville/Toda field theories, and correspondingly compose discrete and continuous families. Acting by the defects of the continuous family on the well-known Fateev-Zamolodchikov-Zamolodchikov boundary states, new boundary states in the Liouville field theory have been obtained. We also have shown in [168] that the well known relation [20] between the OPE structure constants and the fusing matrix with an intermediate entry set to the vacuum proved for rational CFT, holds also for the Liouville theory. We checked that it holds also for the Toda field theory with an external entry set a degenerate field. But we have gathered evidences that it should hold for all primaries also in the Toda field theory.

In paper [144], we study semiclassical limit of the continuous family of the defect two-point functions in the Liouville field theory, derived in [164].

We show that semiclassical limits are in agreement with the recently suggested Lagrangian with topological defects of the continuous family constructed in [3]. In particular we demonstrate that the heavy asymptotic limit is given by the exponential of the Liouville action with defects, evaluated on the solutions of the defect equations of motion with two singular points.

This dissertation is organized in the following way.

The dissertation consists of 8 chapters. In chapter 1 we review the material necessary to present our findings. In chapters 2-7 we deliver our findings. The Chapter 8 contains the list of findings.

In chapter 1 we collect and review the necessary stuff and technique of two-dimensional conformal field theory including bulk as well as boundary aspects. In section 1.1 we review two-dimensional conformal field theory on closed surfaces (bulk aspects). In section 1.2 we collect all the necessary gadgets of conformal field theory on a world-sheet with a boundary. In section 1.3 we study topological defects. In section 1.4 we illustrate the developed technique for the case of free boson theory. In section 1.5 we introduce WZW and gauged WZW models (coset models).

The Chapter 2 is based on papers [54–56, 156–160]. In this chapter we study non-

maximally symmetric branes on WZW model, preserving only part of the diagonal affine symmetry. In section 2.1 we analyze general properties of WZW model on a world-sheet with a boundary. In section 2.2 we study so called parafermionic D-branes. In subsection 2.2.1 we define non-maximally symmetric D-brane, sometimes called also parafermionic, as pointwise product of the conjugacy class and a $U(1)$ subgroup. We construct Lagrangian with the boundary condition constraining group field to take on boundary values in the parafermionic D-brane. We study symmetries of the action and show that it is invariant under the axial combination of the left and right $U(1)$ currents, and vectorial combination of the currents belonging to the subgroup commuting with $U(1)$ group. In subsection 2.2.2 we study geometry of the parafermionic D-branes for $SU(2)$ group and show that generically it is three-dimensional and given by an inequality constraining the values of the second Euler angle. In subsection 2.2.3 we review construction of the boundary state of the parafermionic D-brane for $SU(2)$ group, called MMS (Maldacena-Moore-Seiberg) state, given in [123]. In subsection 2.2.4 we compute the overlap of the MMS boundary state with the graviton wave packet and show that it gives the inequality derived in subsection 2.2.2. In section 2.3 we study permutation branes on a $K + 1$-fold product of group G on a world-sheet with a boundary, with boundary condition constraining product of group fields to take value again in discrete set of conjugacy classes defined in 2.1.2. In subsection 2.3.1 we describe geometry of the permutation branes. In subsection 2.3.2 we write the Lagrangian with these boundary conditions and show that it has symmetries of permutation branes studied in 1.2.4. In subsection 2.3.3 we compute for $SU(2)$ group overlap of the permutation boundary states defined in 1.2.4 with the graviton wave packet and show that in the semiclassical limit they indeed have geometry described in 2.3.1. In section 2.4 we construct type I non-maximally symmetric non-factorizable branes on a product of identical groups. In subsection 2.4.1 we define new branes as product of permutation branes studied in 2.3 with elements of an $U(1)$ subgroup. We construct Lagrangian with these boundary conditions and study their symmetries. In subsection 2.4.2 we study geometry of these branes for $SU(2) \times SU(2)$ group. In subsection 2.4.3 we construct boundary states of the type I branes for $SU(2) \times SU(2)$ group, compute the overlap with the graviton wave packet and show that

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it is in agreement with the calculations in 2.4.2. We also check that type I boundary states satisfy the Cardy criteria.

In section 2.5 we study type II non-maximally symmetric non-factorizable branes on product of identical groups. In subsection 2.5.1 we define new branes as product of permutation branes studied in 2.3 with elements of two of $U(1)$ subgroups. We construct Lagrangian with these boundary conditions and study their symmetries. In subsection 2.5.2 we study geometry of these branes for $SU(2) \times SU(2)$ group. In subsection 2.5.3 we construct boundary states of the type II branes for $SU(2) \times SU(2)$ group, compute the overlap with the graviton wave packet and show that it is in agreement with the calculations in 2.5.2. We also check that type II boundary states satisfy the Cardy criteria.

The chapter 3 is based on papers [56, 158, 160, 161]. In chapter 3 branes and defects in gauged WZW models are constructed. In section 3.1 we study branes in the vectorially gauged WZW model G/H. In subsection 3.1.1 we construct D-branes in vectorially gauged WZW model using the representation of the gauged WZW model Lagrangian via the auxiliary fields reviewed in 1.5.5 and the action of the WZW model with a boundary presented in 2.1.1.

Analysing global issues mentioned in 2.1.2 we find correspondence of the found D-branes with the Cardy states of the coset models in the absence of the common center of G and H.

In subsection 3.1.2 we analyze special case of the cosets when G and H has common center. We show that found D-branes satisfy present in this case field identification and selection rules of the primary fields of coset models.

In section 3.2 we present the Lagrangian approach to defects in WZW models. In section 3.3 we construct Cardy defects and permutation branes in vectorially gauged WZW model. In subsection 3.3.1, using again the representation of the gauged WZW model Lagrangian via the auxiliary fields presented in 1.5.5 and the Lagrangian of the WZW model with defects reviewed in 3.2, the geometry and action of the topological defects and permutation branes in GWZW are constructed. We show that they are in one-to-one correspondence with primary fields of coset models. In subsection 3.3.2 we consider overlap of the permutation brane boundary state on product of cosets $SU(2)_k/U(1) \times SU(2)_k/U(1)$ with the graviton wave packet and show that

it has geometry found in 3.3.1.

In section 3.4 we cosnsider D-branes in asymmetrically gauged cosmological Nappi-Witten model and in the Guadagnini-Martellini-Mintchev model. In subsection 3.4.1 we present D-branes in the Nappi-Witten model, construct the action with these boundary conditions and check gauge invariance. In subsection 3.4.2 we study in detail D-branes in the Nappi-Witten cosmological model $SL(2,R) \times SU(2)/U(1) \times U(1)$ and present the explicit equations of the corresponding D-brane hypersurfaces. In subsection 3.4.3, in a similar way D-branes in the Guadagnini-Martellini-Mintchev model are considered . In subsection 3.4.4 we consider in detail D-branes in the $SU(2) \times SU(2)/U(1)$ GMM model.

The chapter 4 is based on papers [165–167]. In chapter 4 we establish symplectomorphisms between certain phase space of the Chern-Simons and double Chern-Simons theory and that of WZW and gauged models with branes and defects. In section 4.1 we review the symplectic phase space of three-dimensional Chern-Simons theory with sources on a product of a Riemann surface Σ and a time line R.

In section 4.2 we establish symplectomorphisms between certain phase space of the Chern-Simons theory and that of WZW models with branes and defects. In subsection 4.2.1 we compare the Hilbert spaces of the Chern-Simons theory with Wilson lines on certain spaces and that of WZW models with branes and defects and list the statements which we prove here. In subsection 4.2.2 we review bulk WZW model and establish that the symplectic phase space of the WZW model on circle coincides with that of CS theory on annulus [52]. In subsection 4.2.3 we recall that the symplectic phase space of the WZW model on the strip coincides with that of CS theory on the disc with two Wilson lines [88]. In subsection 4.2.4 we establish that the symplectic phase space of the WZW model with a defect is symplectomorphic to that of Chern-Simons theory on an annulus with a Wilson line. In subsection 4.2.5 we establish that the symplectic phase space of the WZW model on a strip with a defect inserted is symplectomorphic to that of CS theory on a disc with three Wilson lines. In subsection 4.2.6 we establish that the symplectic phase space of the WZW model $G \times G$ on a strip with boundary conditions specified by permutation branes coincides with that of CS on an annulus with two Wilson lines.

In section 4.3 we perform canonical quantization of the vectorially gauged WZW model G/H with the defects and boundaries and establish symplectomorphisms between their phase spaces and certain phase spaces of the double Chern-Simons theories. In subsection 4.3.1 we present short summary of the statements proved in this section. In subsection 4.3.2 we review bulk gauged WZW model and show that its phase space on a cylinder coincides with that of double Chern-Simons theory [89, 131] on product of annulus $\mathcal{A}$ and time-line R. In subsection 4.3.3 we show that the phase space of the gauged WZW model on a cylinder with a defect line coincides with that of double Chern-Simons theory on $\mathcal{A} \times R$ with gauge fields of groups G and H coupled to a Wilson line. In subsection 4.3.4 we show that the phase space of the gauged WZW model on a strip with a defect line coincides with that of the double Chern-Simons theory on disc D times time-line R with gauge fields of groups G and H coupled to three Wilson lines. In section 4.4 we analyze especially interesting case of the topological coset G/G. In subsection 4.4.1 we analyze bulk G/G coset and show that the phase space of a bulk G/G theory on a cylinder is symplectomorphic to that of a Chern-Simons theory on $T^2 \times R$, where T^2 is a torus. In subsection 4.4.2 we show that the topological coset G/G on a cylinder with a defect line is symplectomorphic with that of a Chern-Simons theory on $T^2 \times R$ with two Wilson lines. In subsection 4.4.3 we demonstrate the symplectomorphism of the phase space of G/G topological coset on a strip with a defect and a Chern-Simons theory on $S^2 \times R$ with six Wilson lines. In section 4.5 we analyze a product of cosets $G/H \times G/H$ on a strip with boundary conditions specified by permutation branes and show that its phase space is symplectomorphic to the phase space of the double Chern-Simons theory on an annulus times the time-line and with G and H gauge fields both coupled to two Wilson lines. In section 4.6 we establish symplectomorphism of the phase space of product of topological cosets $G/G \times G/G$ on a strip with the boundary conditions given by the permutation branes and that of Chern-Simons theory on a torus times the time-line with four Wilson lines.

The chapter 5 is based on papers [57, 93, 163].

In chapter 5 we study topological defects implementing various dualities.

In section 5.1 we review some basic facts concerning topological defects and their relation

to T-duality. It is established that the defect implementing bosonic T-duality is given by the Poincaré line bundle. We demonstrate in the simple example of a scalar field compactified on a circle how the defect equations of motion reproduce the appropriate duality transformations. In section 5.2 we generalize this to the factorized T-duality in non-linear sigma models with isometries. We also present a defect generating a combined action of the $\mathbb{Z}_k$ orbifolding together with a T-duality transformation. In section 5.3 we explain how the T-duality transformation of the Ramond-Ramond charges can be written as the Fourier-Mukai transform with the kernel given by the exponent of the gauge invariant flux on the corresponding topological defect.

In section 5.4, we study T-dualities in the special case of SU(2) WZW model and a lens space. In subsection 5.4.1 we review [32] kernel of the Fourier-Mukai transform of the T-duality between $SU(2)$ WZW model and lens space implementing the map between the corresponding twisted cohomology groups. In subsection 5.4.2, we construct several families of defects by using T-duality transformation and orbifoldig. In subsection 5.4.3 for one such family we determine the geometry of the underlying bibranes. We recover structure familiar from Fourier-Mukai transformations studied in 5.4.1.

In section 5.5 we construct defects between axial and vector gauging of $G/U(1)$ gauged WZW models [16, 85] for a general group G. For the case of $G = SU(2)$ [17] the geometrical construction is translated to the algebraic parafermionic language.

In subsection 5.5.1 we present geometry and flux of the defects gluing axially-vectorially gauged models. In subsection 5.5.2 we specialize to group $SU(2)$ and show that for level k parafermions there are $k + 1$ topological defects mapping axially gauged $SU(2)/U(1)$ cosets to the vectorially gauged $SU(2)/U(1)$ coset, labeled by the integrable spin $j = 0, \ldots, \frac{k}{2}$. In subsection 5.5.3 we construct them as the appropriate operators in the parafermion Hilbert space. We show that the defect corresponding to $j = 0$ implements $\mathbb{Z}_k$ orbifolding together with T-duality. These defects project $A_{j,n}$ Cardy branes in $SU(2)/U(1)$ coset to the B_j branes constructed in [123].

In section 5.6 we study the defect performing the fermionic T-duality [24]. In subsection 5.6.1 we review the necessary information on pseudodifferential forms integration. In subsection 5.6.2

we review the fermionic T-duality [24]. In subsection 5.6.3 we show that the defect inducing the fermionic T-duality is given by the fermionic generalization of the Poincaré line bundle, which we denote as Super-Poincaré line bundle. We demonstrate that the defect equations of motion reproduce the fermionic T- duality transformation rules found in [24]. In subsection 5.6.4 using the exponent of the gauge invariant flux on this defect as a kernel of the Fourier-Mukai transform with a pushforward map given by the fiberwise integration on supermanifold, we derive the transformation of the Ramond-Ramond fields under the fermionic T-duality.

In section 5.7 we construct topological defects producing non-abelian T-duality.

In subsection 5.7.1 we review non-abelian T-duality. In particular we recall the duality relations and demonstrate general formulas for the case of $SU(2)$ principal chiral model. In subsection 5.7.2 we present defect performing non-abelian T-duality, and show that the defect equations of motion reproduce the duality relations derived in subsection 5.7.1. In subsection 5.7.3 using the flux of non-abelian T-duality defect derived in subsection 5.7.2, we derive the Fourier-Mukai transform formula for non-abelian T-duality, and compute the RR fields transformation for $SU(2)$ isometry group. We obtain that our results are in agreement with that of [109, 175].

The chapter 6 is based on papers [144, 164, 168].

In chapter 6 we study topological defects in the Liouville and Toda field theories. In section 6.1 we write down topological defects in the Liouville field theory. It is shown that defects are labelled by the physical and degenerate primaries of Liouville field theory, and correspondingly compose discrete and continuous families. We also have shown that the well known relation [20] between the OPE structure constants and the fusing matrix with an intermediate entry set to the vacuum proved for rational CFT, holds also for the Liouville theory. In section 6.2 we write down topological defects in the Toda field theory. We have shown that topological defects in Toda field theory are labelled by the physical, semi-degenerate and fully degenerate primaries. In section 6.3 we analyze classical Liouville theory with defects. In subsection 6.3.1 we review the general solution of the Liouville equation. In subsection 6.3.2 we present general solution of the defect equations of motion. In section 6.4 we review the heavy asymptotic

semiclassical limit. In section 6.5 we calculate the defect two-point function in the heavy asymptotic limit. In subsection 6.5.1 we calculate heavy asymptotic limit of defect two-point functions. In subsection 6.5.2 we show that heavy asymptotic limit of defect two-point functions found in the previous section is given by exponential of the action with defects evaluated on solution of defect equations of motion with two singularities.

The chapter 7 is based on papers [10, 162].

In chapter 7 we study Cardy states in $(2, 2, 2, 2)$ Gepner model. In section 7.1 we review necessary background material on the simple current extensions. In section 7.2 we review Gepner models via simple current extension formalism. In section 7.3 we write down all the necessary information on the $(2, 2, 2, 2)$ model: orbit representatives, characters, conformal weights. Using the resolved characters we compute the torus partition function and show that it coincides with the one computed in the appendix 5 as an orbifold partition function at the $SU(2)^4$ point. Using the general formulae of section 7.1 we also derive the annulus partition functions between different Cardy states, paying special attention to the peculiarities caused by the presence of the fixed points. In section 7.4 we study $D0$ branes on the orbifold T^4/Z_4. We compute all the annulus partition functions between $D0$ branes located at points in T^4/Z_4 orbifold that are fully or partially fixed under the orbifold group action. Using previously derived formulae for the annulus partition functions between Cardy states of the $(2,2,2,2)$ model we establish a partial dictionary between Cardy states and $D0$ branes.

In chapter 8 we presented list of our main findings.

In five appendices some technical points are collected. In appendix 1 double Gamma and Sinus functions are reviewed. In appendix 2 asymptotic behaviour of Gamma function is reviewed. In appendices 3 and 4 some identities on Theta function are collected. In appendix 5 some technical points on calculation of the partition function of T^4/Z_4 orbifold are delivered.

All results of this dissertation are **published** in papers [10, 54–57, 93, 144, 156–168].

Chapter 1

Two-dimensional conformal field theories: bulk and boundary aspects

1.1 General facts 2D CFT on closed surfaces

Here we review necessary facts on two-dimensional conformal field theories on closed surfaces (bulk aspects). The standard references here are [79, 95].

1.1.1 Conformal group in two dimensions

Let us consider the conformal transformations in two dimensions $D = 2$. Denote by $g_{\mu\nu}$ the metric tensor. By the definition conformal transformation of the coordinates is the invertible map $x \to x'$ which leaves metric tensor invariant up to scale:

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x) \tag{1.1}$$

where

$$g'_{\mu\nu}(x')\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\rho}} = g_{\lambda\rho} \tag{1.2}$$

Let us investigate the consequences of definition (1.1) on the infinitesimal transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \tag{1.3}$$

It follows from (1.1)

$$g_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} = \Lambda^{-1} g_{\lambda\rho} \tag{1.4}$$

and inserting (1.3) we obtain in the first order by ϵ:

$$g_{\mu\nu} \left(\delta_{\lambda}^{\mu} + \frac{\partial \epsilon^{\mu}}{\partial x^{\lambda}} \right) \left(\delta_{\rho}^{\nu} + \frac{\partial \epsilon^{\nu}}{\partial x^{\rho}} \right) = g_{\lambda\rho} + \partial_{\lambda}\epsilon_{\rho} + \partial_{\rho}\epsilon_{\lambda} \tag{1.5}$$

Therefore the requirement that this map is conformal implies that

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = (\Lambda^{-1} - 1)g_{\mu\nu} = f(x)g_{\mu\nu} \tag{1.6}$$

The factor $f(\mathbf{x})$ can be determined by taking trace on both sides:

$$f(\mathbf{x}) = \partial_{\rho}\epsilon^{\rho} \tag{1.7}$$

Equation (1.6) for $g_{\mu\nu} = \delta_{\mu\nu}$ becomes Cauchy-Riemann condition

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \tag{1.8}$$

Thus it is natural to write $\epsilon(z) = \epsilon_1 + i\epsilon_2$ and $\bar{\epsilon}(\bar{z}) = \epsilon_1 - i\epsilon_2$ in complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. Two dimensional conformal transformations then coincide with the holomorphic coordinate transformations

$$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{1.9}$$

The metric in the complex coordinates is

$$ds^2 = dz d\bar{z} \tag{1.10}$$

Under the analytic coordinate transformations

$$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{1.11}$$

$$ds^2 = dz d\bar{z} \to \left| \frac{\partial f}{\partial z} \right|^2 dz d\bar{z} \tag{1.12}$$

The holomorphic infinitesimal transformation can be expressed as:

$$z' = z + \epsilon(z) \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1} \tag{1.13}$$

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The effect of such a mapping on the field $\phi(z, \bar{z})$ living on plane is:

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_n \{c_n l_n \phi(z, \bar{z}) + \bar{c}_n \bar{l}_n \phi(z, \bar{z})\} \tag{1.14}$$

where we have defined generators

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{1.15}$$

These generators obey commutation relations:

$$[l_n, l_m] = (n - m)l_{n+m} \tag{1.16}$$

$$[\bar{l}_n, \bar{l}_m] = (n - m)\bar{l}_{n+m} \tag{1.17}$$

$$[l_n, \bar{l}_m] = 0 \tag{1.18}$$

$$\tag{1.19}$$

We see that the conformal algebra is direct sum of two isomorphic algebras. The algebra (1.16) is the Witt algebra.

Note that $l_0 = -z\partial_z$ and $\bar{l}_0 = -\bar{z}\partial_{\bar{z}}$ and hence introducing the polar coordinates $z = re^{i\theta}$ we obtain

$$r\frac{\partial}{\partial r} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = -(l_0 + \bar{l}_0) \tag{1.20}$$

and

$$\frac{\partial}{\partial \theta} = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}} = -i(l_0 - \bar{l}_0) \tag{1.21}$$

Thus $(l_0 + \bar{l}_0)$ generates dilatations and $i(l_0 - \bar{l}_0)$ generates rotations.

Let us look for the generators well-defined globally on the Riemann sphere $S^2 = C \cup \infty$. The analytic conformal transformations are generated by the vector fields:

$$v(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1}\partial_z \tag{1.22}$$

The non-singularity of $v(z)$ as $z \to 0$ allows $a_n \neq 0$ only for $n \geq -1$. To understand behavior of $v(z)$ as $z \to \infty$, we perform a transformation $z = -\frac{1}{\omega}$,

$$v(z) = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n+1} \left(\frac{dz}{d\omega}\right)^{-1} \partial_\omega = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n-1} \partial_\omega \tag{1.23}$$

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The non-singularity as $\omega \to 0$ allows $a_n \neq 0$ only for $n \leq 1$. We can see that only conformal transformations generated by $a_n l_n$ for $n = 0, \pm 1$ are globally defined. The same considerations work to the anti-holomorphic transformations.

These generators satisfy the commutation relation:

$$[l_0, l_{-1}] = l_{-1} \tag{1.24}$$

$$[l_0, l_1] = -l_1 \tag{1.25}$$

$$[l_1, l_{-1}] = 2l_0 \tag{1.26}$$

$$\tag{1.27}$$

and similar for antiholomorphic components. This is $sl(2, C)$ algebra.

Let us examine also the group structure. We identify l_{-1} and $\bar{l}_{-1}$ as generators of translations (globally $z \to z + \alpha$), l_0 and $\bar{l}_0$ as generators of dilatations (globally $z \to \lambda z$), and l_1 and $\bar{l}_1$ as generators of the special conformal transformations (globally $z \to \frac{1}{1-\beta z}$). The combined form of these transformations is

$$z \to \frac{az + b}{cz + d} \qquad \bar{z} \to \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \tag{1.28}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. This is the group $SL(2, \mathbb{C})/\mathbb{Z}_2$. The quotient by $\mathbb{Z}_2$ is due to fact that (1.28) is unchanged by taking $a, b.c, d$ to minus of themselves.

1.1.2 Tensor energy-momentum, radial quantization, OPE

Under the coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, the action changes in the following way:

$$\delta S = \int d^2 x \, T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} \int d^2 x \, T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{1.29}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor. The definition (1.6) of the infinitesimal conformal mapping implies that corresponding variation of the action reads

$$\delta S = \frac{1}{2} \int d^2 x \, T^\mu_\mu \partial_\rho \epsilon^\rho \tag{1.30}$$

The vanishing of the trace of the energy-momentum tensor thus implies the invariance of the action under the conformal transformation. The current of conformal symmetry is

$$J_\mu = T_{\mu\nu}\epsilon^\nu \tag{1.31}$$

This current is conserved because

$$\partial^\mu J_\mu = \partial^\mu T_{\mu\nu}\epsilon^\nu + T_{\mu\nu}\partial^\mu\epsilon^\nu = 0 \tag{1.32}$$

which vanishes because the tensor energy-momentum is conserved and traceless.

To implement the conservation equations in the complex plane we compute the components of tensors in the complex coordinates. Since the flat Euclidean metric $ds^2 = dx^2 + dy^2$ in the complex coordinates $z = x + iy$ has the form $ds^2 = dzd\bar{z}$ one has

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad \text{and} \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} \tag{1.33}$$

and

$$g^{zz} = g^{\bar{z}\bar{z}} = 0 \quad \text{and} \quad g^{z\bar{z}} = g^{\bar{z}z} = 2 \tag{1.34}$$

The components of the energy-momentum tensor in this frame are

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \tag{1.35}$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})$$
$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^\mu_\mu$$

Therefore the tracelessness implies

$$T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{1.36}$$

The conservation law $g^{\alpha\mu}\partial_\alpha T_{\mu\nu} = 0$ gives two equations

$$\partial_{\bar{z}}T_{zz} + \partial_z T_{\bar{z}z} = 0 \quad \text{and} \quad \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}}T_{z\bar{z}} = 0 \tag{1.37}$$

Using (1.36) we obtain

$$\partial_{\bar{z}}T_{zz} = 0 \qquad \text{and} \qquad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{1.38}$$

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The two non-vanishing components of the energy-momentum tensor

$$T(z) \equiv T_{zz}(z) \qquad \text{and} \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \tag{1.39}$$

then have only the holomorphic and anti-holomorphic dependence.

Take the system on a cylinder $\Sigma = R \times S^1 = (t, x \mod 2\pi)$, where t is world-sheet time, and x is compactified space coordinate.

Consider now conformal map $w \to z = e^w = e^{t+ix}$, that maps a cylinder to complex plane. Then infinite past and future on a cylinder, $t = \pm\infty$ are mapped to points $z = 0, \infty$ on a plane. The equal time surfaces, $t = \text{const}$ becomes circles of the constant radius on z-plane. Dilatation on the plane e^a becomes time translation $t + a$ on the cylinder, and rotation on the plane $e^{i\alpha}$ is space translation $x + \alpha$ on the cylinder. Therefore the dilatation generator on the conformal plane can be considered as the Hamiltonian, and the rotation generator on the conformal plane can be considered as momentum.

The current of conformal transformations takes the form:

$$J_z = T(z)\epsilon(z) \quad \text{and} \quad J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \tag{1.40}$$

The conserved charge of the conformal transformations takes the form

$$Q = \frac{1}{2\pi i} \oint dz T(z)\epsilon(z) + \frac{1}{2\pi i} \oint d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \tag{1.41}$$

Radial ordering

Product of operators make sense if they are radially ordered. This is an analogue of time ordering for quantum field theory on a cylinder. Recall the time ordering rule:

$$TA(t_a)B(t_b) = A(t_a)B(t_b) \text{ for } t_a > t_b \text{ and } B(t_b)A(t_a) \text{ for } t_a < t_b \tag{1.42}$$

Passing from a cylinder to a plane, Euclidean time coordinate is mapped to radial coordinate, and the time ordering becomes the radial ordering

$$RA(z)B(w) = A(z)B(w) \text{ for } |z| > |w| \text{ and } B(w)A(z) \text{ for } |z| < |w| \tag{1.43}$$

The variation of any field is given by commutator with the charge (1.41):

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = [Q,\Phi(w,\bar{w})] = \tag{1.44}$$

$$\frac{1}{2\pi i}\oint dz\epsilon(z)(T(z)\Phi(w,\bar{w}) - \Phi(w,\bar{w})T(z)) + \frac{1}{2\pi i}\oint d\bar{z}\bar{\epsilon}(\bar{z})(\bar{T}(\bar{z})\Phi(w,\bar{w}) - \Phi(w,\bar{w})\bar{T}(\bar{z}))$$

Let us now analyze the order of operators in the second and the third lines in (1.44). We will discuss the holomorphic part, the similar discussion holds for antiholomorphic part. We have seen that the first term in the commutator is defined only if $|z| > |w|$, whereas the second one requires $|z| < |w|$. Therefore we should use different contours in two terms in commutator

$$\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \frac{1}{2\pi i}\left(\oint_{|z|>|w|} dz\epsilon(z)(T(z)\Phi(w,\bar{w}) - \oint_{|z|<|w|}\Phi(w,\bar{w})T(z)\right) \tag{1.45}$$

Using the definition of the radial ordering (1.43) one can write

$$\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\epsilon(z)R(T(z)\Phi(w,\bar{w})) \tag{1.46}$$

Deforming the contours the result is

$$\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \frac{1}{2\pi i}\oint_{w} dz\epsilon(z)R(T(z)\Phi(w,\bar{w})) \tag{1.47}$$

where integration contour encircles a point w. Collecting all we obtain:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{w} dz\epsilon(z)R(T(z)\Phi(w,\bar{w})) + \frac{1}{2\pi i}\oint_{w} d\bar{z}\bar{\epsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(w,\bar{w})) \tag{1.48}$$

Primary fields possess the following transformation rule:

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^{h}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}}\Phi(f(z),\bar{f}(\bar{z})) \tag{1.49}$$

The infinitesimal transformation of the primary fields of the weight h and $\bar{h}$ is:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = h\partial\epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\partial\Phi(w,\bar{w}) + \bar{h}\bar{\partial}\bar{\epsilon}(\bar{w})\Phi(w,\bar{w}) + \bar{\epsilon}(\bar{w})\bar{\partial}\Phi(w,\bar{w}) \tag{1.50}$$

Comparing (1.48) and (1.50) we get OPE of the energy-momentum tensor with the primary field of the weights h, $\bar{h}$

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_{w}\Phi(w,\bar{w}) \tag{1.51}$$

$$\bar{T}(\bar{z})\Phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) \tag{1.52}$$

1.1.3 Virasoro algebra

Schwarzian derivative

OPE of the tensor energy-momentum with itself takes the form:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{1.53}$$

The term on the rhs, with coefficient c a constant, is allowed by the analicity, Bose symmetry, and the scale invariance. Besides of this term, (1.53) is just a statement that $T(z)$ is conformal field of the weight $(2,0)$. According to (1.48) the variation of T under infinitesimal conformal transformation is

$$\delta_\epsilon T(w) = \frac{1}{2\pi i} \oint \epsilon(z)T(z)T(w) = \frac{1}{12}c\partial_w^3\epsilon(w) + 2T(w)\partial_w\epsilon(w) + \epsilon(w)\partial_w T(w) \tag{1.54}$$

The exponentiation of this infinitesimal variation to the finite transformation $z \to w(z)$ reads

$$T(z) \to \left(\frac{dw}{dz}\right)^2 T(w(z)) + \frac{c}{12}S(w;z) \tag{1.55}$$

where we have introduced so called Schwarzian derivative:

$$S(w;z) = \frac{(d^3w/dz^3)}{(dw/dz)} - \frac{3}{2}\left(\frac{(d^2w/dz^2)}{(dw/dz)}\right)^2 \tag{1.56}$$

It is in fact unique weight two object that vanishes when restricted to the global $SL(2,C)$ subgroup of 2D conformal group. It satisfies a composition law:

$$S(w,z) = \left(\frac{df}{dz}\right)^2 S(w,f) + S(f,z) \tag{1.57}$$

The energy-momentum tensor is example of the field that is quasi-primary, i.e. $SL(2,C)$ primary, but not Virasoro primary. For exponential map $w \to z = e^w$ one has

$$S(e^w, w) = -1/2 \tag{1.58}$$

so

$$T_{\text{cyl}}(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12}S(z,w) = z^2 T(z) - \frac{c}{24} \tag{1.59}$$

Using mode expansion $T(z) = \sum L_n z^{-n-2}$ one finds

$$T_{\text{cyl}}(w) = \sum L_n z^{-n} - \frac{c}{24} = \sum_n \left(L_n - \frac{c}{24} \delta_{n0} \right) e^{-nw} \tag{1.60}$$

The translation generator $(L_0)_{\text{cyl}}$ on a cylinder is then given in the terms of the generator L_0 on plane as

$$(L_0)_{\text{cyl}} = L_0 - \frac{c}{24} \tag{1.61}$$

Virasoro generators

We introduced a current $J(z) = T(z)\epsilon(z)$. Since $\epsilon(z)$ is an arbitrary holomorphic function, it is natural to expand it in modes. We expect that the current $T(z)z^{n+1}$ generates the transformation $z \to z + c_n z^{n+1}$. The corresponding charges are:

$$L_n = \frac{1}{2\pi i} \oint dz T(z) z^{n+1} \tag{1.62}$$

This relation can be inverted:

$$T(z) = \sum_n z^{-n-2} L_n \tag{1.63}$$

The commutator of the charges is

$$[L_n, L_m] = \frac{1}{(2\pi i)^2} \oint_0 dw w^{m+1} \oint_w dz z^{n+1} T(z) T(w) = \frac{1}{12} cn(n^2 - 1)\delta_{n+m,0} + (n-m)L_{m+n} \tag{1.64}$$

Identical consideration for $\bar{T}$ implies

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{c/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial\bar{T}(\bar{w}) \tag{1.65}$$

$$\bar{T}(\bar{z}) = \sum_n \bar{z}^{-n-2} \bar{L}_n \tag{1.66}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{1}{12}\bar{c}n(n^2-1)\delta_{n+m,0} \tag{1.67}$$

Since $T(z)$ and $\bar{T}(\bar{z})$ have no power law singularity in their OPE, we have

$$[L_n, \bar{L}_m] = 0 \tag{1.68}$$

Highest weight state

Consider now state

$$|h, \bar{h}\rangle = \phi(0,0)|0\rangle \tag{1.69}$$

created by the holomorphic field $\phi(z)$ of the weight h. From the OPE (1.51) between the energy-momentum tensor T and the primary field one finds:

$$[L_n, \phi(w, \bar{w})] = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w, \bar{w}) = h(n+1)w^n \phi(w, \bar{w}) + w^{n+1}\partial_w \phi(w, \bar{w}) \tag{1.70}$$

so that $[L_n, \phi(0,0)] = 0$, $n > 0$.

The anti-holomorphic counterpart of this equation is

$$[\bar{L}_n, \phi(w, \bar{w})] = \bar{h}(n+1)\bar{w}^n \phi(w, \bar{w}) + \bar{w}^{n+1}\partial_w \phi(w, \bar{w}) \tag{1.71}$$

Applying this relation to the state (1.69) we conclude:

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle \qquad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle \tag{1.72}$$

and

$$L_n|h, \bar{h}\rangle = 0 \qquad \bar{L}_n|h, \bar{h}\rangle = 0 \qquad n > 0 \tag{1.73}$$

The state satisfying (1.72) and (1.73) is known as a highest weight state.

Correlation functions

Since global conformal group $SL(2, C)$ preserves vacuum and anomaly free we have for $f(z)$ in the form (1.28):

$$\langle \Phi_1(z_1, \bar{z}_1) \ldots \Phi_n(z_n, \bar{z}_n)\rangle = \prod_j (\partial f(z_j))^{h_j} \left(\bar{\partial}\bar{f}(\bar{z}_j)\right)^{\bar{h}_j} \langle \Phi_1(f(z_1), \bar{f}(\bar{z}_1)) \ldots \Phi_n(f(z_n), \bar{f}(\bar{z}_n))\rangle \tag{1.74}$$

These equations completely fix the coordinate dependence of the two and three- point functions

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\rangle = \frac{C}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \tag{1.75}$$

where $h_1 = h_2 = h$, $\bar{h}_1 = \bar{h}_2 = \bar{h}$, and

$$\langle \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \Phi_3(z_3, \bar{z}_3) \rangle = \tag{1.76}$$

$$\frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

where $z_{ij} = z_i - z_j$.

1.1.4   CFT on torus and Modular transformation

Torus

The torus can be defined by specifying two linearly independent lattice vectors on a plane and identifying points that differ by the integer combination of these vectors. On a complex plane these lattice vectors can be represented by two complex numbers ω_1 and ω_2 which we shall call periods of lattice and hence we have

$$w \approx w + n\omega_1 + m\omega_2 \tag{1.77}$$

Clearly properties of the conformal field theories defined on the torus should not depend on overall scale of a lattice, nor on the absolute orientation of lattice vectors. The relevant parameter is a ratio $\tau = \omega_2/\omega_1$, called modular parameter.

Partition function on torus

CFT on the cylinder parameterized by w can now be transferred to the torus. Let H and P be energy and momentum operators, namely the operators that implement translations in the space and time directions $\mathrm{Re}w$ and $\mathrm{Im}w$ respectively. Remember that on a plane $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$ generate dilatations and rotations respectively, so according to discussion of the radial quantization one has $H = (L_0)_{\mathrm{cyl}} + (\bar{L}_0)_{\mathrm{cyl}}$ and $P = (L_0)_{\mathrm{cyl}} - (\bar{L}_0)_{\mathrm{cyl}}$. To define the torus we should identify two periods in w. Let us redefine $w \to iw$ and, as we discussed before, choose $w \equiv w + 2\pi$ and $w \equiv w + 2\pi\tau$. Denote by τ_1 and τ_2 real and imaginary parts of τ

$$\tau = \tau_1 + i\tau_2 \tag{1.78}$$

This implies that surfaces $\mathrm{Im}w = 2\pi\tau_2$ and $\mathrm{Im}w = 0$ should be identified after the shift by $\mathrm{Re}w \to \mathrm{Re}w + 2\pi\tau_1$. Because we define time translation of $\mathrm{Im}w$ by the period $2\pi\tau_2$ to be

accompanied by the spatial translation of $\mathrm{Re}w$ by $2\pi\tau_1$, the operator expression for the partition function of the theory on the torus with the modular parameter τ is

$$Z = \int e^{-S} = \mathrm{Tr}e^{2\pi i\tau_1 P}e^{-2\pi\tau_2 H} = \mathrm{Tr}e^{2\pi i\tau_1\left((L_0)_{\mathrm{cyl}}-(\bar{L}_0)_{\mathrm{cyl}}\right)}e^{-2\pi\tau_2\left((L_0)_{\mathrm{cyl}}+(\bar{L}_0)_{\mathrm{cyl}}\right)} \tag{1.79}$$

$$= \mathrm{Tr}e^{2\pi i\tau(L_0)_{\mathrm{cyl}}}e^{-2\pi i\bar{\tau}(\bar{L}_0)_{\mathrm{cyl}}} = \mathrm{Tr}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} = (q\bar{q})^{-\frac{c}{24}}\mathrm{Tr}q^{L_0}\bar{q}^{\bar{L}_0}$$

where $q = e^{2\pi i\tau}$.

Modular Invariance

Main point of studying conformal field theories on the torus is imposition of the constraints on operator content of a theory from requirement that the partition function should be independent of choice of periods ω_1 and ω_2 for the given torus.

Assume that ω_1' and ω_2' are two periods describing same lattice as ω_1 and ω_2. Since the points ω_1' and ω_2' belong to lattice, they should be written as integer combinations of ω_1 and ω_2:

$$\omega_1' = a\omega_1 + b\omega_2 \tag{1.80}$$

$$\omega_2' = c\omega_1 + d\omega_2$$

where $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$.

These transformations (1.80) form group $SL(2, \mathbb{Z})$.

Under the change of period (1.80) the modular parameter transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{1.81}$$

The generators of the transformations (1.81) are

$$T : \tau \to \tau + 1 \tag{1.82}$$

and

$$S : \tau \to -\frac{1}{\tau} \tag{1.83}$$

The Hilbert space of the conformal field theory has the form:

$$\mathcal{H} = \oplus_{i,\bar{i}}R_i(c) \otimes R_{\bar{i}}(c) \tag{1.84}$$

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$R_i(c)$ is the chiral algebra highest weight i representation. Hence defining the character

$$\chi_i(\tau) = \text{Tr}_{R_i} q^{L_0 - c/24} \tag{1.85}$$

one can write

$$Z(\tau) = \sum_{i, \bar{h}} N_{i,\bar{\imath}} \chi_i(\tau) \bar{\chi}_{\bar{\imath}}(\bar{\tau}) \tag{1.86}$$

where $N_{i,\bar{\imath}}$ denotes multiplicity of occurrence of $R_i(c) \otimes R_{\bar{\imath}}(c)$ in $\mathcal{H}$. The first obvious condition for the partition function to be modular invariant is that the characters $\chi_i(\tau)$ define a representation space of the modular transpositions:

$$\chi_i(\tilde{q}) = \sum_j S_{ij} \chi_j(q), \qquad \tilde{q} = \exp(-2\pi i/\tau) \tag{1.87}$$

$$\chi_i(\tau + 1) = \sum_j T_{ij} \chi_i(\tau) \tag{1.88}$$

It is easy to see that

$$T_{ij} = \delta_{ij} e^{2\pi i (h_i - c/24)} \tag{1.89}$$

where h_i is the conformal weight of the highest weight i. The matrix $N_{i,\bar{\imath}}$ in the partition function is determined by demanding modular invariance of the partition function of the model.

1.1.5 Orbifold model

In CFT the notion of orbifold acquires the following meaning. We start by taking a given modular invariant theory $\mathcal{T}$, whose Hilbert space possesses discrete symmetry G consistent with operator algebra of a theory, and constructing a modded-out theory $\mathcal{T}/G$ that is modular invariant as well.

Orbifold CFT's have the geometric interpretation as σ-models whose target space is the geometrical orbifold. But there are examples where the geometrical interpretation is non-existent. Therefore it is preferable to consider orbifold CFT's from the more abstract point of modding out the modular invariant theory by the Hilbert space symmetry. We will consider here the case of the abelian symmetry group G.

The construction of the orbifold CFT $\mathcal{T}/G$ starts with the Hilbert space projection on the G invariant states.

Therefore the first part of the partition function has the form:

$$Z_{\text{proj}} = |q|^{-c/12} \frac{1}{|G|} \text{Tr} \sum_{g \in G} g q^{L_0} \bar{q}^{\bar{L}_0} = \frac{1}{|G|} \sum_{g \in G} Z[0, g] \tag{1.90}$$

This means that we sum over all insertions of the operator realization of group element g in the trace over states, or alternatively this can be understood as twisting in the time direction. To have modular invariant partition function we should add contribution of the configurations twisted in the space direction which should be derived performing modular transformation $\tau \rightarrow -\frac{1}{\tau}$:

$$S : Z[0, g] \rightarrow Z[g, 0] \tag{1.91}$$

To obtain full modular invariant partition function we should perform projection also in twisted sectors to G invariant states and sum all of them:

$$Z_{\text{orb}} = |q|^{-c/12} \frac{1}{|G|} \sum_{g,h \in G} \text{Tr}_h g q^{L_0} \bar{q}^{\bar{L}_0} = \frac{1}{|G|} \sum_{g,h \in G} Z[h, g] \tag{1.92}$$

1.1.6 Structure constants and conformal bootstrap

Let us study the holomorphic part of the three-point function (1.76) in the limit $z_1 \rightarrow z_2$. The leading singularity is:

$$\langle 0|\Phi_i(z_1)\Phi_j(z_2)\Phi_k(z_3)|0\rangle = C_{ijk}(z_1 - z_2)^{h_3 - h_1 - h_2}(z_1 - z_3)^{-2h_3} \tag{1.93}$$

The last term resembles the propagator of the field Φ_3 and this expression assumes that the two primary fields Φ_i and Φ_j contain in their product the field Φ_3, with the strength C_{ijk}. The precise statement of this fact is the OPE , which states that the product of two operators $O_i(x)$ and $O_j(y)$ in field theory can be expanded in the complete set of operators $O_k(x)$

$$O_i(x)O_j(y) = \sum_k \mathcal{C}_{ijk}(x - y)O_k(x) \tag{1.94}$$

In CFT one can take as the basis all primaries and the complete set of descendants. Thus the OPE has the form [20, 21]:

$$\Phi_{(i\bar{i})}(z_1, \bar{z}_1)\Phi_{(j\bar{j})}(z_2, \bar{z}_2) = \sum_{k,\bar{k},a,\bar{a}} \frac{C^{(k\bar{k})}_{(i\bar{i})(j\bar{j})a\bar{a}}}{(z_1 - z_2)^{h_i + h_j - h_k}(\bar{z}_1 - \bar{z}_2)^{h_{\bar{i}} + h_{\bar{j}} - h_{\bar{k}}}}\Phi_{(k\bar{k})}(z_2, \bar{z}_2) + \text{descendants}.$$

(1.95)

Let us explain notations used in this formula. Each primary has two indices referring to the left i and right $\bar{i}$ highest weight representations. Note that in general the primary field $\Phi_{(k\bar{k})}$ may appear more than one time in the OPE of the fields $\Phi_{(i\bar{i})}$ and $\Phi_{(j\bar{j})}$. In this case we have different channels of the fusion of the fields $\Phi_{(i\bar{i})}$ and $\Phi_{(j\bar{j})}$ producing the field $\Phi_{(k\bar{k})}$. The number of the different channels is called fusion number and usually denoted as N_{ij}^k. To take them into account the structure constants are provided with additional indices $a = 1 \ldots N_{ij}^k$ and $\bar{a} = 1 \ldots N_{\bar{i}\bar{j}}^{\bar{k}}$. We denote by $i = 0$ the vacuum representation having the property $N_{i0}^k = \delta_{ik}$, and by i^* the conjugate representation in a sense $N_{ii^*}^0 = 1$. Denote by $\mathcal{R}$ the set of all primary fields of the theory, or by other words, the set of values of indices i, j, k in (1.95). If the OPE algebra is closed with the finite set $\mathcal{R}$ the theory is called rational. The name is due to fact that for rational theories the conformal weights h_i take rational values [181].

The structure constants satisfy the bootstrap equation [21].

To derive this constraint one consider the four-point correlation function $\langle \Phi_{i\bar{i}}\Phi_{k\bar{k}}\Phi_{j\bar{j}}\Phi_{l\bar{l}} \rangle$. It can be computed in two ways, so called s and t channels. In s channel we use at the beginning the OPE of the fields $\Phi_{j\bar{j}}$ and $\Phi_{l\bar{l}}$, producing in the fusion the field $\Phi_{p\bar{p}}$, and afterwards computing the three-point function $\langle \Phi_{i\bar{i}}\Phi_{k\bar{k}}\Phi_{p\bar{p}} \rangle$. This procedure brings to the expression

$$\sum_{p\bar{p}} \sum_{\rho\tau\bar{\rho}\bar{\tau}} C^{p\bar{p}}_{j\bar{j}l\bar{l}(\tau\bar{\tau})} C^{i\bar{i}}_{k\bar{k}p\bar{p}(\rho\bar{\rho})} \mathcal{F}^s_{p\rho\tau}\begin{bmatrix} k & j \\ i & l \end{bmatrix} \mathcal{F}^s_{\bar{p}\bar{\rho}\bar{\tau}}\begin{bmatrix} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{bmatrix} \qquad (1.96)$$

The function $\mathcal{F}^s_{p\rho\tau}\begin{bmatrix} k & j \\ i & l \end{bmatrix}$ is so called s channel conformal block giving contribution of descendant fields. The conformal blocks carry four indices of the in- and out- fields, the index p of the intermediate field, and two indices $\rho = 1 \ldots N_{kp}^i$, $\tau = 1 \ldots N_{jl}^p$ to disentangle different fusion channels. Note the order of indices of in- and out- fields in brackets. The lower left index l fused with upper left index j, producing intermediate state p, which fused lower and upper

right indices.

In t channel we use at the beginning the OPE of the fields $\Phi_{j\bar{j}}$ and $\Phi_{k\bar{k}}$, producing in the fusion the field $\Phi_{q\bar{q}}$, and afterwards computing the three-point function $\langle\Phi_{i\bar{i}}\Phi_{q\bar{q}}\Phi_{l\bar{l}}\rangle$. This procedure brings to the expression

$$\sum_{q\bar{q}}\sum_{\mu\nu\bar{\mu}\bar{\nu}}C^{q\bar{q}}_{k\bar{k}j\bar{j}(\mu\bar{\mu})}C^{i\bar{i}}_{q\bar{q}l\bar{l}(\nu\bar{\nu})}\mathcal{F}^t_{q\nu\mu}\begin{bmatrix}l & j \\ i & k\end{bmatrix}\mathcal{F}^t_{\bar{q}\bar{\nu}\bar{\mu}}\begin{bmatrix}\bar{l} & \bar{j} \\ \bar{i} & \bar{k}\end{bmatrix},\tag{1.97}$$

The t channel conformal blocks as well carry additional indices $\mu = 1\ldots N^q_{kj}$ $\nu = 1\ldots N^i_{ql}$ to disentangle different fusion channels. Conformal blocks in s and t channels are related by the fusing matrix

$$\mathcal{F}^s_{p\rho\tau}\begin{bmatrix}k & j \\ i & l\end{bmatrix} = \sum_q\sum_{\nu\mu}F_{p,q}\begin{bmatrix}k & j \\ i & l\end{bmatrix}^{\nu\mu}_{\rho\tau}\mathcal{F}^t_{q\nu\mu}\begin{bmatrix}l & j \\ i & k\end{bmatrix},\tag{1.98}$$

and hence we obtain famous bootstrap constraint relating fusing matrix and structure constants:

$$\sum_{p\bar{p}}\sum_{\rho\tau\bar{\rho}\bar{\tau}}C^{p\bar{p}}_{j\bar{j}l\bar{l}(\tau\bar{\tau})}C^{i\bar{i}}_{k\bar{k}p\bar{p}(\rho\bar{\rho})}F_{p,q}\begin{bmatrix}k & j \\ i & l\end{bmatrix}^{\nu\mu}_{\rho\tau}F_{\bar{p},\bar{q}}\begin{bmatrix}\bar{k} & \bar{j} \\ \bar{i} & \bar{l}\end{bmatrix}^{\bar{\nu}\bar{\mu}}_{\bar{\rho}\bar{\tau}} = C^{q\bar{q}}_{k\bar{k}j\bar{j}(\mu\bar{\mu})}C^{i\bar{i}}_{q\bar{q}l\bar{l}(\nu\bar{\nu})}.\tag{1.99}$$

Using the relation [20]

$$\sum_{\bar{q},\bar{\nu},\bar{\mu}}F_{\bar{p},\bar{q}^*}\begin{bmatrix}\bar{k} & \bar{j} \\ \bar{i} & \bar{l}\end{bmatrix}^{\bar{\nu}\bar{\mu}}_{\bar{\rho}\bar{\tau}}F_{\bar{q},s}\begin{bmatrix}\bar{j} & \bar{l} \\ \bar{k}^* & \bar{i}^*\end{bmatrix}^{\gamma_1\gamma_2}_{\bar{\mu}\bar{\nu}} = \delta_{\bar{p}s}\delta_{\bar{\rho}\gamma_1}\delta_{\bar{\tau}\gamma_2},\tag{1.100}$$

one can write the bootstrap equation (1.99) in the form:

$$\sum_p\sum_{\rho\tau}C^{p\bar{p}}_{j\bar{j}l\bar{l}(\tau\bar{\tau})}C^{i\bar{i}}_{k\bar{k}p\bar{p}(\rho\bar{\rho})}F_{p,q}\begin{bmatrix}k & j \\ i & l\end{bmatrix}^{\nu\mu}_{\rho\tau} = \sum_{\bar{q},\bar{\mu},\bar{\nu}}C^{q\bar{q}}_{k\bar{k}j\bar{j}(\mu\bar{\mu})}C^{i\bar{i}}_{q\bar{q}l\bar{l}(\nu\bar{\nu})}F_{\bar{q}^*,\bar{p}}\begin{bmatrix}\bar{j} & \bar{l} \\ \bar{k}^* & \bar{i}^*\end{bmatrix}^{\bar{\rho}\bar{\tau}}_{\bar{\mu}\bar{\nu}}.\tag{1.101}$$

In this work we will analyze the models with diagonal partition function

$$Z = \sum_{i,\bar{i}}Z_{i,\bar{i}}\chi_i(q)\bar{\chi}_{\bar{i}}(\bar{q}),\quad Z_{i,\bar{i}} = \delta_{i,i^*},\quad q = \exp(2i\pi\tau)\tag{1.102}$$

These models called diagonal. Diagonal models satisfy the relation:

$$C^{p\bar{p}}_{k\bar{k}i\bar{i}(\rho\bar{\rho})} = C^p_{ki(\rho\bar{\rho})}\delta_{\bar{p}p^*}\delta_{\bar{k}k^*}\delta_{\bar{i}i^*}\tag{1.103}$$

Now we will show that for the diagonal model one can solve the bootstrap equation to get the explicit expression for structure constant via fusion matrix.

For diagonal model eq. (1.101) takes the form:

$$\sum_{\rho\tau} C^i_{kp(\rho\bar\rho)} C^p_{jl(\tau\bar\tau)} F_{p,q} \begin{bmatrix} k & j \\ i & l \end{bmatrix}^{\nu\mu}_{\rho\tau} = \sum_{\bar\mu\bar\nu} C^q_{kj(\mu\bar\mu)} C^i_{ql(\nu\bar\nu)} F_{q,p} \begin{bmatrix} k^* & i \\ j & l^* \end{bmatrix}^{\bar\tau\bar\rho}_{\bar\mu\bar\nu} . \tag{1.104}$$

It is shown in [168] that the pentagon equation for fusing matrix [20, 129, 132] implies the following important relation:

$$\sum_{\rho,\tau} F_{0,i} \begin{bmatrix} p & k \\ p & k^* \end{bmatrix}^{\bar\rho\rho}_{00} F_{p,q} \begin{bmatrix} k & j \\ i & l \end{bmatrix}^{\nu\mu}_{\rho\tau} F_{0,p} \begin{bmatrix} l & j \\ l & j^* \end{bmatrix}^{\bar\tau\tau}_{00} = \tag{1.105}$$

$$\sum_{\bar\mu,\bar\nu} F_{0,q} \begin{bmatrix} j & k \\ j & k^* \end{bmatrix}^{\bar\mu\mu}_{00} F_{q,p} \begin{bmatrix} k^* & i \\ j & l^* \end{bmatrix}^{\bar\tau\bar\rho}_{\bar\mu\bar\nu} F_{0,i} \begin{bmatrix} q & l \\ q & l^* \end{bmatrix}^{\bar\nu\nu}_{00} .$$

Comparing (1.105) and (1.104) we see that (1.104) can be solved by an ansatz

$$C^p_{ij(\mu\bar\mu)} = \frac{\eta_i \eta_j}{\eta_0 \eta_p} F_{0,p} \begin{bmatrix} j & i \\ j & i^* \end{bmatrix}^{\bar\mu\mu}_{00} \tag{1.106}$$

with arbitrary η_i. To find η_i we set $p = 0$

$$C^0_{ii^*} = \frac{\eta_i \eta_{i^*}}{\eta_0^2} F_i , \tag{1.107}$$

where

$$F_i \equiv F_{0,0} \begin{bmatrix} i & i^* \\ i & i \end{bmatrix} . \tag{1.108}$$

Using

$$C^0_{ii^*} = \frac{C_{ii^*}}{C_{00}} , \tag{1.109}$$

where C_{ii^*} are two-point functions and that $F_0 = 1$ one can solve (1.107) setting

$$\eta_i = \epsilon_i \sqrt{C_{ii^*}/F_i} , \tag{1.110}$$

were ϵ_i a sign factor. We assume that ϵ_i can be chosen to satisfy $\epsilon_i = \epsilon_{i^*}$.

For diagonal models without multiplicities, i.e. all $N_{ij}^k = 0, 1$, we can derive the relation (1.106) in the different way. For these models, we don't need the Greek indices in the structure constants, and fusion matrix to disentangle fusion channels, and the bootstrap equation (1.104) takes the form

$$
C_{ki^*}^{p^*} C_{jl}^p C_{pp^*}^0 F_{p,q} \begin{bmatrix} k & j \\ i & l \end{bmatrix} = C_{kj}^q C_{i^*l}^{q^*} C_{qq^*}^0 F_{q,p} \begin{bmatrix} k^* & i \\ j & l^* \end{bmatrix}. \tag{1.111}
$$

Setting $q = 0$, $k = j^*$, $i = l$ in (1.111) we obtain:

$$
\left(C_{ij}^p \right)^2 = \frac{C_{jj^*} C_{ii^*} F_{0,p} \begin{bmatrix} j & i \\ j & i^* \end{bmatrix}}{C_{00} C_{pp^*} F_{p,0} \begin{bmatrix} j^* & j \\ i & i \end{bmatrix}}. \tag{1.112}
$$

Using the relation [129, 132]

$$
F_{0,i} \begin{bmatrix} j & k \\ j & k^* \end{bmatrix} F_{i,0} \begin{bmatrix} k^* & k \\ j & j \end{bmatrix} = \frac{F_j F_k}{F_i}, \tag{1.113}
$$

we can write (1.112) in two forms

$$
C_{ij}^p = \frac{\eta_i \eta_j}{\eta_0 \eta_p} F_{0,p} \begin{bmatrix} j & i \\ j & i^* \end{bmatrix}, \tag{1.114}
$$

and

$$
C_{ij}^p = \frac{\xi_i \xi_j}{\xi_0 \xi_p} \frac{1}{F_{p,0} \begin{bmatrix} j^* & j \\ i & i \end{bmatrix}}, \tag{1.115}
$$

where η_i is defined in (1.110) and

$$
\xi_i = \eta_i F_i = \epsilon_i \sqrt{C_{ii^*} F_i}. \tag{1.116}
$$

Eq. (1.112) determines (1.114) and (1.115) only up to sign, but comparison with (1.106) shows that the sign ambiguity can be absorbed in factors ϵ_i.

In rational conformal field theory one has the relation

$$F_k = \frac{S_{00}}{S_{0k}},$$

(1.117)

and two-points functions can be normalized to 1. Therefore in rational conformal field theory

$$\xi_k = \frac{\sqrt{S_{00}}}{\sqrt{S_{0k}}}.$$

(1.118)

Dimension of the space of conformal blocks

It is easy to see that the number of conformal blocks, which can be derived in the process of the fusion via all the different channels and intermediate states is

$$\mathcal{N}^{ikjl} = \sum_p N^p_{jl} N^i_{pk}$$

(1.119)

It is possible to show that the number of conformal blocks is the same in s or t channel, namely

$$\sum_p N^p_{jl} N^i_{pk} = \sum_q N^q_{kj} N^i_{ql}$$

(1.120)

The notion of conformal blocks can be generalized to n-point conformal blocks. We should start with n-point function. Repeatedly using OPE we can as before to write the n-point function as product of structure constants and n-point holomorphic and anti-holomorphic conformal blocks. Schematically this can be written as

$$\langle \Phi_{\kappa_1 \bar{\kappa}_1}(z_1, \bar{z}_1) \cdots \Phi_{\kappa_n \bar{\kappa}_n}(z_n, \bar{z}_n) \rangle =$$

(1.121)

$$\sum_{\mu_1, \ldots \mu_{n-3}} \sum_{\bar{\mu}_1, \ldots \bar{\mu}_{n-3}} h_{\mu_1, \ldots \mu_{n-3}; \bar{\mu}_1, \ldots \bar{\mu}_{n-3}} \mathcal{F}^{\kappa_1 \ldots \kappa_n}_{\mu_1, \ldots \mu_{n-3}}(z_1, \cdots z_n) \mathcal{F}^{\bar{\kappa}_1 \ldots \bar{\kappa}_n}_{\bar{\mu}_1, \ldots \bar{\mu}_{n-3}}(\bar{z}_1, \cdots \bar{z}_n)$$

where $h_{\mu_1, \ldots \mu_{n-3}; \bar{\mu}_1, \ldots \bar{\mu}_{n-3}}$ is built out of the structure constants. Again counting all different fusion channels and intermediate states leads to the following formula for dimension of the space of conformal blocks

$$\mathcal{N}^{\kappa_1 \ldots \kappa_n} = \sum_{\mu_1, \ldots \mu_{n-3}} N^{\mu_1}_{\kappa_1 \kappa_2} N^{\mu_2}_{\mu_1 \kappa_3} \cdots N^{\kappa_n}_{\mu_{n-3} \kappa_{n-1}}$$

(1.122)

The fusion coefficients are related to the matrix of modular transformation by the famous Verlinde formula [183]:

$$N^k_{ij} = \sum_l \frac{S_{il} S_{jl} S^*_{lk}}{S_{0l}}$$

(1.123)

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1.2 Boundary rational conformal field theory

In this section we review general aspects on conformal field theory on a world-sheet with boundary [42, 43].

1.2.1 Cardy Condition

Let us consider a conformal field theory on the $\sigma - \tau$ strip, $0 \leq \sigma \leq \pi$, periodic in the τ-direction with a period T. The manifold is an annulus with the modular parameter $q \equiv \exp(-2\pi i T)$. Given certain boundary conditions on the boundaries of the annulus, labelled α and β, the partition function is:

$$Z_{\alpha\beta} = \text{Tr} \exp(-2\pi i T H_{\alpha\beta}) \, , \tag{1.124}$$

where $H_{\alpha\beta}$ is the Hamiltonian corresponding to these boundary conditions. This is the open-string loop channel.

For rational CFT the condition (1.124) can be elaborated further. The eigenstates of $H_{\alpha\beta}$ can be organized into highest weight representations R_i^o of the algebra $\mathcal{A}_{\alpha\beta}$ preserved by the pair of the boundary conditions α and β. These representations R_i^o (superscipt o refers to open) will be labelled by an index i whose specification includes the L_0-eigenvalue of the highest weight state. We then define the non-negative integer $n_{\alpha\beta}^i$ to be the number of times that the representation R_i^o occurs in the spectrum of $H_{\alpha\beta}$. The partition function in the open string channel (1.124) is then

$$Z_{\alpha\beta} = \text{Tr} \exp(-2\pi i T H_{\alpha\beta}) = \sum_i n_{\alpha\beta}^i \chi_i(q) \tag{1.125}$$

where $\chi_i(q)$ is the character of the representation R_i^o.

One may also calculate the partition function using the Hamiltonian acting in the σ-direction . This will be the Hamiltonian $H^{(P)}$ for the cylinder, which is related by the exponential mapping $\zeta = \exp(-i(t + i\sigma))$ to the Virasoro generators in the whole ζ-plane by $H^{(P)} = L_0^{(P)} + \overline{L}_0^{(P)} - c/12$, where we have used the superscript to stress that they are not the same as the generators of the boundary Virasoro algebra. To every boundary condition α,

there corresponds a particular boundary state $|\alpha\rangle$ in the Hilbert space of the closed strings; this enables us to compute the partition function by the following formula:

$$Z_{\alpha\beta} = \langle\alpha|\exp(-\pi i H^{(P)}/T)|\beta\rangle = \langle\alpha|(\tilde{q}^{1/2})^{L_0^{(P)}+\overline{L_0}^{(P)}-c/12}|\beta\rangle \ , \tag{1.126}$$

where $\tilde{q} \equiv e^{-2\pi i/T}$.

This is the closed-string tree channel.

The equality of (1.126) with (1.125) for some set of non-negative integers $n^i_{\alpha\beta}$ is known as Cardy condition for boundary states. This condition promises that a boundary state $|\alpha\rangle$ in the closed string Hilbert space can be interpreted as open string boundary condition α on fields in question.

1.2.2 Maximally-Symmetric Cardy state

Suppose we have a rational diagonal theory with extended holomorphic chiral algebra $\mathcal{A}_L$ containing besides tensor energy-momentum T the set of conserved currents $W^{(r)}$, and similarly the antiholomorphic algebra $\mathcal{A}_R$ with components $\bar{T}$ and $\bar{W}^{(r)}$. Denote as before the representation of the chiral algebra R_i and characters χ_i. We denote by $A^{(m)}$ the set of all generators: $A^{(m)} = \{T, W^{(r)}\}$ and similarly for the right part: $\bar{A}^{(m)} = \{\bar{T}, \bar{W}^{(r)}\}$. The maximally symmetric boundary conditions impose constrints for all the generators and can be chosen as:

$$T(z) = \bar{T}(\bar{z})|_{z=\bar{z}} \qquad W^{(r)}(z) = \overline{W^{(r)}}(\bar{z})|_{z=\bar{z}} \tag{1.127}$$

The first of these conditions according to (1.35) has the direct physical meaning of the absence of energy-momentum flow across the boundary $T_{\sigma\tau} = 0$.

Imposing at the both ends of the strip the boundary condition (1.127) preserves the diagonal subalgebra isomorphic to $\mathcal{A}$ of the full algebra $\mathcal{A}_L \times \mathcal{A}_R$. Therefore the open string Hilbert space will be organised as sum of the representations R_i of $\mathcal{A}$.

Corresponding boundary states should satisfy

$$\left(A_n^{(m)} - (-)^{h_A}\overline{A}_{-n}^{(m)}\right)|\alpha\rangle = 0 \tag{1.128}$$

Define the anti-unitary operator U acting in the way

$$U\overline{A}_{-n}^{(m)} = (-)^{h_A}\overline{A}_{-n}^{(m)}U \tag{1.129}$$

Using $|j, N\rangle$, $N \in \mathbb{N}$, to denote an orthonormal basis of R_i, one can define Ishibashi states:

$$|j\rangle\rangle = \sum_{N=0}^{\infty} |j, N\rangle \otimes U\overline{|j, N\rangle} \tag{1.130}$$

It is shown in [42] that the Ishibashi states are solutions of (1.128). and satisfy

$$\langle\langle j|\tilde{q}^{L_0 - c/24}|i\rangle\rangle = \delta_{i,j}\chi_i(\tilde{q}) \tag{1.131}$$

The boundary states are linear combinations of the Ishibashi states:

$$|\alpha\rangle = \sum B_\alpha^i |i\rangle\rangle \tag{1.132}$$

Inserting expansions (1.132) in the expression (1.126) for the partition function in the closed string channel we obtain:

$$Z_{\alpha\beta} = \sum_i (B_\alpha^i)^* B_\beta^i \chi_i(\tilde{q}) \tag{1.133}$$

Performing modular transformation we get for partition function in the open string channel:

$$Z_{\alpha\beta} = \sum_{i,j} (B_\alpha^i)^* B_\beta^i S_{ij} \chi_j(q) \tag{1.134}$$

Equating (1.125) and (1.134) we derive

$$\sum_i (B_\alpha^i)^* B_\beta^i S_{ij} = n_{\alpha\beta}^j \tag{1.135}$$

The eq. (1.135) was soved by Cardy in [42].

In Cardy's solution, the index α takes the values in the same set $\mathcal{R}$ as the index i of the irreducible highest weight states and

$$B_\alpha^i = \frac{S_{\alpha i}}{\sqrt{S_{0i}}}, \tag{1.136}$$

where S_{ij} is the matrix of the modular transformations. Inserting (1.136) in (1.135) and using the Verlinde formula (1.123) we obtain that $n_{\alpha\beta}^j$ are integer and coincide with the fusion numbers:

$$n_{\alpha\beta}^j = N_{\alpha\beta}^j \tag{1.137}$$

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and

$$Z_{\alpha\beta} = \sum_j N_{\alpha\beta}^j \chi_j(q) \tag{1.138}$$

We proved that in every rational conformal field theory we have at least as much maximally-symmetric boundary states, as we have irreducible highest weight representations. The boundary states $|\alpha\rangle$ carry the same labels as the irreducible representations, and their expansion into Ishibashi states is

$$|\alpha\rangle = \sum_i \frac{S_{\alpha i}}{\sqrt{S_{0i}}} |i\rangle\rangle \tag{1.139}$$

Formula (1.139) describes the famous Cardy states.

The second part of the condition (1.127) may be generalized to incorporate a possible "gluing automorphism" Ω

$$W(z) = \Omega \overline{W}(\bar{z})|_{z=\bar{z}} \tag{1.140}$$

The corresponding boundary state $|\alpha\rangle_\Omega$ satisfies the conditions

$$\left(L_n - \bar{L}_{-n} \right) |\alpha\rangle_\Omega = 0 \qquad \text{and} \qquad \left(W_n^{(r)} - (-)^{h_W} \Omega(\overline{W}_{-n}^{(r)}) \right) |\alpha\rangle_\Omega = 0 \tag{1.141}$$

The state $|\alpha\rangle_\Omega$ is given by a linear combination of twisted Ishibashi states $|i\rangle\rangle_\Omega$:

$$|i\rangle\rangle_\Omega = (\mathrm{Id} \otimes V_\Omega)|i\rangle\rangle \tag{1.142}$$

where V_Ω is the representation of Ω on Hilbert space.

1.2.3 Cardy-Lewellen cluster condition

The coefficients of the expansion of the boundary states into the Ishibashi states should satisfy also the boundary version of the bootstrap constraint. This condition was derived in [43, 118] and known as Cardy-Lewellen cluster condition. Note that if the Cardy condition is true only in rational theories, the Cardy-Lewellen condition should be fulfilled in all theories, including non-rational. Now we will explain how Cardy-Lewellen cluster condition is obtained.

Consider a boundary state

$$|\alpha\rangle = \sum_i B_\alpha^i |i\rangle\rangle \tag{1.143}$$

where i runs over primaries, and $|i\rangle\rangle$ are Ishibashi states. Recall the relation [20, 43] between coefficients B_α^i and one-point functions

$$\langle \Phi_{(i\bar{i})}(z, \bar{z})\rangle_\alpha = \frac{U_\alpha^i \delta_{i*\bar{i}}}{(z - \bar{z})^{2h_i}} \tag{1.144}$$

in the presence of the boundary condition α:

$$U_\alpha^i = \frac{B_\alpha^i}{B_\alpha^0} e^{i\pi h_i} \tag{1.145}$$

Considering now two-point function $\langle \Phi_i(z_1, \bar{z}_1)\Phi_j(z_2, \bar{z}_2)\rangle_\alpha$ in the presence of boundary in two pictures it was shown in [43,118] that the one-point functions U_α^i in the presence of boundary satisfy the equation

$$\sum_{k,a,\bar{a}} C_{(ii^*)(jj^*)a\bar{a}}^{(k,k^*)} U_\alpha^k B_{k^*0}^{(+)} \begin{bmatrix} j & i^* \\ i^* & j^* \end{bmatrix}_{a\bar{a}}^{11} = U_{(\alpha)}^i U_{(\alpha)}^j \tag{1.146}$$

We should note that here we used reflection amplitudes as they defined in [20]. The traditionally used reflection amplitudes [43, 118] differ by phase

$$U_{(\alpha)}^i = \tilde{U}_{(\alpha)}^i e^{i\pi h_i} \tag{1.147}$$

They have the advantage, that related to boundary states coefficients without phase factor:

$$\tilde{U}_{(\alpha)}^i = \frac{B_\alpha^i}{B_\alpha^0} \tag{1.148}$$

Recalling relation between braiding and fusion matrices:

$$B_{pq}^{(+)} \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{ab}^{cd} = e^{i\pi(\Delta_k + \Delta_l - \Delta_p - \Delta_q)} F_{pq} \begin{bmatrix} i & l \\ k & j \end{bmatrix}_{ab}^{cd} \tag{1.149}$$

and symmetry properties of fusion matrix [20]

$$F_{pq} \begin{bmatrix} k & j \\ i & l \end{bmatrix}_{ab}^{cd} = F_{p^*q^*} \begin{bmatrix} l & i^* \\ j^* & k \end{bmatrix}_{ab}^{cd} \tag{1.150}$$

we receive that $\tilde{U}_{(\alpha)}^i$ obey the equation:

$$\sum_{k,a,\bar{a}} C_{(ii^*)(jj^*)a\bar{a}}^{(k,k^*)} \tilde{U}_\alpha^k F_{k0} \begin{bmatrix} i^* & i \\ j & j \end{bmatrix}_{a\bar{a}}^{11} = \tilde{U}_{(\alpha)}^i \tilde{U}_{(\alpha)}^j \tag{1.151}$$

This is Cardy-Lewellen cluster equation.

For diagonal models eq.(1.151) can be significantly simplified. Putting (1.106) in (1.151), and using formulas [20]

$$
\sum_{t_2} F_{0,l} \begin{bmatrix} a & b^* \\ a & b \end{bmatrix}_{00}^{\gamma_1 t_2} F_{b^*,0} \begin{bmatrix} a & a^* \\ l & l \end{bmatrix}_{t_2 \alpha_3}^{00} = F_{0,0} \begin{bmatrix} a & a^* \\ a & a \end{bmatrix}_{00}^{00} \delta_{\gamma_1,\alpha_3} \equiv F_a \delta_{\gamma_1,\alpha_3} \, . \tag{1.152}
$$

and

$$
F_i F_{0,i^*} \begin{bmatrix} j & k \\ j & k^* \end{bmatrix}_{00}^{\nu u_3} = F_{k^*} F_{0,k^*} \begin{bmatrix} i & j \\ i & j^* \end{bmatrix}_{00}^{\nu u_3} \, . \tag{1.153}
$$

to perform the sums by a and $\bar{a}$, we obtain

$$
\sum_k \tilde{U}^k N_{ij}^k \frac{\xi_i \xi_j}{\xi_0 \xi_k} = \tilde{U}^i \tilde{U}^j \, , \tag{1.154}
$$

where N_{ij}^k are the fusion coefficients. Defining

$$
\tilde{U}^k = \Psi^k \frac{\xi_k}{\xi_0} \tag{1.155}
$$

one can write (1.154) in the form:

$$
\sum_k \Psi^k N_{ij}^k = \Psi^i \Psi^j \, . \tag{1.156}
$$

In rational conformal field theory eq. (1.156) is solved by

$$
\Psi_a^k = \frac{S_{ak}}{S_{0a}} \, . \tag{1.157}
$$

Taking into account the relation between one-point functions $\tilde{U}^k$ and coefficients of the boundary state B^k (1.148), and using (1.118), we obtain that the Cardy solution (1.136) indeed satisfy the Cardy-Lewellen constraint.

1.2.4 Permutation branes

Consider N-fold tensor product of a CFT with chiral symmetry algebra $A_L(A_R)$.

On such a product one can consider brane with gluing automorphism given by a cycle $(1 \ldots N)$, or by other words, satisfying following equations:

$$A_L^{(r)}(z) = A_R^{(r+1)}(\bar{z})|_{z=\bar{z}}, \quad r = 1 \ldots N-1 \tag{1.158}$$

$$A_L^{(N)}(z) = A_R^{(1)}(\bar{z})|_{z=\bar{z}}$$

For diagonal rational conformal field theories permutation branes were constructed in [151]. It is shown in [151] that for such a CFT permutation branes are labeled by primaries of single copy and have boundary states:

$$|a\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{(S_{0j})^{N/2}} |j, j\rangle\rangle_{\mathcal{P}} \tag{1.159}$$

where S_{ij} is the matrix of the modular transformations of single copy, and $|j, j\rangle\rangle_{\mathcal{P}}$ permuted Ishibashi state satisfying (1.158). As we have discussed the boundary states should satisfy two criteria: Cardy condition [42], requiring the annulus partition functions to be expressed as sum of some characters with non-negative integer numbers, and Cardy-Lewellen cluster condition [43, 118]. It is shown in [151] that states (1.159) indeed satisfy the both criteria.

For further use we write down the Cardy and Cardy-Lewellen conditions in detail in case of two-fold product $N = 2$. Generalization to generic N is straightforward and corresponding formulae can be found in [151]. For two-fold product permutation boundary state (1.159) satisfies the relations:

$$L_n^{(1)} - \bar{L}_{-n}^{(2)} = 0, \quad W_n^{(1)} - (-1)^{hW} \bar{W}_{-n}^{(2)} = 0 \tag{1.160}$$

$$L_n^{(2)} - \bar{L}_{-n}^{(1)} = 0, \quad W_n^{(2)} - (-1)^{hW} \bar{W}_{-n}^{(1)} = 0$$

and takes the form:

$$|a\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{S_{0j}} |j, j\rangle\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{S_{0j}} \sum_{N,M} |j, N\rangle_0 \otimes U\overline{|j, N\rangle}_1 \otimes |j, M\rangle_1 \otimes U\overline{|j, M\rangle}_0. \tag{1.161}$$

where 0 and 1 labels first and second copy of the CFT in question, sums over N and M run over orthonormal basis of the highest weight representation R_j, and operator U in front of right-movers is chiral CPT operator as usual.

One can show that partition function between permutation branes is:

$$Z_{a_1,a_2} = \sum_{j,k,l} \frac{S_{a_1j}}{S_{0j}} \frac{S_{a_2j}}{S_{0j}} S_{jk} S_{jl} \chi_k(q) \chi_l(q) = \sum_{r,k,l} N^r_{a_1a_2} N^k_{rl} \chi_k(q) \chi_l(q) \, . \tag{1.162}$$

what has indeed the required form.

Note that integers in front of product of characters coincide with the dimension of the space of four-point conformal blocks given by (1.119). The generalization of (1.162) to N-fold product is straightforward. It is shown in [151] that the annulus partition function between two permutation branes corresponding to single copy primaries a_1 ad a_2 on N-fold product is

$$Z_{a_1,a_2} = \sum_{i_1,\ldots,i_N} \mathcal{N}^{a_1,a_2,i_1,\ldots,i_N} \chi_{i_1}(q) \cdots \chi_{i_N}(q) \tag{1.163}$$

where $\mathcal{N}^{a_1,a_2,i_1,\ldots,i_N}$ is dimension of the space of $N + 2$-point conformal blocks given by (1.122).

1.2.5 Cardy-Lewellen condition for permutation branes

The Cardy-Lewellen cluster condition for permutation branes was elaborated in detail in [164].

The primary fields of two-fold product are products of primary fields $\Phi^{(1)}_i \Phi^{(2)}_j$. The form of the gluing relations (1.160) implies that for permutation branes two-point functions have the form:

$$\langle \Phi^{(1)}_{(i\bar{i})}(z_1) \Phi^{(2)}_{(j\bar{j})}(z_2) \rangle_{\mathcal{P}} = \frac{U^{i,\bar{i}}_{(\mathcal{P})} \delta_{i\bar{j}^*} \delta_{\bar{i}j^*}}{(z_1 - \bar{z}_2)^{2h_i}(\bar{z}_1 - z_2)^{2h_{\bar{i}}}} \tag{1.164}$$

The cluster condition for permutation branes was obtained in [151, 164]:

$$\sum_{k,\bar{k},a,\bar{a},c,\bar{c}} C^{(k,\bar{k})}_{(i_1\bar{i}_1)(j_1\bar{j}_1)a\bar{a}} C^{(\bar{k}^*,k^*)}_{(\bar{i}_1^*i_1^*)(\bar{j}_1^*j_1^*)c\bar{c}} B^{(+)}_{k^*0} \begin{bmatrix} j_1 & i_1^* \\ i_1^* & j_1^* \end{bmatrix}^{11}_{a\bar{c}} B^{(+)}_{\bar{k}0} \begin{bmatrix} \bar{j}_1^* & \bar{i}_1 \\ \bar{i}_1 & \bar{j}_1 \end{bmatrix}^{11}_{c\bar{a}} U^{k,\bar{k}}_{(\mathcal{P})} = \tag{1.165}$$

$$U^{i_1,\bar{i}_1}_{(\mathcal{P})} U^{j_1,\bar{j}_1}_{(\mathcal{P})}$$

Again defining new amplitudes

$$\tilde{U}^{i_1,\bar{i}_1}_{(\mathcal{P})} = U^{i_1,\bar{i}_1}_{(\mathcal{P})} e^{i\pi(h_i+h_{\bar{i}})} \tag{1.166}$$

and using (1.149) and (1.150) we derive:

$$\sum_{k,\bar{k},a,\bar{a},c,\bar{c}} C^{(k,\bar{k})}_{(i_1\bar{i}_1)(j_1\bar{j}_1)a\bar{a}} C^{(\bar{k}^*,k^*)}_{(\bar{i}_1^*i_1^*)(\bar{j}_1^*j_1^*)c\bar{c}} F_{k0} \begin{bmatrix} i_1^* & i_1 \\ j_1^* & j_1 \end{bmatrix}^{11}_{a\bar{c}} F_{\bar{k}^*0} \begin{bmatrix} \bar{i}_1 & \bar{i}_1^* \\ \bar{j}_1^* & \bar{j}_1^* \end{bmatrix}^{11}_{c\bar{a}} \tilde{U}^{k,\bar{k}}_{(\mathcal{P})} = \tag{1.167}$$
$$\tilde{U}^{i_1,\bar{i}_1}_{(\mathcal{P})} \tilde{U}^{j_1,\bar{j}_1}_{(\mathcal{P})}$$

This is Cardy-Lewellen constraint for permutation branes.

One can check, that for diagonal models, by use of eq. (1.106), (1.152) and (1.153), the Cardy-Lewellen condition (1.167) simplifies to

$$\sum_k U^k_{(2)\mathcal{P}} N^k_{ij} \left(\frac{\xi_i \xi_j}{\xi_0 \xi_k} \right)^2 = U^i_{(2)\mathcal{P}} U^j_{(2)\mathcal{P}} . \tag{1.168}$$

where we set $U^k_{(2)\mathcal{P}} \equiv \tilde{U}^{k,k^*}_{(\mathcal{P})}$.

Eq. (1.168) can be solved by the relation

$$U^k_{(2)\mathcal{P}} = \Psi^k \left(\frac{\xi_k}{\xi_0} \right)^2 , \tag{1.169}$$

with Ψ^k satisfying (1.156). It is straightforward to generalize (1.167) to general N-fold product.

It can be shown that for permutation branes on the N-fold product, permuted by a cycle $(1\ldots N)$, the corresponding equation has the form:

$$\sum_k U^k_{(N)\mathcal{P}} N^k_{ij} \left(\frac{\xi_i \xi_j}{\xi_0 \xi_k} \right)^N = U^i_{(N)\mathcal{P}} U^j_{(N)\mathcal{P}} , \tag{1.170}$$

and therefore can be solved by the relation

$$U^k_{(N)\mathcal{P}} = \Psi^k \left(\frac{\xi_k}{\xi_0} \right)^N , \tag{1.171}$$

with Ψ^k again satisfying (1.156).

Remembering that for rational theories Ψ_k is given by (1.157), ξ_k by (1.118), and the relation between boundary one-point function and coefficients of the expansion of the Cardy states by Ishiabshi states (1.148), we see that that permutation states (1.159) indeed satisfy the Cardy-Lewellen condition for permutation branes.

1.3 Topological defects in RCFT

Recall basic facts on topological defects in RCFT [75, 99, 138, 139]. The construction of defects lines is analogous to that of boundary condition.

Maximally-symmetric topological defect lines are defined by the conditions:

$$T^{(1)} = T^{(2)} \qquad W^{(1)} = W^{(2)} \tag{1.172}$$

$$\bar{T}^{(1)} = \bar{T}^{(2)} \qquad \bar{W}^{(1)} = \bar{W}^{(2)}$$

Following [138] we can define defect lines also as operators X, satisfying the relations:

$$[L_n, X] = [\bar{L}_n, X] = 0 \tag{1.173}$$

$$[W_n, X] = [\bar{W}_n, X] = 0$$

As in the case of the boundary conditions, there are also consistency conditions, analogous to the Cardy and Cardy-Lewellen constraints, which must be satisfied by the operator X. For simplicity we shall write all the formulae for diagonal models (1.102). To formulate these conditions, one first note that as consequence of (1.173) X is a sum of projectors

$$X = \sum_{i,\bar{i}} \mathcal{D}^{(i,\bar{i})} P^{(i,\bar{i})} \tag{1.174}$$

where

$$P^{(i,\bar{i})} = \sum_{N,\bar{N}} (|i, N\rangle \otimes |\bar{i}, \bar{N}\rangle)(\langle i, N| \otimes \langle \bar{i}, \bar{N}|) \tag{1.175}$$

An analogue of the Cardy condition for defects requires that partition function with insertion of a pair defects after modular transformation can be expressed as sum of characters with non-negative integers. It is found in [138] that for diagonal models one can solve this condition taking for each primary a

$$\mathcal{D}_a^{(i,\bar{i})} = \frac{S_{ai}}{S_{0i}} \tag{1.176}$$

leading to the operators:

$$X_a = \sum_i \frac{S_{ai}}{S_{0i}} P^{(i,\bar{i})} \tag{1.177}$$

For operators (1.177) one has:

$$Z_{ab} = \text{Tr}\left(X_a^\dagger X_b \tilde{q}^{L_0 - \frac{c}{24}} \tilde{\bar{q}}^{\bar{L}_0 - \frac{c}{24}}\right) = \sum_{k,i\bar{i}} N_{bk}^a N_{i\bar{i}}^k \chi_i(q) \chi_{\bar{i}}(\bar{q}) \tag{1.178}$$

Note that coefficient in front of product of characters is dimension of the space of four-point blocks (1.119).

Topological defects can be fused. For defects (1.176) again using the Verlinde formula one derives:

$$X_a X_b = \sum_c N_{ab}^c X_c \tag{1.179}$$

Using (1.179) the formula (1.178) can be generalized to the insertion of N defects: the torus partition function with insertion of N defects corresponding to primaries a_i is

$$Z_{a_1...a_N} = \sum_{i,\bar{i}} \mathcal{N}^{a_1...a_N,i,\bar{i}} \chi_i(q) \chi_{\bar{i}}(\bar{q}) \tag{1.180}$$

where $\mathcal{N}^{a_1...a_N,i,\bar{i}}$ is dimension of the space of $N+2$-conformal blocks (1.122). Topological defects can act on boundary states producing new boundary states. The action of defects (1.176) on Cardy states (1.139) is easily obtained using the Verlinde formula:

$$X_a|b\rangle = \sum_d N_{ab}^d |d\rangle \tag{1.181}$$

Using (1.181) one can compute the annulus partition function between Cardy states corresponding to primaries a and b with insertion of a defect corresponding to primary c:

$$Z_{ab,c} = \text{Tr}_{H_{ab}}\left(X_c \tilde{q}^{L_0 - \frac{c}{24}}\right) = \sum_{d,i} N_{bc}^d N_{id}^a \chi_i(q) \tag{1.182}$$

Note that coefficient in front of product of characters is again dimension of the space of four-point blocks (1.119). This result can be generalized to the insertion of any number N of defects as well: the annulus partition function between Cardy states corresponding to primaries a and b with insertion of N defects corresponding to primaries d_i is

$$Z_{ab,d_1...d_N} = \sum_i \mathcal{N}^{ab,d_1...d_N,i} \chi_i(q) \tag{1.183}$$

where $\mathcal{N}^{ab,d_1...d_N,i}$ is dimension of the space of $N+3$-point conformal blocks (1.122).

Now we turn to the cluster condition for defects [139, 164]. Here we should consider two-point functions

$$\langle \Phi_{i^*}(z_1, \bar{z}_1) X \Phi_i(z_2, \bar{z}_2) \rangle = \frac{D^{(i,\bar{i})}}{(z_1 - z_2)^{2h_i}(\bar{z}_1 - \bar{z}_2)^{h_{\bar{i}}}} \tag{1.184}$$

where

$$D^{(i,\bar{i})} = \mathcal{D}^{(i,\bar{i})} C_{i^*i} \tag{1.185}$$

It was shown in [139, 164] that $D^{(i,\bar{i})}$ satisfies:

$$\sum_k D^0 D^{(k\bar{k})} C^k_{ij,a\bar{a}} C^{k^*}_{i^*j^*,c\bar{c}} F_{k0} \begin{bmatrix} j^* & j \\ i & i \end{bmatrix}^{00}_{ac} F_{\bar{k}0} \begin{bmatrix} \bar{j}^* & \bar{j} \\ \bar{i} & \bar{i} \end{bmatrix}^{00}_{\bar{a}\bar{c}} = D^{(i\bar{i})} D^{(j\bar{j})} \tag{1.186}$$

This is Cardy-Lewellen cluster condition for defects.

By use of eq. (1.106), (1.152) and (1.153), the Cardy-Lewellen condition for defects (1.186) simplifies to

$$\sum_k D^0 D^{k,\bar{k}} N^k_{ij} \left(\frac{\xi_i \xi_j}{\xi_0 \xi_k} \right)^2 = D^{i,\bar{i}} D^{j,\bar{j}}. \tag{1.187}$$

Eq. (1.187) can be brought to eq. (1.156):

$$\frac{D^{k,\bar{k}}}{D^0} = \Psi^k \left(\frac{\xi^k}{\xi^0} \right)^2 \tag{1.188}$$

where Ψ^k satisfies (1.156). And finally to find the coefficient $\mathcal{D}^{i,i^*}$ of the defects expansion to projectors we should according to (1.185) to divide $D^{k,\bar{k}}$ by the two-point function. Remembering that for rational theories ξ^k and Ψ^k are given by (1.118) and (1.157) respectively we see that the ansatz (1.176) indeed satisfies the condition (1.186).

Comparing formulae (1.160) and (1.173), (1.161) and (1.174)-(1.176), (1.167) and (1.186) one reveals deep connection between permutation branes on two-fold product from one side, and defects on other side, known as folding trick [12, 14, 135, 189]. We see that mentioned relations for permutation branes become corresponding relations for defects after performing two-steps operation (folding) on the second copy of the CFT in question: left-right exchange and then hermitian conjugation, turning boundary state to operator. Eq. (1.185) shows that the hermitian conjugation requires inclusion of the two-point functions C_{i^*i}.

1.4   Free boson

1.4.1   Action of 2D free boson

The action of a free massless boson ϕ is

$$S = \frac{1}{8\pi} \int d^2x \eta_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \tag{1.189}$$

In this and next section we will analyze the Euclidean action with the Cartesian metric $\eta_{\mu\nu} = \delta_{\mu\nu}$.

The equation of motion for the field ϕ is:

$$\Box \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0 \tag{1.190}$$

The propagator of free scalar is

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle = -\log(\mathbf{x} - \mathbf{y})^2 \tag{1.191}$$

or in complex coordinates $z = x + iy$, $\bar{z} = x - iy$:

$$\langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle = -(\log(z - w) + \log(\bar{z} - \bar{w})) \tag{1.192}$$

It follows from eq. (1.192)

$$\langle \partial_z \phi(z, \bar{z}) \partial_w \phi(w, \bar{w}) \rangle = -\frac{1}{(z - w)^2} \quad \text{and} \quad \langle \partial_{\bar{z}} \phi(z, \bar{z}) \partial_{\bar{w}} \phi(w, \bar{w}) \rangle = -\frac{1}{(\bar{z} - \bar{w})^2} \tag{1.193}$$

The energy-momentum tensor of the free boson is

$$T_{\mu\nu} = \frac{1}{4\pi} (\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \phi \partial^\rho \phi) \tag{1.194}$$

Note that $\eta^{\mu\nu} T_{\mu\nu} = 0$, as we expect in conformal field theory.

Denoting $\partial\phi \equiv \partial_z \phi$ and $\bar{\partial}\phi \equiv \partial_{\bar{z}}\phi$, the holomorphic and anti-holomorphic components of the energy-momentum tensor are

$$T(z) = -\frac{1}{2} : \partial\phi\partial\phi : \quad \text{and} \quad \bar{T}(\bar{z}) = -\frac{1}{2} : \bar{\partial}\phi\bar{\partial}\phi : \tag{1.195}$$

The normal ordering means:

$$T(z) = -\frac{1}{2} \lim_{w \to z} (\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle) \tag{1.196}$$

The OPE of $T(z)$ with $\partial\phi$ can be calculated from the Wick theorem:

$$T(z)\partial\phi(w) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \sim \frac{\partial\phi(z)}{(z-w)^2} \tag{1.197}$$

By expanding $\partial\phi(z)$ around w we obtain OPE

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} \tag{1.198}$$

Equations (1.193) and (1.198) imply that $\partial\phi(z)$ is a primary field with conformal dimension $h = 1$. The presence of the primary field of the highest weight $h = 1$ means, that in fact we have theory with extended chiral algebra, which besides Virasoro algebra contains the $U(1)$ current $\partial\phi$. This is the reason that often the two-dimensional free boson theory called theory of $U(1)$ affine algebra.

The Wick theorem also allows us to obtain OPE of energy-momentum tensor with itself:

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{1.199}$$

The OPE (1.199) shows that two-dimensional free boson theory is conformal field theory with the central charge $c = 1$.

Let us introduce the vertex operators:

$$\mathcal{V}_\alpha(z, \bar{z}) =: e^{i\alpha\phi(z,\bar{z})} : \tag{1.200}$$

The vertex operators have the following OPE with the $U(1)$ current $\partial\phi$:

$$\partial\phi(z)\mathcal{V}_\alpha(w, \bar{w}) \sim -i\alpha\frac{\mathcal{V}_\alpha(w, \bar{w})}{z-w} \tag{1.201}$$

From here we can derive:

$$\left[\frac{1}{2\pi i}\oint_0 i\partial\phi(z)dz, \mathcal{V}_\alpha(w, \bar{w})\right] = \frac{1}{2\pi i}\oint_w \partial\phi(z)\mathcal{V}_\alpha(w, \bar{w})dz = \alpha\mathcal{V}_\alpha(w, \bar{w}) \tag{1.202}$$

Next we need the OPE of $\mathcal{V}_\alpha$ with the energy-momentum tensor:

$$T(z)\mathcal{V}_\alpha(w,\bar{w}) \sim \frac{\alpha^2}{2}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \mathcal{V}_\alpha(w,\bar{w})}{z-w} \tag{1.203}$$

This implies that vertex operators are primary with dimensions:

$$h_\alpha = \frac{\alpha^2}{2} \tag{1.204}$$

Let us also write down the OPE of two vertex operators [79] :

$$\mathcal{V}_\alpha(z)\mathcal{V}_\alpha(w) \sim (z-w)^{\alpha\beta}\mathcal{V}_{\alpha+\beta}(w) \tag{1.205}$$

1.4.2 Quantization of the compactified free boson on the cylinder

Consider the scalar field $\phi(\tau,\sigma)$ compactified on circle of radius R, on a cylinder of circumference L:

$$\phi(\sigma+L,\tau) \equiv \phi(\sigma,\tau) + 2\pi m R \tag{1.206}$$

Here m is an integer called winding number.

The mode expansion of the solution of the wave equation (1.190) with the boundary condition (1.206) is

$$\phi(\sigma,\tau) = \phi_0 + \frac{4\pi p}{RL}\tau + \frac{2\pi Rm}{L}\sigma + i\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi n(\tau-i\sigma)/L} + \bar{a}_n e^{2\pi n(\tau+i\sigma)/L}\right) \tag{1.207}$$

From reality of ϕ we have

$$a_n^\dagger = a_{-n} \quad \text{and} \quad \bar{a}_n^\dagger = \bar{a}_{-n} \tag{1.208}$$

Commutation relations follows from the equal-time commutation rules

$$[\phi(\sigma),\phi(\sigma')] = 0, \quad [\partial_\tau\phi(\sigma),\partial_\tau\phi(\sigma')] = 0\,, \quad \left[\frac{1}{4\pi}\partial_\tau\phi(\sigma),\phi(\sigma')\right] = i\delta(\sigma-\sigma') \tag{1.209}$$

which imply

$$[a_n,a_k] = n\delta_{n+k} \quad [\bar{a}_n,\bar{a}_k] = n\delta_{n+k} \quad [a_n,\bar{a}_k] = 0 \tag{1.210}$$

The total momentum of the string is

$$\int_0^L \frac{1}{4\pi}\partial_\tau\phi(\sigma)d\sigma = \frac{p}{R} \tag{1.211}$$

and p should be an integer since the vertex operator $e^{ip\phi/R}$ should be single valued under the identification $\phi \equiv \phi + 2\pi R$. Using the conformal coordinates:

$$z = e^{2\pi(\tau - i\sigma)/L} \qquad \bar{z} = e^{2\pi(\tau + i\sigma)/L} \tag{1.212}$$

we find

$$\phi(z, \bar{z}) = \phi_0 - i\left(\frac{p}{R} + \frac{1}{2}Rm\right)\log(z) + i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n} \tag{1.213}$$

$$-i\left(\frac{p}{R} - \frac{1}{2}Rm\right)\log(\bar{z}) + i\sum_{n\neq 0}\frac{1}{n}\bar{a}_n \bar{z}^{-n}$$

and

$$i\partial\phi(z) = \left(\frac{p}{R} + \frac{1}{2}Rm\right)\frac{1}{z} + \sum_{n\neq 0}a_n z^{-n-1} \tag{1.214}$$

$$i\bar{\partial}\phi(\bar{z}) = \left(\frac{p}{R} - \frac{1}{2}Rm\right)\frac{1}{\bar{z}} + \sum_{n\neq 0}\bar{a}_n \bar{z}^{-n-1} \tag{1.215}$$

Using the expressions (1.195) and (1.62) we obtain for L_0 and $\bar{L}_0$

$$L_0 = N + P_L^2 \quad \text{and} \quad \bar{L}_0 = \bar{N} + P_R^2 \tag{1.216}$$

where we introduced the left and right momenta

$$P_L = \left(\frac{p}{R} + \frac{1}{2}Rm\right) \tag{1.217}$$

$$P_R = \left(\frac{p}{R} - \frac{1}{2}Rm\right) \tag{1.218}$$

and the number operators:

$$N = \sum_{n>0}a_{-n}a_n \quad \text{and} \quad \bar{N} = \sum_{n>0}\bar{a}_{-n}\bar{a}_n \tag{1.219}$$

We conclude that the Hilbert space consists of the infinite number sectors $|p, m\rangle$ labelled by momenta and winding $p, m = -\infty, \infty$, for which

$$L_0|p, m\rangle = \frac{1}{2}P_L^2|p, m\rangle \quad \text{and} \quad \bar{L}_0|p, m\rangle = \frac{1}{2}P_R^2|p, m\rangle \tag{1.220}$$

Sometimes the state $|p, m\rangle$ is also denoted as $|P_L\rangle \otimes |P_R\rangle$. The bosonic Fock space generated by α_{-n} consists of all states of the form $|p, m\rangle$, $\alpha_{-n}|p, m\rangle$, $\alpha_{-n}^2|p, m\rangle$,.... Hence calculating trace

in the left $|p,m\rangle$ sector we obtain:

$$\chi_{p,m}(q) = q^{-1/24}\text{Tr}q^{L_0} = \frac{q^{\frac{1}{2}\left(\frac{p}{R}+\frac{mR}{2}\right)^2}}{\eta} \tag{1.221}$$

where

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{1.222}$$

is the Dedekind function.

The function $\chi_{p,m}(q)$ defined in (1.221) is a $U(1)$ character.

Collecting all we have for the partition function (1.79):

$$Z = \frac{1}{\eta\bar{\eta}}\sum_{p,m}q^{\frac{1}{2}\left(\frac{p}{R}+\frac{mR}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{p}{R}-\frac{mR}{2}\right)^2} \tag{1.223}$$

It can be checked [79] that (1.223) is modular invariant.

$\mathbb{Z}_k$ orbifold of $U(1)$ scalar

It is well-known that modding out the circle theory at radius R by the $\mathbb{Z}_k$ shift $\phi \to \phi + 2\pi R/k$ produces the circle theory, but the radius decreased to R/k. Geometrically the $\mathbb{Z}_k$ group generated by the rotation of the circle by $2\pi/k$ is the example of the group action without fixed points, therefore the resulting orbifold $S^1/\mathbb{Z}_k$ is the manifold, in this case topologically again S^1, but at smaller radius. From Hilbert space point of view, projection in untwisted sector removes momentum states allowed at a bigger radius, and twisted sectors provide windings proper to a smaller radius.

T-duality

Note that under the transformation

$$p \leftrightarrow m \quad \text{and} \quad R \to \frac{2}{R} \tag{1.224}$$

the right momentum (1.217) remains unchanged and the left momentum (1.218) flips the sign.

$$P_L \to P_L \quad \text{and} \quad P_R \to -P_R \tag{1.225}$$

Combined with the oscillator's transformation

$$a_n \to a_n \quad \text{and} \quad \bar{a}_n \to -\bar{a}_n \tag{1.226}$$

this implies

$$\partial\phi(z) \rightarrow \partial\phi(z) \quad \text{and} \quad \bar{\partial}\phi(\bar{z}) \rightarrow -\bar{\partial}\phi(\bar{z}) \tag{1.227}$$

Both transformations (1.225) and (1.226) leave the Virasoro modes L_0 and $\bar{L}_0$ and therefore the Hamiltonian unchanged. This is the simplest example of the famous T-duality discovered in [114, 153] (see for review [96]).

1.4.3 $U(1)_k$ theory

Consider the free scalar field ϕ compactified on a circle of radius $\sqrt{2k}$. This special case is called $U(1)_k$ theory. The $U(1)_k$ chiral algebra ($k \in \mathbb{N}$) contains, besides the Gaussian $U(1)$ current $J = i\sqrt{2k}\partial\phi$, two additional generators

$$\Gamma^{\pm} = e^{\pm i\sqrt{2k}\phi} \tag{1.228}$$

of integer dimension k. Eq. (1.202) implies that $J_0 = \frac{1}{2\pi i} \oint J dz$ charges of $\Gamma^{\pm}$ are $\pm 2k$. The primary fields of the extended theory are those vertex operators $e^{i\gamma\phi}$ whose OPEs with the generators (1.228) are local. This requirement together with (1.205) fixes γ to be

$$\gamma = \frac{n}{\sqrt{2k}}, \quad n \in Z \tag{1.229}$$

Their conformal dimensions are $\Delta_n = \frac{n^2}{4k}$. For primary fields, the range of n must be restricted to the fundamental domain $n = -k+1, -k+2, \ldots, k$ since the shift of n by $2k$ in $V_n = e^{inX/\sqrt{2k}}$ amounts to the insertion of the ladder operator Γ^{+}, which produces the descendant field. We see that w.r.t. to the extended symmetry, including also $\Gamma^{\pm}$, the free boson theory compactified on a circle of the radius $\sqrt{2k}$ becomes rational theory, possessing only finite number of primaries. From the point of view of the extended algebra the characters are easily derived. A factor $q^{\Delta_n - 1/24}/\eta(q)$ takes care of the action of the free scalar generators. To account for the effect of the distinct multiple applications of the generators (1.228), which give rise shifts of the momentum n by the integer multiples of $2k$, we should replace n by $n + l2k$ and sum over l. The corresponding character is

$$\psi_n(q) = \frac{1}{\eta(q)} \sum_{l \in Z} q^{k(l+n/2k)^2} . \tag{1.230}$$

The action of the modular transformation S on the characters (1.230) is

$$\psi_n(q') = \frac{1}{\sqrt{2k}} \sum_{n'} e^{\frac{-i\pi nn'}{k}} \psi_{n'}(q) \quad q = e^{2\pi i\tau} \quad \tau' = -\frac{1}{\tau}. \tag{1.231}$$

The diagonal modular invariant is

$$Z = \sum_n |\psi_n(q)|^2 \tag{1.232}$$

The primary fields V_n have J_0 charge n. Consider the symmetry $\mathbb{Z}_k$ generated by element $g = e^{\frac{2\pi i}{2k}(J_0 + \bar{J}_0)}$. This is in fact shift symmetry $X \to X + 2\pi \frac{\sqrt{2k}}{k}$ discussed above. Let us mod out this symmetry. According to general prescription we should form at the beginning the projected sector (1.90):

$$Z_{\text{proj}} = \sum_{s=0}^{k-1} Z[0,s] \tag{1.233}$$

where

$$Z[0,s] = \sum_n e^{\frac{2\pi ins}{k}} |\psi_n(q)|^2 \tag{1.234}$$

To get twisted sectors we should perform in (1.234) modular transformation $\tau \to -\frac{1}{\tau}$. Remembering (1.231) we obtain:

$$Z[s,0] = \sum_{n,n'n''} e^{\frac{\pi in}{k}(2s - n' + n'')} \psi_{n'}(q) \bar{\psi}_{n''}(q) \tag{1.235}$$

Performing sum over n we get

$$Z[s,0] = \sum_n \psi_n(q) \bar{\psi}_{n-2s}(q) \tag{1.236}$$

Therefore twisted sectors are given by the relative shifts of the left and right charges. Taking this into account we obtain the orbifold partition function (1.92):

$$Z_{\text{orb}} = \frac{1}{k} \sum_{r,s} Z[r,s] = \frac{1}{k} \sum_{r,s} \sum_n e^{\frac{2\pi is(2n-2r)}{2k}} \psi_n(q) \bar{\psi}_{n-2r}(q) = \sum_n \psi_n(q) \bar{\psi}_{-n}(q) \tag{1.237}$$

Remembering (1.227) we see that $\mathbb{Z}_k$ orbifolding leads to T dual theory. Note that this consistent with the mentioned above fact that the $\mathbb{Z}_k$ shift modding decreases the radius $\sqrt{2k}$ to $\frac{\sqrt{2k}}{k} = \sqrt{2/k}$, which is the radius of the T-dual theory (1.224): $\frac{2}{\sqrt{2k}} = \sqrt{2/k}$.

1.4.4 Boundary c=1 systems

A scalar compactified on a circle of generic radius

Here we review for future use the free boson theory on a world-sheet with a boundary. We find convenient to use in this part the Minkowski metric on a world-sheet. Introducing the light-cone coordinate

$$x^+ = \tau + \sigma \quad \text{and} \quad x^- = \tau - \sigma \tag{1.238}$$

and using the relations

$$\partial_\tau = \partial + \bar\partial \quad \text{and} \quad \partial_\sigma = \partial - \bar\partial \tag{1.239}$$

we can write the action in the form

$$S = \frac{1}{4\pi} \int \partial\phi \bar\partial\phi \, dx^+ dx^- \tag{1.240}$$

The variation of the action in the presence of the boundary takes the form:

$$\delta S = -\frac{1}{2\pi} \int \partial\bar\partial\phi \delta\phi \, dx^+ dx^- + \frac{1}{4\pi} \int (\partial\phi dx^+ - \bar\partial\phi dx^-)\delta\phi \tag{1.241}$$

Assume that boundary located at $\sigma = 0$ (open string loop channel).

In this case the boundary term takes the form:

$$\frac{1}{4\pi} \int (\partial\phi dx^+ - \bar\partial\phi dx^-)\delta\phi = \frac{1}{4\pi} \int \partial_\sigma\phi\delta\phi d\tau \tag{1.242}$$

and we have two kinds of boundary conditions: the Neumann boundary condition

$$\partial\phi = \bar\partial\phi \quad \text{or} \quad \partial_\sigma\phi|_{\sigma=0} = 0 \tag{1.243}$$

and the Dirichlet boundary condition:

$$\partial\phi = -\bar\partial\phi \quad \text{or} \quad \phi|_{\sigma=0} = \text{const} \tag{1.244}$$

We see that either the Neumann and Dirichlet boundary conditions preserve diagonal $U(1)$ affine symmetry, but the Dirichlet boundary condition includes the gluing automorphism (1.140) Ω given by the reflection $\phi \to -\phi$.

If the boundary located at $\tau = 0$ (closed string tree channel), the boundary term takes the form:

$$\frac{1}{4\pi} \int (\partial\phi dx^+ - \bar{\partial}\phi dx^-)\delta X = \frac{1}{4\pi} \int \partial_\tau \phi \delta \phi d\sigma \qquad (1.245)$$

and the Neumann boundary condition takes the form:

$$\partial\phi = -\bar{\partial}\phi \quad \text{or} \quad \partial_\tau\phi|_{\tau=0} = 0 \qquad (1.246)$$

while the Dirichlet boundary condition is

$$\partial\phi = \bar{\partial}\phi \quad \text{or} \quad \phi|_{\tau=0} = \text{const} \qquad (1.247)$$

Note that the Neumann and Dirichlet boundary conditions are mapped to each other by the T-duality (1.227).

Neumann boundary states

The action with the Neumann boundary condition can include also the Wilson line term at the boundary :

$$S = \frac{1}{8\pi} \int_0^\pi d\sigma \int d\tau \partial_\alpha\phi\partial^\alpha\phi + \sum_B \frac{ia_B}{4\pi} \int_B d\phi , \qquad (1.248)$$

where B labels boundaries and a_B are the constant modes of the $U(1)$ gauge potential coupling to the boundaries (and are periodic, with periods $4\pi/R$). The Wilson line term being topological, does not change neither bulk nor the boundary equations of motion, but can have contribution to the action.

In the closed-string channel the task is to find the boundary states $|N_i\rangle$, with Chan-Paton factor i, which are found by imposing the corresponding boundary conditions. The boundary is located at $\tau = 0$ and one has the boundary condition (1.246).

Inserting in (1.246) the mode expansion (1.207), where we performed the Wick rotation $\tau \to i\tau$, and set $L = \pi$,

$$\phi(\sigma, \tau) = \phi_0 + \frac{4p}{R}\tau + 2Rm\sigma + i\sum_{n\neq 0} \frac{1}{n}\left(a_n e^{2in(\tau-\sigma)} + \bar{a}_n e^{2in(\tau+\sigma)}\right) \qquad (1.249)$$

we get:

$$p = 0, \qquad \alpha_n = -\tilde{\alpha}_{-n} . \qquad (1.250)$$

Taking into account the properties of coherent state and the $U(1)$ modes a_i we get for $|N_i\rangle$:

$$|N_i\rangle = g_N \sum_w e^{-ia_i mR/2} \exp\left(\sum_{n>0} -\frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right) |0, m\rangle , \qquad (1.251)$$

where the phase factor comes from the Wilson line term:

$$\frac{ia_B}{4\pi} \int d\sigma \partial_\sigma \phi = \frac{1}{2} ia_B m_B R , \qquad (1.252)$$

where m_B is the winding number of the boundary. Here g_N is the normalization factor, which is not determined by the boundary condition (1.246). Note that the vacuum $|0, m\rangle$ has the structure $\left|\frac{Rm}{2}\right\rangle \otimes \left|-\frac{Rm}{2}\right\rangle$, consistent with the fact, that the Neumann boundary state has vanishing momentum.

Let us closer look at the state (1.251). Expanding the exponential we can write

$$\exp\left(\sum_{n>0} -\frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right) |0, m\rangle = \qquad (1.253)$$

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \prod_{n=1}^{\infty} \frac{1}{\sqrt{k_n!}} \left(\frac{a_{-n}}{\sqrt{n}}\right)^{k_n} \left|\frac{Rm}{2}\right\rangle \otimes \frac{1}{\sqrt{k_n!}} \left(\frac{-\bar{a}_{-n}}{\sqrt{n}}\right)^{k_n} \left|-\frac{Rm}{2}\right\rangle$$

The states

$$|k_1, k_2 \ldots\rangle = \prod_{n=1}^{\infty} \frac{1}{\sqrt{k_n!}} \left(\frac{a_{-n}}{\sqrt{n}}\right)^{k_n} \left|\frac{Rm}{2}\right\rangle \qquad (1.254)$$

form orthonormal basis basis of the Fock space built over the vacuum $\left|\frac{Rm}{2}\right\rangle$. We see that the coherent state appearing in (1.251) has the form of the Ishibashi state (1.130) for the Fock space. To obtain the normalization factor $g_N = \langle 0|N_i\rangle$ we should equate the partition function in the closed and open string channels [55]:

$$g_N^2 = \frac{R}{2} , \qquad (1.255)$$

Dirichlet boundary states

The boundary condition (1.247) leads to:

$$m = 0, \qquad \alpha_n = \tilde{\alpha}_{-n} . \qquad (1.256)$$

From these conditions, for the boundary state located at the point y we get

$$|D_y\rangle = g_{D_y} \delta(\phi_0 - y) \exp\left(\sum_{n>0} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right) |0\rangle = g_{D_y} \sum_p e^{\frac{-ipy}{R}} \exp\left(\sum_{n>0} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right) |p, 0\rangle . \quad (1.257)$$

Note that the Dirichlet boundary state is sum of sectors with the coinciding left and right momenta: $|p, 0\rangle = |\frac{p}{2R}\rangle \otimes |\frac{p}{2R}\rangle$. It is consistent with the fact the Dirichlet boundary state has full momentum $\frac{p}{R}$.

The normalization g_D again can be derived equating partition functions calculated in closed and open string channels [55]:

$$g_D^2 = \frac{1}{R} \ . \tag{1.258}$$

It is easy to see that the Dirichlet boundary state (1.257) can be obtained from the Neuamnn boundary state (1.251) by the T-duality, if an addition to the rules (1.224) and (1.226) we also map Wilson line factors a_i to coordinates y_i.

Boundary states in $U(1)_k$ theory

Since $U(1)_k$ is a rational theory we can use the Cardy formula (1.139). For $U(1)_k$ theory it takes the form:

$$|A, \hat{n}\rangle_C = \frac{1}{(2k)^{1/4}} \sum_{n=0}^{2k-1} e^{\frac{-i\pi \hat{n}n}{k}} |An\rangle\rangle^{U(1)} \tag{1.259}$$

Let us elaborate the Ishibashi state $|An\rangle\rangle^{U(1)}$. As we discussed in 1.4.3 the space of the highest weight representation of the $U(1)_k$ theory is generated by oscillators and Γ operators, and hence is sum of Fock spaces $\sum_l \mathcal{F}_{r+2kl}$. The part of the Ishibashi state over single Fock space we have elaborated in the previous section, and shown that it is given by coherent state. Therefore the extended symmetry (1.229) implies that the Ishibashi states of the $U(1)_k$ theory should have the form [123]:

$$|Ar\rangle\rangle^{U(1)} = \exp\left[\sum_{n=1}^{\infty} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right] \sum_{l \in Z} |\frac{r+2kl}{\sqrt{2k}}\rangle \otimes |\overline{\frac{r+2kl}{\sqrt{2k}}}\rangle \ . \tag{1.260}$$

This shows that the Cardy states (1.260) are Dirichlet boundary states (1.257) at the special quantized positions. We arrive to the conclusion that the extended symmetry leads to the quantization of the $D0$-branes positions. This is easy to understand, since the extended symmetry requires that all vacuums of the form $|r+2kl\rangle$ should appear with same exponetial factor.

Let us consider Neumann boundary conditions. To derive them we will us the equivalence of the T-duality and $\mathbb{Z}_k$ orbifolding proved in (1.237). The relation (1.237) implies that starting from the Dirichlet Cardy states, one can obtain the Neumann boundary state, implementing at the beginning orbifolding, namely sum over $\mathbb{Z}_k$ images, taking us to the boundary state on the orbifold $U(1)_k/\mathbb{Z}_k$, and consecutively performing T-duality yielding the Neumann boundary state on $U(1)_k$. Doing so we obtain two Neumann states :

$$|B, \eta \pm 1\rangle = \left(\frac{k}{2}\right)^{1/4}\left(|B,0\rangle\rangle^{U(1)} + \eta|B,k\rangle\rangle^{U(1)}\right) \tag{1.261}$$

where

$$|Br\rangle\rangle^{U(1)} = \exp\left[-\sum_{n=1}^{\infty}\frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right]\sum_{l\in Z}|\frac{r+2kl}{\sqrt{2k}}\rangle \otimes \overline{|-\frac{r+2kl}{\sqrt{2k}}\rangle}, \tag{1.262}$$

is a B-type Ishibashi state of $U(1)_k$ theory satisfying the Neumann boundary conditions. The parameter η is the two-valued Wilson line.

1.5 WZW model

1.5.1 Action

The world-sheet action of the bulk WZW model is [185]

$$\begin{aligned}S^{\text{WZW}}(g) &= \frac{k}{4\pi}\int_{\Sigma}\text{Tr}(\partial_z g^{-1}\partial_{\bar{z}}g)dzd\bar{z} + \frac{k}{4\pi}\int_B\frac{1}{3}\text{Tr}(g^{-1}dg)^3 \\ &\equiv \frac{k}{4\pi}\left[\int_{\Sigma}dzd\bar{z}L^{\text{kin}} + \int_B\omega^{\text{WZ}}\right],\end{aligned} \tag{1.263}$$

B is a 3-manifold such that $\partial B = \Sigma$. This action describes a bosonic field living on the semisimple group manifold G associated with the Lie algebra $\mathcal{A}$. The action (1.263) depends on the extension of the field on three-manifold B. However given that k is integer the quantum amplitude is unambiguously defined.

Let us obtain the equations of motion. The variation of the kinetic term yields

$$\delta(\text{Tr}(\partial_z g^{-1}\partial_{\bar{z}}g)) = \tag{1.264}$$

$$= \text{Tr}\left(\delta gg^{-1}[\partial_{\bar{z}}(\partial_z gg^{-1}) + \partial_z(\partial_{\bar{z}}gg^{-1})] - \partial_z(\delta gg^{-1}\partial_{\bar{z}}gg^{-1}) - \partial_{\bar{z}}(\delta gg^{-1}\partial_z gg^{-1})\right)$$

The variation of the ω^{WZ} is given by:

$$\delta\omega^{WZ} = d[\text{Tr}(\delta gg^{-1}(dgg^{-1})^2)] \tag{1.265}$$

Eq. (1.265) implies

$$\int_B \delta\omega^{WZ} = \int_\Sigma \text{Tr}(\delta gg^{-1}[\partial_{\bar{z}}gg^{-1}\partial_z gg^{-1} - \partial_z gg^{-1}\partial_{\bar{z}}gg^{-1}] \tag{1.266}$$

$$= \int_\Sigma \text{Tr}(\delta gg^{-1}[\partial_{\bar{z}}(\partial_z gg^{-1}) - \partial_z(\partial_{\bar{z}}gg^{-1})]$$

Taking the sum of (1.264) and (1.266) and omitting the full derivative terms we obtain:

$$\delta S^{\text{WZW}}(g) = \frac{k}{2\pi} \int_\Sigma dzd\bar{z}\,\text{Tr}[\delta gg^{-1}\partial_{\bar{z}}(\partial_z gg^{-1})] \tag{1.267}$$

Using that

$$g\partial_z(g^{-1}\partial_{\bar{z}}g)g^{-1} = \partial_{\bar{z}}(\partial_z gg^{-1}) \tag{1.268}$$

one can equivalently write (1.267) in the form

$$\delta S^{\text{WZW}}(g) = \frac{k}{2\pi} \int_\Sigma dzd\bar{z}\,\text{Tr}[g^{-1}\delta g\partial_z(g^{-1}\partial_{\bar{z}}g)] \tag{1.269}$$

Taking δgg^{-1} arbitrary we get from equations (1.267), that EOM of the WZW model is

$$\partial_{\bar{z}}(\partial_z gg^{-1}) = 0 \tag{1.270}$$

or equivalently

$$\partial_z(g^{-1}\partial_{\bar{z}}g) = 0 \tag{1.271}$$

On the other hand taking $\delta gg^{-1} \equiv \omega(z)$ holomorphic we see from (1.267) using the integration by parts that $\delta_\omega S = 0$ identically. Therefore the WZW action (1.263) has the symmetry

$$\delta_\omega g = \omega(z)g \tag{1.272}$$

and the corresponding conserved current is

$$J(z) = -k\partial_z gg^{-1} \tag{1.273}$$

The EOM in the form (1.270) coincides with the condition of of the conservation of the current (1.273). Therefore the current (1.273) is holomorphic. Similarly taking $-g^{-1}\delta g \equiv \bar{\omega}(\bar{z})$ anti-holomorphic we receive from (1.269) using the integration by parts that $\delta_{\bar{\omega}}S = 0$ identically. Hence the action (1.263) has additionally the symmetry

$$\delta_{\bar{\omega}}g = -g\bar{\omega}(\bar{z}) \tag{1.274}$$

and the corresponding conserved current is

$$\bar{J}(\bar{z}) = kg^{-1}\partial_{\bar{z}}g \tag{1.275}$$

Again the EOM in the form (1.271) coincides with the condition of the conservation of the current (1.275). Therefore the current (1.275) is anti-holomorphic. Thus we have shown that the action (1.263) is invariant under the transformation:

$$g(z, \bar{z}) \rightarrow h_L(z)g(z, \bar{z})h_R(\bar{z}) \tag{1.276}$$

Classically the components of the energy-momentum tensor are

$$T = \frac{1}{2k}\mathrm{Tr}J^2 \quad \text{and} \quad \bar{T} = \frac{1}{2k}\mathrm{Tr}\bar{J}^2 \tag{1.277}$$

The symmetries of the WZW model can be also derived using the Polyakov-Wiegmann identities

$$L^{\mathrm{kin}}(gh) = L^{\mathrm{kin}}(g) + L^{\mathrm{kin}}(h) - \left(\mathrm{Tr}(g^{-1}\partial_z g \partial_{\bar{z}} h h^{-1}) + \mathrm{Tr}(g^{-1}\partial_{\bar{z}} g \partial_z h h^{-1})\right) \tag{1.278}$$

$$\omega^{\mathrm{WZ}}(gh) = \omega^{\mathrm{WZ}}(g) + \omega^{\mathrm{WZ}}(h) - \mathrm{d}\left(\mathrm{Tr}(g^{-1}\mathrm{d}g\mathrm{d}h h^{-1})\right), \tag{1.279}$$

Indeed taking the sum of (1.278) and (1.279) we obtain

$$S(gh) = S(g) + S(h) - \frac{k}{2\pi}\int d^2z \left(\mathrm{Tr}(g^{-1}\partial_{\bar{z}} g \partial_z h h^{-1})\right) \tag{1.280}$$

Eq. (1.280) implies that the action (1.263) is indeed invariant under the transformation (1.276), namely under the left multiplication by a holomorphic element and right multiplication by an anti-holomorphic element.

1.5.2 WZW model-Quantization

Remembering (1.48) and denoting by T^a generators of the Lie algebra $\mathcal{A}$ we can write for the transformation of a field X under (1.272) and (1.274)

$$\delta_{\omega,\bar\omega}X = -\frac{1}{2\pi i}\oint dz \sum_a \omega^a J^a X + \frac{1}{2\pi i}\oint d\bar z \sum_a \bar\omega^a \bar J^a X \tag{1.281}$$

where

$$J = \sum_a J^a T^a, \quad \omega = \sum_a \omega^a T^a, \quad \text{and} \quad \text{Tr}(T^a T^b) = \delta^{ab} \tag{1.282}$$

Denote by f_{abc} structure constants:

$$[T^a, T^b] = if_{abc}T^c \tag{1.283}$$

The transformation laws for currents follow from (1.272) and (1.273)

$$\delta_\omega J = -k(\partial_z(\delta_\omega g)g^{-1} - \partial_z g g^{-1}\delta_\omega g g^{-1}) = [\omega, J] - k\partial_z\omega \tag{1.284}$$

Using (1.283), (1.284) can be rewritten as

$$\delta_\omega J^a = \sum_{b,c} if_{abc}\omega^b J^c - k\partial_z\omega^a \tag{1.285}$$

Comparing (1.281) and (1.285) we arrive

$$J^a(z)J^a(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{(z-w)} \tag{1.286}$$

This is OPE of the affine algebra. Introducing the modes J_n^a from the expansion

$$J^a(z) = \sum_{n\in\mathbf{Z}} z^{-n-1}J_n^a \tag{1.287}$$

we can obtain the commutation relations of the affine algebra Lie at the level k:

$$[J_n^a, J_m^b] = \sum_c if_{abc}J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0} \tag{1.288}$$

Transformation property of $\bar J$ is

$$\delta_{\bar\omega}\bar J = [\bar\omega, \bar J] - k\partial_z\bar\omega \tag{1.289}$$

This gives rise another copy of affine algebra for the modes $\bar{J}_m^b$. Since the $\bar{\omega}(\bar{z})$ is independent of z

$$\delta_{\bar{\omega}} J = 0 \tag{1.290}$$

This implies

$$[J_n^a, \bar{J}_m^b] = 0 \tag{1.291}$$

The components of the energy-momentum tensor are given by the Sugawara formula

$$L_n = \frac{1}{2(k + h_G)} \sum_a \sum_m : J_m^a J_{n-m}^a : \tag{1.292}$$

where h_G is the dual Coxeter number, which is half of the Casimir is the adjoint representation:

$$\sum_{b,c} f_{abc} f_{dbc} = 2 h_G \delta_{ad} \tag{1.293}$$

They satisfy the relations:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{1.294}$$

$$[L_n, J_m^a] = -m J_{n+m}^a$$

with $c = \frac{k|G|}{k+h_G}$.

1.5.3 Representations of the affine algebras

Cartan-Weyl basis of Lie algebra

Here we will review the Cartan-Weyl basis of the algebra and the general facts on the highest weight representations.

The set of the commutation relations in the Cartan-Weyl basis is

$$[\mathcal{H}^i, \mathcal{H}^j] = 0 \tag{1.295}$$

$$[\mathcal{H}^i, E^\alpha] = \alpha^i E^\alpha$$

$$[E^\alpha, E^\beta] = N_{\alpha,\beta} E^{\alpha+\beta}, \qquad \text{if } \alpha + \beta \in \Delta$$

$$= 2\alpha \cdot \mathcal{H}/|\alpha|^2 \qquad \text{if } \alpha = -\beta$$

$$= 0 \qquad \text{otherwise}$$

The vectors α^i are called roots and E^α are the corresponding ladder operators. Δ denotes the set of all roots. The set of generators $\mathcal{H}^i$ form the Cartan subalgebra h.

Positive roots

Let us fix the basis in the space of roots $\{\beta_1, \cdots \beta_r\}$. Any root can be expanded in this basis:

$$\alpha = \sum_1^r n_i \beta_i \tag{1.296}$$

The α is said positive if the first nonzero number in the sequence $(n_1, n_2, \cdots, n_r)$ is positive. The set of positive roots we denote by Δ_+. The simple root α_i is defined to be the root that can't be written as a sum of two positive roots. There are r simple roots and their set $\{\alpha_1, \cdots \alpha_r\}$ provides very convenient basis for the r-dimensional space of roots. It is convenient to introduce notion

$$\alpha_i^\vee = \frac{2\alpha_i}{|\alpha_i|^2} \tag{1.297}$$

The $\alpha_i^\vee$ is the coroot associated with the root.

Highest root

The distinguished element of Δ is highest root θ. It is unique object and which, in expansion $\sum_i m_i \alpha_i$ the sum $\sum_i m_i$ gets maximized.

Highest weights of the affine algebra

In the Cartan-Weyl basis the commutation relation of the affine algebra takes the form:

$$[\mathcal{H}_n^i, \mathcal{H}_m^j] = kn\delta^{ij}\delta_{n+m,0} \tag{1.298}$$

$$[\mathcal{H}_n^i, E_m^\alpha] = \alpha^i E_{n+m}^\alpha$$

$$[E_n^\alpha, E_m^\beta] = N_{\alpha,\beta} E_{n+m}^{\alpha+\beta}, \qquad \text{if } \alpha + \beta \in \Delta$$

$$= \frac{2}{|\alpha|^2}\left(\alpha \cdot \mathcal{H}_{n+m} + kn\delta_{n+m,0}\right) \quad \text{if } \alpha = -\beta$$

$$= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise}$$

The highest weight state now is defined to satisfy:

$$\mathcal{H}_n^i |\lambda\rangle = E_n^{\pm\alpha} |\lambda\rangle = 0, \qquad n > 0 \tag{1.299}$$

$$\mathcal{H}_0^i |\lambda\rangle = \lambda^i |\lambda\rangle, \qquad \text{and} \qquad E_0^\alpha |\lambda\rangle = 0, \qquad \alpha > 0$$

Consider the $su(2)$ subalgebra generated by: E_0^α, $E_0^{-\alpha}$, $\frac{2}{|\alpha|^2}\alpha \cdot \mathcal{H}_0$. Commutation relations imply:

$$\langle\lambda| E_0^\alpha E_0^{-\alpha} |\lambda\rangle = \langle\lambda| [E_0^\alpha E_0^{-\alpha}] |\lambda\rangle = \frac{2}{|\alpha|^2}\alpha \cdot \lambda \langle\lambda|\lambda\rangle \geq 0 \tag{1.300}$$

Hence we must have $\alpha \cdot \lambda \geq 0$.

Now look another $su(2)$ subalgebra generated by: E_{-1}^α, $E_1^{-\alpha}$, $\frac{2}{|\alpha|^2}(-\alpha \cdot \mathcal{H}_0 + k)$. From (1.298) we have

$$\langle\lambda| E_1^{-\alpha} E_{-1}^\alpha |\lambda\rangle = \langle\lambda| [E_1^{-\alpha} E_{-1}^\alpha] |\lambda\rangle = \frac{2}{|\alpha|^2}(-\alpha \cdot \lambda + k) \langle\lambda|\lambda\rangle \geq 0 \tag{1.301}$$

Restrict ourself for simplicity to the case of unitary algebras for which all roots normalized to 2.

Since the component of the J_3 generatot of $su(2)$ are integer , and we know that for any weight λ the $\alpha \cdot \lambda$ is integer, we obtain that k is integer.

Then it follows from (1.301) that any highest weight should satisfy the inequility

$$\alpha \cdot \lambda \leq k \tag{1.302}$$

The condition (1.302) is stringent for the highest root θ

$$\theta \cdot \lambda \leq k \tag{1.303}$$

Using the expression (1.294) for L_0 we derive the conformal weight of the highest weight state:

$$L_0 |\lambda\rangle = \frac{C_\lambda}{2(k + h_G)} |\lambda\rangle \tag{1.304}$$

where C_λ is the quadratic Casimir of the representation λ.

Let us now specialize to the $SU(2)$ group.

Since the quadratic Casimir in the representation j has value $C_j = 2j(j+1)$, therefore in the adjoint representation $j = 1$, $C_{\text{adj}} = 4$, $h_{SU(2)} = 2$, and the central charge of the corresponding affine algebra is

$$c = \frac{3k}{k+2} \tag{1.305}$$

Here we have one root and all weights are given by half-integer j and the highest weights of the $su(2)$ affine algebra are given by the half-integer j satisfying the inequality:

$$2j \leq k \tag{1.306}$$

The conformal weight of these states are:

$$h_j = \frac{j(j+1)}{k+2} \tag{1.307}$$

The matrix of the modular transformation is

$$S_{aj} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2a+1)(2j+1)\pi}{k+2}\right). \tag{1.308}$$

Characters are

$$\chi_l^{SU(2)} = \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}} \tag{1.309}$$

$$\Theta_{m,k}(\tau, z, u) = e^{-2\pi i m u} \sum_{n \in \mathbb{Z} + m/2k} e^{2\pi i m(n^2 \tau - nz)} \tag{1.310}$$

1.5.4   Coset models

GKO construction

Assume we have subgroup H of group G: $H \subset G$. We denote the G currents by J_G^a and the H currents by J_H^i, where i runs from 1 to $|H| \equiv \dim H$. We can now construct two energy-momentum tensors

$$T_G(z) = \frac{1}{2(k_G + h_G)} \sum_{a=1}^{|G|} : J_G^a(z) J_G^a(z) : \tag{1.311}$$

and

$$T_H(z) = \frac{1}{2(k_H + h_H)} \sum_{a=1}^{|H|} : J_H^i(z) J_H^i(z) : \tag{1.312}$$

Now we have:

$$T_G(z)J_H^i(w) \sim \frac{J_H^i(w)}{(z-w)^2} + \frac{\partial J_H^i(w)}{(z-w)} \tag{1.313}$$

but also that

$$T_H(z)J_H^i(w) \sim \frac{J_H^i(w)}{(z-w)^2} + \frac{\partial J_H^i(w)}{(z-w)} \tag{1.314}$$

Note that the OPE of $(T_G - T_H)$ with J_H^i is non-singular. Since T_H above is constructed only from H currents J_H^i it follows that $T_{G/H} = T_G - T_H$ has the nonsingular OPE with all of T_H. This implies that

$$T_G = (T_G - T_H) + T_H = T_{G/H} + T_H \tag{1.315}$$

gives the orthogonal decomposition of the Virasoro algebra generated by T_G into two commuting Virasoro subalgebras, $[T_{G/H}, T_H] = 0$. To calculate central charge of the Virasoro subalgebra generated by $T_{G/H}$, we note that the highest negative degree in OPE of two T_G's decomposes as

$$T_G T_G = \frac{c_G/2}{(z-w)^4} \sim T_{G/H} T_{G/H} + T_H T_H \sim \frac{c_{G/H}/2 + c_H/2}{(z-w)^4} \tag{1.316}$$

This results

$$c_{G/H} = c_G - c_H = \frac{k_G|G|}{k_G + h_G} - \frac{k_H|H|}{k_H + h_H} \tag{1.317}$$

To describe the states that arise in a G/H theory, we should consider how representation of G decompose under (1.315). Let us denote representation space of affine G at level k_G by $|c_G, \lambda_G\rangle$, where c_G is the central charge corresponding to k_G, and λ_G is the highest weight representation. Under orthogonal decomposition of the Virasoro algebra $T_G = T_{G/H} + T_H$, this space will decompose as direct sum of the irreducible representations

$$|c_G, \lambda_G\rangle = \oplus_j |c_{G/H}, h_{G/H}^j\rangle \otimes |c_H, \lambda_H^j\rangle \tag{1.318}$$

where $|c_{G/H}, h_{G/H}^j\rangle$ denotes the irreducible representation of $T_{G/H}$ with the lowest L_0 eigenvalue $h_{G/H}^j$. It follows from decomposition (1.318) that a character of the affine G representation with the highest weight λ^a satisfies:

$$\chi_{\lambda_G^a}^{k_G}(\tau) = \sum_j \chi_{h_{G/H}(\lambda_G^a, \lambda_H^j)}^{c_{G/H}}(\tau) \chi_{\lambda_H^j}^{k_H} \equiv \chi_{G/H} \cdot \chi_{\lambda_H}^H \tag{1.319}$$

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In (1.319) L_0 eigenvalues $h_{G/H}$ characterizing the $T_{G/H}$ Virasoro representation depend on highest weights λ_G^a and λ_H^j characterizing corresponding G and H representation. On the r.h.s. of (1.319) we have introduced the matrix notation.

Under the modular transformation:

$$\zeta : \tau \to \frac{a\tau + b}{c\tau + d} \tag{1.320}$$

characters allowed at given fixed level k_G of the affine algebra transform as the unitary representation

$$\chi^{k_G}(\tau') = M^{k_G}(\zeta)\chi^{k_G}(\tau) \tag{1.321}$$

with M^{k_G} unitary matrix. From (1.319) we have

$$\chi^{k_G}(\tau') = \chi^{k_{G/H}}(\tau')M^{k_H}(\zeta)\chi^{k_H}(\tau) \tag{1.322}$$

The linear independence of G and H characters thus enables us to solve for modular transformation properties of the $T_{G/H}$ characters as

$$\chi^{k_{G/H}}(\tau') = M^{k_G}(\zeta)\chi^{k_{G/H}}(\tau)M^{k_H}(\zeta)^{-1} \tag{1.323}$$

1.5.5 Lagrangian of coset model: Gauged WZW model

Let G be a compact, simply connected, non-abelian group. The G/H coset CFT, where H is a subgroup of G, can be described in terms of a gauged WZW action, where the symmetry

$$g \to h(z, \bar{z})gh^{-1}(z, \bar{z}) \tag{1.324}$$

$g \in G$, $h \in H$ is gauged away. An H Lie algebra valued world sheet vector field A is added to the system, and the G/H action on a world-sheet without boundary becomes [16, 83, 85, 113]:,

$$S^{G/H} = S^{\text{WZW}} + S^{\text{gauge}} \tag{1.325}$$
$$= \frac{k_G}{4\pi}\left[\int_\Sigma d^2z L^{\text{kin}} + \int_B \omega^{WZW}\right]$$
$$+ \frac{k_G}{2\pi}\int_\Sigma d^2z \text{Tr}[A_{\bar{z}}\partial_z gg^{-1} - A_z\partial_{\bar{z}}gg^{-1} + A_{\bar{z}}gA_zg^{-1} - A_{\bar{z}}A_z]$$

The action (1.325) is invariant under the transformation (1.324) together with transformation of gauge fields

$$A_z \rightarrow hA_zh^{-1} + \partial_z hh^{-1} \tag{1.326}$$

$$A_{\bar{z}} \rightarrow hA_{\bar{z}}h^{-1} + \partial_{\bar{z}} hh^{-1}$$

We can write the action (1.325) also in other form which make clear why this action describes GKO coset models.

Introduce H group element valued world sheet fields U and $\tilde{U}$ as

$$A_z = \partial_z U U^{-1} \tag{1.327}$$

$$A_{\bar{z}} = \partial_{\bar{z}} \tilde{U} \tilde{U}^{-1} \tag{1.328}$$

Using the Polyakov-Wiegmann identities (1.278) and (1.279) the coset action becomes:

$$S^{G/H} = S^{G/H}(U^{-1}g\tilde{U}) - S^H(U^{-1}\tilde{U}) \tag{1.329}$$

The level k_H of the S^H term is related to k_G through the embedding index of H in G.

The form (1.329) demonstrates that the gauge invariant action (1.325) indeed gives the GKO model. The form (1.329) also makes obvious the local gauge transformation

$$g(z, \bar{z}) \rightarrow h(z, \bar{z}) g(z, \bar{z}) h^{-1}(z, \bar{z}) \tag{1.330}$$

$$U(z, \bar{z}) \rightarrow h(z, \bar{z}) U(z, \bar{z})$$

$$\tilde{U}(z, \bar{z}) \rightarrow h(z, \bar{z}) \tilde{U}(z, \bar{z})$$

with $h(z, \bar{z}) \in H$.

The parafermion $\mathcal{A}^{PF(k)} = \frac{SU(2)_k}{U(1)_k}$

The chiral algebra of this theory has a set of irreducible representations described by pairs (j, n) where $j \in \frac{1}{2}Z$, $0 \leq j \leq k/2$, and n is an integer defined modulo $2k$. The pairs are subject to a constraint $2j + n = 0 \mod 2$, and an equivalence relation $(j, n) \sim (k/2 - j, k + n)$. The character of the representation (j, n), denoted by $\chi_{j,n}(q)$, is determined implicitly by the decomposition

$$\chi_j^{SU(2)}(q) = \sum_{n=-k}^{k+1} \chi_{j,n}^k(q)\psi_n(q). \tag{1.331}$$

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where $\chi_j^{SU(2)}(q)$ is $SU(2)$ affine characters given by (1.309) and $\psi_n(q)$ is $U(1)_k$ affine characters (1.230). The action of the S element of the modular group on the character is

$$\chi_{j,n}^k(\tilde{q}) = \sum_{(j',n')} S_{(j,n),(j'n')}^{PF} \chi_{j',n'}^k(q) \tag{1.332}$$

and the PF S-matrix according to (1.323) is

$$S_{(j,n),(j'n')}^{PF} = \sqrt{\frac{2}{k}} e^{\frac{i\pi nn'}{k}} S_{jj'}, \tag{1.333}$$

where $S_{jj'}$ defined in (1.308).

When combining left and right-movers, the simplest modular invariant partition function of the parafermion theory is obtained by summing over all distinct representations

$$Z = \sum_{(j,n)\in PF_k} |\chi_{j,n}|^2. \tag{1.334}$$

The parafermion theory has a global Z_k symmetry under which the fields $\psi_{j,n}$ generating the representation (j,n) transform as

$$g: \quad \psi_{j,n} \to \omega^n \psi_{j,n}, \quad \omega = e^{\frac{2\pi i}{k}}. \tag{1.335}$$

Therefore we can orbifold the theory by this group. Taking the symmetric orbifold by Z_k of (1.334) similar to $U(1)_k$ case leads to the partition function

$$Z = \sum_{(j,n)\in PF_k} \chi_{j,n} \bar{\chi}_{j,-n}. \tag{1.336}$$

We see that effect of the orbifold is to change the relative sign between the left and right movers of the $U(1)$ group with which we orbifold. Therefore the Z_k orbifold of the parafermion theory at level k is T-dual to the original theory. This fact will be the basis of many constructions in the main text.

Chapter 2

Non-maximally symmetric D-branes in the WZW models

2.1 WZW model on a world-sheet with boundary

2.1.1 Action of Boundary WZW model

Here we analyze the action of the WZW model on a world-sheet with a boundary. In particular we present geometric realization [8, 87] of the Cardy boundary states (1.139). Let us consider the maximally-symmetric boundary conditions preserving a diagonal affine symmetry:

$$J^a = \bar{J}^a, \qquad a = 1, \cdots \dim G \tag{2.1}$$

As we explained before in the absence of the boundary the WZW action possesses the affine $G_L \times G_R$ symmetry:

$$g(z, \bar{z}) \to h_L(z) g(z, \bar{z}) h_R^{-1}(\bar{z}) \tag{2.2}$$

The boundary condition (2.1) implies that the symmetry (2.2) is broken to the diagonal symmetry, requiring that $h_L = h_R = h$ on the boundary. The presence of this symmetry constraints the boundary conditions that can be placed on g. Allowing $g|_{\text{boundary}} = f$ for some $f \in G$ we must also allow $g|_{\text{boundary}} = h f h^{-1}$ for every $h \in G$. This means that g on the boundary takes

value in the conjugacy class $\mathcal{C}_f$ containing f:

$$g|_{\text{boundary}} = \mathcal{C}_f = \{g \in G \quad | \quad \text{there exists} \quad h \in G \quad \text{with} \quad g = hfh^{-1}\}. \qquad (2.3)$$

Now we are going to write down the corresponding boundary Lagrangian. Recall that to write the WZW model we used the three-manifold B satisfying the condition $\partial B = \Sigma$. When the world-sheet Σ has itself boundaries, it cannot be the boundary of a three dimensional manifold, since a boundary cannot have boundary. To define the WZW term for this case, one should fill holes in the worldsheet by adding auxiliary discs, and extend the mapping from the worldsheet into the group manifold to these discs. One further demands that the whole disc D is mapped into a region inside the conjugacy class in which the corresponding boundary lies. B will then be defined as a three-manifold bounded by the union $\Sigma \cup D$, which now has no boundaries. To make the action independent on the location of the auxiliary disc inside conjugacy class we should demand that

$$\omega^{\text{WZW}}(g)|_{g \in \mathcal{C}_f} = d\omega_f \qquad (2.4)$$

and modify the action by the boundary term

$$S^{\text{bndry}-\text{WZW}} = S^{\text{WZW}} - \frac{k_G}{4\pi} \int_D \omega_f \qquad (2.5)$$

First of all using the Polyakov-Wiegmann identities it is easy to check that indeed (2.4) for $\mathcal{C}_f = kfk^{-1}$ fulfilled with:

$$\omega_f(k) = \text{Tr}(k^{-1}dkfk^{-1}dkf^{-1}) \qquad (2.6)$$

Note that $\omega_f(k)$ is in fact depends only on elements of conjugacy class.

Now we can check that the action (2.5) is invariant under the transformation

$$g(z, \bar{z}) \to h_L(z)g(z, \bar{z})h_R^{-1}(\bar{z}) \qquad (2.7)$$

with the boundary condition $h_L(z)|_{\text{boundary}} = h_R(\bar{z})_{\text{boundary}} = h(\tau)$.

Under this transformation, the change in the L^{kin} term is canceled by the corresponding Σ integral of the boundary term from the change in the ω^{WZW} term. In the presence of a

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world-sheet boundary there remains the contribution from D to the latter change. And since according to the Polyakov-Wiegmann identity (1.279)

$$\omega^{\text{WZW}}(hgh^{-1}) - \omega^{\text{WZW}}(g) = -d\Psi(h, g) \tag{2.8}$$

where

$$\Psi(h, g) = \text{Tr}[h^{-1}dh(g^{-1}h^{-1}dhg + g^{-1}dg + dgg^{-1})] \tag{2.9}$$

we have

$$\Delta(S^{\text{WZW}}) = -\frac{k_G}{4\pi} \int_D \Psi(h, g) \tag{2.10}$$

On the other hand under this transformation $k \to hk$ and

$$\omega_f(hk) - \omega_f(k) = -\Psi(h, \mathcal{C}_f) \tag{2.11}$$

where $\mathcal{C}_f = kfk^{-1}$.

Equations (2.10) and (2.11) imply invariance of the action (2.5) under (2.7).

Let us now elaborate boundary equation of motion. The full derivatives terms from (1.264) gives the following contribution to the boundary terms:

$$\int \text{Tr}[\delta gg^{-1}\partial_z gg^{-1}dz - \delta gg^{-1}\partial_{\bar{z}}gg^{-1}d\bar{z}] \tag{2.12}$$

To find contribution from the ω^{WZW} and $\omega_f(k)$ terms note the identity [54]:

$$\text{Tr}(g^{-1}\delta g(g^{-1}dg)^2)|_{g=C} - \delta\omega_f(k) = dA_f(k). \tag{2.13}$$

$$A_f(k) = \text{Tr}[k^{-1}\delta k(f^{-1}k^{-1}dkf - fk^{-1}dkf^{-1})]. \tag{2.14}$$

Using the parametrization

$$z = \tau + i\sigma \qquad \bar{z} = \tau - i\sigma \tag{2.15}$$

and taking boundary at the $\sigma = 0$ we get

$$\int \text{Tr}\left[\delta gg^{-1}\partial_z gg^{-1} - \delta gg^{-1}\partial_{\bar{z}}gg^{-1} + k^{-1}\delta kf^{-1}k^{-1}\frac{dk}{d\tau}f - k^{-1}\delta kfk^{-1}\frac{dk}{d\tau}f^{-1}\right]d\tau \tag{2.16}$$

Remembering that $g = kfk^{-1}$, after some transformation we obtain:

$$\int \mathrm{Tr}\left[2\delta kk^{-1}\left(g^{-1}\partial_{\bar{z}}g + \partial_z gg^{-1}\right)\right]d\tau \qquad (2.17)$$

Therefore boundary equations of motion imply

$$g^{-1}\partial_{\bar{z}}g + \partial_z gg^{-1} = 0 \qquad (2.18)$$

or recalling the definition of currents

$$J = \bar{J} \qquad (2.19)$$

as expected.

We have chosen the boundary conditions (2.3) such that the particular diagonal subgroup G of $G \times G$ defined in eq. (2.2) will survive them. This is of course not a unique choice. One can act on these boundary conditions with any element of the $G \times G$ symmetry group to get equivalent boundary conditions which preserve a different diagonal subgroup. Thus we can multiply the conjugacy class $\mathcal{C}_f$ in eq. (2.3) from the right (which is the same as multiplying from the left) by any group element f to get modified boundary conditions

$$g|_{\mathrm{boundary}} \in \mathcal{C}_f m \qquad (2.20)$$

These boundary conditions also preserve a diagonal subgroup, since the set $\mathcal{C}_f m$ satisfies

$$\mathcal{C}_f m = h(\mathcal{C}_f m)m^{-1}h^{-1}m \qquad (2.21)$$

for any $h \in G$. Therefore, the boundary conditions (2.20) preserve the diagonal subgroup of $G_L \times G_R$ defined by $h_R = m^{-1}h_L^{-1}m$. In terms of the infinitesimal generators of $G_L \times G_R$, *i.e.* the left and right handed currents, the invariance of (2.20) under this subgroup implies for the corresponding boundary state the condition

$$J^a = (Ad_{m^{-1}}\bar{J})^a \qquad (2.22)$$

which modifies (2.1) by conjugating the right handed currents by m^{-1}.

2.1.2 Global issues

The modified action (2.5) is independent, by construction, of continuous deformation of D inside $\mathcal{C}_f$. However, in general, the second homotopy of a conjugacy class $\pi_2(\mathcal{C}_f)$ is non-trivial. We should compare then the value of the action for D and D', two different choices of embedding the disc in $\mathcal{C}_f$ with the same boundary, where D' may not be a continuous deformation of D in $\mathcal{C}_f$. The union of two such embedded discs is not the boundary of a three volume inside the conjugacy class, where (2.4) is valid. In that case the above analysis does not imply that the two ways to evaluate the action (2.5) agree. Since there is no natural way to choose between the two embeddings, (2.5) is not yet a well defined action. In particular, for $G = SU(2)$ the conjugacy clases $\mathcal{C}_f$ have the topology of S^2, the two-sphere generated by all possible axes of rotation by a fixed angle in three dimensions. One may then choose D and D' such that their union covers the whole of S^2. In that case the difference between the action S_D, the value of (2.5) with embedding D, and $S_{D'}$ with embedding D' is

$$\Delta S^{\text{bndry}-\text{WZW}} = \frac{k_G}{4\pi}\left[\int_B \omega^{\text{WZW}} - \int_{\mathcal{C}_f} \omega_f\right] \tag{2.23}$$

where B is the three-volume in $SU(2)$ bounded by the two-sphere $\mathcal{C}_f$. For the case of $SU(2)$, which has the topology of S^3, the form ω^{WZW} 4 times the volume form on the unit three sphere. For $\mathcal{C}_f$ with $f = e^{i\psi\sigma_3}$, the first term in (2.23) is

$$\int_B \omega^{\text{WZW}} = 8\pi(\psi - \frac{1}{2}\sin(2\psi)) \tag{2.24}$$

As to the two-form ω^f it is proportional to the volume form of the unit two-sphere. We can directly compute for $\mathcal{C}_\psi$

$$\omega^f = \sin(2\psi)\text{vol}S^2 \tag{2.25}$$

This gives for the change in the action for two topologically different embeddings

$$\Delta S^{\text{bndry}-\text{WZW}} = 2k_G\psi \tag{2.26}$$

Although this is non-zero, the quantum theory is still well-defined if ΔS is an multiple of 2π. We find that the possible conjugacy classes on which a boundary state live are quantized, the

corresponding ψ should take the values:

$$\psi_0 = 2\pi \frac{a}{k_G} \tag{2.27}$$

with a integer or half-integer satisfying $0 \le a \le k/2$.

For an arbitrary group G we can argue in the following way. Since h in (2.3) is defined modulo right multiplication by any element commuting with f and the group of such elements for a generic f is isomorphic to T^G, the Cartan torus of G, the conjugacy class (2.3) can be described as G/T^G. Its second homotopy group is therefore,

$$\Pi^2(C_f^G) = \Pi^1(T^G) \tag{2.28}$$

If r is the rank of G, a topologically non trivial embedding of S^2 in C_f^G is characterized by an r dimensional vector in the coroot lattice of G. Namely, if one embedding D of the disc into C_f^G is given by hfh^{-1} and another embedding D' sends it into $h'fh'^{-1}$, then on the topologically circular boundary the two embeddings should coincide. This implies

$$h(\tau)h'(\tau)^{-1} = t(\tau) \tag{2.29}$$

where $t(\tau)$ is an element of the subgroup isomorphic to T^G which commutes with f. Eq. (2.29) determines a mapping from the circular boundary of a given hole in the world sheet into the torus T^G. Since T^G is R^r modulo 2π times the coroot lattice, every such mapping belongs to a topological sector parameterized by a vector in the coroot lattice describing the winding of this circle on the torus T^G. This lattice vector determines, by (2.28) , the element of $\Pi^2(C_f^G)$ corresponding to the union of D and D'.

Let the element f in (2.3) chosen in the Cartan torus be of the form $f = e^{i\lambda \cdot \mathcal{H}}$ where $\mathcal{H}$ are Cartan generators. The change in the action resulting from a topological change in the embedding of the disc which is characterized by a coroot lattice vector s, is given by [87]

$$\Delta S^{\text{bndry}-\text{WZW}} = k_G(\lambda \cdot s) \tag{2.30}$$

where the length of long roots is normalized to 2 . Consistency of the model then implies the condition

$$\lambda \cdot \alpha^{\vee} \in 2\pi Z/k_G \tag{2.31}$$

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for any coroot $\alpha^\vee$. In this normalization the weight lattice is the set of points in R^r whose scalar product with any coroot takes integral values. Eq. (2.31) implies then, that λ should be $2\pi/k_G$ times a vector in the weight lattice. As a point in T^G, λ is defined modulo 2π times the coroot lattice. The allowed conjugacy classes correspond then to points in the weight lattice divided by k_G, modulo the coroot lattice. This is also the characterization of the integrable representations of $\hat{G}$, the affine G algebra at level k_G, which correspond to the primary fields of the WZW model.

We proved that geometrical realization on the group manifold of the Cardy states, is given by the following set of the conjugacy classes

$$C_\mu = \{ h f_\mu h^{-1} = h e^{2i\pi\mu/k_G} h^{-1}, \quad h \in G \}, \tag{2.32}$$

where $\mu \equiv \boldsymbol{\mu} \cdot \boldsymbol{\mathcal{H}}$ is a highest weight representation integrable at level k_G, taking value in the Cartan subalgebra of the G Lie algebra.

2.1.3 Boundary states geometry

In the previous section we checked that maximally symmetric boundary conditions of the WZW model are given by the quantized set of the conjugacy classes. Here we will show for the $SU(2)$ group that Cardy boundary states indeed has the geometry of the above described conjugacy classes.

Given a boundary state (1.139),

$$|a\rangle_C = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} |j\rangle\rangle \tag{2.33}$$

the shape of the brane can be deduced by considering the overlap of the boundary state with the localised bulk state $|\vec{\theta}\rangle$, with $\vec{\theta}$ denoting the three $SU(2)$ angles in some coordinate system. As we will see, the boundary state wave function over the configuration space of all localised bulk states peaks precisely at those states which are localised at positions derived by the effective methods in the previous sections. In the large k limit, the eigen-position bulk state

is given by

$$|\vec{\theta}\rangle = \sum_{j,m,m'} \sqrt{2j+1} \mathcal{D}^j_{mm'}(\vec{\theta})|j,m,m'\rangle\,, \qquad (2.34)$$

where $\mathcal{D}^j_{mm'}$ are the Wigner $\mathcal{D}$-functions:

$$\mathcal{D}^j_{mm'} = \langle jm|g(\vec{\theta})|jm'\rangle, \quad \langle jm|jm'\rangle = \delta_{m,m'} \qquad (2.35)$$

where $|jm\rangle$ are a basis for the spin j representation of $SU(2)$. Using the matrix of the modular transformation S of $SU(2)$ at level k (1.308), we can compute in the large-k limit the ratio of S-matrix elements appearing in the boundary state

$$\frac{S_{aj}}{\sqrt{S_{0j}}} \sim \frac{(2(k+2))^{1/4}}{\sqrt{\pi(2j+1)}} \sin[(2j+1)\hat{\psi}]\,, \qquad (2.36)$$

where, to shorten the notation, we have introduced

$$\hat{\psi} = \frac{(2a+1)\pi}{k+2}\,. \qquad (2.37)$$

Note that in the limit of the large a and k $\hat{\psi} \to \psi_0$ defined in (2.27). Using these results, the overlap between the boundary state and the localised bulk state becomes

$$\langle\vec{\theta}|a\rangle_C \sim \sum_{j,m} \frac{(2(k+2))^{1/4}}{\sqrt{\pi}} \sin[(2j+1)\hat{\psi}]\mathcal{D}^j_{mm}(g(\vec{\theta})\,. \qquad (2.38)$$

Finally, one needs the property of the Wigner D-functions that

$$\sum_n \mathcal{D}^j_{nn}(g) = \frac{\sin(2j+1)\psi}{\sin\psi}\,, \qquad (2.39)$$

where ψ is defined by the relation $\text{Tr}g = 2\cos\psi$. The overlap (2.38) becomes

$$\langle\vec{\theta}|a\rangle_C \sim \frac{(2k+4)^{1/4}}{\sqrt{\pi}\sin\psi} \sum_j \sin[(2j+1)\hat{\psi}]\sin[(2j+1)\psi] \qquad (2.40)$$

and from the completeness of $\sin(n\psi)$ on the interval $[0,\pi]$ one concludes

$$\langle\vec{\theta}|a\rangle_C \sim \frac{\sqrt{\pi}(k+2)^{1/4}}{2^{7/4}\sin\psi}\delta(\psi-\hat{\psi})\,. \qquad (2.41)$$

Hence we see that the brane wave function in the large k limit is localized on $\psi = 2\pi\frac{a}{k}$. as required by (2.27).

2.2 Parafermionic branes

2.2.1 Lagrangian construction

Let us consider D-brane in the WZW model with a group G as a product of the conjugacy class with the $U(1)$ subgroup [156]:

$$g|_{\text{boundary}} = LC = Lhfh^{-1} \tag{2.42}$$

where f is defined in (2.32), $L = e^{i\alpha Y} \in U(1)_Y$ and Y is an arbitrary but fixed generator in the Cartan subalgebra of G. We should check that on this subset exists a two-form $\omega^{(2)}$ satisfying the condition (2.4):

$$d\omega^{(2)} = \omega^{\text{WZW}}|_{\text{boundary}} \tag{2.43}$$

It may be easily checked using the Polyakov-Wiegmann identity (1.279):

$$\omega^{\text{WZW}}(LC) = \omega^{\text{WZW}}(L) + \omega^{\text{WZW}}(C) - d\operatorname{Tr}(L^{-1}dLdCC^{-1}) \tag{2.44}$$

Using that for the abelian group, L, $\omega^{\text{WZW}}(L) = 0$, and

$$\omega^{\text{WZW}}(C) = d\omega^f(h) = d\left(\operatorname{Tr}\left(h^{-1}dhfh^{-1}dhf^{-1}\right)\right) \tag{2.45}$$

we get that indeed

$$\omega^{\text{WZW}}|_{\text{boundary}} = d\omega^{(2)}(L, h) \tag{2.46}$$

where

$$\omega^{(2)}(L, h) = \omega_f(h) - \operatorname{Tr}(L^{-1}dLdCC^{-1}) \tag{2.47}$$

where $\omega_f(h)$ is defined in (2.6). Now the action is

$$S^{\text{pf}-\text{br}} = S^{\text{WZW}}(g) - \frac{k}{4\pi}\int_D \omega^{(2)}(L, h) \tag{2.48}$$

Here we use the setup and notations of the previous section.

Let us show that the action (2.48) is invariant under the following symmetries

1.

$$g(z, \bar{z}) \rightarrow h_L(z)g(z, \bar{z})h_R(\bar{z}) \tag{2.49}$$

with $h_L(z)|_{\text{boundary}} = h_R(\bar{z})_{\text{boundary}} = k(\tau)$, $k \in U(1)_Y$. Under this transformation $L \rightarrow kLk$ and $C \rightarrow k^{-1}Ck$ and $h \rightarrow k^{-1}h$.

2.

$$g(z, \bar{z}) \rightarrow h_L(z)g(z, \bar{z})h_R^{-1}(\bar{z}) \tag{2.50}$$

with $h_L(z)|_{\text{boundary}} = h_R(\bar{z})_{\text{boundary}} = k(\tau)$, where k satisfies the conditions $[k, L] = 0$, $\text{Tr}(Yk^{-1}dk) = 0$. Under this transformation $C \rightarrow kCk^{-1}$.

Under the transformation (2.49), as before the change in the L^{kin} term is canceled by the corresponding Σ integral of the boundary term from the change in the ω^{WZW} term. In the presence of a world sheet boundary there remains the contribution from D to the latter change

$$\Delta(S^{\text{WZW}}) = \frac{k}{4\pi} \int_D \text{Tr}[k^{-1}dk(g^{-1}dg - gk^{-1}dkg^{-1} - dgg^{-1})] \tag{2.51}$$

where $g = LC$. Substituting this value in (2.51) we get

$$\Delta S^{\text{WZW}} = \frac{k}{4\pi} \int_D \text{Tr}[k^{-1}dk(C^{-1}dC - Ck^{-1}dkC^{-1} + C^{-1}L^{-1}dLC - dLL^{-1} - dCC^{-1})] \tag{2.52}$$

Now we compute $\omega^{(2)}(kLk, k^{-1}h) - \omega^{(2)}(L, h)$ using that

$$\omega_f(k^{-1}h) - \omega_f(h) = \text{Tr}[k^{-1}dk(Ck^{-1}dkC^{-1} + C^{-1}dC + dCC^{-1})] \tag{2.53}$$

and

$$\text{Tr}[(kLk)^{-1}d(kLk)d(k^{-1}Ck)k^{-1}C^{-1}k - L^{-1}dLdCC^{-1}] \tag{2.54}$$
$$= \text{Tr}[k^{-1}dk(2dCC^{-1} + 2Ck^{-1}dkC^{-1} + L^{-1}dL - C^{-1}L^{-1}dLC)]$$

resulting in

$$\omega^{(2)}(kLk, k^{-1}h) - \omega^{(2)}(L, h) = \text{Tr}[k^{-1}dk(C^{-1}dC - Ck^{-1}dkC^{-1} - \tag{2.55}$$
$$dCC^{-1} - L^{-1}dL + C^{-1}L^{-1}dLC)]$$

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which cancels (2.52).

By the same arguments under (2.50):

$$\Delta(S^{\mathrm{WZW}}) = \frac{k_G}{4\pi} \int_D \mathrm{Tr}[k^{-1}dk(gk^{-1}dkg^{-1} - g^{-1}dg - dgg^{-1})], \qquad (2.56)$$

where $g = LC$. Substituting this value of g in (2.56) we get:

$$\Delta(S^{\mathrm{WZW}}) = \frac{k_G}{4\pi} \int_D \mathrm{Tr}[k^{-1}dk(LCk^{-1}dkC^{-1}L^{-1} - \qquad (2.57)$$
$$C^{-1}L^{-1}(dLC + LdC) - (dLC + LdC)C^{-1}L^{-1})],$$

and using $[k, L] = 0$ and cyclic permutation under the trace we obtain:

$$\Delta(S^{\mathrm{WZW}}) = \frac{k_G}{4\pi} \int_D \mathrm{Tr}[k^{-1}dk(Ck^{-1}dkC^{-1} - C^{-1}L^{-1}dLC - C^{-1}dC - dLL^{-1} - dCC^{-1})]. \quad (2.58)$$

Now we compute $\omega^{(2)}(L, kh) - \omega^{(2)}(L, h)$, using that

$$\omega_f(kh) - \omega_f(h) = \mathrm{Tr}[k^{-1}dk(Ck^{-1}dkC^{-1} - C^{-1}dC - dCC^{-1})] \qquad (2.59)$$

and

$$\mathrm{Tr}[L^{-1}dLd(kCk^{-1})kC^{-1}k^{-1} - L^{-1}dLdCC^{-1}] = \mathrm{Tr}[L^{-1}dLdkk^{-1} - L^{-1}dLCk^{-1}dkC^{-1}], \quad (2.60)$$

resulting in

$$\omega^{(2)}(L, kh) - \omega^{(2)}(L, h) = \mathrm{Tr}[k^{-1}dk(Ck^{-1}dkC^{-1} - C^{-1}dC - dCC^{-1} \qquad (2.61)$$
$$+ L^{-1}dL - C^{-1}L^{-1}dLC)].$$

Collecting (2.58) and (2.61) we obtain:

$$\Delta S^{\mathrm{pf-br}} = \frac{k_G}{2\pi} \int_D \mathrm{Tr}(L^{-1}dLk^{-1}dk). \qquad (2.62)$$

Noting, that $\mathrm{Tr}(Yk^{-1}dk) = 0$, we prove that the action (2.48 possesses by the vectorially diagonal symmetry commuting with $U(1)_Y$. For $G = SU(N + 1)$ this commuting symmetry is $SU(N)$ composed from generators commuting with Y. We also see from (2.62) that the vectorially diagonal $U(1)_Y$ symmetry is broken. Thus we obtained the preserved symmetries of the parafermionic brane:

$$J^a = \bar{J}^a, \quad T^a \in SU(N) \qquad (2.63)$$

$$J^Y = -\bar{J}^Y \qquad (2.64)$$

2.2.2 Geometry

Here we elaborate geometry of the parafermionic branes (2.42) for $SU(2)$ group [157]. Brane is given by the conjugacy class multiplied by the $U(1)_{\sigma_3}$ group: $g = hfh^{-1}e^{i\alpha\frac{\sigma_3}{2}} \equiv \mathcal{CL}$. The geometry of the image can be determined as in [157]. Using the fact that $\mathrm{Tr}\,\mathcal{C} = \mathrm{Tr}f = \mathrm{const} = 2\cos\psi_0$, where ψ_0 is defined in (2.27), we can write

$$\mathrm{Tr}\left(ge^{-i\alpha\frac{\sigma_3}{2}}\right) = 2\cos\psi_0\,. \tag{2.65}$$

From here we see that the element g belongs to the image of the brane surface if and only if there is a $U(1)$ element $(e^{i\alpha\frac{\sigma_3}{2}})$ such that the equation (2.65) is satisfied. So let us determine for which g this equation admits solutions for α. Introduce the Euler parametrization

$$g = e^{i\chi\frac{\sigma_3}{2}}e^{i\tilde{\theta}\frac{\sigma_1}{2}}e^{i\varphi\frac{\sigma_3}{2}} \tag{2.66}$$

Expanding the exponentials one obtains

$$g = \begin{pmatrix} \cos\theta e^{i\tilde{\phi}} & \sin\theta e^{i\phi} \\ -\sin\theta e^{-i\phi} & \cos\theta e^{-i\tilde{\phi}} \end{pmatrix} \tag{2.67}$$

where

$$\chi = \tilde{\phi} + \phi\,, \quad \varphi = \tilde{\phi} - \phi\,, \quad \theta = \frac{\tilde{\theta}}{2}\,. \tag{2.68}$$

The equation (2.65) takes the form

$$\cos\theta\cos(\tilde{\phi} - \frac{\alpha}{2}) = \cos\psi_0\,, \tag{2.69}$$

or equivalently,

$$0 \leq \cos^2(\tilde{\phi} - \frac{\alpha}{2}) = \frac{\cos^2\psi_0}{\cos^2\theta} \leq 1\,. \tag{2.70}$$

Hence, equation (2.70) can be solved for α only when $\cos^2\theta \geq \cos^2\psi_0$, or equivalently when

$$\cos\tilde{\theta} \geq \cos 2\psi_0\,, \quad \tilde{\theta} = 2\theta\,. \tag{2.71}$$

We see that generic parafermionic brane on $SU(2)$ is a three-dimensional surface defined by the inequality (2.71). For $\psi_0 = 0$ it is one-dimensional circle.

2.2.3   Boundary state

Boundary state for the parafermionic branes were constructed in [123, 124].

Let us start by reviewing the T-duality between a Lens space and the $SU(2)$ theory. Geometrically, a Lens space is obtained by quotienting the group manifold by the right action of the subgroup Z_k of the $U(1)$, and in the Euler coordinates (2.66) it corresponds to the identification $\varphi \sim \varphi + \frac{4\pi}{k}$. In terms of the $SU(2)$ WZW model this is the orbifold $SU(2)/Z_k^R$, where Z_k^R is embedded in the right $U(1)$. The partition function for this theory can be derived using the technique elaborated in 1.4.3 and it is

$$Z = \sum_j \chi_j^{SU(2)}(q)\chi_{jn}^{PF}(\bar{q})\psi_{-n}^{U(1)}(\bar{q}) \tag{2.72}$$

Partition function (2.72) coincides with the one for the $SU(2)$ group, up to T-duality. This relation enables one to construct new D-branes in the $SU(2)$ theory starting from the known ones. As a first step one constructs the brane in the Lens theory. As is usual for orbifolds, this is achieved by summing over images of D-branes under the right Z_k multiplications. Performing then the T-duality on the Lens theory brings us back to the $SU(2)$ theory and maps the orbifolded brane to a new $SU(2)$ brane.

Our starting point is a maximally symmetric Cardy state. If we shift the brane by the right multiplication with some element $\omega^l = e^{\frac{2\pi l i}{k}\sigma_3}$ of the Z_k^R group, then the symmetries preserved by this brane, as we explained at the end of section 2.1.1, are[*]

$$J^a + \omega^l \bar{J}^a \omega^{-l} = 0\,, \quad (a = 1, 2, 3)\,, \tag{2.73}$$

while the brane is described by the Cardy state with rotated Ishibashi state

$$|A, a\rangle_C^{\omega^l} = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} \sum_N |j, N\rangle \otimes (\omega^l \overline{|j, N\rangle})\,. \tag{2.74}$$

Summing over the images one obtains a Z_k^R invariant state, present in the Lens theory

$$\sum_{l=0}^k |A, a\rangle_C^{\omega^l} = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} \sum_{l=0}^k \sum_N |j, N\rangle \otimes (\omega^l \overline{|j, N\rangle})\,. \tag{2.75}$$

---

[*]In this subsection we write gluing conditions in the closed string channel.

To compute the sum of the Ishibashi states on the right-hand side, one next uses the parafermion decomposition of $SU(2)_k$ (1.331). This decomposition implies that Ishibashi states for the maximally symmetric A-brane can be written as

$$|A, j\rangle\rangle^{SU(2)} = \sum_{n=1}^{2k} \frac{1 + (-1)^{2j+n}}{2} |A, j, n\rangle\rangle^{PF} \otimes |A, n\rangle\rangle^{U(1)}, \qquad (2.76)$$

where

$$|A, j, n\rangle\rangle^{PF} = \sum_{N} |j, n, N\rangle \otimes \overline{|j, n, N\rangle}, \qquad (2.77)$$

is the A-type Ishibashi states for the parafermion and $|A, n\rangle\rangle^{U(1)}$ is defined in (1.260).

Under the action of element $\omega^l \in Z_k^R$ the expression (2.76) transform as

$$|A, j\rangle\rangle^{SU(2)} \rightarrow \sum_{n=1}^{2k} \frac{1 + (-1)^{2j+n}}{2} e^{\frac{2\pi i l n}{k}} |A, j, n\rangle\rangle^{PF} \otimes |A, n\rangle\rangle^{U(1)}. \qquad (2.78)$$

Hence summing over images projects onto the Z_k^R-invariant Ishibashi states for which n is restricted to the two values 0 and k. Performing T-duality, flips the sign of the right moving $U(1)$ sector and one gets a B-type Ishibashi state of the original $SU(2)$ theory,

$$|B, j\rangle\rangle^{SU(2)} = \left[\frac{1 + (-1)^{2j}}{2} |A, j, 0\rangle\rangle^{PF} \otimes |B, 0\rangle\rangle^{U(1)} + \right. \qquad (2.79)$$
$$\left. \frac{1 + (-1)^{2j+k}}{2} |A, j, k\rangle\rangle^{PF} \otimes |B, k\rangle\rangle^{U(1)} \right],$$

where $|Br\rangle\rangle^{U(1)}$ is a B-type Ishibashi state of $U(1)_k$ theory satisfying the Neumann boundary conditions defined in (1.262). Knowing the T-dual expression of the (2.75) allows one to write down the boundary state for the B-type brane

$$|B, a\rangle_C^{SU(2)} = \sum_{j \in Z} \frac{\sqrt{k} S_{aj}}{\sqrt{S_{0j}}} |Aj, 0\rangle\rangle^{PF} \otimes (|B0\rangle\rangle^{U(1)} + \eta |Bk\rangle\rangle^{U(1)}). \qquad (2.80)$$

where $\eta = (-1)^{2a}$. In deriving this expression one uses the field identification rule $(j, n) \sim (k/2 - j, k + n)$ and the following property of the matrix of modular transformation (1.308)

$$S_{a,k/2-j} = (-1)^{2a} S_{aj}. \qquad (2.81)$$

To derive the symmetries preserved by the B-brane, one observes from (2.73) that a Z_k^R invariant superposition of the A-branes preserves only the current $J^3 + \bar{J}^3$ and breaks all other currents;

namely, any two Z_k^R images only have this preserved current in common. Performing further T-duality in the $\bar{J}^3$ direction flips the relative sign between the two terms in this current and hence implies that the only current preserved by the B-brane is

$$J^3 - \bar{J}^3 = 0 \,. \tag{2.82}$$

2.2.4 Overlap of the state and the coordinate wave function

We will now show that the boundary state (2.80) reproduces the effective brane geometry (2.71). In the large k limit the second term in (2.80) can be ignored. As in the case of Cardy state one should compute the overlap $\langle \vec{\theta} | B, a \rangle_C^{SU(2)}$. We will again use the formula (2.34), but taking into account that the matrix $\mathcal{D}$ has left and right indices 0. Therefore, the overlap is again given by formula (2.38), but with m set to zero. Hence we arrive at the equation

$$\langle \vec{\theta} | B, a \rangle_C^{SU(2)} \sim \sum_{j \in \mathbb{Z}} \frac{k^{3/2}}{\pi} \sin[(2j+1)\hat{\psi}] \, \mathcal{D}_{00}^j(g(\vec{\theta})) \tag{2.83}$$

where $\hat{\psi}$ is defined in (2.37).

Next we will need the relation between the Wigner D-functions and the Legendre polynomials $P_j(\cos \tilde{\theta})$ given by $\mathcal{D}_{00}^j = P_j(\cos \tilde{\theta})$, where $\tilde{\theta}$ introduced in (2.66), as well as the formula for the generating function for Legendre polynomials

$$\sum_n t^n P_n(x) = \frac{1}{\sqrt{1 - 2tx + t^2}} \,. \tag{2.84}$$

Using these expressions equation (2.83) can be simplified to

$$\langle \vec{\theta} | B, a \rangle_C^{SU(2)} \sim \frac{\Theta(\cos \tilde{\theta} - \cos 2\hat{\psi})}{\sqrt{\cos \tilde{\theta} - \cos 2\hat{\psi}}} \,, \tag{2.85}$$

where Θ is the step function. This indeed coincides with the expression for the effective geometry (2.71) in the large k limit.

2.3 Permutation branes

Here we present Lagrangian description of the permutation branes (1.159) and (1.161).

2.3.1 Definition of the brane

Let us consider a group manifold M, which is a product of $K+1$ copies of a group G: $M = G \times \cdots \times G$. We define the *maximally symmetric, permutational* brane by the following formula [66]

$$
\begin{aligned}
(g_0, g_1, \cdots, g_K)|_{\text{brane}} = {}& \left\{ (h_0 f_0 h_1^{-1}, h_1 f_1 h_2^{-1}, h_2 f_2 h_3^{-1} \cdots h_{K-1} f_{K-1} h_K^{-1}, h_K f_K h_{K+1}^{-1}) \right. \\
& \left. \Big| \quad h_0 = h_{K+1}, \forall h_i \in G, \quad (i = 1, \cdots, K+1) \right\}.
\end{aligned}
\tag{2.86}
$$

where g_i denotes an element of the i'th copy of G in target space, and $|_{\text{brane}}$ denotes the restriction to the brane surface. It is easy to see that by redefinition of the elements h_i one can always bring an arbitrary brane $(f_0, \ldots, f_K)$ into the form $(f_0 f_1 \cdots f_K, e, \ldots e)$, where e is the identity element. Hence, another, more convenient form of writing the equation (2.86) is

$$
(g_0, g_1, \cdots, g_K)|_{\text{brane}} =
\tag{2.87}
$$

$$
\left\{ (h_0 f h_0^{-1} g_K^{-1} \cdots g_1^{-1}, g_1, \cdots, g_K) \right|
\tag{2.88}
$$

$$
f \equiv f_0 f_1 \cdots f_k \,, \; \forall h_0, g_i \in G, \, (i = 1, \cdots, K) \Big\}.
$$

The dimension of this brane can easily be determined by looking at the image of (2.86) under the map $m : M = G \times \ldots \times G \to G$, defined by $m(g_0, g_1, \ldots, g_K) = g_0 g_1 \ldots g_K \equiv g$ [66]. The map m maps the brane (2.86) to the conjugacy class

$$
m(g_0, g_1, \ldots, g_k)|_{\text{brane}} = \mathcal{C} = \{ C = h_0 f h_0^{-1} | h_0 \in G \}.
\tag{2.89}
$$

The global issues again require the conjugacy class (2.89) to have the form (2.32).

Next, note that the inverse of a point under m is diffeomorphic to G^K. To see this, observe that for any element of the form

$$
h \equiv (f_0 h_1^{-1}, h_1 f_1 h_2^{-1}, h_2 f_2 h_3^{-1}, \ldots, h_K^{-1} f_K), \quad (\forall h_i \in G)
\tag{2.90}
$$

the relation $m(h) = m(f_0 \ldots f_K)$ holds. Hence altogether we see that the dimension D of a generic brane $(f_0, \ldots f_K)$ is given by

$$
D = \dim \mathcal{C} + K \dim G.
\tag{2.91}
$$

As we explained before to write the Lagrangian of the WZW theory of the product group M with a boundary condition specified by a permutation brane, the restriction of the WZW form to the worldvolume of the brane should satisfy (2.4). Using the Polyakov-Wiegmann identities (1.278) and (1.279) and the relation (2.86) it is easy to see that

$$\sum_{i=0}^{K} \omega^{\mathrm{WZ}}(g_i)\Big|_{\mathrm{brane}} = d\omega^{(2)} \tag{2.92}$$

where

$$\omega^{(2)} = \sum_{i=0}^{K} \mathrm{Tr}\left(f_i^{-1} h_i^{-1} \mathrm{d}h_i f_i h_{i+1}^{-1} \mathrm{d}h_{i+1} \right), \tag{2.93}$$

which will be used in the following sections. The global issues also here constrain the conjugacy class $\mathcal{C}$ in (2.89) to have the form (2.32).

2.3.2 Symmetries of the brane

Next we want to determine the symmetries preserved by the brane (2.86). The boundary conditions (2.86) are invariant under any transformation of the form

$$g_i \to g_i k_i^{-1}, \quad g_{i+1} \to k_i g_{i+1}, \tag{2.94}$$

$$g_0 \to k g_0, \quad g_K \to g_K k^{-1}, \quad (k_i, k \in G), \ (i = 0, 1, \ldots, (K-1)), \tag{2.95}$$

which in our parametrisation correspond to the transformations

$$h_{i+1} \to k_i h_{i+1}, \quad h_{k+1} \to k h_{k+1}, \tag{2.96}$$

respectively. We will now show that the full action

$$S^{\mathrm{per-WZW}} = \sum_{i=0}^{k} S^{\mathrm{WZW}}(g_i) - \frac{k}{4\pi} \int_D \omega^{(2)} \tag{2.97}$$

with boundary condition (2.86) and $\omega^{(2)}$ given in (2.93) is invariant under the following transformations

$$g_i(z, \bar{z}) \to g_i(z, \bar{z}) k_{iR}^{-1}(\bar{z}),$$

$$g_{i+1}(z, \bar{z}) \to k_{iL}(z) g_{i+1}(z, \bar{z}), \quad (k_i \in G), \ (i = 0, 1, \ldots, K-1), \tag{2.98}$$

$$k_{iL}(z)|_{\mathrm{boundary}} = k_{iR}(\bar{z})|_{\mathrm{boundary}} = k_i(\tau),$$

as well as

$$g_0(z, \bar{z}) \to k_L(z) g_0(z, \bar{z}),$$

$$g_K(z, \bar{z}) \to g_K(z, \bar{z}) k_R^{-1}(\bar{z}),$$

$$k_L(z)|_{\text{boundary}} = k_R(\bar{z})|_{\text{boundary}} = k(\tau), \quad (k \in G).$$

(2.99)

For fixed i, in order to determine the variation of the action, we only need to consider the following terms

$$S(g_i, g_{i+1}) = S^{\text{WZW}}(g_i) + S^{\text{WZW}}(g_{i+1}),$$

(2.100)

$$\omega^{(2)}(h_{i+1}) = \text{Tr}\left(f_i^{-1} h_i^{-1} \mathrm{d}h_i f_i h_{i+1}^{-1} \mathrm{d}h_{i+1} + f_{i+1}^{-1} h_{i+1}^{-1} \mathrm{d}h_{i+1} f_{i+1} h_{i+2}^{-1} \mathrm{d}h_{i+2} \right).$$

(2.101)

The variation of the kinetic and Wess-Zumino terms in the action can be read off from (1.278) and (1.279). Using the fact that, due to the (anti-)holomorphicity, $\omega^{WZ}(k_{iR/L}) = 0$, one deduces that the variation of the Wess-Zumino term reduces to a surface integral over the disc D and the string worldsheet Σ. The integral over the string world sheet is canceled by the corresponding Σ integral coming from the variation of the kinetic term. The remaining integral over the disc is

$$\Delta(S(g_i, g_{i+1})) = -\frac{k}{4\pi} \int_D \text{Tr}\left(k_i^{-1} \mathrm{d}k_i (g_i^{-1} \mathrm{d}g_i + \mathrm{d}g_{i+1} g_{i+1}^{-1}) \right).$$

(2.102)

Substituting $g_i = h_i f_i h_{i+1}^{-1}$ and $g_{i+1} = h_{i+1} f_{i+1} h_{i+2}^{-1}$ we obtain

$$\Delta(S(g_i, g_{i+1})) =$$

(2.103)

$$\frac{k}{4\pi} \int_D \text{Tr}\left(k_i^{-1} \mathrm{d}k_i (h_{i+1} f_{i+1} h_{i+2}^{-1} \mathrm{d}h_{i+2} f_{i+1}^{-1} h_{i+1}^{-1} - h_{i+1} f_i^{-1} h_i^{-1} \mathrm{d}h_i f_i h_{i+1}^{-1}) \right).$$

This term is canceled by the variation of the two-form term in (2.101). Computing the change of (2.101) we find

$$\omega^{(2)}(k_i h_{i+1}) - \omega^{(2)}(h_{i+1}) =$$

(2.104)

$$\text{Tr}\left(k_i^{-1} \mathrm{d}k_i (h_{i+1} f_{i+1} h_{i+2}^{-1} \mathrm{d}h_{i+2} f_{i+1}^{-1} h_{i+1}^{-1} - h_{i+1} f_i^{-1} h_i^{-1} \mathrm{d}h_i f_i h_{i+1}^{-1}) \right)$$

which cancels (2.103). The proof of the invariance of the action (2.97) under the variation (2.99) is similar.

Having determined the symmetries of the brane (2.86) we can now turn to the question of which bulk currents are preserved by this brane. The invariance of the manifold M under separate left/right group multiplication in each subgroup gets lifted, on the world sheet of a *closed string*, to a *local* infinite-dimensional symmetry group $M(z) \times M(\bar{z})$. The presence of these symmetries implies the existence of the conserved currents $J_i(z) = -\partial g_i g_i^{-1}$ and $\bar{J}_i(\bar{z}) = g_i^{-1} \bar{\partial} g_i$ $(i = 0, 1, \ldots, K)$. As we have seen, the symmetries under separate left/right group multiplication are, in the presence of the worldsheet boundary, reduced to symmetries under *simultaneous* multiplication (2.98) and (2.99). This implies the following relations between the currents,

$$\bar{J}_i^a = J_{i+1}^a, \quad (i = 0, \ldots K - 1), \tag{2.105}$$

$$J_0^a = \bar{J}_K^a, \quad \forall T^a \in \mathrm{Lie(G)}. \tag{2.106}$$

These are gluing conditions of permutation branes (1.158) for WZW model.

2.3.3   Effective geometry for maximally symmetric, permutation branes

Let us consider an explicit example of the maximally symmetric permutation branes (2.86) for the case of double product with the group $G = SU(2)$, i.e. $M = SU(2) \times SU(2)$. In this case the general formula (2.86) reduces to

$$(g_0, g_1)\Big|_{\mathrm{brane}} = (h_0 f_0 h_1^{-1}, \, h_1 f_1 h_0^{-1}). \tag{2.107}$$

where $f_0 f_1 = e^{\frac{2i\pi a \sigma_3}{k}}$, with a integer or half-integer satisfying $0 \le a \le k/2$.

The preserved currents are

$$J_0^a = \bar{J}_1^a, \quad J_1^a = \bar{J}_0^a \quad (a = 1, 2, 3). \tag{2.108}$$

The general expression for two form $\omega^{(2)}$ given in (2.93) reduces to

$$\omega^{(2)}(h_0, h_1) = \mathrm{Tr}\Big(h_0^{-1} \mathrm{d}h_0 (f_0 h_1^{-1} \mathrm{d}h_1 f_0^{-1} - f_1^{-1} h_1^{-1} \mathrm{d}h_1 f_1)\Big). \tag{2.109}$$

Using the constructed boundary states we then calculate the effective geometries of these branes, recovering the classical results from the previous sections.

Recall the permutation brane boundary state (1.161)

$$|a\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{S_{0j}} |j,j\rangle\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{S_{0j}} |j\rangle\rangle^{SU(2)_0 \times SU(2)_{\bar{1}}} |j\rangle\rangle^{SU(2)_{\bar{0}} \times SU(2)_1}. \tag{2.110}$$

$$|j\rangle\rangle^{SU(2)_0 \times SU(2)_{\bar{1}}} = \sum_N |j,N\rangle_0 \otimes \overline{|j,N\rangle}_1 \tag{2.111}$$

$$|j\rangle\rangle^{SU(2)_{\bar{0}} \times SU(2)_1} = \sum_M |j,M\rangle_1 \otimes \overline{|j,M\rangle}_0 \tag{2.112}$$

Using the S matrix of the modular transformation for $SU(2)_k$ (1.308), one obtains in the large-k limit the ratio of S-matrix elements appearing in the defect operator

$$\frac{S_{aj}}{S_{0j}} \sim \frac{(k+2)}{\pi(2j+1)} \sin[(2j+1)\hat{\psi}], \tag{2.113}$$

where, as before, we have introduced $\hat{\psi} = \frac{(2a+1)\pi}{k+2}$. Using these results, the overlap between the boundary state (2.110) and the localised bulk state (2.34) becomes

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle_{\mathcal{P}} \sim \sum_{j,m,n} \frac{(k+2)}{\pi} \sin[(2j+1)\hat{\psi}] \mathcal{D}^j_{nm}(g_0(\vec{\theta}_0)) \mathcal{D}^j_{mn}(g_1(\vec{\theta}_1)). \tag{2.114}$$

To simplify this expression we need the identity

$$\sum_m \mathcal{D}^j_{nm}(g_1(\vec{\theta}_1)) \mathcal{D}^j_{mn'}(g_2(\vec{\theta}_2)) = \mathcal{D}^j_{nn'}(g_1(\vec{\theta}_1)g_2(\vec{\theta}_2)), \tag{2.115}$$

which follows from the fact that the matrices $\mathcal{D}^j_{nm}$ form a representation of the group. Remembering (2.39) we obtain

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle_{\mathcal{P}} \sim \frac{k+2}{\pi \sin \psi} \sum_j \sin[(2j+1)\hat{\psi}] \sin[(2j+1)\psi] \tag{2.116}$$

where

$$\mathrm{Tr}(g_0 g_1) = 2 \cos \psi \tag{2.117}$$

and from the completeness of $\sin(n\psi)$ on the interval $[0,\pi]$ one concludes

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle_{\mathcal{P}} \sim \frac{k+2}{4 \sin \psi} \delta(\psi - \hat{\psi}). \tag{2.118}$$

Hence we see that the brane wave function has the geometry (2.107).

2.4 The symmetry breaking brane of type I on a product $G \times G$

In this section we construct new non-maximally symmetric non-factorizable branes in the WZW model with a product group $G \times G$ following [160].

2.4.1 Definition and symmetries

We define the boundary conditions of the type I brane as

$$\text{I}: \quad (g_0, g_1)\Big|_{\text{brane}} = \left\{ (h_0 f_0 h_1^{-1}, \, h_1 f_1 h_0^{-1} L) \Big| \,, \, \forall L \equiv e^{i\alpha Y} \in U(1)_Y \right\} \tag{2.119}$$

where Y is an arbitrary (but fixed) generator in the Cartan subalgebra of G. As before, in order to fully specify the consistent D-brane we need to determine the worldvolume two-form $\omega^{(2)}$. We can reduce this calculation to the one which we did for (2.86) by introducing variables $K_0 = h_0 f_0 h_1^{-1}$ and $K_1 = h_1 f_1 h_0^{-1}$. We have already shown that

$$\omega^{\text{WZ}}(K_0)\Big|_{\text{brane}} + \omega^{\text{WZ}}(K_1)\Big|_{\text{brane}} = \mathrm{d}\omega^{(2)}(h_0, h_1)\Big|_{\text{brane}} \tag{2.120}$$

with $\omega^{(2)}(h_0, h_1)$ given in (2.109). Using (1.279) and the property that $\omega^{\text{WZ}}(L) = 0$ for abelian groups, we further get that

$$\omega^{\text{WZ}}(K_1 L)\Big|_{\text{brane}} = \omega^{\text{WZ}}(K_1)\Big|_{\text{brane}} - \mathrm{d}\Big(\mathrm{Tr}(K_1^{-1} \mathrm{d}K_1 \mathrm{d}LL^{-1}) \Big). \tag{2.121}$$

Combining (2.120) and (2.121) we finally obtain

$$\omega^{\text{WZ}}(g)\Big|_{\text{brane}} = \mathrm{d}\omega^{(2)}(h_0, h_1, L) = \mathrm{d}\Big(\omega^{(2)}(h_0, h_1) - \mathrm{Tr}(K_1^{-1} \mathrm{d}K_1 \mathrm{d}LL^{-1}) \Big). \tag{2.122}$$

The full action with the boundary condition (2.119) is:

$$S^{BR-I} = S^{\text{WZW}}(g_0) + S^{\text{WZW}}(g_1) - \int_D \omega^{(2)}(h_0, h_1, L) \tag{2.123}$$

To determine the symmetries preserved by the brane we first look for the symmetries preserved by the boundary (2.119):

1. $g_0 \to g_0 k^{-1}$, $g_1 \to k g_1$ for all $k \in G$; under this transformation $h_1 \to k h_1$ and $K_1 \to k K_1$.

2. $g_0 \to k g_0$, $g_1 \to g_1 k^{-1}$ for all $k \in G$, $k \notin U(1)_Y$ and $[k, L] = 0$. Under this transformation $h_0 \to k h_0$. This means that for example, in the case of $G = SU(N + 1)$ we get that $k \in SU(N)$ generated with isospin generators commuting with Y.

3. $g_0 \to k g_0$, $g_1 \to g_1$ for all $k \in U(1)_Y$. Under this transformation $h_0 \to k h_0$ and $L \to kL$.

4. $g_0 \to g_0$, $g_1 \to g_1 k$ for all $k \in U(1)_Y$. Under this transformation $L \to Lk$.

When extending these transformations to transformations of the action (as in equations (2.98) and (2.99)) one can show that the *full* action (2.123) with the boundary term on the auxiliary disc given by (2.122) is invariant separately under the transformations 1 and 2. On the other hand, only the following *combination* of the transformations 3 and 4 is a real symmetry of the full action:

3'. $g_0 \to k g_0$, $g_1 \to g_1 k$ where $k \in U(1)_Y$. Under this transformation $h_0 \to k h_0$, $L \to kLk$.

The details of all of these calculation can be found in [160]. The set of symmetries listed above implies that the D-brane (2.119) preserves the following set of currents,

$$\bar{J}_0{}^a = J_1^a, \quad \forall T^a \in \mathrm{Lie(G)} \tag{2.124}$$

$$J_0^a = \bar{J}_1{}^a, \quad \forall T^a \in \mathrm{Lie(G)} \quad \text{s.t.} \quad [\mathrm{T^a}, \mathrm{Y}] = 0, \tag{2.125}$$

$$J_0^Y = -\bar{J}_1^Y. \tag{2.126}$$

We see that multiplication of the second group with the $U(1)_L$ subgroup leads to a removal of some of the currents present in the symmetric brane (2.106) and, as expected, also flips the sign of the current in the Y-direction.

2.4.2 Geometry of type I brane on an $SU(2) \times SU(2)$

Next we want to determine the geometry of the type I brane (2.119) on an $SU(2) \times SU(2)$ manifold,

$$(g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1}, h_1 f_1 h_0^{-1} L), \tag{2.127}$$

where we will take L to be of the form $L = e^{i\alpha \frac{\sigma_3}{2}}$. In this case, the preserved currents (2.124)–(2.126) reduce to

$$J_0^3 = -\bar{J}_1^3 , \quad \bar{J}_0^a = J_1^a , \quad (a = 1, 2, 3) . \tag{2.128}$$

Under the map m of (2.89), the type I brane gets mapped to the conjugacy class multiplied by the $U(1)_{\sigma_3}$ group: $\hat{g} \equiv g_0 g_1 = h_0 f_0 f_1 h_0^{-1} e^{i\alpha \frac{\sigma_3}{2}} \equiv \mathcal{C}L$. In what follows we will always denote with hats those quantities which appear in a product of group elements from the first and the second group. The geometry of the image can be determined as in section 2.2.2. Repeating the same steps we easily obtain that the world-volume of the brane is given by inequality

$$\cos \hat{\tilde{\theta}} \geq \cos 2\psi_0 , \quad \hat{\tilde{\theta}} = 2\hat{\theta} , \tag{2.129}$$

where ψ_0 is defined in (2.27). We see that the image of the brane under multiplication m is a three-dimensional surface defined by the inequality (2.129). To determine the geometry of the full brane, let us denote the Euler angles for elements in g_0 and g_1 with "0" and "1" indices. Then the $\hat{\tilde{\theta}}$ and $\hat{\tilde{\phi}}$ angles of their product are given by [184]

$$\cos \hat{\tilde{\theta}} = \cos \tilde{\theta}_0 \cos \tilde{\theta}_1 - \sin \tilde{\theta}_0 \sin \tilde{\theta}_1 \cos(\chi_1 + \varphi_0) , \tag{2.130}$$

$$e^{i\hat{\tilde{\phi}}} = \frac{e^{i\frac{\chi_0 + \varphi_1}{2}}}{\cos \frac{\hat{\tilde{\theta}}}{2}} \left(\cos \frac{\tilde{\theta}_0}{2} \cos \frac{\tilde{\theta}_1}{2} e^{i\frac{\chi_1 + \varphi_0}{2}} - \sin \frac{\tilde{\theta}_0}{2} \sin \frac{\tilde{\theta}_1}{2} e^{-i\frac{\chi_1 + \varphi_0}{2}} \right) . \tag{2.131}$$

Substituting the expression for $\hat{\tilde{\theta}}$ in the equation for the image of the brane, we see that a generic brane (2.119) is six dimensional and given by the inequality

$$\cos \hat{\tilde{\theta}} = \cos \tilde{\theta}_0 \cos \tilde{\theta}_1 - \sin \tilde{\theta}_0 \sin \tilde{\theta}_1 \cos(\chi_1 + \varphi_0) \geq \cos 2\psi_0 . \tag{2.132}$$

As before, the previous discussion was valid in the cases for which $f_0 f_1 \neq e$. If $\psi_0 = 0$, the conjugacy class $\mathcal{C}$ is a point and the total brane is four dimensional, given by the relations

$$\tilde{\theta}_0 = \tilde{\theta}_1, \quad \chi_1 + \varphi_0 = \pi . \tag{2.133}$$

2.4.3 Boundary states for symmetry breaking type I branes

We now want to construct the boundary state for the brane of type I given in (2.127),

$$(g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1}, h_1 f_1 h_0^{-1} e^{i\alpha \frac{\sigma_3}{2}}) . \tag{2.134}$$

Recall that, as we have derived before using the Langrangian approach, this brane preserves the currents[†]

$$J_0^3 \;-\; \bar{J}_1^3 = 0 \,, \tag{2.135}$$

$$\bar{J}_0^a \;+\; J_1^a = 0 \,, \quad (a = 1, 2, 3) \,. \tag{2.136}$$

To construct the boundary state, our starting point is the maximally symmetric permutation brane (2.110) which preserves the symmetries (2.108). In order to reduce these symmetries down to (2.135), we will now show that one should apply the procedure described in the previous section to the second $SU(2)$ group in which the permutation brane lives. Namely, let us shift the brane (2.110) by multiplying it from the right with an element $\omega_{(2)}^l = e^{\frac{2\pi l i}{k}\sigma_3}$ of the Z_k^R subgroup of the second $SU(2)$ group. The shifted brane preserves the symmetries

$$J_0^a \;+\; \omega_{(2)}^l \bar{J}_1^a \omega_{(2)}^{-l} = 0 \,, \tag{2.137}$$

$$\bar{J}_0^a \;+\; J_1^a = 0 \,, \qquad (a = 1, 2, 3) \,, \tag{2.138}$$

and is given by the Cardy state

$$|a\rangle_C^{\omega_{(2)}^l} = \sum_j \frac{S_{aj}}{S_{0j}} |j\rangle\rangle_\tau^{SU(2)_1 \times SU(2)_{\bar{0}}} \otimes |j\rangle\rangle_\tau^{\omega_{(2)}^l} \,, \tag{2.139}$$

where

$$|j\rangle\rangle_\tau^{\omega_{(2)}^l} = \sum_N |j, N\rangle_0 \otimes (\omega_{(2)}^l \overline{|j, N\rangle}_{\bar{1}}) \,. \tag{2.140}$$

As in the previous section, summing over the images and performing the T-duality in the right sector, will reduce the first set of currents (2.137) down to (2.135), as desired. As far as the boundary state is concerned, summing over images will not touch the first Ishibashi state in (2.139) but will project the second Ishibashi state down to the Z_k^R invariant components. Let us introduce parafermion and $U(1)_k$ permuted Ishibshi states

$$|j_1, n_1\rangle\rangle_\tau^{PF_0 \times PF_{\bar{1}}} \;=\; \sum_N |j_1, n_1, N\rangle_0 \otimes \overline{|j_1, n_1, N\rangle}_1 \,, \tag{2.141}$$

$$|j_2, n_2\rangle\rangle_\tau^{PF_1 \times PF_{\bar{0}}} \;=\; \sum_M |j_2, n_2, M\rangle_1 \otimes \overline{|j_2, n_2, M\rangle}_0 \,.$$

[†]Here we write gluing conditions in the closed string channel.

and

$$|r\rangle\rangle_{\tau\pm}^{U(1)_0 \times U(1)_{\bar{1}}} = \exp\left[\pm\sum_{n=1}^{\infty}\frac{\alpha_{-n}^0\tilde{\alpha}_{-n}^1}{n}\right]\sum_{l\in Z}|\frac{r+2kl}{\sqrt{2k}}\rangle_0 \otimes |\pm\overline{\frac{r+2kl}{\sqrt{2k}}}\rangle_1 \qquad (2.142)$$

$$|r'\rangle\rangle_{\tau\pm}^{U(1)_1 \times U(1)_{\bar{0}}} = \exp\left[\pm\sum_{n=1}^{\infty}\frac{\alpha_{-n}^1\tilde{\alpha}_{-n}^0}{n}\right]\sum_{l'\in Z}|\pm\frac{r'+2kl'}{\sqrt{2k}}\rangle_1 \otimes |\overline{\frac{r'+2kl'}{\sqrt{2k}}}\rangle_0 .$$

Using the permuted version of the decomposition (2.76)

$$|j\rangle\rangle^{SU(2)_0 \times SU(2)_{\bar{1}}} = \sum_{n=1}^{2k}\frac{1+(-1)^{2j+n}}{2}|j,n\rangle\rangle_{\tau}^{PF_0 \times PF_{\bar{1}}} \otimes |n\rangle\rangle_{\tau+}^{U(1)_0 \times U(1)_{\bar{1}}} \qquad (2.143)$$

and applying T-duality to it, one obtains the permuted B-type Ishibashi state of the initial $SU(2)$ theory,

$$|B,j\rangle\rangle_{\tau}^{0\bar{1}} = \frac{1+(-1)^{2j}}{2}|j,0\rangle\rangle_{\tau}^{PF_0 \times PF_{\bar{1}}} \otimes |0\rangle\rangle_{\tau-}^{U(1)_0 \times U(1)_{\bar{1}}} \qquad (2.144)$$
$$+\frac{1+(-1)^{2j+k}}{2}|j,k\rangle\rangle_{\tau}^{PF_0 \times PF_{\bar{1}}}|k\rangle\rangle_{\tau-}^{U(1)_0 \times U(1)_{\bar{1}}} .$$

Here the permutation $U(1)$ and the permutation parafermion Ishibashi states are given in formulas (2.142) and (2.141). Using this expression the Cardy state for a new brane can be written as:

$$|a\rangle_C^{(1)} = \sqrt{k}\sum_{j}\frac{S_{aj}}{S_{0j}}|j\rangle\rangle_{\tau}^{SU(2)_1 \times SU(2)_{\bar{0}}} \otimes |B,j\rangle\rangle_{\tau}^{0\bar{1}} . \qquad (2.145)$$

Note also that since the boundary state (2.145) is "derived" from the maximally symmetric boundary state (2.110), it is characterised with a *single* primary j as was the case for the brane (2.110). This is again related to the fact that in the effective description (2.134), there is only one independent parameter $(f \equiv f_0 f_1)$.[‡]

To check the consistency of the proposed boundary state, one should check, as usual, the Cardy condition. Since we are in a theory which admits several different types of branes, one should in principle check these conditions for the type I brane with any of the other branes in the spectrum. We have done the calculation involving two branes of type I, with one of type I and a permutational brane, and with a brane which is direct product of two $SU(2)$ A-branes. The tree-level amplitude between two Cardy states for two type I branes reduces, after the

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[‡]This can be easily be seen by changing coordinates as $h_0 \to h_0 f_1^{-1}$.

S-modular transformation reduces, to

$$Z_{a_1 a_2} = \sum_{r,j',j''} \sum_{n_1,n_2} N^r_{a_1 a_2} N^{j'}_{rj''} \chi_{j'}(q) \chi_{j'',n_1}(q) \psi_{n_2}(q) \frac{1 + (-1)^{n_1+n_2}}{2} \,, \tag{2.146}$$

hence satisfying the Cardy requirement. The annulus amplitude between the type I and the maximally symmetric permutation brane (2.110) reduces, after the S-modular transformation, to

$$Z_{a_1 a_2} = \sum_{r,j',j''} \sum_{n_1} N^r_{a_1 a_2} N^{j'}_{rj''} \chi_{j'}(q) \chi_{j'',n_1}(q) \frac{q^{1/48}}{\prod_m (1 - q^{m-1/2})} \,, \tag{2.147}$$

Here the factor $\frac{q^{1/48}}{\prod_m (1-q^{m-1/2})}$ is the partition function of a scalar with mixed Neumann-Dirichlet type boundary conditions. The details of calculations of (2.146) and (2.147) can be found in [160].

We will now show that the boundary state (2.145) reproduces the effective brane geometry (2.132). In the large k limit the second term in (2.144) can be ignored. As in section 2.3.3 one should compute the overlap $\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle^{(1)}_C$. We will again use the formula (2.34), but taking into account that the matrix $\mathcal{D}^{(1)}$ derived for the first group has left index 0 and the right index m, whereas the $\mathcal{D}^{(2)}$ matrix derived for the second group has left index m and the right index 0. Therefore, the overlap is again given by formula (2.114), but with n set to zero. Using furthermore (2.115) we arrive at the equation

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle^{(1)}_C \sim \sum_{j \in \mathbb{Z}} \frac{k^{3/2}}{\pi} \sin[(2j+1)\hat{\psi}] \, \mathcal{D}^j_{00}(g_0(\vec{\theta}_0) g_1(\vec{\theta}_1)) \tag{2.148}$$

where $\hat{\psi}$ is defined by (2.37). Again using (2.84) equation (2.148) can be simplified to

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle^{(1)}_C \sim \frac{\Theta(\cos\hat{\tilde{\theta}} - \cos 2\hat{\psi})}{\sqrt{\cos\hat{\tilde{\theta}} - \cos 2\hat{\psi}}} \,, \tag{2.149}$$

where Θ is the step function. This indeed coincides with the expression for the effective geometry (2.132) in the large k limit.

2.5 Symmetry-breaking branes of type II

2.5.1 Definition

The number of preserved affine symmetries can be further reduced by implementing the procedure from the previous section on both groups in $M = G \times G$. More precisely, let us consider the brane

$$\text{II}: \ (g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1} L_0, \quad h_1 f_1 h_0^{-1} L_1), \tag{2.150}$$

where L_0, L_1 belong to two *different* $U(1)$ groups in M: $L_0, \in U(1)_{Y_0}$, $L_1, \in U(1)_{Y_1}$, $L_0 = e^{i\beta Y_0}$, $L_1 = e^{i\alpha Y_1}$. Using the same technique as for brane I, we can write the full action with the boundary conditions (2.150). Having the action, it is easy to show that this brane preserves the currents (2.125). On the other hand, the equations (2.124) and (2.126) get modified in an obvious manner,

$$\bar{J}_0{}^a = J_1^a, \quad \forall T^a \in \text{Lie}(G) \quad \text{s.t.} \quad [\text{T}^a, \text{Y}_0] = 0, \tag{2.151}$$

$$J_0^{Y_1} = -\bar{J}_1^{Y_1}, \quad \bar{J}_0^{Y_0} = -J_1^{Y_0}. \tag{2.152}$$

The details of calculations can be found in [160].

2.5.2 The symmetry-breaking brane of type II on an $SU(2) \times SU(2)$

As explained before, the symmetries preserved by the brane of type I can be broken further by multiplying its first term by a $U(1)$ subgroup,

$$(g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1} e^{i\beta \frac{\sigma_3}{2}}, h_1 f_1 h_0^{-1} e^{i\alpha \frac{\sigma_3}{2}}). \tag{2.153}$$

Here we have taken both $U(1)$ groups to be along the same generator, but we take them to be parametrised by two *independent* parameters α and β. The symmetries of brane I, given in (2.128) are now reduced to

$$J_0^3 = -\bar{J}_1^3, \quad J_1^3 = -\bar{J}_0^3. \tag{2.154}$$

Note that the following equation holds

$$\mathrm{Tr}\left(g_0\, e^{-i\beta\frac{\sigma_3}{2}}\, g_1 e^{-i\alpha\frac{\sigma_3}{2}}\right) = \mathrm{Tr}\left(f_0 f_1\right). \tag{2.155}$$

Using the same arguments which, in the previous case, led to the inequality (2.132), one now concludes that

$$\cos\tilde\theta_0 \cos\tilde\theta_1 - \sin\tilde\theta_0 \sin\tilde\theta_1 \cos(\chi_1 + \varphi_0 - \beta) \geq \cos 2\psi_0\,, \tag{2.156}$$

where $\mathrm{Tr}(f_0 f_1) = 2\cos\psi_0$ and ψ_0 is defined by (2.27). As before, the elements g_0 and g_1 will belong to the brane surface if and only if this inequality admits a solution for the parameter β. This will happen if and only if the maximum of the left hand side of (2.156) is larger than $\cos 2\psi_0$. It is easy to see that this maximum is equal to $\cos(\tilde\theta_0 - \tilde\theta_1)$. Therefore, the generic brane (2.153) is six dimensional and given by an inequality

$$\cos(\tilde\theta_0 - \tilde\theta_1) \geq \cos 2\psi_0\,. \tag{2.157}$$

When $\psi_0 = 0$ the brane is five dimensional and given by the equation $\tilde\theta_0 = \tilde\theta_1$.

2.5.3 Boundary states for symmetry breaking type II branes

Let us now turn to the type II brane (2.153)

$$(g_0, g_1)\Big|_{\mathrm{brane}} = (h_0 f_0 h_1^{-1} e^{i\beta\frac{\sigma_3}{2}}, h_1 f_1 h_0^{-1} e^{i\alpha\frac{\sigma_3}{2}})\,, \tag{2.158}$$

which preserves the currents[§]

$$J_0^3 - \bar{J}_1^3 = 0\,, \quad \bar{J}_0^3 - J_1^3 = 0\,. \tag{2.159}$$

This brane has a structure which is very similar to the type I brane. It can be derived from this brane by applying the described procedure (with right cosetting) to the *first* $SU(2)$ group in which brane I lives. As for brane I, this procedure will reduce the currents (2.135) and (2.136) down to (2.159). At the level of the boundary state, the Ishibashi state in the $0\bar{1}$ sector will remain unchanged, while the $\bar{0}1$ Ishibashi state (2.112) will be projected down to a $Z_{k,1}^R$ invariant

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[§]Here we again work in the closed string channel.

state (where subscript 1, indicates that this action is taken in the first $SU(2)$ group). Finally, applying the T-duality we obtain the boundary state

$$|a\rangle_C^{(2)} = k \sum_j \frac{S_{aj}}{S_{0j}} |B, j\rangle\rangle_\tau^{0\bar{1}} \otimes |B, j\rangle\rangle_\tau^{1\bar{0}} \tag{2.160}$$

where $|B, j\rangle\rangle_\tau^{1\bar{0}}$ is defined as in (2.144) with 0 and 1 exchanged, and the coefficients in the linear combination are fixed by the Cardy condition. For even k the tree-level amplitude between the states (2.160) reduces to

$$Z_{a_1 a_2} = \sum_{r,j',j''} \sum_{n_1,n_2} \sum_{n_3,n_4} N_{a_1 a_2}^r N_{rj''}^{j'} \chi_{j',n_1}(q) \chi_{j'',n_3}(q) \psi_{n_2}(q) \psi_{n_4}(q) \tag{2.161}$$

$$\times \frac{(1 + (-1)^{n_1+n_2})(1 + (-1)^{n_3+n_4})}{4} .$$

For an odd k (2.160) can be simplified and written as

$$|a\rangle_C^{(2)} = k \sum_j \frac{S_{aj}}{S_{0j}} \left[\frac{1 + (-1)^{2j}}{2} |j, 0\rangle\rangle_\tau^{PF_0 \times PF_{\bar{1}}} \otimes |j, 0\rangle\rangle_\tau^{PF_1 \times PF_{\bar{0}}} \right. \tag{2.162}$$

$$\otimes |0\rangle\rangle_{\tau-}^{U(1)_0 \times U(1)_{\bar{1}}} \otimes |0\rangle\rangle_{\tau-}^{U(1)_1 \times U(1)_{\bar{0}}}$$

$$+ \frac{1 + (-1)^{2j+k}}{2} |j, k\rangle\rangle_\tau^{PF_0 \times PF_{\bar{1}}} \otimes |j, k\rangle\rangle_\tau^{PF_1 \times PF_{\bar{0}}}$$

$$\left. \otimes |k\rangle\rangle_{\tau-}^{U(1)_0 \times U(1)_{\bar{1}}} \otimes |k\rangle\rangle_{\tau-}^{U(1)_1 \times U(1)_{\bar{0}}} \right],$$

The tree level amplitude between the states (2.162) is

$$Z_{a_1 a_2} = \sum_{r,j',j''} \sum_{n_1,n_2} \sum_{n_3,n_4} N_{a_1 a_2}^r N_{rj''}^{j'} \chi_{j',n_1}(q) \chi_{j'',n_3}(q) \psi_{n_2}(q) \psi_{n_4}(q) \frac{1 + \eta(-1)^{n_2+n_4}}{4} \tag{2.163}$$

where $\eta = (-1)^{2a_1+2a_2}$. The Cardy condition is satisfied in this case because, after taking into account field identification in the parafermionic sector, one can show that each state in the sum appears twice, and therefore all states appear with integer coefficient. The calculations leading to (2.161) and (2.163) follow closely to that of for (2.146) outlined in [160].

To derive the effective geometry of this brane, one follows the same arguments we as presented for the type I brane. The overlap with the localised bulk probe is given by

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a \rangle_C^{(2)} \sim \sum_{j \in \mathbb{Z}} \frac{k^2}{\pi} \sin[(2j+1)\hat{\psi}] P_j(\cos \tilde{\theta}_0) P_j(\cos \tilde{\theta}_1) . \tag{2.164}$$

where $\hat{\psi}$ is defined by (2.37). Using now the formula [184]

$$P_j(\cos\tilde{\theta}_0)P_j(\cos\tilde{\theta}_1) = \frac{1}{\pi}\int_{|\tilde{\theta}_0-\tilde{\theta}_1|}^{\tilde{\theta}_0+\tilde{\theta}_1} P_j(\cos\theta)\frac{\sin\theta d\theta}{\sqrt{[\cos\theta-\cos(\tilde{\theta}_0+\tilde{\theta}_1)][\cos(\tilde{\theta}_0-\tilde{\theta}_1)-\cos\theta]}} \quad (2.165)$$

and equation (2.84) we obtain

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle_C^{(2)} \sim \frac{1}{\pi}\int_{|\tilde{\theta}_0-\tilde{\theta}_1|}^{\tilde{\theta}_0+\tilde{\theta}_1} \frac{\Theta(\cos\theta-\cos 2\hat{\psi})}{\sqrt{\cos\theta-\cos 2\hat{\psi}}}\frac{\sin\theta d\theta}{\sqrt{[\cos\theta-\cos(\tilde{\theta}_0+\tilde{\theta}_1)][\cos(\tilde{\theta}_0-\tilde{\theta}_1)-\cos\theta]}} \ .$$

$$(2.166)$$

The integral (2.166) is different from zero if $\cos(\tilde{\theta}_0-\tilde{\theta}_1) \geq \cos 2\hat{\psi}$ which is precisely the condition (2.157) in the large k limit.

Chapter 3

Defects and branes in gauged WZW models

3.1 Branes in vectorially gauged WZW model

3.1.1 Open strings in gauged WZW model

Here we construct boundary conditions for the vectorially gauged WZW model, corresponding to the Cardy states [56].

As we explained in subsection 1.5.5, the action of the gauged WZW model using the Polyakov-Wiegamnn identities can be written in the form:

$$S^{G/H} = S^{G/H}(U^{-1}g\tilde{U}) - S^H(U^{-1}\tilde{U}) \tag{3.1}$$

Consider the action (3.1) on a world-sheet with a boundary. Following the corresponding discussion of the WZW model on a world-sheet with a boundary in section 2.1.1 we suggest the following boundary conditions:

$$U^{-1}g\tilde{U}|_{\text{boundary}} = (U^{-1}n)f(U^{-1}n)^{-1}, \qquad n, f \ \in G \tag{3.2}$$

and

$$U^{-1}\tilde{U}|_{\text{boundary}} = (U^{-1}p)l^{-1}(U^{-1}p)^{-1} \qquad p, l \ \in H \tag{3.3}$$

Conditions (3.2) and (3.3) imply

$$g|_{\text{boundary}} = nfn^{-1}plp^{-1} = c_1 c_2 \tag{3.4}$$

where $c_1 = nfn^{-1}$ and $c_2 = plp^{-1}$, and also on the boundary

$$\tilde{U} = pl^{-1}p^{-1}U \tag{3.5}$$

Now we can write the action of the gauged WZW model in the presence of a boundary:

$$S^{\text{bndry}-\text{G/H}} = S^{\text{G/H}}(U^{-1}g\tilde{U}) - S^{H}(U^{-1}\tilde{U}) - \frac{k}{4\pi}\int_{D}\omega_f(U^{-1}n) + \frac{k}{4\pi}\int_{D}\omega_{l^{-1}}(U^{-1}p) \tag{3.6}$$

where $\omega_f(k)$ is defined in (2.6). Using again PW identities (1.278) and (1.279) we obtain

$$S^{\text{bndry}-\text{G/H}} = S^{\text{WZW}} + S^{\text{gauge}} \tag{3.7}$$

$$= \frac{k_G}{4\pi}\left[\int_{\Sigma} d^2 z L^{\text{kin}} + \int_{B}\omega^{WZW}\right]$$

$$+ \frac{k_G}{2\pi}\int_{\Sigma} d^2 z \text{Tr}[A_{\bar{z}}\partial_z g g^{-1} - A_z \partial_{\bar{z}} g g^{-1} + A_{\bar{z}} g A_z g^{-1} - A_{\bar{z}} A_z] - \frac{k}{4\pi}\int_{D}\Omega$$

with

$$\Omega = \omega_f(U^{-1}n) - \omega_{l^{-1}}(U^{-1}p) \tag{3.8}$$

$$+ \text{Tr}\left[g^{-1}dg d\tilde{U}\tilde{U}^{-1} - dUU^{-1}dg g^{-1} - dUU^{-1}g d\tilde{U}\tilde{U}^{-1}g^{-1} + dUU^{-1}d\tilde{U}\tilde{U}^{-1}\right]$$

After some straightforward calculations we obtain that is

$$\Omega(c_1, c_2) = \omega_f(n) + \omega_l(p) + \text{Tr}(dc_2 c_2^{-1} c_1^{-1} dc_1) \tag{3.9}$$

It is easy to check that:

$$\omega^{\text{WZW}}(c_1 c_2) = d\Omega \tag{3.10}$$

As in section 2.1.2, the embedding of the disc D into $C_f^G C_l^H$ involves a topological choice. Holding plp^{-1} in (3.4) fixed on the disc while performing a topological change corresponding to a G coroot lattice vector s_G in the definition on the interior of D, of the factor nfn^{-1}, will induce in $S^{\text{bndry}-\text{G/H}}$ in (3.7) the same change as that of section 2.1.2

$$\Delta_G S^{\text{bndry}-\text{G/H}} = k_G(\theta_G \cdot s_G) \tag{3.11}$$

where $f = e^{i\theta_G \cdot \mathcal{H}_G}$. The consistency of the action requires then the same quantization condition (2.31) on the G conjugacy class

$$\theta_G \cdot \alpha_G \in 2\pi Z/k_G \tag{3.12}$$

Similarly, a topological change corresponding to an H coroot vector s_H in the continuation to D of the factor plp^{-1} in (3.4) with the nfn^{-1} held fixed, will also change $S^{\text{bndry}-\text{G/H}}$. For $l = e^{i\theta_H \cdot \mathcal{H}_H}$ this change will be

$$\Delta_H S^{\text{bndry}-\text{G/H}} = k_H(\theta_H \cdot s_H) \tag{3.13}$$

The consistency of the action (3.7) then also constrains the H conjugacy class factor by

$$\theta_H \cdot \alpha_H \in 2\pi Z/k_H. \tag{3.14}$$

3.1.2 The case of a common center

When H contains some subgroup C of the center of G, the above discussion gets modified in two ways. First, for $z \in C$ the region $C_f^G C_l^H$ is identical to the region $C_{zf}^G C_{z^{-1}l}^H$. The brane corresponding to the pair (f, l) of conjugacy classes is then identical to the brane corresponding to the pair $(zf, z^{-1}l)$. This is the geometrical origin of the phenomena known in the context of coset CFT without boundary as "field identification" [79, 90, 91]. It is again consistent with Cardy's identification of boundary states with primary fields.

Since the gauge transformation takes g into hgh^{-1}, it does not distinguish between the transformations h and zh for any $z \in C$. We can then think of the gauge group as H/C. Recall that the element n in (3.4) is defined modulo right multiplication by G group elements from the torus T_f^G commuting with f. Similarly p in that equation can be multiplied from the right by any element of T_l^H. Let the boundary of the hole be parameterized by $0 \leq \tau \leq 2\pi$. We have seen in (2.29) that upon replacing the boundary value $n(\tau)$ by

$$n'(\tau) = n(\tau)t(\tau) \tag{3.15}$$

with

$$t(\tau) = e^{\frac{i}{2\pi}\tau(s \cdot \mathcal{H})} \tag{3.16}$$

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s being a coroot lattice vector and $\mathcal{H}$ a vector of generators commuting with f, continuing n' rather than n into the disc, the change (3.11) is induced in the action. This gave rise to the quantization condition (3.12). A similar independent change in $p(\tau)$ induces the change (3.13) leading to the condition (3.14). Recall also that a gauge transformation $h \in H$ multiplies both n and p by h from the left. Let $z \in C$ be represented as

$$z = e^{i(w \cdot \mathcal{H})}. \tag{3.17}$$

Notice that w is a common weight vector of G and H. Consider an H/C gauge transformation $h(z, \bar{z}) \in H$, which satisfies on the boundary of the hole

$$h(0) = z^{-1} h(2\pi) \tag{3.18}$$

Let this transformation act on a configuration with a given continuous choice of n and p on the boundary and inside the disc. On the world sheet Σ the action density, being gauge invariant, does not change. The representation (3.4) of g on the boundary is changed, the transformed n and p satisfy

$$n(0) = z^{-1} n(2\pi) \tag{3.19}$$

$$p(0) = z^{-1} p(2\pi)$$

In this form n and p are discontiuous in H. They are continuous in H/C, but the paths $n(\tau)$ and $p(\tau)$ of (3.19) are non contractible in H/C and cannot be continued into the interior of the disc to be substituted in the action (3.7). To define the action we must, before continuing into the disc, to redefine n and p according to (3.15), multiplying them from the right by an appropriate Cartan element, changing n into n' and p into p' defined as

$$n'(\tau) = n'(\tau) e^{\frac{i}{2\pi} \tau (w \cdot \mathcal{H})} \tag{3.20}$$

$$p'(\tau) = p'(\tau) e^{\frac{i}{2\pi} \tau (w \cdot \mathcal{H})}$$

The redefined n' and p' are contractible and can be continued into the disc. The redefinition (3.20), like (3.15), induces a change in the disc term of the action, according to (3.11) and

(3.13). Notice that, unlike (3.16), (3.20) contains a weight vector rather than a root vector, and that this twist is done together on n and on p. Equations (3.11) and (3.13) give then for the change of the action induced by (3.20)

$$\Delta S^{\text{bndry}-\text{G/H}} = (k_G \theta_G + k_H \theta_H) \cdot w \tag{3.21}$$

where $f = e^{i(\theta_G \cdot \mathcal{H})}$ and $l = e^{i(\theta_H \cdot \mathcal{H})}$. Invariance under the gauge transformation (3.18) requires this change to be a multiple of 2π leading to a further condition, a correlation between G and H conjugacy classes,

$$(k_G \theta_G + k_H \theta_H) \cdot w \in 2\pi Z \tag{3.22}$$

for every common weight of G and H. This is again in accordance with Cardy's correspondence of boundary states with primary fields of the CFT without boundary. The condition (3.22) for coset CFT is known as the selection rule [79,90,91] , demanding the same behavior of members of the pair of G and H representations under the common center.

3.2 Lagrangian of the WZW model with defects

In this section we review that action of the WZW model with defect [78] and for future applications consider geometrical realization of the Cardy defects (1.177).

Let us assume that one has a defect line S separating the world-sheet into two regions Σ_1 and Σ_2. In such a situation the WZW model is defined by pair of maps g_1 and g_2. On the defect line itself one has to impose conditions that relate the two maps. The necessary data are captured by the geometrical structure of a bibrane: a bibrane is in particular a submanifold of the Cartesian product of the group G with itself : $Q \subset G \times G$. The pair of maps (g_1, g_2) are restricted by the requirement that the combined map

$$S \rightarrow (G \times G) : s \rightarrow (g_1(s), g_2(s) \in Q \tag{3.23}$$

takes its value in the submanifold Q. Additionally one should require, that on the submanifold Q a two-form $\varpi(g_1, g_2)$ exists satisfying the relation

$$d\varpi(g_1, g_2) = \omega^{WZ}(g_1)|_Q - \omega^{WZ}(g_2)|_Q . \tag{3.24}$$

To write the action of the WZW model with defect one should introduce an auxiliary disc D satisfying the conditions:

$$\partial B_1 = \Sigma_1 + D \ \text{ and } \ \partial B_2 = \Sigma_2 + \bar{D} \,, \tag{3.25}$$

and continue the fields g_1 and g_2 on this disc always holding the condition (3.23). After this preparations the topological part of the action takes the form :

$$S^{\text{top-def}} = \frac{k}{4\pi} \int_{B_1} \omega^{WZ}(g_1) + \frac{k}{4\pi} \int_{B_2} \omega^{WZ}(g_2) - \frac{k}{4\pi} \int_D \varpi(g_1, g_2) \,. \tag{3.26}$$

Equation (3.24) guarantees that (3.26) is well defined.

The full action is

$$S^{\text{def-WZW}} = S^{\text{kin-def}} + S^{\text{top-def}} \tag{3.27}$$

where

$$S^{\text{kin-def}}(g_1, g_2) = \frac{k}{4\pi} \int_{\Sigma_1} L^{\text{kin}}(g_1) d^2z + \frac{k}{4\pi} \int_{\Sigma_2} L^{\text{kin}}(g_2) d^2z \tag{3.28}$$

Let us consider as the bibrane Q the submanifold:

$$(g_1, g_2) = (C_\mu p, p) \tag{3.29}$$

or alternatively

$$g_1 g_2^{-1} = C_\mu \tag{3.30}$$

where C_μ is defined in (2.32).

We can easily check that the equation (3.24) is satisfied with

$$\varpi(C_\mu, p) = \omega_f(h) - \text{Tr}(C_\mu^{-1} dC_\mu dp p^{-1}) \tag{3.31}$$

where $\omega_f(h)$ is defined in (2.6).

It is straightforward to prove that

$$\text{Tr}(g_1^{-1}\delta g_1(g_1^{-1}dg_1)^2) - \text{Tr}(g_2^{-1}\delta g_2(g_2^{-1}dg_2)^2) - \delta\varpi = dB_\mu \,. \tag{3.32}$$

where

$$B_\mu = A_\mu(h) - \text{Tr}(\delta p p^{-1} C_\mu^{-1} dC_\mu) + \text{Tr}(C_\mu^{-1} \delta C_\mu dp p^{-1}) \tag{3.33}$$

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A_μ is defined by (2.14) for f defined in (2.32).

Recalling that the first two terms come from the equation

$$\delta\omega^{WZ} = d[\text{Tr}(g^{-1}\delta g(g^{-1}dg)^2)]\,, \tag{3.34}$$

we see that the existence of the one-form B satisfying (3.32) is a consequence of the equation (3.24).

The defect equation of motion is

$$\text{Tr}\Big[\delta g_1 g_1^{-1}(\partial_z g_1 g_1^{-1} - \partial_{\bar{z}} g_1 g_1^{-1})\Big]d\tau - \text{Tr}\Big[\delta g_2 g_2^{-1}(\partial_z g_2 g_2^{-1} - \partial_{\bar{z}} g_2 g_2^{-1})\Big]d\tau + B_\mu = 0 \tag{3.35}$$

After some calculation one can show that (3.35) implies:

$$J_1 = J_2 \qquad \text{and} \qquad \bar{J}_1 = \bar{J}_2 \tag{3.36}$$

These are indeed the topological defect gluing conditions (1.172) for WZW model. To take C_μ in the form (2.32) is forced again by the global issues discussed in the previous section.

Now we show for $SU(2)_k$ model that the geometry of the defect operator (1.177) is indeed that of described in (3.30). Consider a defect operator corresponding to a primary a:

$$X_a = \sum_j \frac{S_{aj}}{S_{0j}} \sum_{N,\bar{N}} (|j, N\rangle \otimes |j, \bar{N}\rangle)(\langle j, N| \otimes \langle j, \bar{N}|) \tag{3.37}$$

We will show that it has geometry of the form (3.30) with $\mu = a\sigma_3$ as it should be for $SU(2)$ group.

Using the S matrix of the modular transformation for $SU(2)_k$ (1.308), one obtains in the large-k limit the ratio of S-matrix elements appearing in the defect operator

$$\frac{S_{aj}}{S_{0j}} \sim \frac{(k+2)}{\pi(2j+1)} \sin[(2j+1)\hat{\psi}]\,, \tag{3.38}$$

where, as before, we have introduced $\hat{\psi} = \frac{(2a+1)\pi}{k+2}$. Using these results, the overlap between the boundary state and the localised bulk state (2.34) becomes

$$\langle\vec{\theta}_1|X_a|\vec{\theta}_2\rangle \sim \sum_{j,m,n} \frac{(k+2)}{\pi} \sin[(2j+1)\hat{\psi}]\mathcal{D}^j_{nm}(g_1(\vec{\theta}_1))\mathcal{D}^j_{mn}(g_2^{-1}(\vec{\theta}_2))\,. \tag{3.39}$$

To simplify this expression we need the identity

$$\sum_m \mathcal{D}^j_{nm}(g_1(\vec{\theta}_1))\mathcal{D}^j_{mn'}(g_2^{-1}(\vec{\theta}_2)) = \mathcal{D}^j_{nn'}(g_1(\vec{\theta}_1)g_2^{-1}(\vec{\theta}_2)), \tag{3.40}$$

which follows from the fact that the matrices $\mathcal{D}^j_{nm}$ form a representation of the group. Finally, recalling (2.39), the overlap (3.39) becomes

$$\langle \vec{\theta}_1|X_a|\vec{\theta}_2\rangle \sim \frac{k+2}{\pi \sin\psi}\sum_j \sin[(2j+1)\hat{\psi}]\sin[(2j+1)\psi] \tag{3.41}$$

where ψ defined by

$$\mathrm{Tr}(g_0 g_1^{-1}) = 2\cos\psi \tag{3.42}$$

and from the completeness of $\sin(n\psi)$ on the interval $[0,\pi]$ one concludes

$$\langle \vec{\theta}_1|X_a|\vec{\theta}_2\rangle \sim \frac{k+2}{4\sin\psi}\delta(\psi - \hat{\psi}). \tag{3.43}$$

We see that the defect world-volume indeed has the required form.

3.3 Defects in vectorially gauged WZW model

3.3.1 Geometry and action

Let us consider the action (3.1)

$$S^{G/H} = S^{G/H}(U^{-1}g\tilde{U}) - S^H(U^{-1}\tilde{U}) \tag{3.44}$$

on a world-sheet with a defect. The analysis of the WZW theory with defects in section 3.2 implies that we should impose the following constraints:

$$U_1^{-1}g_1\tilde{U}_1 = \tilde{p}_1\tilde{C}_1 = \tilde{U}_1^{-1}p_1\tilde{U}_1\tilde{U}_1^{-1}C_1\tilde{U}_1 \tag{3.45}$$

$$U_2^{-1}g_2\tilde{U}_2 = \tilde{p}_1 = \tilde{U}_1^{-1}p_1\tilde{U}_1 \tag{3.46}$$

$$U_1^{-1}\tilde{U}_1 = \tilde{p}_2\tilde{C}_2 = \tilde{U}_1^{-1}p_2\tilde{U}_1\tilde{U}_1^{-1}C_2^{-1}\tilde{U}_1 \tag{3.47}$$

$$U_2^{-1}\tilde{U}_2 = \tilde{p}_2 = \tilde{U}_1^{-1}p_2\tilde{U}_1 \tag{3.48}$$

These equations imply

$$g_1 = C_2 p_2^{-1} p_1 C_1 \tag{3.49}$$

and

$$g_2 = L p_2^{-1} p_1 L^{-1} \tag{3.50}$$

where

$$L = \tilde{U}_2 \tilde{U}_1^{-1} \tag{3.51}$$

Defining $p = p_2^{-1} p_1$ and redefining $C_1 \to p^{-1} C_1 p$ we obtain for bibrane the following ansatz:

$$(g_1, g_2) = (C_2 C_1 p, L p L^{-1}) \,. \tag{3.52}$$

Here $C_1 \in C_G^{\mu_1}$, $C_2 \in C_H^{\mu_2}$, $p \in G$ and $L \in H$.

The Polyakov-Wiegmann identity (1.279) implies that the bibrane (3.52) satisfies the condition (3.24) with the following ϖ:

$$\varpi(L, p, C_2, C_1) = \Omega^{(2)}(C_2, C_1) - \mathrm{tr}((C_2 C_1)^{-1} d(C_2 C_1) dp p^{-1}) + \Psi(L, p) \,, \tag{3.53}$$

where $\Omega^{(2)}(C_2, C_1)$ is defined in (3.9), $\omega_\mu(C)$ is defined in (2.6) and $\Psi(L, p)$ is defined in (2.9).

Permutation branes are given by the folded version of (3.52):

$$(g_1, g_2) = (C_2 C_1 p, L p^{-1} L^{-1}) \,. \tag{3.54}$$

Alternatively (3.52) and (3.54) can be written as

$$(g_1, g_2) = (n r n^{-1} h_1 f h_2^{-1}, L^{-1} h_1 h_2^{-1} L) \tag{3.55}$$

and

$$(g_1, g_2) = (n r n^{-1} h_1 f h_2^{-1}, L^{-1} h_2 h_1^{-1} L) \tag{3.56}$$

3.3.2 Permutation branes on $SU(2)_k/U(1) \times SU(2)_k/U(1)$

In this section we consider permutation branes on product of $SU(2)_k/U(1) \times SU(2)_k/U(1)$ coset. We show that the geometrical description given above coincide with the permutation boundary state (1.161) overlap with the graviton wave packet.

Taking as H the $U(1)$ subgroup generated by σ_3, the brane (3.56) takes the form:

$$(g_1, g_2)\Big|_{\text{brane}} = (h_1 f h_2^{-1},\ e^{i\alpha \frac{\sigma_3}{2}} h_2 h_1^{-1} e^{-i\alpha \frac{\sigma_3}{2}} e^{i\frac{\pi M}{k}\frac{\sigma_3}{2}}). \qquad (3.57)$$

where as before $f = e^{i\psi_0 \frac{\sigma_3}{2}}$, $\psi_0 = \frac{2j\pi}{k}$, $j = 0, \ldots, \frac{k}{2}$, and M is an integer. The factor $e^{i\frac{\pi M}{k}\frac{\sigma_3}{2}}$ reflects $\mathcal{Z}_k$ symmetry of an abelian coset [123]. One can multiply with this factor also the first element in (3.57) , but performing the redefinition of h_1, one gets again (3.57). We see that all the branes are labelled by two indices ψ_0 and M, exactly as the permutation states on the parafermions product. The elements g_1 and g_2 belong to the brane surface if the following equation admits a solution for the parameter α,

$$\text{tr}\left(g_1 e^{-i\alpha \frac{\sigma_3}{2}} g_2 e^{i\alpha \frac{\sigma_3}{2}} e^{-i\frac{\pi M}{k}\frac{\sigma_3}{2}}\right) = 2\cos\psi_0. \qquad (3.58)$$

This equation can be further elaborated in the Euler coordinates (2.66).

Denoting by $\tilde{\Theta}$, $\tilde{\Phi}$ Euler angles $\tilde{\theta}$ and $\tilde{\phi}$ of the product $g_1 e^{-i\alpha \frac{\sigma_3}{2}} g_2$ and using (2.130) and (2.131) we can rewrite (3.58) as

$$\cos\frac{\tilde{\Theta}}{2}\cos(\gamma/2 - \xi/2 - \tilde{\phi}_1 - \tilde{\phi}_2 + \frac{\pi M}{2k}) = \cos\psi_0, \qquad (3.59)$$

where

$$\cos\tilde{\Theta} = \cos\tilde{\theta}_1 \cos\tilde{\theta}_2 - \sin\tilde{\theta}_1 \sin\tilde{\theta}_2 \cos\gamma, \qquad (3.60)$$

and we have introduced new labels $\gamma = \chi_2 + \varphi_1 - \alpha$ and $\xi/2 = \tilde{\Phi} - \frac{\chi_1 + \varphi_2}{2}$. The variables ξ and γ are related to each other by the equation

$$e^{i\frac{\xi}{2}} = \frac{1}{\cos\frac{\tilde{\Theta}}{2}}\left(\cos\frac{\tilde{\theta}_1}{2}\cos\frac{\tilde{\theta}_2}{2}e^{i\frac{\gamma}{2}} - \sin\frac{\tilde{\theta}_1}{2}\sin\frac{\tilde{\theta}_2}{2}e^{-i\frac{\gamma}{2}}\right). \qquad (3.61)$$

Let us recall that the vectorial gauging of $U(1)$ symmetry is given by the translation of ϕ and the resulting target space of the $SU(2)_k/U(1)$ model, derived after the gauge fixing $\phi = 0$ and integrating out of the gauge field, is the two-dimensional disc, parameterized by θ and $\tilde{\phi}$. In the case of product the target space is parameterized by $\theta_1, \theta_2, \tilde{\phi}_1, \tilde{\phi}_2$. Hence the brane consists of those points for which equation (3.59) admits a solution for γ. Θ and ξ are considered here as the complicated functions of $\tilde{\theta}_1, \tilde{\theta}_2$ and γ given by (3.60) and (3.61) respectively. For $\psi_0 = 0$

there are additional constraints, which imply that in this case the brane is two dimensional and given by the equations

$$\tilde{\theta}_1 = -\tilde{\theta}_2, \quad \tilde{\phi}_1 = -\tilde{\phi}_2 + \frac{\pi M}{2k} . \tag{3.62}$$

Now we calculate the effective geometry corresponding to the permutation boundary state (1.161), which recalling the matrix of the modular transformation for the parafermions (1.333), takes the form:

$$|L, M\rangle = \sum_{j,m} \frac{S_{Lj}}{S_{0j}} e^{i\pi Mm/k} \sum_{N_1, N_2} |j, m, N_1\rangle_1 \otimes \overline{|j, m, N_1\rangle_2} \otimes |j, m, N_2\rangle_2 \otimes \overline{|j, m, N_2\rangle_1} \tag{3.63}$$

where S_{Lj} is matrix of the modular transformation of $SU(2)_k$ (1.308)

$$S_{Lj} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2L+1)(2j+1)\pi}{k+2}\right) . \tag{3.64}$$

To obtain the effective geometry, one should compute the overlap $\langle \theta_1, \tilde{\phi}_1, \theta_2, \tilde{\phi}_2 | L, M\rangle$. At the beginning we should find the wave-functions of the parafermion disc theory [123]:

$$\Psi_{j,m}(\theta, \tilde{\phi}) = \langle \theta, \tilde{\phi} | j, m \rangle\rangle \tag{3.65}$$

The wave-functions of the disc are the $SU(2)$ wave-functions that are invariant under translation of ϕ. (Note that in [123] axial gauging is considered, and as a consequence the roles of ϕ and $\tilde{\phi}$ are interchanged). Recalling that the $SU(2)$ wave-functions are the normalized Wigner functions

$$\sqrt{2j+1}\mathcal{D}^j_{nm}(g(\vec{\theta})) = \sqrt{2j+1}e^{-i(n\chi+m\varphi)}d^j_{nm}(\cos\tilde{\theta}) , \tag{3.66}$$

we see that the function on disc are those of them with $m = n$. Using the eq. (3.38) for the ratio of the elements of the $SU(2)_k$ matrix of the modular transformations for the large k:

$$\frac{S_{Lj}}{S_{0j}} \sim \frac{(k+2)}{\pi(2j+1)} \sin[(2j+1)\hat{\psi}] , \tag{3.67}$$

where $\hat{\psi} = \frac{(2L+1)\pi}{k+2}$, one obtains that in the large-k limit the overlap reduces to

$$\langle \vec{\theta}_1, \vec{\theta}_2 | L, M\rangle \sim \sum_j \sum_m \sin[(2j+1)\hat{\psi}] e^{i\pi Mm/k} \mathcal{D}^j_{mm}(g_1(\vec{\theta}_1)) \mathcal{D}^j_{mm}(g_2(\vec{\theta}_2)) . \tag{3.68}$$

It is known [184] that d^j_{nm} are satisfying the relation (note that there is no summation assumed for the repeated indices)

$$d^j_{mm}(\cos\tilde{\theta}_1)d^j_{mm}(\cos\tilde{\theta}_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im(\gamma-\xi)}d^j_{mm}(\cos\tilde{\Theta})d\gamma , \tag{3.69}$$

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The functions $\tilde{\Theta}$ and ξ are functions of $\tilde{\theta}_1, \tilde{\theta}_2$ and γ defined in equations (3.60) and (3.61). Using (3.69) the overlap of the boundary state with the bulk probe can be written as

$$\langle \vec{\theta}_1, \vec{\theta}_2 | L, M \rangle \sim \sum_j \sum_m \int_{-\pi}^{\pi} \sin[(2j+1)\hat{\psi}] e^{im(\gamma - \xi - 2\tilde{\phi}_1 - 2\tilde{\phi}_2 + \frac{\pi M}{k})} d_{mm}^j (\cos \tilde{\Theta}) d\gamma \qquad (3.70)$$

Now using that $\sum_m \mathcal{D}_{mm}^j(g) = \frac{\sin(2j+1)\psi}{\sin \psi}$, where ψ is defined by the relation $\text{Tr} g = 2\cos \psi$, and the completeness of $\sin[(2j+1)\psi]$ on the interval $[0, \pi]$ we get

$$\langle \vec{\theta}_1, \vec{\theta}_2 | L, M \rangle \sim \int_{-\pi}^{\pi} \frac{\delta(\psi - \hat{\psi})}{\sin \hat{\psi}} d\gamma, \qquad (3.71)$$

where

$$\cos \psi = \cos \frac{\tilde{\Theta}}{2} \cos(\gamma/2 - \xi/2 - \tilde{\phi}_1 - \tilde{\phi}_2 + \frac{\pi M}{2k}) \qquad (3.72)$$

From this equation it follows that the brane consist of all those points for which the expression in the argument of the δ function has a root for γ. This is the same condition as the one coming from equation (3.59), obtained in the Langrangian approach.

3.4 D-branes in asymmetrically gauged WZW model

3.4.1 D-branes in the Nappi-Witten model

Let us consider the gauged WZW model G/H defined in the following way [133]. One takes $G = G_1 \times G_2$ and chooses two U(1) subgroups $U(1)_1 \in G_1$ and $U(1)_2 \in G_2$. As gauge group H one takes a product of the two U(1) groups, parametrized by ρ and τ, $H = U(1)_\rho \times U(1)_\tau$, with embeddings $\text{em}_{\rho,1} : U(1)_\rho \to U(1)_1$, $\text{em}_{\rho,2} : U(1)_\rho \to U(1)_2$, $\text{em}_{\tau,1} : U(1)_\tau \to U(1)_1$, $\text{em}_{\tau,2} : U(1)_\tau \to U(1)_2$. We assume that $U(1)_1$ is generated by a_1, $U(1)_1 = e^{i\lambda_1 a_1}$ and $U(1)_2$ by a_2: $U(1)_2 = e^{i\lambda_2 a_2}$ and the generators are normalized in the usual way, $\text{Tr} a_1^2 = \text{Tr} a_2^2 = 2$. The action of H we take in the form

$$(g_1, g_2) \to (h_1 g_1 h_2, h_2' g_2 h_1')), \qquad (3.73)$$

where

$$h_1 = \text{em}_{\rho,1}(h_\rho) = e^{ip\rho a_1}, \qquad (3.74)$$

$$h_1' = \text{em}_{\rho,2}(h_\rho) = e^{iq\rho a_2},$$

$h_\rho \in U(1)_\rho$, and

$$h_2 = \text{em}_{\tau,1}(h_\tau) = e^{ip\tau a_1}, \tag{3.75}$$

$$h_2' = \text{em}_{\tau,2}(h_\tau) = e^{iq\tau a_2},$$

where $h_\tau \in U(1)_\tau$.

The action of the model in the absence of a boundary is

$$S^{\text{NW}} = S^{\text{WZW}}(g_1, k_1) + S^{\text{WZW}}(g_2, k_2) + S(g_1, g_2, A_1, A_2), \tag{3.76}$$

where $S^{\text{WZW}}(g_i, k_i)$, $i = 1, 2$ are the usual WZW actions given by (1.263) and $S(g_1, g_2, A_1, A_2)$ makes the action gauge invariant. Its explicit form is not important here for us and can be found in [133] . For gauge invariance, the levels k_1, k_2, and embedding coefficients p, q should satisfy

$$k_1 p^2 = k_2 q^2. \tag{3.77}$$

Now we consider the model in the presence of a boundary. We take the $U(1)_\alpha$ group parametrized by α and consider embeddings $\text{em}_{\alpha,1} : U(1)_\alpha \to U(1)_1$, and $\text{em}_{\alpha,2} : U(1)_\alpha \to U(1)_2$. We define the boundary conditions

$$g = (g_1, g_2)|_{\text{boundary}} = (m_1 C_1, m_2 C_2), \tag{3.78}$$

where

$$m_1 = \text{em}_{\alpha,1}(m_\alpha) = e^{ip(\alpha+\gamma_1)a_1}, \tag{3.79}$$

$$m_2 = \text{em}_{\alpha,2}(m_\alpha) = e^{iq(\alpha+\gamma_2)a_2},$$

and $m_\alpha \in U(1)_\alpha$, $C_1 = l_1 f_1 l_1^{-1}$ and $C_2 = l_2 f_2 l_2^{-1}$. The parameters p and q are the same as in (3.108) and (3.75). γ_1 and γ_2 are possibly quantized [123] constants.

In other words, we take as the D-branes diagonally embedded U(1)s multiplied by the conjugacy classes. These boundary conditions were suggested in [148]. Our description (3.78) is slightly different from that in [148], and more convenient for present purposes. Note that boundary conditions on each group coincide with (2.42).

Let us check that the boundary conditions (3.78) are invariant under the gauge transformation (3.73) :

$$g = (g_1, g_2) \rightarrow (h_1 g_1 h_2, h_2' g_2 h_1') = (h_1 m_1 l_1 f_1 l_1^{-1} h_2, h_2' m_2 l_2 f_2 l_2^{-1} h_1') = \tag{3.80}$$

$$((h_1 m_1 h_2)(h_2^{-1} l_1) f_1 (h_2^{-1} l_1)^{-1}, (h_2' m_2 h_1')(h_1'^{-1} l_2) f_2 (h_1'^{-1} l_2)^{-1}).$$

We see that the boundary conditions preserve their form under the gauge transformation, with modified parameters:

$$\alpha \rightarrow \alpha + \rho + \tau, \ \ l_1 \rightarrow h_2^{-1} l_1, \ \ l_2 \rightarrow h_1^{-1} l_2. \tag{3.81}$$

As explained in section 2.2, in the presence of a boundary the action should be modified by adding the boundary two-form (2.47 :

$$S^{\text{bndry}-\text{NW}} = S^{\text{WZW}}(g_1, k_1) - \frac{k_1}{4\pi} \int_D \omega^{(2)}(m_1, l_1) + S^{\text{WZW}}(g_2, k_2) - \frac{k_2}{4\pi} \int_D \omega^{(2)}(m_2, l_2) + S(g_1, g_2, A_1, A_2),$$
$$\tag{3.82}$$

where $\omega^{(2)}(m, l)$ is

$$\omega^{(2)}(m, h) = \omega_f(l) - \text{Tr}(m^{-1} dm dC C^{-1}). \tag{3.83}$$

We now check that (3.82) is invariant under (3.73) accompanied by (3.81) .

First we compute the change of

$$S_1 = S^{\text{WZW}}(g_1, k_1) - \frac{k_1}{4\pi} \int_D \omega^{(2)}(m_1, l_1) \tag{3.84}$$

under the transformations $g_1 \rightarrow h_1 g_1 h_2$, $m_1 \rightarrow h_1 m_1 h_2$ and $l_1 \rightarrow h_2^{-1} l_1$, resulting from the presence of the boundary. From the Polyakov-Wiegmann identity (1.279) we get

$$\Delta^{\text{bound}} S^{\text{WZW}}(g_1, k_1) = -\frac{k_1}{4\pi} \int_D \text{Tr}(h_1^{-1} dh_1 dm_1 m_1^{-1} + h_1^{-1} dh_1 dC_1 C_1^{-1} \tag{3.85}$$

$$+ h_1^{-1} dh_1 C_1 dh_2 h_2^{-1} C_1^{-1} + C_1^{-1} m_1^{-1} dm_1 C_1 dh_2 h_2^{-1} + C_1^{-1} dC_1 dh_2 h_2^{-1}).$$

Then we have

$$\Delta \omega^f(l) = \text{Tr}(dh_2 h_2^{-1} C_1^{-1} dC_1 + dh_2 h_2^{-1} dC_1 C_1^{-1} + dh_2 h_2^{-1} C_1 dh_2 h_2^{-1} C_1^{-1}), \tag{3.86}$$

and

$$\Delta(\text{Tr}(m_1^{-1} dm_1 dC_1 C_1^{-1})) = \text{Tr}(-h_1^{-1} dh_1 h_2^{-1} dh_2 + h_1^{-1} dh_1 dC_1 C_1^{-1} \tag{3.87}$$

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$$+h_1^{-1}dh_1C_1dh_2h_2^{-1}C_1^{-1} - m_1^{-1}dm_1h_2^{-1}dh_2 + m_1^{-1}dm_1C_1dh_2h_2^{-1}C_1^{-1}$$

$$+h_2^{-1}dh_2dC_1C_1^{-1} + h_2^{-1}dh_2C_1dh_2h_2^{-1}C_1^{-1}).$$

Collecting (3.85) ,(3.86) and (3.87) we obtain

$$\Delta^{\text{bound}}S_1 = \frac{k_1}{4\pi}\int_D \text{Tr}(h_2^{-1}dh_2m_1^{-1}dm_1 - h_1^{-1}dh_1dm_1m_1^{-1} - h_1^{-1}dh_1h_2^{-1}dh_2). \tag{3.88}$$

Similarly for

$$S_2 = S^{\text{WZW}}(g_2, k_2) - \frac{k_2}{4\pi}\int_D \omega^{(2)}(m_2, l_2) \tag{3.89}$$

we obtain

$$\Delta^{\text{bound}}S_2 = \frac{k_2}{4\pi}\int_D \text{Tr}(h_1'^{-1}dh_1'm_2^{-1}dm_2 - h_2'^{-1}dh_2'dm_2m_2^{-1} - h_2'^{-1}dh_2'h_1'^{-1}dh_1'). \tag{3.90}$$

Taking into account (3.74) , (3.75) ,(3.79) and (3.77) we find that $\Delta^{\text{bound}}S_1 + \Delta^{\text{bound}}S_2 = 0$, proving the gauge invariance of the action (3.82).

3.4.2 $SL(2, R) \times SU(2)/U(1) \times U(1)$ **NW model**

Let us consider the $SL(2, R) \times SU(2)/U(1) \times U(1)$ Nappi-Witten model.

Here $G_1 = SL(2, R)$, $G_2 = SU(2)$, $k_1 = -k_2$, $p = -i$, $q = 1$, and the $U(1)_\rho \times U(1)_\tau$ gauge group acts in the following way:

$$(g_1, g_2) \rightarrow (e^{\rho\sigma_3}g_1e^{\tau\sigma_3}, e^{i\tau\sigma_3}g_2e^{i\rho\sigma_3}). \tag{3.91}$$

The D-branes proposed in section 3.4.1 have the form

$$g_{\text{boundary}} = (e^{(\alpha+\gamma_1)\sigma_3}C_1, e^{i(\alpha+\gamma_2)\sigma_3}C_2), \tag{3.92}$$

where $C_1 = l_1 f_1 l_1^{-1}$ and $C_2 = l_2 f_2 l_2^{-1}$ are conjugacy classes, and $f_2 = e^{i\hat{\psi}\sigma_3}$, where $\hat{\psi}$ belongs to the set (2.27). γ_1 and γ_2 are possibly quantized constants. Now we describe this hypersurface in detail. For this purpose we introduce Euler angles for $SL(2, R)$ and $SU(2)$,

$$g_1 = e^{\chi_1\frac{\sigma_3}{2}}e^{i\theta_1\frac{\sigma_2}{2}}e^{\phi_1\frac{\sigma_3}{2}}, \tag{3.93}$$

$$g_1 = e^{\chi_1\frac{\sigma_3}{2}}e^{\tau_1\frac{\sigma_1}{2}}e^{\phi_1\frac{\sigma_3}{2}}, \tag{3.94}$$

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$$g_2 = e^{i\chi_2 \frac{\sigma_3}{2}} e^{i\tilde{\theta}_2 \frac{\sigma_1}{2}} e^{i\phi_2 \frac{\sigma_3}{2}}, \tag{3.95}$$

where the first two formulae describe different patches of $SL(2, R)$ and the last one is the usual Euler parametrisation for $SU(2)$. It is shown in [156] that in the Euler angle parametrisations the product of a $U(1)$ subgroup and a conjugacy class can be described by inequalities: $e^{\alpha \sigma_3} C_1$ in the patch given by (3.93) is described by the condition

$$\cos \frac{\theta_1}{2} \leq \frac{\mathrm{Tr} f_1}{2}, \tag{3.96}$$

and in the patch (3.94) by the condition

$$\cosh \frac{\tau_1}{2} \leq \frac{\mathrm{Tr} f_1}{2}, \tag{3.97}$$

and $e^{i\alpha \sigma_3} C_2$ in the parametrisation (3.95) is given by the condition

$$\cos \frac{\tilde{\theta}_1}{2} \geq \frac{\mathrm{Tr} f_2}{2}. \tag{3.98}$$

In order to find the equation of the D-brane hypersurface we should find α on the $SL(2, R)$ and $SU(2)$ sides and equate them to each other. It is easy to find the angle α in each case. Writing the boundary condition in the form $e^{-\alpha \sigma_3} g_1 = C_1$ and taking the trace on both sides we easily obtain in the first patch:

$$\cosh(\alpha + \gamma_1 - \frac{\chi_1 + \phi_1}{2}) = \frac{\mathrm{Tr} f_1}{2 \cos \frac{\theta_1}{2}}, \tag{3.99}$$

in the second patch:

$$\cosh(\alpha + \gamma_1 - \frac{\chi_1 + \phi_1}{2}) = \frac{\mathrm{Tr} f_1}{2 \cosh \frac{\tau_1}{2}}, \tag{3.100}$$

and for $SU(2)$:

$$\cos(\alpha + \gamma_2 - \frac{\chi_2 + \phi_2}{2}) = \frac{\mathrm{Tr} f_2}{2 \cos \frac{\tilde{\theta}_2}{2}}. \tag{3.101}$$

We see that the conditions (3.96) , (3.97) and (3.98) are necessary for the existence of solutions to eq. (3.99) ,(3.100) and (3.101) respectively. Now using gauge fixing conditions $\chi_1 = 0$ and $\phi_1 = 0$ we can explicitly write down the D-brane hypersurface equation. In the first patch we have

$$\cosh \left(\arccos \left(\frac{\mathrm{Tr} f_2}{2 \cos \frac{\tilde{\theta}_2}{2}} \right) + \frac{\chi_2 + \phi_2}{2} + \gamma_2 - \gamma_1 \right) = \frac{\mathrm{Tr} f_1}{2 \cos \frac{\theta_1}{2}}, \tag{3.102}$$

and in the second patch

$$\cosh\left(\arccos\left(\frac{\operatorname{Tr} f_2}{2\cos\frac{\tilde{\theta}_2}{2}}\right) + \frac{\chi_2 + \phi_2}{2} + \gamma_2 - \gamma_1\right) = \frac{\operatorname{Tr} f_1}{2\cosh\frac{\tau_1}{2}}. \tag{3.103}$$

3.4.3 D-branes in the Guadagnini-Martellini-Mintchev Model

We begin by reviewing the model introduced in [102, 103] (see also [136]). This model is a kind of gauged WZW model based on a group $G_1 \times G_2$. The gauge group H acts in the folowing way: we choose subgroups $H_1 \in G_1$ and $H_2 \in G_2$ and take embeddings $em_1 \;:\; H \to H_1$ and $em_2 \;:\; H \to H_2$. It is assumed that H_1 and H_2 are the same subgroups of G_1 and G_2 :$H = H_1 = H_2$. The group H acts by the formula

$$(g_1, g_2) \to (g_1 em_1(h^{-1}), em_2(h)g_2). \tag{3.104}$$

It was shown in [102] that the following action is invariant under (3.120) :

$$S_{\text{GMM}} = S^{\text{WZW}}(g_1, k_1) + S^{\text{WZW}}(g_2, k_2) + S_{\text{int}}(g_1, g_2, k), \tag{3.105}$$

where $S^{\text{WZW}}(g_i, k_i)$, $i = 1, 2$ are the usual WZW actions (1.263) and

$$S_{\text{int}}(g_1, g_2, k) = -\frac{k}{2\pi} \int d^2x (\operatorname{Tr}(R_\alpha g_1^{-1}\partial_\mu g_1)\operatorname{Tr}(R'_\alpha \partial^\mu g_2 g_2^{-1}) \tag{3.106}$$
$$+\epsilon^{\mu\nu}\operatorname{Tr}(R_\alpha g_1^{-1}\partial_\mu g_1)\operatorname{Tr}(R'_\alpha \partial_\nu g_2 g_2^{-1})).$$

Here R_α and R'_α are the generators of the Lie algebra of the subgroup H in G_1 and G_2 respectively. It is shown in [102] that for gauge invariance the coefficients entering in (3.105) should satisfy

$$k_1 = kr', \quad k_2 = kr, \tag{3.107}$$

where r and r' are given by the embeddings:

$$\operatorname{Tr}(R_\alpha R_\beta) = r\delta_{\alpha\beta}, \quad \operatorname{Tr}(R'_\alpha R'_\beta) = r'\delta_{\alpha\beta}. \tag{3.108}$$

The conformal field theory defined by this sigma model was discussed in [103] , where the current algebra and the Virasoro algebra with a central charge value coinciding with that of the GKO construction for the coset $(G_1 \times G_2)/H$ were found.

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Here we consider the case when the gauge group is an abelian group, parametrized by ρ: $H = U(1)_\rho$. As before we assume that H_1 is generated by a generator a_1, $H_1 = e^{i\lambda_1 a_1}$ and H_2 by a_2: $H_2 = e^{i\lambda_2 a_2}$, and that the generators are normalized as usual: $\mathrm{Tr}a_1^2 = \mathrm{Tr}a_2^2 = 2$. In this case the gauge group acts as

$$(g_1, g_2) \to (g_1 h_1, h_2 g_2), \tag{3.109}$$

where

$$h_1 = em_1(h_\rho^{-1}) = e^{-ip\rho a_1}, \tag{3.110}$$

$$h_2 = em_2(h_\rho) = e^{iq\rho a_2},$$

$h_\rho \in U(1)_\rho$ and p and q satisfy the relation

$$k_1 p^2 = k_2 q^2. \tag{3.111}$$

Now we consider the model in the presence of a boundary. We take the $U(1)_\alpha$ group parametrized by α and consider embeddings $em_{\alpha,1} : U(1)_\alpha \to U(1)_1$, and $em_{\alpha,2} : U(1)_\alpha \to U(1)_2$. We suggest the following boundary conditions:

$$(g_1, g_2)|_{\text{boundary}} = (m_1 C_1, m_2 C_2), \tag{3.112}$$

where

$$m_1 = em_{\alpha,1}(m_\alpha) = e^{-ip(\alpha+\gamma_1)a_1}, \tag{3.113}$$

$$m_2 = em_{\alpha,2}(m_\alpha) = e^{iq(\alpha+\gamma_2)a_2}$$

and $m_\alpha \in U(1)_\alpha$, $C_1 = l_1 f_1 l_1^{-1}$, $C_2 = l_2 f_2 l_2^{-1}$. The parameters p and q are the same as in (3.110). γ_1 and γ_2 are possibly quantized [123] constants. These boundary conditions are invariant under (3.109) :

$$(m_1 l_1 f_1 l_1^{-1}, m_2 l_2 f_2 l_2^{-1}) \to (m_1 l_1 f_1 l_1^{-1} h_1, h_2 m_2 l_2 f_2 l_2^{-1}) \tag{3.114}$$

$$= ((h_1 m_1)(h_1^{-1} l_1) f_1 (h_1^{-1} l_1)^{-1}, (h_2 m_2) l_2 f_2 l_2^{-1}).$$

We see that boundary conditions keep the form with modified parameters

$$\alpha \to \alpha + \rho, \quad l_1 \to h_1^{-1} l_1. \tag{3.115}$$

In the presence of a boundary we suggest the following action:

$$S_{\mathrm{GMM}}^{\mathrm{bndry}} = S^{\mathrm{WZW}}(g_1, k_1) - \frac{k_1}{4\pi} \int_D \omega^{(2)}(m_1, l_1) + S^{\mathrm{WZW}}(g_2, k_2) - \frac{k_2}{4\pi} \int_D \omega^{(2)}(m_2, l_2) + S_{\mathrm{int}}(g_1, g_2, k).$$

(3.116)

Now we check that the action is invariant under (3.109) accompanied by (3.115) . We easily derive the change of S_1 and S_2 defined in (3.84) and (3.89) correspondingly under a gauge transformation,

$$\Delta^{\mathrm{boundary}} S_1 = \frac{k_1}{4\pi} \int_D \mathrm{Tr}(h_1^{-1} dh_1 dm_1 m_1^{-1}),$$

(3.117)

$$\Delta^{\mathrm{boundary}} S_2 = -\frac{k_2}{4\pi} \int_D \mathrm{Tr}(h_2^{-1} dh_2 dm_2 m_2^{-1}).$$

(3.118)

which cancel each other as a consequence of the conditions (3.110) , (3.113) and (3.111) .

3.4.4 $SU(2) \times SU(2)/U(1)$ GMM model

We begin by describing this model following [136].

The $SU(2)$ group elements are parametrized as

$$g_1 = \exp(i\phi_1 \sigma_3) \exp(i\theta_1 \sigma_2) \exp(i\psi_1 \sigma_3),$$

(3.119)

$$g_2 = \exp(i\phi_2 \sigma_3) \exp(i\theta_2 \sigma_2) \exp(i\psi_2 \sigma_3).$$

The gauge action of the $U(1)$ subgroup is defined by

$$\psi_1 \to \psi_1 - p\varepsilon(z, \bar{z}), \quad \phi_2 \to \phi_2 + q\varepsilon(z, \bar{z}).$$

(3.120)

In the parametrization (3.119) the action (3.105) is

$$S = \frac{1}{4\pi} \int d^2 x [k_1 (\partial_\mu \theta_1 \partial^\mu \theta_1 + \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \psi_1 \partial^\mu \psi_1 + \cos(2\theta_1) \partial_\mu \phi_1 \partial_\nu \psi_1 (\eta^{\mu\nu} + \epsilon^{\mu\nu}))$$ (3.121)

$$+ k_2 (\partial_\mu \theta_2 \partial^\mu \theta_2 + \partial_\mu \phi_2 \partial^\mu \phi_2 + \partial_\mu \psi_2 \partial^\mu \psi_2 + \cos(2\theta_2) \partial_\mu \phi_2 \partial_\nu \psi_2 (\eta^{\mu\nu} + \epsilon^{\mu\nu}))$$

$$+ k_3 (\cos(2\theta_1) \partial_\mu \phi_1 + \partial_\mu \psi_1)(\cos(2\theta_2) \partial_\nu \psi_2 + \partial_\mu \phi_2)(\eta^{\mu\nu} + \epsilon^{\mu\nu})].$$

For the action to be invariant under (3.120) one needs to impose the following algebraic constraints:

$$k_1 p = k_3 q, \quad k_2 q = k_3 p.$$

(3.122)

Multiplying these equation we obtain

$$k_3 = \sqrt{k_1 k_2}, \quad p/q = \sqrt{k_2/k_1}. \tag{3.123}$$

Fixing the gauge by setting $\phi_2 = 0$ one gets a background whose metric is of the (non-Einstein) $T^{1,Q}$ type

$$ds^2 = k[d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + Q^2(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + (d\psi + \cos \theta_1 d\phi_1 + Q \cos \theta_2 d\phi_2)^2], \tag{3.124}$$

where we have rescaled all variables by $1/2$, renamed $\psi_2 \to \phi_2$, $\psi_1 \to \psi$ and introduced

$$Q = p/q = \sqrt{k_2/k_1}, \quad k = k_1. \tag{3.125}$$

The background also includes the antisymmetric tensor field

$$B_{\phi_1 \psi} = k \cos \theta_1, \quad B_{\phi_1 \phi_2} = kQ \cos \theta_1 \cos \theta_2, \quad B_{\phi_2 \psi} = -kQ \cos \theta_2. \tag{3.126}$$

The D-branes proposed in section 3.4.3 have the form

$$(g_1, g_2)_{\text{boundary}} = (e^{-ip(\alpha+\gamma_1)\sigma_3} C_1, e^{iq(\alpha+\gamma_2)\sigma_3} C_2), \tag{3.127}$$

where $C_1 = h_1 f_1 h_1^{-1}$ and $C_2 = h_2 f_2 h_2^{-1}$ are conjugacy classes, $f_1 = e^{i\hat{\psi}_1 \sigma_3}$ and $f_2 = e^{i\hat{\psi}_2 \sigma_3}$, and $\hat{\psi}_1$, $\hat{\psi}_2$ belong to the set (2.27) . Let us now find the equation describing this hypersurface. As before, we should find in the parametrization (3.119) the angle α and equate both sides. Writing the boundary conditions as

$$\text{Tr}(e^{ip(\alpha+\gamma_1)\sigma_3} g_1) = 2 \cos \hat{\psi}_1, \tag{3.128}$$

$$\text{Tr}(e^{-iq(\alpha+\gamma_2)\sigma_3} g_2) = 2 \cos \hat{\psi}_2, \tag{3.129}$$

from (3.128) and (3.129) we obtain

$$\cos(p(\alpha + \gamma_1) + \phi_1 + \psi_1) = \frac{\cos \hat{\psi}_1}{\cos \theta_1}, \tag{3.130}$$

and

$$\cos(-q(\alpha + \gamma_2) + \phi_2 + \psi_2) = \frac{\cos \hat{\psi}_2}{\cos \theta_2}. \tag{3.131}$$

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Eliminating α from (3.130) and (3.131) we get

$$\frac{1}{p}\arccos\left(\frac{\cos\hat{\psi}_1}{\cos\theta_1}\right) - \frac{\phi_1+\psi_1}{p} - \gamma_1 = -\frac{1}{q}\arccos\left(\frac{\cos\hat{\psi}_2}{\cos\theta_2}\right) + \frac{\phi_2+\psi_2}{q} - \gamma_2. \tag{3.132}$$

Using now the gauge fixing condition $\phi_2 = 0$, and rescaling and renaming all the variables as before, we get the D-brane hypersurface on this $T^{1,Q}$ type space,

$$\phi_2 = 2\arccos\left(\frac{\cos\hat{\psi}_2}{\cos\frac{\theta_2}{2}}\right) + \frac{2}{Q}\arccos\left(\frac{\cos\hat{\psi}_1}{\cos\frac{\theta_1}{2}}\right) - \frac{\phi_1+\psi}{Q} + 2q(\gamma_2-\gamma_1), \tag{3.133}$$

where Q is defined in (3.125) . As before θ_1 and θ_2 satisfy the inequalities

$$\cos\frac{\theta_1}{2} \geq \cos\hat{\psi}_1, \quad \cos\frac{\theta_2}{2} \geq \cos\hat{\psi}_2. \tag{3.134}$$

The presence of the constant term $q(\gamma_2-\gamma_1)$ reflects the invariance of the action (3.121) under the rotations $\phi_i \to \phi_i + \beta_i$, $\psi_i \to \psi_i + \delta_i$, where β_i and δ_i are constant angles, $i = 1,2$. But, as noted in [123] , in the gauged WZW models these symmetries are broken to some discrete subgroups. In the case in question we have

$$\gamma_1 = \frac{n_1}{k_1 p^2}, \quad \gamma_2 = \frac{n_2}{k_2 q^2}, \tag{3.135}$$

where n_1 and n_2 are integers, and using (3.123) we have for the last part

$$2q(\gamma_1-\gamma_2) = \frac{2n}{qk_2}, \tag{3.136}$$

where $n = n_1 - n_2$. We see that the branes (3.133) are specified by the three parameters $\hat{\psi}_1, \hat{\psi}_2$ and n, in one-to-one correspondence with the primaries of the corresponding GKO coset model $(SU(2) \times SU(2))/U(1)$.

Chapter 4

Canonical quantization of the WZW and gauged WZW models with defects and boundaries

4.1 3D Chern-Simons theory

4.1.1 Action of Chern-Simons theory

Consider Chern-Simons theory with sources on a product of a Riemann surface Σ and a time line R [52]:

$$S^{\text{CS}} = \frac{k}{4\pi} \int_{\Sigma \times R} \text{tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right) \tag{4.1}$$
$$+ i \sum_i \int_{R_i} dt_i \text{tr} \lambda_i v_i(t)^{-1} (\partial_0 + \mathcal{A}_0) v_i(t)$$

where $\mathcal{A}$ is a three-dimensional connection, $v_i(t)$ is a group valued map on the line R_i, $\lambda_i \equiv \boldsymbol{\lambda_i} \cdot \boldsymbol{\mathcal{H}}$ is a highest weight representation integrable at level k, taking value in the Cartan subalgebra.

Writing $\mathcal{A} = A + \mathcal{A}_0 dt$, where A is tangent to the surface Σ, one can use gauge freedom to impose the condition $\mathcal{A}_0 = 0$. In this gauge equations of motion are

$$\frac{k}{2\pi}F(z) + i\sum_{i=1}^{n}T_i\delta(z - z_i) = 0 \tag{4.2}$$

where $F = dA + A^2$, and z_i are points where Wilson lines hit M. T_i are conjugacy classes in the Lie algebra g

$$T_i = v_i\lambda_i v_i^{-1}, \quad v_i \in G \tag{4.3}$$

Introducing local angular coordinate ϕ_i on discs D_i around point z_i one can locally write

$$A_i = -\frac{i}{k}\eta_i\lambda_i\eta_i^{-1}d\phi_i - d\eta_i\eta_i^{-1} \tag{4.4}$$

where $\eta_i \in G$ is single-valued on the disc and $\eta_i(z_i) = v_i$.

The solution (4.4) implies that holonomy M_i of flat connection around point z_i takes value in conjugacy classes $\mathcal{C}_i$:

$$M_i = \eta_i e^{2\pi i\lambda_i/k}\eta_i^{-1} \tag{4.5}$$

The phase space of the Chern-Simons theory is given by the moduli space of flat connections on the Riemann surface Σ punctured at the points z_i where Wilson lines hit M, with the holonomies around punctures belonging to the conjugacy classes M_i.

4.1.2 genus 0

Here we present details on symplectic form on moduli space of flat connections on sphere $S^2_{n,m}$ with n Wilson line and m holes.

Holonomies are subject to the relation

$$M_n\cdots M_1 = 1 \tag{4.6}$$

The symplectic form on the moduli space of flat connections on 2-dimensional manifold M with n sources is given by formula

$$\Omega = \frac{k}{4\pi}\text{tr}\int_M(\delta A)^2 + i\sum_{i=1}^{n}\text{tr}(\lambda_i(v_i^{-1}\delta v_i)^2) \tag{4.7}$$

where A satisfies (4.2). The δ denotes here exterior derivative on moduli space.

In the terms of holonomies M_i it takes the form [7]:

$$\Omega_{S_n^2} = \frac{k}{4\pi} \sum_{i=1}^{n} \omega_{\lambda_i}(M_i) + \frac{k}{4\pi} \sum_{i=1}^{n} \text{tr}(K_{i-1}^{-1}\delta K_{i-1}K_i^{-1}\delta K_i) \tag{4.8}$$

where $\omega_{\lambda_i}(M_i)$ is the same two-form which appeared in (2.6):

$$\omega_{\lambda_i}(M_i) = \text{tr}(\eta_i^{-1}\delta\eta_i e^{2\pi i\lambda_i/k}\eta_i^{-1}\delta\eta_i e^{-2\pi i\lambda_i/k}) \tag{4.9}$$

Here $K_0 = K_n = I$ and

$$K_i = M_i \cdots M_1 \tag{4.10}$$

Let us briefly explain how quantization of the moduli space of flat connection on $S_{n,0}^2$ with form (4.8) leads to the space of conformal blocks. By a change of variables symplectic form (4.8) can be written as sum of Poisson-Lie Ω^{PL} forms,

$$\Omega_{S_n^2} = \sum_{i=1}^{n} \Omega^{PL}(M_i) \tag{4.11}$$

where

$$\Omega^{PL}(M) = \omega_\lambda(M) + \text{Tr}(L_+^{-1}\delta L_+ L_-^{-1}\delta L_-) \tag{4.12}$$

L_+ and L_- here are components of the Gauss decomposition $L_+L_- = M$. On the other side it is known that quantization with Ω^{PL} leads to the highest weight representations $\Upsilon_{q,\lambda}$ of the deformed enveloping algebra $\mathcal{U}_q(\text{g})$. Hence quantizing $\mathcal{P}_{S_{n,0}^2}$ with the form $\Omega_{S_n^2}$ leads to the tensor product $\otimes_i \Upsilon_{q,\lambda_i}$. Gauge transformation of gauge connections give rises on the quantum level to the diagonal action of $\mathcal{U}_q(\text{g})$ on $\otimes_i \Upsilon_{q,\lambda_i}$. Therefore, in the first approximation, we obtain the subspace of invariant tensors of that action. More precisely, the subspace of invariants may be equipped with a semipositive scalar product and one should divide by the subspace of null-vectors. The quotient spaces are isomorphic to the spaces of conformal blocks of the WZW theory.

Consider now the case of sphere with m holes.

The symplectic form on moduli space of flat connections on sphere with n sources and m holes can be decomposed as sum of symplectic forms on moduli space of flat connections on

sphere $S^2_{n+m,0}$ with $n+m$ sources and m copies of the symplectic form on moduli space of flat connections on the two-dimensional disc with one source D_1:

$$\Omega_{S^2_{n,m}} = \Omega_{S^2_{n+m,0}} + \sum_{i=1}^{m} \Omega_{D_{1_i}} \tag{4.13}$$

$$\Omega_{D_1} = \Omega^{\mathrm{LG}}(\eta, \lambda) \tag{4.14}$$

where

$$\Omega^{\mathrm{LG}}(\eta, \lambda) = \int_{\partial D} \frac{k}{4\pi} \mathrm{tr}(\eta^{-1}\delta\eta) d(\eta^{-1}\delta\eta) + \frac{1}{2\pi} \mathrm{tr}(i\lambda(\eta^{-1}\delta\eta)^2) d\phi \tag{4.15}$$

Geometrical quantization of the coadjoint orbits of $\hat{LG}$ with the form (4.15) leads to the integrable representation H_λ of the affine algebra $\hat{\mathfrak{g}}$ at level k.

We obtain that Hilbert space of quantized Chern-Simons theory on $S^2_{n,m} \times R$, were n time-like Wilson lines assigned with representations $\lambda_1, \ldots \lambda_n$ must be of the form

$$\mathcal{H}_{n,m} = \sum_{\tau_1,\ldots\tau_m} V_{\lambda_1,\ldots\lambda_n,\tau_1,\ldots\tau_m} \otimes H_{\tau_1} \otimes \ldots H_{\tau_m} \tag{4.16}$$

where H_{τ_i} are the representation spaces of $\hat{LG}$ corresponding to the highest weights τ_i, and $V_{\kappa_1,\ldots\kappa_l}$ is space of conformal blocks of the WZW model with group G.

We finish this section by writing explicitly formula (4.8) for the cases $n = 3$ and $n = 4$, which we need in next sections. For the case of $n = 3$

$$\Omega_{S^2_{3,0}} = \frac{k}{4\pi} \sum_{i=1}^{3} \omega_{\lambda_i}(M_i) + \frac{k}{4\pi} \mathrm{tr}(\delta M_1 M_1^{-1} M_2^{-1} \delta M_2) \tag{4.17}$$

For the case of $n = 4$ the second term in (4.8) can be written in two equivalent forms:

$$\Omega_{S^2_{4,0}} = \frac{k}{4\pi} \sum_{i=1}^{4} \omega_{\lambda_i}(M_i) + \frac{k}{4\pi} \mathrm{tr}(\delta M_1 M_1^{-1} M_2^{-1} \delta M_2 + \delta M_3 M_3^{-1} M_4^{-1} \delta M_4) \tag{4.18}$$

or

$$\Omega_{S^2_{4,0}} = \frac{k}{4\pi} \sum_{i=1}^{4} \omega_{\lambda_i}(M_i) + \frac{k}{4\pi} \mathrm{tr}(\delta M_1 M_1^{-1} M_2^{-1} \delta M_2 + \delta M_1 M_1^{-1} M_2^{-1} M_3^{-1} \delta M_3 M_2 + \delta M_2 M_2^{-1} M_3^{-1} \delta M_3) \tag{4.19}$$

4.1.3 genus g

Denoting holonomies around handles a_j and b_j by A_j and B_j, and around punctures by $M_i \in C_G^{\lambda_i}$ we arrive at the conclusion that the moduli space of the flat connections on a Riemann surface of the genus g with punctures is

$$\mathcal{F}_{g,n} = G^{2g} \times \prod_{i=1}^{n} C_G^{\lambda_i} \tag{4.20}$$

subject to the relation

$$[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] M_n \cdots M_1 = I \,, \tag{4.21}$$

where

$$[B_j, A_j] = B_j A_j B_j^{-1} A_j^{-1} \,, \tag{4.22}$$

and to the adjoint group action.

The symplectic form on $\mathcal{F}_{g,n}$ was derived in [7] and has the form:

$$\Omega_{\mathcal{M}_{g,n}} = \sum_{i=1}^{n} \Omega_{M_i} + \sum_{j=1}^{g} \Omega_{H_j} \,, \tag{4.23}$$

where

$$\Omega_{M_i} = \frac{k}{4\pi} \omega_{\lambda_i}(M_i) + \frac{k}{4\pi} \mathrm{tr}(K_{i-1}^{-1} \delta K_{i-1} K_i^{-1} \delta K_i) \,, \tag{4.24}$$

$$\begin{aligned} \Omega_{H_j} &= \frac{k}{4\pi} \Psi(B_j, A_j) + \frac{k}{4\pi} (\mathrm{tr}(K_{n+2j-2}^{-1} \delta K_{n+2j-2} K_{n+2j-1}^{-1} \delta K_{n+2j-1}) \\ &+ \mathrm{tr}(K_{n+2j-1}^{-1} \delta K_{n+2j-1} K_{n+2j}^{-1} \delta K_{n+2j})) \,, \end{aligned} \tag{4.25}$$

and where

$$K_i = M_i \cdots M_1 \qquad i \le n \,, \tag{4.26}$$

$$K_{n+2j-1} = A_j [B_{j-1}, A_{j-1}^{-1}] \cdots [B_1, A_1^{-1}] K_n \,, \tag{4.27}$$

$$K_{n+2j} = [B_j, A_j^{-1}] \cdots [B_1, A_1^{-1}] K_n \qquad 1 \le j \le g \,.$$

K_0 can be chosen to be equal to the unity element. According to (4.21) also

$$K_{n+2g} = I \,. \tag{4.28}$$

$\omega_\lambda(M)$ and $\Psi(B, A)$ are defined in equations (4.9) and (2.9) correspondingly.

It was also proved in [7] that quantization of the moduli space $\mathcal{F}_{g,n}$ with the symplectic form (4.23) leads to the space of n-point conformal blocks on a Riemann surface of the genus g.

4.1.4 Double Chern-Simons theory

The last piece which we need is the double CS theory [131] with a pair $(\mathcal{A}, \mathcal{B})$ of the respectively group G and group $H \subset G$ gauge fields. The action functional of the double theory is the difference of the CS actions for group G and H:

$$S^{2CS}(\mathcal{A}, \mathcal{B}) = S^{CS}(\mathcal{A}) - S^{CS}(\mathcal{B}) \tag{4.29}$$

The symplectic form is

$$\Omega^{2CS} = \frac{k}{4\pi} \int_\Sigma \mathrm{Tr} \left[(\delta A)^2 - (\delta B)^2 \right] \tag{4.30}$$

Clearly, both gauge fields may be coupled to time-like Wilson lines with labels in the Cartan subalgebras of g and h respectively.

4.2 Canonical quantization of the WZW model with defects and boundaries

4.2.1 Statements to prove

Let us now explain how the Hilbert space described by eq.(4.16) appears in the different instances of the WZW model.

Comparing $H_{0,2}$ with (1.102) we see that it is the Hilbert space of the WZW model on cylinder. This implies that the symplectic phase space of the WZW model on a cylinder is symplectomorphic to that of the Chern-Simons theory on an annulus. This was observed in [52].

Comparing $H_{2,1}$ with (1.138) shows that it is the Hilbert space of the WZW model on a strip with the boundary conditions specified by the Cardy states. It was proved in [88] that the symplectic phase space of the WZW model on a strip is indeed symplectomorphic to that of Chern-Simons theory on a disc with two Wilson lines.

Inclusion of defects allows to extend these results [165].

Comparing $H_{N,2}$ with (1.180) shows that it is the Hilbert space of the WZW model on cylinder with N defects.

Comparing $H_{N+2,1}$ with (1.183) we see that it is the Hilbert space of the WZW model on a strip with N defects.

Comparing $H_{2,N}$ with (1.163) shows that it is the Hilbert space of N-fold product of WZW models on a strip with boundary conditions given by the permutation branes.

This leads to the following statements [165] proved in the next section:

1. The symplectic phase space of the WZW model with N defects on a cylinder is isomorphic to that of Chern-Simons gauge theory on an annulus $\mathcal{A}$ times the time-line R with N time-like Wilson lines.

2. The symplectic phase space of the WZW model with N defects on a strip is isomorphic to that of Chern-Simons gauge theory on a disc D times the time-line R with $N + 2$ time-like Wilson lines.

3. The symplectic phase space of N-fold product of WZW models on a strip with boundary conditions given by the permutation branes is isomorphic to that of Chern-Simons gauge theory on a sphere with N holes times the time-line R and with two time-like Wilson lines.

4.2.2   Bulk WZW model

In this section we review canonical quantization of the WZW model on the cylinder $\Sigma = R \times S^1 = (t, x \mod 2\pi)$ [44, 58, 86]. The world-sheet action of the bulk WZW model is studied

in section 1.5.1, is given by (1.263), but here for the purpose of the canonical quantization we will perform Wick rotation, and use instead of the complex the light cone coordinates $x^{\pm} = x \pm t$:

$$S^{\text{bulk}}(g) = \tag{4.31}$$

$$\frac{k}{4\pi} \int_{\Sigma} \text{Tr}(g^{-1}\partial_+ g)(g^{-1}\partial_- g) dx^+ dx^- + \frac{k}{4\pi} \int_B \omega^{WZ}(g)$$

Note also that we use here and in the next chapter the WZW kinetic term with the opposite sign than in section 1.5.1. The phase space of solutions $\mathcal{P}$ can be described by the Cauchy data at $t = 0$.

$$g(x) = g(0, x) \quad \text{and} \quad \xi_0(x) = g^{-1}\partial_t g(0, x) \tag{4.32}$$

The corresponding symplectic form is

$$\Omega^{\text{bulk}} = \frac{k}{4\pi} \int_0^{2\pi} \Pi(g) dx \tag{4.33}$$

where

$$\Pi(g) = \text{tr}\left(-\delta\xi_0 g^{-1}\delta g + (\xi_0 + g^{-1}\partial_x g)(g^{-1}\delta g)^2\right) \tag{4.34}$$

The δ denotes here as before exterior derivative on the phase space $\mathcal{P}$. It is easy to check that the symlectic form density $\Pi(g)$ has the following exterior derivative

$$\delta\Pi(g) = \partial_x \omega^{WZ}(g) \tag{4.35}$$

what implies closedness of the Ω

$$\delta\Omega^{\text{bulk}} = 0 \tag{4.36}$$

The general solution of equations of motion (1.270) satisfying the periodicity conditions

$$g(t, x + 2\pi) = g(t, x) \tag{4.37}$$

is

$$g(t, x) = g_L(x^+) g_R^{-1}(x^-) \tag{4.38}$$

with $g_{L,R}$ satisfying monodromy conditions

$$g_L(x^+ + 2\pi) = g_L(x^+)\gamma \tag{4.39}$$

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$$g_R(x^- + 2\pi) = g_R(x^-)\gamma \tag{4.40}$$

with the same matrix γ. Expressing the symlectic form density $\Pi(g)$ in the terms of $g_{L,R}$ we obtain

$$\Pi = \text{tr}[g_L^{-1}\delta g_L \partial_x (g_L^{-1}\delta g_L) - g_R^{-1}\delta g_R \partial_x (g_R^{-1}\delta g_R) + \partial_x (g_L^{-1}\delta g_L g_R^{-1}\delta g_R)] \tag{4.41}$$

Using (4.41) and (4.39), (4.40) one derives for Ω

$$\Omega^{\text{bulk}} = \Omega^{\text{chiral}}(g_L, \gamma) - \Omega^{\text{chiral}}(g_R, \gamma) \tag{4.42}$$

where

$$\Omega^{\text{chiral}}(g_L, \gamma) = \frac{k}{4\pi} \int_0^{2\pi} \text{tr}\left(g_L^{-1}\delta g_L \partial_x (g_L^{-1}\delta g_L)\right) dx + \frac{k}{4\pi}\text{tr}(g_L^{-1}\delta g_L(0)\delta\gamma\gamma^{-1}) \tag{4.43}$$

The chiral field g_L can be decomposed into the product of a closed loop in G, a multivalued field in the Cartan subgroup and a constant element in G:

$$g_L(x) = h(x)e^{i\tau x/k}g_0^{-1} \tag{4.44}$$

where $h \in LG$, $\tau \in t$ (the Cartan algebra) and $g_0 \in G$. For the monodromy of g_L we obtain

$$\gamma = g_0 e^{2i\pi\tau/k}g_0^{-1} \tag{4.45}$$

Parametrization (4.44) induces the following decomposition of $\Omega^{\text{chiral}}(g_L, \gamma)$

$$\Omega^{\text{chiral}}(g_L, \gamma) = \Omega^{LG}(h, \tau) + \frac{k}{4\pi}\omega_\tau(\gamma) \tag{4.46}$$

where $\Omega^{LG}(h, \tau)$ is defined in (4.15), and $\omega_\tau(\gamma)$ is defined in (4.9). Recalling (4.13), and (4.42) we see that the symplectic phase of the WZW model on circle coincides with that of CS theory on annulus.

4.2.3 Boundary WZW model

Consider the WZW model on a strip $R \times [0, \pi]$ for the Cardy boundary conditions. From the analysis of the section 2.1.1 follows that for the case of strip we should impose the boundary conditions (2.32):

$$g(t, 0) \in \mathcal{C}_{\mu_0}, \quad g(t, \pi) \in \mathcal{C}_{\mu_\pi} \tag{4.47}$$

The boundary equations of motion (2.18)*

$$\partial_+ g g^{-1} = g^{-1} \partial_- g \tag{4.48}$$

force g_L and g_R to satisfy the constraint

$$g_L(y + 2\pi) = g_L(y)\gamma, \quad \text{and} \quad g_R(y) = g_L(-y)h_0^{-1} \tag{4.49}$$

The equations (4.49) imply

$$g(t, 0) = g_L(t)g_R^{-1}(-t) = g_L(t)h_0 g_L^{-1}(t) \tag{4.50}$$

and

$$g(t, \pi) = g_L(\pi + t)g_R^{-1}(\pi - t) = g_L(-\pi + t)\gamma h_0 g_L^{-1}(-\pi + t) \tag{4.51}$$

Therefore to be in agreement with (4.47) one should require

$$h_0 \in \mathcal{C}_{\mu_0}, \quad \text{and} \quad \gamma h_0 = h_\pi \in \mathcal{C}_{\mu_\pi} \tag{4.52}$$

The symplectic form on the phase space of the WZW model on the strip is:

$$\Omega^{\text{strip}} = \frac{k}{4\pi} \left[\int_0^\pi \Pi(g)dx + \omega_{\mu_0}(g(0,0)) - \omega_{\mu_\pi}(g(0,\pi)) \right] \tag{4.53}$$

where ω_μ is defined in (2.6).

The equations (4.35), (2.4) imply that the form (4.53) is closed. Using the relations above it is obtained in [88] that

$$\Omega^{\text{strip}} = \Omega^{LG}(h, \tau) + \Omega^{\text{bndry}} \tag{4.54}$$

where

$$\Omega^{\text{bndry}} = \frac{k}{4\pi} \left[\omega_\tau(\gamma) + \omega_{\mu_0}(h_0) - \omega_{\mu_\pi}(\gamma h_0) + \text{tr}(\delta h_0 h_0^{-1}\gamma^{-1}\delta\gamma) \right] \tag{4.55}$$

Note that (4.55) has the form (4.17). Therefore comparing (4.52) with (4.6), and (4.54) with (4.13) for $n = 2$ and $m = 1$ we obtain that symplectic phase space of the WZW model on the strip coincides with that of CS theory on the disc with two Wilson lines.

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*The sign difference comes from the choice of the light cone coordinates $x^\pm$.

4.2.4 WZW model with Topological defects

Recall briefly the basic facts on defects studied in section 3.2. Assume that one has defect line separating world-sheet on two regions Σ_1 and Σ_2. In such a situation WZW model defined by a pair of maps g_1 and g_2. Maximally-symmetric topological defects defined as defect lines satisfying conditions:

$$J_1 = J_2|_{\text{defect line}} \quad \text{and} \quad \bar{J}_1 = \bar{J}_2|_{\text{defect line}} \tag{4.56}$$

The conditions (4.56) imply that on the defect line fields g_1 and g_2 satisfy the constraint

$$g_1 g_2^{-1}|_{\text{defect line}} = F \in \mathcal{C}_\mu = \beta e^{2i\pi\mu/k}\beta^{-1}, \quad \beta \in G \tag{4.57}$$

where $\mu \equiv \boldsymbol{\mu} \cdot \mathcal{H}$, as before, is a highest weight representation integrable at level k, taking value in the Cartan subalgebra. To write action of the WZW model with defect one again should introduce auxiliary disc satisfying conditions

$$\partial B_1 = \Sigma_1 + \bar{D} \quad \text{and} \quad \partial B_2 = \Sigma_2 + D \tag{4.58}$$

and continue fields g_1 and g_2 on this disc always holding the condition (4.57). After this preparations the action takes the form:

$$S = S^{\text{bulk}}(g_1) + S^{\text{bulk}}(g_2) + \frac{k}{4\pi} \int_D \varpi(g_1, g_2) \tag{4.59}$$

where

$$\varpi(g_1, g_2) = \omega_\mu(F) - \text{Tr}(g_1^{-1}dg_1 g_2^{-1}dg_2) \tag{4.60}$$

The form (4.60) satisfies the equation:

$$d\varpi(g_1, g_2) = \omega^{WZ}(g_1)|_{\text{defect}} - \omega^{WZ}(g_2)|_{\text{defect}} \tag{4.61}$$

Equation (4.61) guarantees that the action (4.59) is well defined.

Now consider WZW model on the same cylinder as in section 4.2.2, and put defect line at $x = a$ in parallel to the time line. The defect gluing conditions (4.56) constrain $g_{L_1}, g_{R_1}, g_{L_2}, g_{R_2}$ to satisfy the following relations:

$$g_{L_2}(y) = g_{L_1}(y)h_a^{-1} \tag{4.62}$$

$$g_{R_2}(y) = g_{R_1}(y)m_a$$

The equations (4.62) imply

$$F(t, a) = g_1 g_2^{-1}(t, a) = \tag{4.63}$$

$$g_{L_1}(a + t) g_{R_1}^{-1}(a - t) g_{R_2}(a - t) g_{L_2}^{-1}(a + t) =$$

$$g_{L_1}(a + t) m_a h_a g_{L_1}^{-1}(a + t)$$

Therefore to satisfy the boundary condition (4.57) we should require

$$m_a h_a = d_a \in \mathcal{C}_{\mu_a} \tag{4.64}$$

Given that we consider WZW model on cylinder we should additionally require

$$g_2(t, 2\pi) = g_1(t, 0) \tag{4.65}$$

The condition (4.65) imposes the following relation on monodromies γ_L, γ_R of g_{L_1} and g_{R_1}:

$$g_{L_1}(y + 2\pi) = g_{L_1}(y)\gamma_L \tag{4.66}$$

$$g_{R_1}(y + 2\pi) = g_{R_1}(y)\gamma_R$$

and

$$\gamma_R^{-1}\gamma_L = m_a h_a = d_a \tag{4.67}$$

It is instructive to compare (4.67) to (4.39) and (4.40). We have seen in section 4.2.2, that in the absence of defect left and right monodromies are equal, whereas presence of defect creates relative shift between them equal to the defect conjugacy class. The symplectic form now is:

$$\Omega^{\mathrm{d}} = \frac{k}{4\pi} \left[\int_0^a \Pi(g_1) dx + \int_a^{2\pi} \Pi(g_2) dx - \varpi(g_1(0, a), g_2(0, a)) \right] \tag{4.68}$$

The conditions (4.35) and (4.61) imply that

$$\delta\Omega^{\mathrm{d}} = 0 \tag{4.69}$$

Substituting in (4.68) the relations above, we can show that [165]

$$\Omega^{\mathrm{d}} = \Omega^{LG}(h_L, \tau_L) - \Omega^{LG}(h_R, \tau_R) + \Omega^{\mathrm{dl}} \tag{4.70}$$

where

$$\Omega^{\text{dl}} = \frac{k}{4\pi}[\omega_{\tau_L}(\gamma_L) - \omega_{\tau_R}(\gamma_R) - \omega_{\mu_a}(d_a)] + \text{tr}(\delta\gamma_R\gamma_R^{-1}\delta\gamma_L\gamma_L^{-1}) \tag{4.71}$$

Note that (4.71) has the form (4.17). Comparing (4.67) with (4.6) and (4.70) with (4.13) for $n = 1$ and $m = 2$ we obtain that the phase space of the WZW model with a defect is symplectomorphic with that of Chern-Simons theory on an annulus with a Wilson line. Generalization to an arbitrary number of defects is straightforward.

4.2.5   Defects in open string

In this section we consider WZW model with defect on a strip. Assume again that we have defect at point $x = a$ in parallel to the time line. The strip is divided to two parts with fields g_1 and g_2. We should impose here boundary conditions (2.32) at $x = 0$ on g_1, requiring

$$g_1(t,0) \in \mathcal{C}_{\mu_0} = \beta_0 e^{2i\pi\mu_0/k}\beta_0^{-1}, \quad \beta_0 \in G \tag{4.72}$$

then defect condition (4.57) at $x = a$, requiring

$$g_1 g_2^{-1}(t,a) \in \mathcal{C}_{\mu_a} = \beta_a e^{2i\pi\mu_a/k}\beta_a^{-1}, \quad \beta_a \in G \tag{4.73}$$

and finally boundary condition (2.32) at $x = \pi$ on g_2, requiring

$$g_2(t,\pi) \in \mathcal{C}_{\mu_\pi} = \beta_\pi e^{2i\pi\mu_\pi/k}\beta_\pi^{-1}, \quad \beta_\pi \in G \tag{4.74}$$

Equations (4.72) and (4.73) as before yield:

$$g_{R_1}(y) = g_{L_1}(-y)h_0^{-1} \tag{4.75}$$

$$g_1(0,t) = g_{L_1}(t)g_{R_1}^{-1}(-t) = g_{L_1}(t)h_0 g_{L_1}^{-1}(t) \tag{4.76}$$

$$h_0 \in \mathcal{C}_{\mu_0} \tag{4.77}$$

$$g_{L_2}(y) = g_{L_1}(y)h_a^{-1} \tag{4.78}$$

$$g_{R_2}(y) = g_{R_1}(y)m_a$$

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$$m_a h_a = d_a \in \mathcal{C}_{\mu_a} \tag{4.79}$$

To solve the last boundary condition (4.74) we assume that g_{L_1} has monodromy matrix γ:

$$g_{L_1}(y + 2\pi) = g_{L_1}(y)\gamma \tag{4.80}$$

Using (4.75) and (4.78) one obtains:

$$g_{L_2}(y + 2\pi) = g_{L_2}(y)h_a \gamma h_a^{-1} \tag{4.81}$$

$$g_{R_2}(y) = g_{L_2}(-y)h_a h_0^{-1} m_a \tag{4.82}$$

Equations (4.81) and (4.82) imply

$$g_2(\pi, t) = g_{L_2}(\pi + t)g_{R_2}^{-1}(\pi - t) = \tag{4.83}$$

$$g_{L_2}(-\pi + t)h_a \gamma h_a^{-1} m_a^{-1} h_0 h_a^{-1} g_{L_2}^{-1}(-\pi + t)$$

To satisfy (4.74) one should require

$$\gamma h_a^{-1} m_a^{-1} h_0 = \gamma d_a^{-1} h_0 = h_\pi \in \mathcal{C}_{\mu_\pi} \tag{4.84}$$

It is again instructive to compare (4.84) to (4.52). We see that presence of defect again requires to include defect conjugacy class. This is classical analogue of the defect-boundary fusion (1.181). The symplectic form is

$$\Omega^{\text{sd}} = \frac{k}{4\pi}\left[\int_0^a \Pi(g_1)dx + \int_a^\pi \Pi(g_2)dx - \right. \tag{4.85}$$

$$\left. \varpi(g_1(0, a), g_2(0, a)) + \omega_{\mu_0}(g_1(0, 0)) - \omega_{\mu_\pi}(g_2(0, \pi))\right]$$

Using the relations above we obtain [165]:

$$\Omega^{\text{sd}} = \Omega^{LG}(h, \tau) + \Omega^{\text{b-d}} \tag{4.86}$$

where

$$\Omega^{\text{b-d}} = \frac{k}{4\pi}[\omega_\tau(\gamma) + \omega_{\mu_0}(h_0) - \omega_{\mu_\pi}(h_\pi) - \tag{4.87}$$

$$\omega_{\mu_a}(d_a) + \text{tr}(d_a^{-1}\delta h_0 h_0^{-1} d_a \gamma^{-1}\delta\gamma) + \text{tr}(\gamma^{-1}\delta\gamma d_a^{-1}\delta d_a)$$

$$+\text{tr}(\delta d_a d_a^{-1}\delta h_0 h_0^{-1})]$$

Note that (4.87) has the form (4.19). Therefore comparing (4.84) to (4.6) and (4.86) to (4.13) for $n = 3$ and $m = 1$ we obtain that the phase space of the WZW model on a strip with a defect inserted is symplectomorphic to that of CS theory on disc with three Wilson lines.

4.2.6 Permutation branes

Recall basic facts on the permutation branes studied in section 2.3.

Maximally symmetric permutation branes on two-fold product of the WZW models $G \times G$ is defined as boundary conditions satisfying the relations (2.108)[†]:

$$J_1 = -\bar{J}_2|_{\partial M} \tag{4.88}$$

and

$$J_2 = -\bar{J}_1|_{\partial M} \tag{4.89}$$

Here label 1 and 2 refer two the first and the second copy. The conditions (4.88) and (4.89) imply that values of g_1 and g_2 on the boundary constrained by the relation:

$$g_1 g_2|_{\partial M} = \tilde{F} \in \mathcal{C}_\mu = \beta e^{2i\pi\mu/k} \beta^{-1}, \quad \beta \in G. \tag{4.90}$$

The corresponding Lagrangian is:

$$S = S^{\text{bulk}}(g_1) + S^{\text{bulk}}(g_2) - \frac{k}{4\pi} \int_D \omega_{\mathcal{P}}(g_1, g_2) \tag{4.91}$$

where

$$\omega_{\mathcal{P}}(g_1, g_2) = \omega_\mu(\tilde{F}) + \text{Tr}(g_1^{-1} dg_1 dg_2 g_2^{-1}) \tag{4.92}$$

The form (4.92) satisfies the equation:

$$d\omega_{\mathcal{P}}(g_1, g_2) = \omega^{WZ}(g_1)|_{\text{boundary}} + \omega^{WZ}(g_2)|_{\text{boundary}} \tag{4.93}$$

Equation (4.93) guarantees that the action (4.91) is well defined. Consider now two-fold product on a strip with boundary conditions (4.88) and (4.89) imposed at points $x = 0$ and $x = \pi$. It is possible to show that these boundary conditions can be solved with $g_{L_1}, g_{R_1}, g_{L_2}, g_{R_2}$ satisfying:

$$g_{L_1}(y + 2\pi) = g_{L_1}(y)\gamma_1 \tag{4.94}$$

[†]The sign difference comes from the choice of the light cone coordinates $x^\pm$

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$$g_{L_2}(y + 2\pi) = g_{L_2}(y)\gamma_2 \tag{4.95}$$

$$g_{R_2}(y) = g_{L_1}(-y)h_0^{-1} \tag{4.96}$$

$$g_{R_1}(y) = g_{L_2}(-y)m_0^{-1} \tag{4.97}$$

From (4.96), (4.97) we obtain:

$$\tilde{F}(0, t) = g_{L_1}(t)m_0 h_0 g_{L_1}^{-1}(t) \tag{4.98}$$

Therefore to be in agreement with (4.90) we should require:

$$m_0 h_0 = p_0 \in \mathcal{C}_{\mu_0} \tag{4.99}$$

Equations (4.94) and (4.95) further imply

$$\tilde{F}(\pi, t) = g_{L_1}(-\pi + t)\gamma_1 m_0 \gamma_2 h_0 g_{L_1}^{-1}(-\pi + t) \tag{4.100}$$

Therefore we additionally should require:

$$\gamma_1 m_0 \gamma_2 h_0 = \gamma_1 p_0 h_0^{-1}\gamma_2 h_0 = \gamma_1 p_0 \tilde{\gamma}_2 = p_\pi \in \mathcal{C}_{\mu_\pi} \tag{4.101}$$

where

$$\tilde{\gamma}_2 = h_0^{-1}\gamma_2 h_0 \tag{4.102}$$

The symplectic form corresponding to the action (4.91) on the strip is

$$\Omega_{\mathcal{P}} = \frac{k}{4\pi}\left[\int_0^\pi (\Pi(g_1) + \Pi(g_2))dx + \omega_{\mathcal{P}}(g_1(0,0), g_2(0,0)) - \omega_{\mathcal{P}}(g_1(0,\pi), g_2(0,\pi))\right] \tag{4.103}$$

Repeating the same steps as explained in the previous sections we obtain [165]:

$$\Omega_{\mathcal{P}} = \Omega^{LG}(h_1, \tau_{1,}) + \Omega^{LG}(h_2, \tau_{2,}) + \Omega^{\text{bndry−perm}} \tag{4.104}$$

where

$$\Omega^{\text{bndry−perm}} = \frac{k}{4\pi}\left[\omega_{\tau_1}(\gamma_1) + \omega_{\tau_2}(\tilde{\gamma}_2) + \omega_{\mu_0}(p_0) - \omega_{\mu_\pi}(p_\pi)\right.$$
$$\left. - \text{tr}(p_0^{-1}\delta p_0 \delta\tilde{\gamma}_2\tilde{\gamma}_2^{-1}) - \text{tr}(\gamma_1^{-1}\delta\gamma_1\delta p_0 p_0^{-1}) - \text{tr}(p_0^{-1}\gamma_1^{-1}\delta\gamma_1 p_0 \delta\tilde{\gamma}_2\tilde{\gamma}_2^{-1})\right] \ . \tag{4.105}$$

Comparing (4.6) to (4.101) and (4.105) to (4.19), and finally (4.104) with (4.13) for $n = 2$ and $m = 2$, we see that symplectic phase space of the WZW model $G \times G$ on strip with boundary conditions specified by permutation branes coincides with that of CS on annulus with two Wilson lines. The generalization to the case of permutation branes on N-fold product is again cumbersome but straightforward.

4.3 Canonical quantization of the Gauged WZW model with boundaries and defects

4.3.1 Main Statements

In this chapter we prove the following statements [166, 167]:

1. The symplectic phase space of the gauged WZW G/H model on a cylinder with N defects is symplectomorphic to the symplectic phase space of the double Chern-Simons theory on an annulus $\mathcal{A}$ times the time-line R with G and H gauge fields both coupled to N Wilson lines.

2. The symplectic phase space of the gauged WZW G/H model on a strip with N defects is symplectomorphic to the symplectic phase space of the double Chern-Simons theory on a disc D times the time-line R with G and H gauge fields both coupled to $N + 2$ time-like Wilson lines.

3. The symplectic phase space of the N-fold product of the gauged WZW models on a strip with boundary conditions given by permutation branes is symplectomorphic to the symplectic phase space of the double Chern-Simons theory on a sphere with N holes times the time-line R with G and H gauge fields both coupled to two Wilson lines.

   In the special case of topological coset G/G these isomorphisms take the form:

4. The symplectic phase space of the gauged WZW G/G model on a cylinder with N defects

is symplectomorphic to the symplectic phase space of the Chern-Simons theory on $T^2 \times R$ with $2N$ Wilson lines.

5. The symplectic phase space of the gauged WZW G/G model on a strip with N defects is symplectomorphic to the symplectic phase space of the Chern-Simons theory on $S^2 \times R$ with $2N + 4$ time-like Wilson lines.

6. The symplectic phase space of the N-fold product of the topological coset G/G on a strip with boundary conditions given by permutation branes is symplectomorphic to the symplectic phase space of the Chern-Simons theory on a Riemann surface of the genus $N - 1$ times the time-line with four Wilson lines.

The isomorphisms 4 and 5 allow us to achieve to a very detailed picture of defects in this particular example of topological field theory. This picture enables us to infer that in general defects in semisimple 2D TFT should be described by means of a 2-category of matrices of vector spaces [82] and that the action of defects on boundary states is given by the discrete Fourier-Mukai transform.

4.3.2   Bulk gauged WZW model

Here we review quantization of the gauged WZW model on the cylinder $\Sigma = R \times S^1 = (t, x \bmod 2\pi)$ as it is done in [89].

The action of the gauged WZW model is studied in section 1.5.5:

$$S^{G/H}(g, A) = S^{\mathrm{WZW}}(g) + S^{\mathrm{gauge}}(g, A), \tag{4.106}$$

where

$$
\begin{aligned}
S^{\mathrm{WZW}}(g) &= \frac{k}{4\pi} \int_\Sigma \mathrm{Tr}(g^{-1}\partial_+ g)(g^{-1}\partial_- g) dx^+ dx^- + \frac{k}{4\pi} \int_B \frac{1}{3} \mathrm{tr}(g^{-1}dg)^3 \\
&\equiv \frac{k}{4\pi} \left[\int_\Sigma dx^+ dx^- L^{\mathrm{kin}} + \int_B \omega^{\mathrm{WZ}} \right],
\end{aligned}
\tag{4.107}
$$

$$S^{\mathrm{gauge}}(g, A) = \frac{k}{2\pi} \int_\Sigma L^{\mathrm{gauge}}, \tag{4.108}$$

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$$L^{\text{gauge}}(g, A) = -\text{tr}[-\partial_+ g g^{-1} A_- + g^{-1}\partial_- g A_+ + g A_+ g^{-1} A_- - A_+ A_-]. \qquad (4.109)$$

Here as in the previous section $x^{\pm} = x \pm t$. and since the kinetic term has the opposite sign than in section 1.5.5, the terms in (4.109) also have different signs than in 1.5.5.

With the help of the Polyakov-Wiegmann identities (1.279) and (1.278) it is easy to check that the action (4.106) is invariant under the gauge transformation:

$$g \rightarrow hgh^{-1}, \qquad A \rightarrow hAh^{-1} - dhh^{-1} \qquad (4.110)$$

for $h : \Sigma \rightarrow H$.

The equations of motions are:

$$D_+(g^{-1}D_- g) = 0, \quad \text{Tr}(g^{-1}D_- g T_H) = \text{Tr}(g D_+ g^{-1} T_H) = 0, \quad F(A) = 0, \qquad (4.111)$$

where $D_{\pm} g = \partial_{\pm} g + [A_{\pm}, g]$ and T_H is any element in the H Lie algebra.

The flat gauge field A can be written as $h^{-1}dh$ for $h : R^2 \rightarrow H$ and satisfying:

$$h(t, x + 2\pi) = \rho^{-1} h(t, x) \qquad (4.112)$$

for some $\rho \in H$.

Define $\tilde{g} = hgh^{-1}$. Note that $\tilde{g}$ satisfies

$$\tilde{g}(t, x + 2\pi) = \rho^{-1} \tilde{g}(t, x) \rho. \qquad (4.113)$$

In the terms of $\tilde{g}$ equations (4.111) take the form:

$$\partial_+(\tilde{g}^{-1}\partial_- \tilde{g}) = 0, \quad \text{Tr}(\tilde{g}^{-1}\partial_- \tilde{g} T_H) = \text{Tr}(\tilde{g}\partial_+ \tilde{g}^{-1} T_H) = 0. \qquad (4.114)$$

The canonical symplectic form density, obtained following the general prescription [38, 39, 86], is given by:

$$\Pi^{G/H}(g, h) = \Pi^G(\tilde{g}) + \partial_x \Psi(h, g), \qquad (4.115)$$

where $\Psi(h, g)$ is two-form defined in (2.9), $\Pi^G(\tilde{g})$ is defined in (4.34).

Integrating (4.115) we get the canonical symplectic form:

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$$\Omega^{G/H} = \frac{k}{4\pi} \int_0^{2\pi} \Pi^G(\tilde{g}) dx + \frac{k}{4\pi} \Psi(\rho^{-1}, hgh^{-1}(0)) \,. \tag{4.116}$$

Collecting (4.35), (4.113) and using (2.8) one can show that the form (4.116) is closed.

Equations (4.114) can be solved in the terms of the chiral fields:

$$\tilde{g} = g_L(x^+) g_R^{-1}(x^-) \,, \qquad \text{Tr}(\partial_y g_L g_L^{-1} T_H) = \text{Tr}(\partial_y g_R g_R^{-1} T_H) = 0 \tag{4.117}$$

with the monodromy properties:

$$g_L(y + 2\pi) = \rho^{-1} g_L(y) \gamma \,, \qquad g_R(y + 2\pi) = \rho^{-1} g_R(y) \gamma \,. \tag{4.118}$$

The monodromy properties (4.118) imply that the chiral fields $g_{L,R}$ should be written as products of fields as well:

$$g_L = h_B^{-1} g_A \,, \qquad g_R = h_D^{-1} g_C \,, \tag{4.119}$$

where $h_B, h_D \in H$ and $g_A, g_C \in G$. The fields in (4.119) should additionally satisfy[‡]:

$$\text{tr}[T_H(\partial_y h_B h_B^{-1} - \partial_y g_A g_A^{-1})] = 0 \,, \qquad \text{tr}[T_H(\partial_y h_D h_D^{-1} - \partial_y g_C g_C^{-1})] = 0 \tag{4.120}$$

and

$$h_B(y + 2\pi) = h_B(y)\rho \,, \qquad g_A(y + 2\pi) = g_A(y)\gamma \,, \tag{4.121}$$

$$h_D(y + 2\pi) = h_D(y)\rho \,, \qquad g_C(y + 2\pi) = g_C(y)\gamma \,. \tag{4.122}$$

Using (4.120) one can show:

$$\text{tr}[g_L^{-1} \delta g_L \partial_y (g_L^{-1} \delta g_L)] = \text{tr}[g_A^{-1} \delta g_A \partial_y (g_A^{-1} \delta g_A) - h_B^{-1} \delta h_B \partial_y (h_B^{-1} \delta h_B) + \partial_y (\delta h_B h_B^{-1} \delta g_A g_A^{-1})] \tag{4.123}$$

and similarly for g_R and h_D, g_C.

Collecting (4.117)-(4.123) and (4.41) one can show that

$$\Omega^{G/H} = \Omega^{\text{chiral}}(g_A, \gamma) - \Omega^{\text{chiral}}(g_C, \gamma) - \Omega^{\text{chiral}}(h_B, \rho) + \Omega^{\text{chiral}}(h_D, \rho) \tag{4.124}$$

[‡]One can arrive at the decomposition (4.119) with the properties (4.121) and (4.122) in the following way: taking, say, a field h_B satisfying the first part of (4.121), one can then define g_A as $g_A \equiv h_B g_L$, satisfying the second part of (4.121).

Comparing (4.124) with (4.42), remembering that the latter is the symplectic form of the Chern-Simons theory on $\mathcal{A} \times R$, and recalling (4.30), we arrive at the conclusion that the phase space of the gauged WZW model on a cylinder coincides with that of double Chern-Simons theory [89, 131] on $\mathcal{A} \times R$.

4.3.3 Quantization of GWZW with defects

Recall the action of the gauged WZW model with defect studied in section 3.3. The bibrane has the form (3.52):

$$(g_1, g_2) = (C_2 C_1 p, L p L^{-1}) \,. \tag{4.125}$$

where $C_1 \in C_G^{\mu_1}$, $C_2 \in C_H^{\mu_2}$, $p \in G$ and $L \in H$ and action is

$$S^{G/H-def}(g_1, g_2, A_1. A_2) = S^{\text{kin}-\text{def}}(g_1, g_2) + S^{\text{gauge}-\text{def}}(g_1, g_2, A_1, A_2) + S^{\text{top}-\text{def}} \,, \tag{4.126}$$

where

$$S^{\text{kin}-\text{def}}(g_1, g_2) = \frac{k}{4\pi} \int_{\Sigma_1} L^{\text{kin}}(g_1) dx^+ dx^- + \frac{k}{4\pi} \int_{\Sigma_2} L^{\text{kin}}(g_2) dx^+ dx^- \tag{4.127}$$

and

$$S^{\text{gauge}}(g_1, g_2, A_1, A_2) = \frac{k}{2\pi} \int_{\Sigma_1} L^{\text{gauge}}(g_1, A_1) + \frac{k}{2\pi} \int_{\Sigma_2} L^{\text{gauge}}(g_2, A_2) \,. \tag{4.128}$$

$$S^{\text{top}-\text{def}} = \frac{k}{4\pi} \int_{B_1} \omega^{WZ}(g_1) + \frac{k}{4\pi} \int_{B_2} \omega^{WZ}(g_2) - \frac{k}{4\pi} \int_D \varpi(g_1, g_2) \,. \tag{4.129}$$

where $\varpi(g_1, g_2)$ is given by (3.53):

$$\varpi(L, p, C_2, C_1) = \Omega^{(2)}(C_2, C_1) - \text{tr}((C_2 C_1)^{-1} d(C_2 C_1) dp p^{-1}) + \Psi(L, p) \,, \tag{4.130}$$

The gauge fields A_1 and A_2 are not restricted on the defect line.

One can check that the action (4.126) is invariant under the gauge transformations:

$$g_1 \;\rightarrow\; h_1 g_1 h_1^{-1} \,, \qquad A_1 \rightarrow h_1 A_1 h_1^{-1} - dh_1 h_1^{-1} \,, \tag{4.131}$$

$$g_2 \;\rightarrow\; h_2 g_2 h_2^{-1} \,, \qquad A_2 \rightarrow h_2 A_2 h_2^{-1} - dh_2 h_2^{-1} \,,$$

where $h_1 : \Sigma \to H$, $h_2 : \Sigma \to H$. For this purpose note that under (4.131) the boundary parameters transform in the following way:

$$p \to h_1 p h_1^{-1} \,, \quad C_1 \to h_1 C_1 h_1^{-1} \,, \quad C_2 \to h_1 C_2 h_1^{-1} \,, \quad L \to h_2 L h_1^{-1} \,. \tag{4.132}$$

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The gauge invariance follows from the Polyakov-Wiegmann identities and the transformation properties of $\varpi(L, p, C_2, C_1)$:

$$\varpi(h_2 L h_1^{-1}, h_1 p h_1^{-1}, h_1 C_2 h_1^{-1}, h_1 C_1 h_1^{-1}) - \varpi(L, p, C_2, C_1), = -\Psi(h_1, C_2 C_1 p) + \Psi(h_2, L p L^{-1}) \tag{4.133}$$

Now we consider the gauged WZW model on the cylinder $\Sigma = R \times S^1 = (t, x \bmod 2\pi)$ and put defect line at $x = a$ in parallel to the time line.

The variational equation $\delta S^{G/H-def}(g_1, g_2, A_1.A_2) = 0$ implies the bulk equations (4.111) for g_1, A_1 and g_2, A_2 separately supplemented by the defect equations at $x = a$:

$$g_1^{-1} D_- g_1 - L^{-1} g_2^{-1} D_- g_2 L = 0, \tag{4.134}$$

$$C_2^{-1} g_1 D_+ g_1^{-1} C_2 - L^{-1} g_2 D_+ g_2^{-1} L = 0, \tag{4.135}$$

$$L^{-1} D_t L = 0, \quad C_2^{-1} D_t C_2 = 0, \tag{4.136}$$

where $D_t = D_+ - D_-$, $D_\pm L = \partial_\pm L + A_{2\pm} L - L A_{1\pm}$, $D_\pm g_1 = \partial_\pm g_1 + [A_{1\pm}, g_1]$, $D_\pm g_2 = \partial_\pm g_2 + [A_{2\pm}, g_2]$, $D_\pm C_2 = \partial_\pm C_2 + [A_{1\pm}, C_2]$.

The equations (4.134), (4.135), (4.136) are derived in [166].

Flat gauge fields can be parameterised as before:

$$A_1 = h_1^{-1} dh_1, \qquad A_2 = h_2^{-1} dh_2. \tag{4.137}$$

Defining as before:

$$\tilde{g}_1 = h_1 g_1 h_1^{-1}, \qquad \tilde{g}_2 = h_2 g_2 h_2^{-1}, \tag{4.138}$$

$$\tilde{C}_1 = h_1 C_1 h_1^{-1}, \qquad \tilde{C}_2 = h_1 C_2 h_1^{-1},$$

$$\tilde{p} = h_1 p h_1^{-1}, \qquad \tilde{L} = h_2 L h_1^{-1},$$

we have the bulk equations (4.114) for $\tilde{g}_1$ and $\tilde{g}_2$ and the defect equations (4.134), (4.135), (4.136) take the form:

$$\tilde{g}_1^{-1} \partial_- \tilde{g}_1 - \tilde{L}^{-1} \tilde{g}_2^{-1} \partial_- \tilde{g}_2 \tilde{L} = 0, \tag{4.139}$$

$$\tilde{C}_2^{-1} \tilde{g}_1 \partial_+ \tilde{g}_1^{-1} \tilde{C}_2 - \tilde{L}^{-1} \tilde{g}_2 \partial_+ \tilde{g}_2^{-1} \tilde{L} = 0, \tag{4.140}$$

$$\tilde{L}^{-1}\partial_t\tilde{L} = 0 \,, \quad \tilde{C}_2^{-1}\partial_t\tilde{C}_2 = 0 \,. \tag{4.141}$$

Equation (4.141) implies that $\tilde{L}$ and $\tilde{C}_2$ are constant along the defect line.

Using that, the bulk-defect equations can be solved in the terms of the chiral fields:

$$\tilde{g}_1 = g_{1L}g_{1R}^{-1} \,, \qquad \text{Tr}(\partial_y g_{1L}g_{1L}^{-1}T_H) = \text{Tr}(\partial_y g_{1R}g_{1R}^{-1}T_H) = 0 \,, \tag{4.142}$$

$$\tilde{g}_2 = g_{2L}g_{2R}^{-1} \,, \qquad \text{Tr}(\partial_y g_{2L}g_{2L}^{-1}T_H) = \text{Tr}(\partial_y g_{2R}g_{2R}^{-1}T_H) = 0 \,, \tag{4.143}$$

and

$$g_{2L} = \tilde{L}\tilde{C}_2^{-1}g_{1L}n^{-1} \,, \qquad g_{2R} = \tilde{L}g_{1R}m^{-1} \,, \tag{4.144}$$

with m and $n \in G$. Equations (4.144) imply

$$(\tilde{g}_1(t,a), \tilde{g}_2(t,a)) = (\tilde{C}_2\tilde{C}_1\tilde{p}, \tilde{L}\tilde{p}\tilde{L}^{-1}) \,, \tag{4.145}$$

where

$$\tilde{p} = \tilde{C}_2^{-1}g_{1L}(a+t)n^{-1}mg_{1R}^{-1}(a-t) \,, \tag{4.146}$$

$$\tilde{C}_1 = \tilde{C}_2^{-1}g_{1L}(a+t)m^{-1}ng_{1L}^{-1}(a+t)\tilde{C}_2 \,. \tag{4.147}$$

To have that $\tilde{C}_1 \in C_G^{\mu_1}$ we should require that $d \equiv m^{-1}n \in C_G^{\mu_1}$.

Given that we consider GWZW model on a cylinder we should additionally require:

$$g_1(t,0) = g_2(t,2\pi) \,, \tag{4.148}$$

$$h_1(t,0) = \rho h_2(t,2\pi) \,. \tag{4.149}$$

From (4.148) and (4.149) one obtains:

$$\tilde{g}_1(t,0) = \rho\tilde{g}_2(t,2\pi)\rho^{-1} \,, \tag{4.150}$$

and

$$g_{1L}(y+2\pi) = \tilde{C}_2\tilde{L}^{-1}\rho^{-1}g_{1L}(y)\gamma_L \,, \qquad g_{1R}(y+2\pi) = \tilde{L}^{-1}\rho^{-1}g_{1R}(y)\gamma_R \,, \tag{4.151}$$

with γ_L and γ_R satisfying the relation:

$$\gamma_R^{-1}\gamma_L = d \,. \tag{4.152}$$

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Comparing (4.151) with (4.118) we see that the presence of the defect leads to the relative shifts between the left and right monodromies, equal to the defect conjugacy classes.

The monodromies (4.151) as before can be realized in the terms of the decomposition of the fields g_{1L} and g_{1R} as products:

$$g_{1L} = h_B^{-1} g_A \,, \qquad g_{1R} = h_D^{-1} g_C \tag{4.153}$$

of the new fields h_B, g_A, h_D, g_C possessing the monodromy properties:

$$h_B(2\pi) = h_B(0)\rho\tilde{L}\tilde{C}_2^{-1} \,, \qquad g_A(2\pi) = g_A(0)\gamma_L \,, \tag{4.154}$$

$$h_D(2\pi) = h_D(0)\rho\tilde{L} \,, \qquad g_C(2\pi) = g_C(0)\gamma_R \,, \tag{4.155}$$

and satisfying (4.120).

The symplectic form of the gauged WZW model with a defect can be written using the symplectic form density (4.115) and the form ϖ:

$$\Omega^{G/H-def} = \frac{k}{4\pi}\left[\int_0^a \Pi^{G/H}(g_1, h_1)dx + \int_a^{2\pi} \Pi^{G/H}(g_2, h_2)dx - \varpi(g_1(a), g_2(a))\right] \,. \tag{4.156}$$

Substituting in (4.156) the symplectic form density (4.115) and using the transformation property (4.133) we obtain:

$$\Omega^{G/H-def} = \frac{k}{4\pi}\left[\int_0^a \Pi(\tilde{g}_1)dx + \int_a^{2\pi} \Pi(\tilde{g}_2)dx - \varpi(\tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_1) - \Psi(\rho, \tilde{g}_2(2\pi))\right] \,, \tag{4.157}$$

where $\tilde{p}$ and $\tilde{C}_1$ defined in (4.146) and (4.147).

Performing similar steps as before we arrive at the following expression for the symplectic form of the gauged WZW model with defects:

$$\Omega^{G/H-def} = \Omega^{\text{chiral}}(g_A, \gamma_L) - \Omega^{\text{chiral}}(g_C, \gamma_R) - \Omega^{\text{chiral}}(h_B, \rho\tilde{L}\tilde{C}_2^{-1}) + \Omega^{\text{chiral}}(h_D, \rho\tilde{L}) \tag{4.158}$$

$$+ \frac{k}{4\pi}\left[-\omega_{\mu_2}(\tilde{C}_2) - \omega_{\mu_1}(d) - \text{tr}(d^{-1}\delta d\gamma_L^{-1}\delta\gamma_L) - \text{tr}(\tilde{C}_2^{-1}\delta\tilde{C}_2(\rho\tilde{L})^{-1}\delta(\rho\tilde{L}))\right] \,.$$

Recalling the decomposition (4.46) of Ω^{chiral} and (4.17), (4.30), we arrive at the conclusion that the phase space of the gauged WZW model on a cylinder with a defect line coincides with

that of double Chern-Simons theory on $\mathcal{A} \times R$ with gauge fields of groups G and H coupled to a Wilson line. This result can be straightforwardly generalized to the presence of the N defect lines.

4.3.4 Defects in open coset model G/H

Let us at the beginning remind some facts on boundary coset model G/H studied in section 3.1.1.

Boundary condition corresponding to a Cardy state (μ, ν) is given by the product of the conjugacy classes

$$g|_{\text{boundary}} = bc \,, \tag{4.159}$$

where $b \in C_G^\mu$ and $c \in C_H^\nu$. As explained in section 3.1.1 in the presence of the common center C μ and ν should satisfy the selection rule.

To write the action one should introduce an auxiliary disc D satisfying the condition $\partial B = \Sigma + D$, and continue the field g on this disc, always taking value in product of conjugacy classes.

The action with the boundary conditions (4.159) has the form (3.7):

$$S^{G/H-bndry} = S^{G/H} - \frac{k}{4\pi} \int_D \Omega^{(2)}(b, c) \,, \tag{4.160}$$

where $\Omega^{(2)}(b, c)$ is defined in (3.9).

Consider a WZW model with a defect on the strip $R \times [0, \pi]$. Assume again that we have a defect at the point $x = a$ in parallel to the time line. The strip is divided into two parts with the fields g_1, A_1 and g_2, A_2. We impose a Cardy boundary condition (4.159) at $x = 0$ on g_1 requiring:

$$g_1(t, 0) = C_3 C_4 \,, \qquad C_3 \in C_G^{\mu_3} \,, \qquad C_4 \in C_H^{\mu_4} \,, \tag{4.161}$$

a defect condition (3.52) at $x = a$:

$$(g_1, g_2) = (C_2 C_1 p, L p L^{-1}) \,, \tag{4.162}$$

and again a Cardy boundary condition (4.159) at $x = \pi$:

$$g_2(t, \pi) = C_5 C_6 \,, \qquad C_5 \in C_G^{\mu_5} \,, \qquad C_6 \in C_H^{\mu_6} \,. \tag{4.163}$$

Let us analyze first the consequences of the boundary condition (4.161) at the point $x = 0$.

The boundary equations of motion resulting from the action (4.160) at $x = 0$ are derived in [89]:

$$g_1^{-1} D_- g_1 + C_4^{-1} g_1 D_+ g_1^{-1} C_4 = 0 \,, \qquad C_4^{-1} D_t C_4 = 0 \,. \tag{4.164}$$

Representing again the flat gauge field $A_1 = h_1^{-1} dh_1$, and again defining $\tilde{g}_1 = h_1 g_1 h_1^{-1}$, $\tilde{C}_3 = h_1 C_3 h_1^{-1}$, $\tilde{C}_4 = h_1 C_4 h_1^{-1}$ one can write (4.164) as:

$$\tilde{g}_1^{-1} \partial_- \tilde{g}_1 + \tilde{C}_4^{-1} \tilde{g}_1 \partial_+ \tilde{g}_1^{-1} \tilde{C}_4 = 0 \,, \tag{4.165}$$

$$\tilde{C}_4^{-1} \partial_t \tilde{C}_4 = 0 \,. \tag{4.166}$$

The last equation implies that $\tilde{C}_4$ is constant on the boundary. Therefore using the chiral decomposition (4.142) $\tilde{g}_1 = g_{1L} g_{1R}^{-1}$ one can solve (4.165):

$$g_{1R}(y) = \tilde{C}_4^{-1} g_{1L}(-y) R_0^{-1} \tag{4.167}$$

with $R_0 \in G$. Now we get that :

$$\tilde{g}_1(t, 0) = g_{1L}(t) R_0 g_{1L}^{-1}(t) \tilde{C}_4 \,. \tag{4.168}$$

The boundary condition (4.161) implies:

$$\tilde{g}_1(0, t) = \tilde{C}_3 \tilde{C}_4 \,, \qquad \tilde{C}_3 \in C_G^{\mu_3} \,, \qquad \tilde{C}_4 \in C_H^{\mu_4} \,. \tag{4.169}$$

We find that

$$\tilde{C}_3 = g_{1L}(t) R_0 g_{1L}^{-1}(t) \,. \tag{4.170}$$

To be in agreement with the requirement that $\tilde{C}_3 \in C_G^{\mu_3}$ one should demand:

$$R_0 \in C_G^{\mu_3} \,. \tag{4.171}$$

The defect condition as before implies:

$$g_{1L} = \tilde{C}_2 \tilde{L}^{-1} g_{2L} n \,, \qquad g_{1R} = \tilde{L}^{-1} g_{2R} m \,, \tag{4.172}$$

where g_{2L}, g_{2R} are fields of the chiral decomposition (4.143): $\tilde{g}_2 = g_{2L} g_{2R}^{-1}$.

From the boundary condition (4.163) we conclude:

$$\tilde{g}_2(t, \pi) = \tilde{C}_5 \tilde{C}_6 \,, \qquad \tilde{C}_5 \in C_G^{\mu_5} \,, \qquad \tilde{C}_6 \in C_H^{\mu_6} \,, \tag{4.173}$$

where $\tilde{C}_5 = h_2 C_5 h_2^{-1}$, $\tilde{C}_6 = h_2 C_6 h_2^{-1}$.

To satisfy (4.173) we assume the following monodromy behaviour of g_{1L}:

$$g_{1L}(y + 2\pi) = \rho^{-1} g_{1L}(y) \gamma \,. \tag{4.174}$$

From relations (4.167), (4.172) and (4.174) we derive:

$$\tilde{g}_2(t, \pi) = \tilde{L} \tilde{C}_2^{-1} g_{1L}(\pi + t) n^{-1} m R_0 \gamma (\tilde{L} \tilde{C}_2^{-1} g_{1L}(\pi + t))^{-1} \tilde{L} \tilde{C}_2^{-1} \rho^{-1} \tilde{C}_4 \tilde{L}^{-1} \,. \tag{4.175}$$

We see that

$$\tilde{C}_5 = \tilde{L} \tilde{C}_2^{-1} g_{1L}(\pi + t) n^{-1} m R_0 \gamma (\tilde{L} \tilde{C}_2^{-1} g_{1L}(\pi + t))^{-1} \,, \tag{4.176}$$

and

$$\tilde{C}_6 = \tilde{L} \tilde{C}_2^{-1} \rho^{-1} \tilde{C}_4 \tilde{L}^{-1} \,. \tag{4.177}$$

To satisfy (4.173) we should demand:

$$d^{-1} R_0 \gamma = R_\pi \in C_G^{\mu_5} \,, \tag{4.178}$$

$$\tilde{C}_2^{-1} \rho^{-1} \tilde{C}_4 = S_\pi \in C_H^{\mu_6} \,. \tag{4.179}$$

The symplectic form is

$$\Omega^{G/H-def-bndry} = \frac{k}{4\pi} \left[\int_0^a \Pi(\tilde{g}_1) + \int_a^\pi \Pi(\tilde{g}_2) - \varpi(\tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_1) + \Omega(\tilde{C}_3, \tilde{C}_4) - \Omega(\tilde{C}_5, \tilde{C}_6) \right] \,. \tag{4.180}$$

In formula (4.180) $\tilde{p}$, $\tilde{C}_1$, $\tilde{C}_3$, $\tilde{C}_5$, $\tilde{C}_6$ are given by the equations (4.146), (4.147), (4.170), (4.176), (4.177) correspondingly. Representing again

$$g_{1L} = h_B^{-1} g_A \,, \tag{4.181}$$

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with h_B and g_A possessing the monodromy properties:

$$h_B(y + 2\pi) = h_B(y)\rho\,, \tag{4.182}$$

$$g_A(y + 2\pi) = g_A(y)\gamma\,, \tag{4.183}$$

and repeating the same steps as before we obtain:

$$\frac{4\pi}{k}\Omega^{G/H-def-bndry} = \frac{4\pi}{k}\Omega^{\text{chiral}}(g_A, \gamma) - \frac{4\pi}{k}\Omega^{\text{chiral}}(h_B, \rho) + \omega_{\mu_3}(R_0) + \tag{4.184}$$

$$\omega_{\mu_4}(\tilde{C}_4) - \omega_{\mu_5}(R_\pi) - \omega_{\mu_6}(S_\pi) - \omega_{\mu_2}(\tilde{C}_2) - \omega_{\mu_1}(d)$$

$$-\text{tr}(R_0^{-1}\delta R_0\delta\gamma\gamma^{-1}) + \text{tr}(\delta dd^{-1}\delta R_0 R_0^{-1}) + \text{tr}(\delta dd^{-1}R_0\delta\gamma\gamma^{-1}R_0^{-1})$$

$$-\text{tr}(\delta\tilde{C}_4\tilde{C}_4^{-1}\delta\rho\rho^{-1}) - \text{tr}(\delta\tilde{C}_2\tilde{C}_2^{-1}\rho^{-1}\delta\rho) + \text{tr}(\delta\tilde{C}_2\tilde{C}_2^{-1}\rho^{-1}\delta\tilde{C}_4\tilde{C}_4^{-1}\rho)\,.$$

Recalling again the decomposition (4.46) of Ω^{chiral} and (4.19), (4.30), we arrive at the conclusion that the phase space of the gauged WZW model on a strip with a defect line coincides with that of the double Chern-Simons theory on $D \times R$ with gauge fields of groups G and H coupled to three Wilson lines. This result can be straightforwardly generalized to the presence of the N defect lines.

4.4 Defects in Topological G/G coset

4.4.1 Bulk G/G coset

In this section we consider the bulk G/H model studied in section 4.3.2 for the special case $G = H$. It was shown in section 4.3.2 that the phase space of the bulk G/H model is symplectomorphic to that of the double Chern-Simons theory on $R \times \mathcal{A}$. In the special case, when $G = H$ it becomes a Chern-Simons theory on the torus times $R : R \times (\mathcal{A} \cup (-\mathcal{A})) = R \times T^2$. This result can be obtained also by a direct calculation.

In the case when $G = H$ the equations of motion (4.114) imply that $\tilde{g}$ is (t, x) independent and therefore the symplectic form $\Omega^{G/H}$ (4.116) reduces to

$$\Omega^{G/G} = \frac{k}{4\pi}\Psi(\rho, \tilde{g}^{-1})\,. \tag{4.185}$$

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The fact that $\tilde{g}$ is constant on a cylinder and the relation (4.113) also imply

$$\rho \tilde{g} \rho^{-1} \tilde{g}^{-1} = I \, . \tag{4.186}$$

Comparing (4.185) and (4.186) with formulae (4.21) and (4.23) we arrive at the conclusion that the phase space of a bulk G/G theory on a cylinder is symplectomorphic to that of a Chern-Simons theory on $T^2 \times R$. The quantization of the latter gives rise to the space of the 0-point conformal blocks of the WZW theory on the torus. The dimension of the space of conformal blocks on a Riemann surface of genus g with insertion of the primary fields with labels μ_n is:

$$N_{\mu_n}(g) = \sum_\alpha (S_0^\alpha)^{2-2g} \prod_n (S_{\mu_n}^\alpha / S_0^\alpha) \, . \tag{4.187}$$

This implies that the Hilbert space of the quantized G/G theory on a cylinder has dimension equals to the number of the integrable primaries. The equivalence of the topological G/G coset on a cylinder $R \times S^1$ with a Chern-Simons on $R \times T^2$ demonstrated here is actually a particular case of the more general equivalence of the topological G/G coset on a Riemann surface Σ and the Chern-Simons theory on $\Sigma \times S^1$ established in [27, 176, 187].

4.4.2 A defect in a closed topological model G/G

We have established in section 4.3.3 that the phase space of the coset G/H on a cylinder with a defect is symplectomorphic to that of a double Chern-Simons theory on $R \times \mathcal{A}$ with G and H gauge fields both coupled to a time like Wilson line. In the case when $G = H$ we again arrive at the conclusion that the topological coset G/G on a cylinder with a defect line is equivalent to the Chern-Simons theory on $R \times T^2$ with two time like Wilson lines. This again can be verified by a direct calculation. For the case $G = H$ the bulk equations of motion imply that $\tilde{g}_1$ and $\tilde{g}_2$ are (t, x) independent.

Therefore one has:

$$\tilde{g}_1(0) = \tilde{g}_1(a) = \tilde{C}_2 \tilde{C}_1 \tilde{p} \, , \qquad \tilde{L} \tilde{p} \tilde{L}^{-1} = \tilde{g}_2(a) = \tilde{g}_2(2\pi) \, . \tag{4.188}$$

From (4.150) we also obtain:

$$\tilde{g}_1(0) = \rho \tilde{g}_2(2\pi) \rho^{-1} \, . \tag{4.189}$$

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Inserting (4.189) in (4.188) we get:

$$\tilde{C}_2\tilde{C}_1\tilde{p} = \rho\tilde{L}\tilde{p}\tilde{L}^{-1}\rho^{-1}\,. \tag{4.190}$$

The symplectic form (4.157) now takes the form:

$$\Omega^{G/G-def} = -\frac{k}{4\pi}\varpi(\rho\tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_1)\,. \tag{4.191}$$

Comparing (4.190) and (4.191) with (4.21) and (4.23) we arrive at the conclusion that the topological coset G/G on a cylinder with a defect line is symplectomorphic with that of a Chern-Simons theory on $T^2 \times R$ with two Wilson lines. The quantization of the latter gives rise to the space of the 2-point conformal blocks of the WZW theory on a torus. Using equation (4.187) we can compute the dimension of the Hilbert space of the quantized topological coset G/G on cylinder with a defect line (μ_1, μ_2) :

$$\dim H_{d_{\mu_1,\mu_2}} = \sum_{\alpha\beta} N^{\beta}_{\alpha\mu_1} N^{\alpha}_{\beta\mu_2}\,. \tag{4.192}$$

4.4.3   Defects in the open topological model G/G

Previously we have seen that the phase space of G/H coset on a strip with a defect is symplectomorphic to that of the double Chern-Simons theory on $D \times R$ with gauge fields G and H both coupled to three Wilson lines. In the case when $G = H$ we arrive at the conclusion that the G/G topological coset on a strip with a defect line is equivalent to the Chern-Simons theory on sphere times R : $(D \cup (-D)) \times R = S^2 \times R$ with six time-like Wilson lines. This can be verified also directly. In this case $\tilde{g}_1$ and $\tilde{g}_2$ are (t, x) independent and therefore one has:

$$\tilde{g}_1(0) = \tilde{C}_3\tilde{C}_4 = \tilde{C}_2\tilde{C}_1\tilde{p} = \tilde{g}_1(a)\,, \tag{4.193}$$

$$\tilde{g}_2(a) = \tilde{L}\tilde{p}\tilde{L}^{-1} = \tilde{C}_5\tilde{C}_6 = \tilde{g}_2(2\pi)\,. \tag{4.194}$$

From equations (4.193) and (4.194) one obtains:

$$(\tilde{L}\tilde{C}_1^{-1}\tilde{L}^{-1})(\tilde{L}\tilde{C}_2^{-1}\tilde{L}^{-1})(\tilde{L}\tilde{C}_3\tilde{L}^{-1})(\tilde{L}\tilde{C}_4\tilde{L}^{-1})\tilde{C}_6^{-1}\tilde{C}_5^{-1} = I\,, \tag{4.195}$$

and from (4.180) one derives:

$$\Omega^{G/G-def-bndry} = -\frac{k}{4\pi}\varpi(\tilde{L}, \tilde{p}, \tilde{C}_2, \tilde{C}_1) + \frac{k}{4\pi}\Omega(\tilde{C}_3, \tilde{C}_4) - \frac{k}{4\pi}\Omega(\tilde{C}_5, \tilde{C}_6)\,. \tag{4.196}$$

Comparing (4.195) and (4.196) with 4.8 we arrive at the mentioned symplectomorphism of the phase space of G/G topological coset on a strip with a defect and a Chern-Simons theory on $S^2 \times R$ with six Wilson lines. The quantization of the latter gives rise to the space of the 6-point conformal blocks of the WZW theory on a sphere. Using equation (4.187) we can compute the dimension of the Hilbert space of the quantized topological coset G/G on a strip with a defect line:

$$N_{\mu_3\mu_4}^{\lambda_1} N_{\lambda_1\mu_1}^{\lambda_2} N_{\lambda_2\mu_2}^{\lambda_3} N_{\lambda_3\mu_5}^{\mu_6}\,. \tag{4.197}$$

Recall that here (μ_3, μ_4) are labels of the Cardy state on the first end of the strip, (μ_5, μ_6) are labels of the Cardy state on the second end of the strip, and (μ_1, μ_2) are the labels of the defect.

To interpret this result let us remind some general facts on a semisimple 2D topological theory on a world-sheet with boundary [130]. First of all let us recall that the whole content of the 2D topological field theory is encoded in a finite-dimensional commutative Frobenius algebra $\mathcal{C}$. In the case when $\mathcal{C}$ is semisimple it can be realized as the algebra of complex-valued functions on a finite set $X = \mathrm{Spec}\,\mathcal{C}$, which can be considered as a toy "space-time". Using sewing constraints of open topological theory it was proved in [130] that every boundary condition a is realized by a collections of vector spaces corresponding to each point of X: $x \to V_{x,a}$. This can be considered as a vector bundle over finite space-time, in agreement with the K-theory interpretation of boundary conditions. The Hilbert space of open string with boundary conditions specified by a and b is given by the bundle morphism:

$$H_{ab} = \oplus_x \mathrm{Hom}(V_{x,a}; V_{x,b})\,. \tag{4.198}$$

Consider now an open topological G/G coset. Note that in this case the points of X are labelled by integrable primaries. Let us remind first the situation without defect considered in [89]. The dimension of the Hilbert space for this case can be derived from (4.197) putting

there μ_1 and μ_2 equal to vacuum state:

$$N^{\lambda}_{\mu_3\mu_4} N^{\mu_6}_{\lambda\mu_5} . \tag{4.199}$$

This can be interpreted saying that the Hilbert space of the open string with the Cardy boundary conditions (μ_3, μ_4) and (μ_5, μ_6) at the ends is

$$H_{\mu_3,\mu_4;\mu_5,\mu_6} = \oplus_{\lambda} \mathrm{Hom}(W_{\mu_3\mu_4\lambda}; W_{\mu_5\mu_6\lambda}), \tag{4.200}$$

where $W_{\mu\nu\lambda}$ are spaces of three points conformal blocks. This implies that the Cardy state $(\mu.\nu)$ is given by the vector bundle

$$\lambda \to W_{\mu\nu\lambda} . \tag{4.201}$$

Now consider the case with a defect (μ_1, μ_2).

It is well known (see e.g. [69, 99, 111, 141, 174]), that open string propagating with boundary conditions a and b with inserted defect d can be considered, as propagating between one of the original boundary conditions, say a, and the second transformed by defect: $d * b$. According to formula (4.197) the transformed state corresponds to the spaces $V_{\lambda,\mu_1,\mu_2,\mu_5,\mu_6}$ with the dimensions

$$N^{\lambda_2}_{\lambda_1\mu_1} N^{\lambda_3}_{\lambda_2\mu_2} N^{\mu_6}_{\lambda_3\mu_5}, \tag{4.202}$$

and therefore can be considered as transformed by tensoring and summing with the space of 4-point conformal blocks $W_{\mu_1\mu_2\lambda_1\lambda_3}$:

$$V_{\lambda_1,\mu_1,\mu_2,\mu_5,\mu_6} = \oplus_{\lambda_3} W_{\mu_1\mu_2\lambda_1\lambda_3} \otimes W_{\mu_5\mu_6\lambda_3} . \tag{4.203}$$

This suggests the following general description of defects in semisimple 2D TFT's. It seems that to every defect separating 2D TFT's with "space-time"'s X and Y corresponds a collection of spaces $V^D_{x,y}$ where $x \in X$ and $y \in Y$. This can be considered as a fibre bundle over $X \times Y$. Then the boundary condition given by the fibre bundle V_y over Y is transformed to the boundary condition corresponding to the following bundle over X:

$$x \to \oplus_y V^D_{x,y} \otimes V_y . \tag{4.204}$$

It is interesting to note that the transformation (4.204) can be viewed as a discrete Fourier-Mukai transform in agreement with the general interpretation of the defect worldvolume or bi-brane as kernel of the Fourier-Mukai transform suggested in [35, 112, 163].

Let us elaborate now on fusion of defects. For this purpose consider an open string with insertion of two defects. The Hilbert space in this case is given by the space of 8-point conformal blocks. Along the same lines we conclude that the fusion of two defects (μ_1, μ_2) and (ν_1, ν_2) is given by the space of 6-point conformal blocks: $W_{\mu_1,\mu_2,\nu_1,\nu_2,\lambda_1,\lambda_2}$. According to the factorization properties of the space of conformal blocks this space can be expressed through the space of 4-point conformal blocks:

$$W_{\mu_1,\mu_2,\nu_1,\nu_2,\lambda_1,\lambda_2} = \oplus_\gamma W_{\mu_1,\mu_2,\lambda_1,\gamma} \otimes W_{\nu_1,\nu_2,\lambda_2,\gamma} \,. \tag{4.205}$$

This suggests that in general the fusion of two defects given by the bundles $V_{x,y}^{D_1}$ and $V_{y,z}^{D_2}$ over the spaces $X \times Y$ and $Y \times Z$ is given by the equation:

$$V_{x,z}^{D_1*D_2} = \oplus_y V_{x,y}^{D_1} \otimes V_{y,z}^{D_2} \,. \tag{4.206}$$

It is interesting to note that equation (4.206) appeared as a composition rule in the 2-category of matrices of vector spaces (see for example [82]). The relation with 2-categories actually can be traced further.

Note that equation (4.192) for the dimension of the G/G theory on a cylinder with a defect can be written as the dimension of the space $\sum_\lambda W_{\mu,\nu,\lambda,\lambda}$:

$$\dim H_{d_{\mu,\nu}} = \dim \sum_\lambda W_{\mu,\nu,\lambda,\lambda} \,. \tag{4.207}$$

We can conclude that probably in the general case the dimension of the bulk theory with defect given by the collection of the spaces $\{V_{x_1,x_2}, \ x_1, x_2 \in X\}$, is given by the dimension of the space $\oplus_x V_{x,x}^D$:

$$\dim H_d = \dim \ \oplus_x V_{x,x}^D \,. \tag{4.208}$$

The space $\oplus_x V_{x,x}^D$ appears in [82] as categorical trace.

4.5 Permutation branes in gauged WZW model

Worldvolume Q of the permutation branes on product of cosets $G/H \times G/H$ corresponding to a primary (μ, ν) has been constructed in [161] and reviewed in section 3.3) and have the form:

$$(g_1, g_2) = (cbp, Lp^{-1}L^{-1}), \tag{4.209}$$

where $p \in G$, $L \in H$, $c \in C_H^\nu$, $b \in C_G^\mu$ and C_G^μ are the conjugacy classes in G (2.32):

$$C_G^\mu = \{\beta f_\mu \beta^{-1} = \beta e^{2i\pi\mu/k}\beta^{-1}, \quad \beta \in G\}, \tag{4.210}$$

where $\mu \equiv \boldsymbol{\mu} \cdot \boldsymbol{H}$ is a highest weight representation integrable at level k, taking value in the Cartan subalgebra of the G Lie algebra. C_H^ν are the similarly defined conjugacy classes in H. If G and H possess common center, μ and ν should satisfy the selection rules explained in section 3.1.2.

To write the action one should introduce an auxiliary disc D satisfying the condition $\partial B = \Sigma + D$ and continue the fields g_1 and g_2 on this disc always holding the condition (4.209).

The action with the boundary condition (4.209) has the form

$$S_{\mathcal{P}}^{G/H \times G/H} = S^{G/H}(g_1, A_1) + S^{G/H}(g_2, A_2) - \frac{k}{4\pi} \int_D \varpi(L, p, c, b) \tag{4.211}$$

where

$$\varpi(L, p, c, b) = \Omega^{(2)}(c, b) - \operatorname{tr}((cb)^{-1}d(cb)dpp^{-1}) + \Psi(L, p), \tag{4.212}$$

where

$$\Omega^{(2)}(c, b) = \omega_\nu(c) - \operatorname{tr}(c^{-1}dcdbb^{-1}) + \omega_\mu(b) \tag{4.213}$$

is defined in (3.9), and $\omega_\mu(C)$ is defined in (2.6) and $\Psi(L, p)$ is defined in (2.9). The form $\varpi(L, p, c, b)$ satisfies the condition:

$$d\varpi(L, p, c, b) = \omega^{WZ}(g_1)|_Q + \omega^{WZ}(g_2)|_Q. \tag{4.214}$$

One can check that the action (4.211) is invariant under the gauge transformations:

$$g_1 \;\rightarrow\; h_1 g_1 h_1^{-1}, \qquad A_1 \rightarrow h_1 A_1 h_1^{-1} - dh_1 h_1^{-1}, \tag{4.215}$$

$$g_2 \;\rightarrow\; h_2 g_2 h_2^{-1}, \qquad A_2 \rightarrow h_2 A_2 h_2^{-1} - dh_2 h_2^{-1},$$

where $h_1 : \Sigma \to H$, $h_2 : \Sigma \to H$. For this purpose note that under (4.215) the boundary parameters transform in the following way:

$$p \to h_1 p h_1^{-1}, \quad c \to h_1 c h_1^{-1}, \quad b \to h_1 b h_1^{-1}, \quad L \to h_2 L h_1^{-1}. \tag{4.216}$$

The gauge invariance follows from the Polyakov-Wiegmann identities and the transformation properties of $\varpi(L, p, c, b)$:

$$\varpi(h_2 L h_1^{-1}, h_1 p h_1^{-1}, h_1 c h_1^{-1}, h_1 b h_1^{-1}) - \varpi(L, p, c, b) . = -\Psi(h_1, cbp) - \Psi(h_2, Lp^{-1}L^{-1}) \tag{4.217}$$

Consider $G/H \times G/H$ product of coset models on the strip $R \times [0, \pi]$ with boundary conditions on both sides given by the permutation branes:

$$(g_1, g_2)(0) = (C_2 C_1 p_1, L_1 p_1^{-1} L_1^{-1}) \tag{4.218}$$

$$(g_1, g_2)(\pi) = (C_4 C_3 p_2, L_2 p_2^{-1} L_2^{-1}) \tag{4.219}$$

Here $C_1 \in C_G^{\mu_1}$, $C_2 \in C_H^{\mu_2}$, $C_3 \in C_G^{\mu_3}$, $C_4 \in C_H^{\mu_4}$, $L_1, L_2 \in H$, $p_1, p_2 \in G$.

The boundary equation of motion resulting from the action (4.211) at $x = 0$ are [167]:

$$g_1^{-1} D_- g_1 + L_1^{-1} g_2 D_+ g_2^{-1} L_1 = 0 \tag{4.220}$$

$$C_2^{-1} g_1 D_+ g_1^{-1} C_2 + L_1^{-1} g_2^{-1} D_- g_2 L_1 = 0 \tag{4.221}$$

$$L_1^{-1} D_t L_1 = 0 \quad C_2^{-1} D_t C_2 = 0 \tag{4.222}$$

where $D_t = D_+ - D_-$, $D_\pm L = \partial_\pm L + A_{2\pm} L - L A_{1\pm}$, $D_\pm g_1 = \partial_\pm g_1 + [A_{1\pm}, g_1]$, $D_\pm g_2 = \partial_\pm g_2 + [A_{2\pm}, g_2]$, $D_\pm C_2 = \partial_\pm C_2 + [A_{1\pm}, C_2]$.

Parameterising again flat gauge fields as

$$A_1 = h_1^{-1} dh_1 \qquad A_2 = h_2^{-1} dh_2 \tag{4.223}$$

one can define as before

$$\tilde{g}_1 = h_1 g_1 h_1^{-1} \qquad \tilde{g}_2 = h_2 g_2 h_2^{-1} \tag{4.224}$$

$$\tilde{C}_1 = h_1 C_1 h_1^{-1} \qquad \tilde{C}_2 = h_1 C_2 h_1^{-1}$$

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$$\tilde{p}_1 = h_1 p_1 h_1^{-1} \qquad \tilde{L}_1 = h_2 L_1 h_1^{-1}$$

$$\tilde{C}_3 = h_1 C_3 h_1^{-1} \qquad \tilde{C}_4 = h_1 C_4 h_1^{-1}$$

$$\tilde{p}_2 = h_1 p_2 h_1^{-1} \qquad \tilde{L}_2 = h_2 L_2 h_1^{-1}$$

and we have the bulk equations (4.114) for $\tilde{g}_1$ and $\tilde{g}_2$ and boundary equations take the form:

$$\tilde{g}_1^{-1}\partial_-\tilde{g}_1 + \tilde{L}_1^{-1}\tilde{g}_2\partial_+\tilde{g}_2^{-1}\tilde{L}_1 = 0 \tag{4.225}$$

$$\tilde{C}_2^{-1}\tilde{g}_1\partial_+\tilde{g}_1^{-1}\tilde{C}_2 + \tilde{L}_1^{-1}\tilde{g}_2^{-1}\partial_-\tilde{g}_2\tilde{L}_1 = 0 \tag{4.226}$$

$$\tilde{L}_1^{-1}\partial_t\tilde{L}_1 = 0 \quad \tilde{C}_2^{-1}\partial_t\tilde{C}_2 = 0 \tag{4.227}$$

Equation (4.227) implies that $\tilde{L}_1$ and $\tilde{C}_2$ are constant along the boundary. Boundary conditions (4.218) and (4.219) imply

$$(\tilde{g}_1, \tilde{g}_2)(0) = (\tilde{C}_2\tilde{C}_1\tilde{p}_1, \tilde{L}_1\tilde{p}_1^{-1}\tilde{L}_1^{-1}) \tag{4.228}$$

$$(\tilde{g}_1, \tilde{g}_2)(\pi) = (\tilde{C}_4\tilde{C}_3\tilde{p}_2, \tilde{L}_2\tilde{p}_2^{-1}\tilde{L}_2^{-1}) \tag{4.229}$$

Using the chiral decomposition one can solve the boundary equation of motion

$$g_{1R}(y) = \tilde{L}_1^{-1}g_{2L}(-y)m \tag{4.230}$$

$$g_{2R}(y) = \tilde{L}_1\tilde{C}_2^{-1}g_{1L}(-y)n \tag{4.231}$$

Equations (4.230) and (4.231) indeed imply (4.228) with

$$\tilde{p}_1(t) = \tilde{C}_2^{-1}g_{1L}(t)ng_{2L}^{-1}(t)\tilde{L}_1 \tag{4.232}$$

$$\tilde{C}_1 = \tilde{C}_2^{-1}g_{1L}(t)m^{-1}n^{-1}g_{1L}^{-1}(t)\tilde{C}_2 \tag{4.233}$$

To have that $\tilde{C}_1 \in C_G^{\mu_1}$ we should require $m^{-1}n^{-1} \equiv R_0 \in C_G^{\mu_1}$.

To satisfy (4.229) we assume the following monodromy properties of g_{1L} and g_{2L}

$$g_{1L}(y + 2\pi) = \rho_1^{-1}g_{1L}(y)\gamma_1 \qquad g_{2L}(y + 2\pi) = \rho_2^{-1}g_{2L}(y)\gamma_2 \tag{4.234}$$

Now one can show that (4.229) is satisfied with

$$\tilde{p}_2(t) = \tilde{L}_2^{-1}\tilde{L}_1\tilde{C}_2^{-1}\rho_1 g_{1L}(\pi + t)\gamma_1^{-1}ng_{2L}^{-1}(\pi + t)\tilde{L}_2 \tag{4.235}$$

$$\tilde{C}_3 = \tilde{C}_4^{-1} g_{1L}(\pi + t) m^{-1} \gamma_2 n^{-1} \gamma_1 (\tilde{C}_4^{-1} g_{1L}(\pi + t))^{-1} \tag{4.236}$$

if we require

$$\rho_2^{-1} = \tilde{L}_2 \tilde{L}_1^{-1} \tag{4.237}$$

and

$$\rho_1^{-1} \tilde{C}_2 \tilde{L}_1^{-1} \tilde{L}_2 = \rho_1^{-1} \tilde{C}_2 \tilde{\rho}_2^{-1} = \tilde{C}_4 \tag{4.238}$$

where $\tilde{\rho}_2 = \tilde{L}_1^{-1} \rho_2 \tilde{L}_1$.

To have that $\tilde{C}_3 \in C_G^{\mu_3}$ we should require $m^{-1} \gamma_2 n^{-1} \gamma_1 = \tilde{\gamma}_2 R_0 \gamma_1 = R_\pi \in C_G^{\mu_3}$, where $\tilde{\gamma}_2 = m^{-1} \gamma_2 m$.

The monodromies (4.234) as before can be realized in the terms of the decomposition of the fields g_{1L} and g_{2L} as products:

$$g_{1L} = h_B^{-1} g_A \,, \qquad g_{2L} = h_D^{-1} g_C \tag{4.239}$$

of the new fields h_B, g_A, h_D, g_C possessing the monodromy properties:

$$h_B(2\pi) = h_B(0)\rho_1 \,, \qquad g_A(2\pi) = g_A(0)\gamma_1 \,, \tag{4.240}$$

$$h_D(2\pi) = h_D(0)\rho_2 \,, \qquad g_C(2\pi) = g_C(0)\gamma_2 \,, \tag{4.241}$$

and satisfying (4.120).

The symplectic form of product of the gauged WZW models on the strip with boundary conditions specified by the permutation branes can be written using the symplectic form density (4.115) and the form ϖ:

$$\Omega_{\mathcal{P}}^{G/H} = \frac{k}{4\pi} \left[\int_0^\pi \Pi^{G/H}(g_1, h_1) dx + \int_0^\pi \Pi^{G/H}(g_2, h_2) dx + \varpi(g_1(0), g_2(0)) - \varpi(g_1(\pi), g_2(\pi)) \right] . \tag{4.242}$$

Substituting in (4.242) the symplectic form density (4.115) and using the transformation property (4.217) we obtain:

$$\Omega_{\mathcal{P}}^{G/H} = \frac{k}{4\pi} \left[\int_0^\pi \Pi(\tilde{g}_1) dx + \int_0^\pi \Pi(\tilde{g}_2) dx + \varpi(\tilde{L}_1, \tilde{p}_1, \tilde{C}_2, \tilde{C}_1) - \varpi(\tilde{L}_2, \tilde{p}_2, \tilde{C}_4, \tilde{C}_3) \right] , \tag{4.243}$$

where $\tilde{p}_1$ and $\tilde{C}_1$ defined in (4.232) and (4.233) and $\tilde{p}_2$ and $\tilde{C}_3$ defined in (4.235) and (4.236).

Using (4.44) one can obtain for (4.243):

$$\Omega_{\mathcal{P}}^{G/H} = \Omega^{LG}(s_1, \tau_1) + \Omega^{LG}(s_2, \tau_2) - \Omega^{LG}(s_3, \tau_3) - \Omega^{LG}(s_4, \tau_4) + \Omega_1^{\text{bndry}} - \Omega_2^{\text{bndry}} \qquad (4.244)$$

$$\Omega_1^{\text{bndry}} = \frac{k}{4\pi} \left[\omega_{\tau_1}(\gamma_1) + \omega_{\tau_2}(\tilde{\gamma}_2) + \omega_{\mu_1}(R_0) - \omega_{\mu_3}(R_\pi) \right. \qquad (4.245)$$
$$\left. - \operatorname{tr}(R_0^{-1} \delta R_0 \delta \gamma_1 \gamma_1^{-1}) - \operatorname{tr}(\tilde{\gamma}_2^{-1} \delta \tilde{\gamma}_2 \delta R_0 R_0^{-1}) - \operatorname{tr}(R_0^{-1} \tilde{\gamma}_2^{-1} \delta \tilde{\gamma}_2 R_0 \delta \gamma_1 \gamma_1^{-1}) \right]$$

$$\Omega_2^{\text{bndry}} = \frac{k}{4\pi} \left[\omega_{\tau_3}(\rho_1) + \omega_{\tau_4}(\tilde{\rho}_2) - \omega_{\mu_2}(\tilde{C}_2) + \omega_{\mu_4}(\tilde{C}_4) \right. \qquad (4.246)$$
$$\left. + \operatorname{tr}(\delta \tilde{C}_2 \tilde{C}_2^{-1} \delta \rho_1 \rho_1^{-1}) + \operatorname{tr}(\tilde{\rho}_2^{-1} \delta \tilde{\rho}_2 \tilde{C}_2^{-1} \delta \tilde{C}_2) - \operatorname{tr}(\tilde{C}_2 \tilde{\rho}_2^{-1} \delta \tilde{\rho}_2 \tilde{C}_2^{-1} \delta \rho_1 \rho_1^{-1}) \right]$$

Comparing (4.244) with the formulae 4.8) and (4.30) we arrive at the conclusion that the phase space of product of coset models on a strip with boundary conditions specified by permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on an annulus times the time-line and with G and H gauge fields both coupled to two Wilson lines.

4.6 Permutation branes in topological G/G coset

In this section we discuss permutation branes on the product of topological coset $G/G \times G/G$.

In the previous section we have seen that the phase space of the product of the gauged WZW models on a strip with boundary conditions given by the permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on an annulus times the time-line with G and H gauge fields both coupled to two Wilson lines. In the case when $G = H$ we arrive at the conclusion that product of topological cosets $G/G \times G/G$ on a strip with boundary conditions given by the permutation branes is equivalent to the Chern-Simons theory on the torus $T^2 = \mathcal{A} \cup (-\mathcal{A})$ times the time-line R with four Wilson lines. This can be verified by a direct calculation. For the case $G = H$ the bulk equations of motion (4.114) imply that $\tilde{g}_1$ and $\tilde{g}_2$ are (t, x) independent.

Therefore one has:

$$\tilde{g}_1(0) = \tilde{C}_2 \tilde{C}_1 \tilde{p}_1 = \tilde{g}_1(\pi) = \tilde{C}_4 \tilde{C}_3 \tilde{p}_2 \tag{4.247}$$

$$\tilde{g}_2(0) = \tilde{L}_1 \tilde{p}_1^{-1} \tilde{L}_1^{-1} = \tilde{g}_2(\pi) = \tilde{L}_2 \tilde{p}_2^{-1} \tilde{L}_2^{-1} \tag{4.248}$$

From equations (4.247) and (4.248) we get

$$m \tilde{p}_2 m^{-1} \tilde{p}_2^{-1} \tilde{C}_3^{-1} \tilde{C}_4^{-1} \tilde{C}_2 \tilde{C}_1 = 1 \tag{4.249}$$

where

$$m = \tilde{L}_1^{-1} \tilde{L}_2 \tag{4.250}$$

The symplectic form (4.243) in this case reduces to

$$\Omega_{\mathcal{P}}^{G/G} = \frac{k}{4\pi} \left[\varpi(\tilde{L}_1, \tilde{p}_1, \tilde{C}_2, \tilde{C}_1) - \varpi(\tilde{L}_2, \tilde{p}_2, \tilde{C}_4, \tilde{C}_3) \right] , \tag{4.251}$$

Comparing formulae (4.249) and (4.251) with the formulae (4.21) and (4.23) we arrive at the mentioned symplectomorphism of the product of topological cosets $G/G \times G/G$ on a strip with the boundary conditions given by the permutation branes and that of Chern-Simons theory on the torus times the time-line with four Wilson lines.

This construction can be easily generalized to N-fold product of coset models G/H. The ansatz for permutation branes has the form:

$$(g_1, \ldots, g_N) = (C_2 C_1 p_{N-1} \cdots p_1, L_1 p_1^{-1} L_1^{-1}, \ldots, L_{N-1} p_{N-1}^{-1} L_{N-1}^{-1}) \tag{4.252}$$

where $C_1 \in C_G^{\mu_1}$, $C_2 \in C_H^{\mu_2}$, $p_i \in G$, $L_i \in H$. The ansatz is invariant under the N-fold adjoint action : $g_i \to h_i g_i h_i^{-1}$, where $h_i : \Sigma \to H$. Using the Polyakov-Wiegmann identity (1.279) it is straightforward to check the existence of the two-form ϖ_N satisfying the relation:

$$\sum_1^N \omega^{\mathrm{WZ}}(g_i)|_{\mathrm{brane}} = d\varpi_N \tag{4.253}$$

Performing the same steps as before we arrive at the conclusion, that the phase space of the N-fold product of the gauged WZW model G/H on a strip with boundary conditions given by

permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on a sphere with N holes times the time-line and with G and H gauge fields both coupled to two Wilson lines. For the special case of the toplogical coset G/G we get, that the phase space of the N-fold product of the topological cosets G/G on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of Chern-Simons theory on a Riemann surface of genus $N-1$ times the time-line with four Wilson lines.

Chapter 5

Duality Defects

5.1 Preliminaries

We review the construction of an action with defects [78, 163]. We locate the defect at the vertical line S defined by the condition $\sigma = 0$. Denote by Σ_1 the left half-plane ($\sigma \leq 0$), and by Σ_2 the right half-plane ($\sigma \geq 0$), and a pair of maps $X : \Sigma_1 \to M_1$ and $\tilde{X} : \Sigma_2 \to M_2$, where M_1 and M_2 are the target spaces for the two quantum field theories. Suppose we have a submanifold Q of the cartesian product of target spaces: $Q \subset M_1 \times M_2$, with a connection one-form A, and a combined map :

$$\Phi : S \to M_1 \times M_2 \tag{5.1}$$

$$s \mapsto (X(s), \tilde{X}(s))$$

which takes values in the submanifold Q. Q is called the world-volume of the defect.

In this setup we can write the action:

$$I = \int_{\Sigma_1} dx^+ dx^- L_1 + \int_{\Sigma_2} dx^+ dx^- L_2 + \int_S \Phi^* A \tag{5.2}$$

where

$$L_1 = E^{(1)}_{mn} \partial X^m \bar{\partial} X^n, \tag{5.3}$$

$$L_2 = E^{(2)}_{mn} \partial \tilde{X}^m \bar{\partial} \tilde{X}^n. \tag{5.4}$$

In this section we use the light-cone coordinates defined as

$$x^{\pm} = \tau \pm \sigma, \tag{5.5}$$

with $E_{mn}^{(i)}$ being the components of two second rank tensors. The tensors $E^{(i)}$ are split as

$$E^{(i)} = G^{(i)} + B^{(i)}. \tag{5.6}$$

where $G^{(i)}$ are the symmetric target space metrics of the two sigma models and $B^{(i)}$ are the corresponding NS antisymmetric two-forms.

As a warm-up exercise we work out the following simple example, when we have on both sides free scalars compactified on circles S_X^1 and $S_{\tilde{X}}^1$ of radii R_1 and R_2:

$$L_1 = R_1^2 \partial X \bar{\partial} X \tag{5.7}$$

and

$$L_2 = R_2^2 \partial \tilde{X} \bar{\partial} \tilde{X}. \tag{5.8}$$

The world-volume of the defect is a product of the target spaces $S_X^1 \times S_{\tilde{X}}^1$ with the connection $A = -X d\tilde{X}$. The curvature of this connection is $F = d\tilde{X} \wedge dX$. This forms a Poincaré bundle $\mathcal{P}$ [101]. The equations of motion on the defect line are:

$$R_1^2 (\partial X - \bar{\partial} X) - \partial_{\tau} \tilde{X} = 0 \tag{5.9}$$

$$R_2^2 (\partial \tilde{X} - \bar{\partial} \tilde{X}) - \partial_{\tau} X = 0 \tag{5.10}$$

For $R_2 = \frac{1}{R_1}$, (5.9) and (5.10) take the form:

$$R_1^2 (\partial X - \bar{\partial} X) - (\partial \tilde{X} + \bar{\partial} \tilde{X}) = 0 \tag{5.11}$$

$$(\partial \tilde{X} - \bar{\partial} \tilde{X}) - R_1^2 (\partial X + \bar{\partial} X) = 0 \tag{5.12}$$

Equations (5.11) and (5.12) imply

$$R_1^2 \partial X = \partial \tilde{X} \tag{5.13}$$

$$R_1^2 \bar{\partial} X = -\bar{\partial} \tilde{X} \tag{5.14}$$

which are the T-duality relations (1.227). Equations (5.13) and (5.14) also show that the defect given by the Poincaré bundle $\mathcal{P}$ for $R_2 = \frac{1}{R_1}$ is topological.

One generalization that comes to mind is a defect $\mathcal{P}^k$ with the same world-volume but with k units of the flux above: $F = k d\tilde{X} \wedge dX$. In the same way it is possible to show that this defect is topological when the radii satisfy the relation

$$R_1 R_2 = k \tag{5.15}$$

and instead of (5.13) and (5.14) one obtains:

$$R_1^2 \partial X = k \partial \tilde{X} \tag{5.16}$$

$$R_1^2 \bar{\partial} X = -k \bar{\partial} \tilde{X} \tag{5.17}$$

These relations imply that the defect $\mathcal{P}^k$ combines the actions of the Z_k orbifolding and T-duality.

All this is in agreement with [14, 77], where more general submanifolds Q are considered. There the worldvolume Q of the defect is either two dimensional with flux $F = k_1 d\tilde{X} \wedge dX$, but allowed to wrap the product $S_X^1 \times S_{\tilde{X}}^1$ torus k_2 times, or Q is made one dimensional winding around the cycles (k_1, k_2) times. Then the existence of the topological defect is proved for the radii satisfying the relations:

$$R_1 R_2 = \left| \frac{k_1}{k_2} \right| \qquad \text{or} \qquad \frac{R_2}{R_1} = \left| \frac{k_1}{k_2} \right| \tag{5.18}$$

where $k_1, k_2 \in \mathbb{Z}$.

5.2   Factorized T-duality in non-linear sigma model

Consider the action:

$$S = \int dx^+ dx^- (G_{ij} + B_{ij}) \partial X^i \bar{\partial} X^j = \int dx^+ dx^- E_{ij} \partial X^i \bar{\partial} X^j = \tag{5.19}$$

$$\int dx^+ dx^- (E_{11} \partial X^1 \bar{\partial} X^1 + E_{1N} \partial X^1 \bar{\partial} X^N + E_{M1} \partial X^M \bar{\partial} X^1 + E_{MN} \partial X^M \bar{\partial} X^N)$$

Suppose that G_{ij} and B_{ij} do not depend on X^1. We assume here that lower case indices run from 1 to the dimension of the target space, and upper case indices run from 2 to the dimension of the target space. Replace derivatives of X^1 by vector fields $(A, \bar{A})$ and add a lagrange multiplier field $\tilde{X}^1$ to force the vector field to be the derivatives of a scalar:

$$S = \int dx^+ dx^- (E_{11} A \bar{A} + E_{1N} A \bar{\partial} X^N + E_{M1} \partial X^M \bar{A} + E_{MN} \partial X^M \bar{\partial} X^N + \tilde{X}^1 (\partial \bar{A} - \bar{\partial} A)) \quad (5.20)$$

Intergrating out first $\tilde{X}^1$ we obtain $\partial \bar{A} - \bar{\partial} A = 0$ which can be solved setting

$$A = \partial X^1 \quad \text{and} \quad \bar{A} = \bar{\partial} X^1, \quad (5.21)$$

and we go back to the original action. If we first integrate out the vector field:

$$A = \frac{1}{E_{11}} (\partial \tilde{X}^1 - E_{M1} \partial X^M) \text{ and } \bar{A} = -\frac{1}{E_{11}} (\bar{\partial} \tilde{X}^1 + E_{1M} \bar{\partial} X^M) \quad (5.22)$$

we obtain T-dualized action

$$S = \int dx^+ dx^- (\tilde{E}_{11} \partial \tilde{X}^1 \bar{\partial} \tilde{X}^1 + \tilde{E}_{1N} \partial \tilde{X}^1 \bar{\partial} X^N + \tilde{E}_{M1} \partial X^M \bar{\partial} \tilde{X}^1 + \tilde{E}_{MN} \partial X^M \bar{\partial} X^N) \quad (5.23)$$

where

$$\tilde{E}_{11} = \frac{1}{E_{11}} \quad (5.24)$$
$$\tilde{E}_{1M} = \frac{E_{1M}}{E_{11}}$$
$$\tilde{E}_{M1} = -\frac{E_{M1}}{E_{11}}$$
$$\tilde{E}_{MN} = E_{MN} - \frac{E_{M1} E_{1N}}{E_{11}}$$

In components one has:

$$\tilde{G}_{11} = \frac{1}{G_{11}} \quad (5.25)$$
$$\tilde{G}_{1M} = \frac{B_{1M}}{G_{11}}$$
$$\tilde{B}_{1M} = \frac{G_{1M}}{G_{11}}$$
$$\tilde{G}_{MN} = G_{MN} - \frac{1}{G_{11}} (G_{M1} G_{1N} + B_{1N} B_{M1})$$
$$\tilde{B}_{MN} = B_{MN} - \frac{1}{G_{11}} (G_{M1} B_{1N} + G_{1N} B_{M1})$$

The capital latin indices run from 2 to $\dim M$.

The dual coordinate $\tilde{X}^1$ is related to the original X^1 by the relations:

$$\partial \tilde{X}^1 = E_{11}\partial X^1 + E_{M1}\partial X^M \quad \text{and} \quad \bar{\partial}\tilde{X}^1 = -(E_{11}\bar{\partial}X^1 + E_{1M}\bar{\partial}X^M) \tag{5.26}$$

The rest of the coordinates remains unchanged.

Consider the action (5.2) with a defect as in the situation above, where M and $\tilde{M}$ are related by the equations (5.24), Q is the correspondence space, given by the equations

$$X^N = \tilde{X}^N, \qquad N = 2\ldots\dim M \tag{5.27}$$

with the connection

$$A = -X^1 d\tilde{X}^1 \tag{5.28}$$

and the curvature

$$F = d\tilde{X}^1 \wedge dX^1. \tag{5.29}$$

In this case the action (5.2) yields

$$E_{j1}\partial X^j - E_{1j}\bar{\partial}X^j - \partial_\tau \tilde{X}^1 = 0 \tag{5.30}$$

$$E_{jN}\partial X^j - E_{Nj}\bar{\partial}X^j - \tilde{E}_{jN}\partial \tilde{X}^j + \tilde{E}_{Nj}\bar{\partial}\tilde{X}^j = 0, \qquad N = 2\ldots\dim M \tag{5.31}$$

$$\tilde{E}_{j1}\partial \tilde{X}^j - \tilde{E}_{1j}\bar{\partial}\tilde{X}^j - \partial_\tau X^1 = 0. \tag{5.32}$$

The index j runs from 1 to $\dim M$. Additionally the conditions (5.27) imply

$$\partial_\tau X^N = \partial_\tau \tilde{X}^N, \qquad N = 2\ldots\dim M \tag{5.33}$$

or in the coordinates (5.5):

$$\partial X^N + \bar{\partial}X^N = \partial \tilde{X}^N + \bar{\partial}\tilde{X}^N, \qquad N = 2\ldots\dim M \tag{5.34}$$

Solving the equations (5.30), (5.31), (5.32) and (5.34) one obtains:

$$\bar{\partial}\tilde{X}^N = \bar{\partial}X^N \qquad N = 2, \ldots \dim M \tag{5.35}$$

$$\partial\tilde{X}^N = \partial X^N \qquad N = 2, \ldots \dim M$$

$$\partial\tilde{X}^1 = E_{11}\partial X^1 + E_{M1}\partial X^M$$

$$\bar{\partial}\tilde{X}^1 = -(E_{11}\bar{\partial}X^1 + E_{1M}\bar{\partial}X^M)$$

We see that equations (5.35) coincide with the T-duality relations (5.26). Therefore the defect given by the Poincaré bundle on the correspondence space induces T-duality.

One can check that (5.25) and (5.35) imply

$$T = G_{ij}\partial X^i\partial X^j = \tilde{T} = \tilde{G}_{ij}\partial\tilde{X}^i\partial\tilde{X}^j \tag{5.36}$$

and

$$\bar{T} = G_{ij}\bar{\partial}X^i\bar{\partial}X^j = \tilde{\bar{T}} = \tilde{G}_{ij}\bar{\partial}\tilde{X}^i\bar{\partial}\tilde{X}^j \tag{5.37}$$

which means that the defect is topological.

In this general set-up one can also consider the defect with the same world-volume given by equations (5.27) but with the flux

$$F = kd\tilde{X}^1 \wedge dX^1. \tag{5.38}$$

Repeating the calculations above one can show that this defect is topological if E and $\tilde{E}$ are related by the equations

$$\tilde{E}_{11} = \frac{k^2}{E_{11}} \tag{5.39}$$

$$\tilde{E}_{1M} = \frac{kE_{1M}}{E_{11}}$$

$$\tilde{E}_{M1} = -\frac{kE_{M1}}{E_{11}}$$

$$\tilde{E}_{MN} = E_{MN} - \frac{E_{M1}E_{1N}}{E_{11}}$$

Again the effects of the Z_k orbifolding of the first coordinate and the T-duality are combined.

All this can be generalized to T-dualizing of several coordinates. Suppose we T-dualize the first n coordinates, indexed by Greek letters. The matrix E is broken to four pieces:

$$E = \begin{pmatrix} E_{\alpha\beta} & E_{\alpha N} \\ E_{M\beta} & E_{MN} \end{pmatrix} \tag{5.40}$$

The transformed background has the form

$$\tilde{E} = \begin{pmatrix} E_{\alpha\beta}^{-1} & E_{\alpha\beta}^{-1} E_{\beta N} \\ -E_{M\alpha} E_{\alpha\beta}^{-1} & E_{MN} - E_{M\alpha} E_{\alpha\beta}^{-1} E_{\beta N} \end{pmatrix} \tag{5.41}$$

Now we should consider the defect, with the world-volume

$$X^N = \tilde{X}^N, \qquad N = n+1, \ldots \dim M, \tag{5.42}$$

with the connection

$$A = -\sum_1^n X^\alpha d\tilde{X}^\alpha \tag{5.43}$$

and the curvature

$$F = \sum_1^n d\tilde{X}^\alpha \wedge dX^\alpha. \tag{5.44}$$

In the same way it can be shown that for M and $\tilde{M}$ related by equations (5.41) this defect is topological and implies the defect equations:

$$\bar{\partial}\tilde{X}^N = \bar{\partial}X^N \qquad N = n+1, \ldots \dim M \tag{5.45}$$

$$\partial\tilde{X}^N = \partial X^N \qquad N = n+1, \ldots \dim M$$

$$\partial\tilde{X}^\alpha = E_{\beta\alpha}\partial X^\beta + E_{M\alpha}\partial X^M$$

$$\bar{\partial}\tilde{X}^\alpha = -(E_{\alpha\beta}\bar{\partial}X^\beta + E_{\alpha M}\bar{\partial}X^M)$$

We have obtained again T-duality relations for several T-dualized coordinates.

5.3 Defects and Fourier-Mukai transform

As we mentioned, a topological defect can be fused with a boundary, producing new boundary condition from the old one. From the other side boundary conditions correspond to D-branes,

which can be characterized by their RR charges or by elements of the K-theory. Therefore an action of the defect on the Ramond-Ramond charges and K-theory elements can be defined. It is expected [15, 35, 57, 78, 163, 166] that the action should be "Fourier-Mukai" type with a kernel given by the exponent of the gauge invariant flux $\mathcal{F} = \hat{B} - B + F$ on defect, or by the defect bundle $\mathcal{P}$ correspondingly. Saying Fourier-Mukai type transform we mean the following construction.[*] . Suppose we can associate to a target space X a ring $D(X)$ (e.g. cohomology groups, K-theory groups, etc.), in a way that for a map $p : X_1 \to X_2$ exist pullback $p^* : D(X_2) \to D(X_1)$ and pushforward $p_* : D(X_1) \to D(X_2)$ maps. Assume one has an element $K \in D(X \times Y)$. Now we can define the Fourier-Mukai transform, $FM(F): D(X) \to D(Y)$ with the kernel K by the formula:

$$FM(F) = p_*^Y (K \cdot p^{X*}F) \tag{5.46}$$

where $F \in D(X)$, and $p^X : X \times Y \to X$, $p^Y : X \times Y \to Y$ are projections. One can see that usual Fourier transform has this form with the Riemann integral as pushforward map.

To derive the transformation of the RR fields under the T-duality several approaches were developed: via dimensional reduction [23, 127], vertex operators for RR fields [143], the gravitino supersymmetry transformation [104], pure spinor formalism [22]. Dimensional reduction approach brings to the expression:

$$\hat{\mathcal{G}}^{(n)}_{M...NA1} = \mathcal{G}^{(n-1)}_{M...NA} - (n-1)\frac{\mathcal{G}^{(n-1)}_{[M...N|1}G_{|A]1}}{G_{11}} \tag{5.47}$$

$$\hat{\mathcal{G}}^{(n)}_{M...NAB} = \mathcal{G}^{(n+1)}_{M...NAB1} + n\mathcal{G}^{(n-1)}_{[M...NA}B_{B]1} + n(n-1)\frac{\mathcal{G}^{(n-1)}_{[M...N|1}B_{|A|1}G_{|B]1}}{G_{11}} \tag{5.48}$$

Three other approaches bring to the expression:

$$\hat{P} = P\Omega^{-1} \tag{5.49}$$

where $P = \frac{e^{\Phi}}{2}\sum_k \frac{1}{k!}\mathcal{G}_{\mu_1...\mu_k}\Gamma^{\mu_1...\mu_k}$ and k runs the values $k = 1, 3 \dots 9$ in the case of IIB, and the values $k = 0, 2 \dots 10$ in the case of IIA. The curved indices Gamma matrices are defined

---

[*]The paragraph below is neither a rigorous nor a precise definition of the Fourier-Mukai transform, and only has a goal to outline basic ideas. For the rigorous definitions see [19, 108] and references therein.

as usual by contracting with the tetrads e_μ^A. The matrix Ω is spinor representation of the relative twist between the left and right movers. For example for T-duality in the direction of coordinate 1, it is spinor representation of the parity operator in the direction 1: $\Gamma^1\Gamma_{11}$.

Hori suggested in [106] that the RR fields of the theory on $T^n \times M$ and those of the T-dual theory on $\hat{T}^n \times M$ are related by a Fourier-Mukai transform [19, 108]:

$$\hat{\mathcal{G}} = \int_{T^n} \mathcal{G} \wedge e^{\mathcal{F}} = \int_{T^n} \mathcal{G} \wedge e^{\hat{B} - B + \sum_{i=1}^{n} d\hat{t}_i \wedge dt^i} \tag{5.50}$$

Here B is the Neveu-Schwarz B-field and $\mathcal{G} = \sum_p \mathcal{G}_p$ is the sum of gauge invariant RR field strength where the sum is over $p = 0, 2, 4, \ldots$ for Type IIA and $p = 1, 3, \ldots$ for Type IIB. The integrand in (5.50) is considered as a form on the space $M \times T^n \times \hat{T}^n$ and pushforward map is fiberwise integration $\int_{T^n}$ [31], mapping forms on $M \times T^n \times \hat{T}^n$ to forms on $M \times \hat{T}^n$. The integral operates on the forms of the highest degree n in dt_i and sets to zero forms of lower degree in dt_i:

$$f(x, \hat{t}_i, t^i) p^* \omega \wedge dt_{i_1} \wedge \ldots dt_{i_r} \mapsto 0, \quad r < n \tag{5.51}$$

$$f(x, \hat{t}_i, t^i) p^* \omega \wedge dt_1 \wedge \ldots dt_n \mapsto \omega \int_{T^n} f(x, \hat{t}_i, t^i) dt_1 \ldots dt_n$$

Here p is the projection $M \times T^n \times \hat{T}^n \to M \times \hat{T}^n$, ω is a form on $M \times \hat{T}^n$, $f(x, \hat{t}_i, t^i)$ is an arbitrary function and x denotes a point in M. The fiberwise integration is actually Berezin integration, which is not surprising when one remembers that the one-forms dt_i anticommute.

Since the gauge invariant flux $\mathcal{F}$ satisfies the condition

$$d\mathcal{F} = \hat{H} - H, \qquad dB = H, \qquad d\hat{B} = \hat{H} \tag{5.52}$$

and the exterior differentiation d commutes with the fiberwise integration, one can show that the dual forms satisfy the equation [32]:

$$(d - \hat{H}) \wedge \hat{\mathcal{G}} = \int_{T^n} e^{\mathcal{F}} \wedge (d - H) \wedge \mathcal{G} \tag{5.53}$$

This implies that $d_H = d - H$ closed forms mapped to $d_{\hat{H}} = d - \hat{H}$ closed form. This means that if the RR fields $\mathcal{G}$ satisfy the supergravity Bianchi identity, so do the dual RR fields $\hat{\mathcal{G}}$.

Note that the kernel of the Fourier-Mukai transform (5.50) is indeed the exponential of the gauge invariant combination of the B fields and the flux of the T-duality defect (5.44):

$$e^{\mathcal{F}} = e^{\hat{B}-B+\sum_{i=1}^{n} d\hat{t}_i \wedge dt^i} \tag{5.54}$$

Let us check that the formula (5.50) produces the known transformation rules of the Ramond-Ramond fields for the case of the abelian T-dualization in the direction of one coordinate, which we choose to be the first one.

Using transformation rules (5.25), (5.50) takes the form:

$$\hat{\mathcal{G}} = \int_{S^1} \mathcal{G} \wedge e^{(A_1+d\hat{t}^1)\wedge(A_2+dt^1)} = \int_{S^1} \mathcal{G} \wedge \left(1 + (A_1 + d\hat{t}^1) \wedge (A_2 + dt^1)\right) \tag{5.55}$$

where

$$A_1 = B_{1N}dX^N \quad \text{and} \quad A_2 = \frac{G_{1N}}{G_{11}}dX^N \tag{5.56}$$

Taking $\mathcal{G}$ in the form

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(1)} \wedge dt^1 \tag{5.57}$$

and using the rules (5.51), one obtains

$$\widehat{\mathcal{G}} = \widehat{\mathcal{G}}^{(0)} + \widehat{\mathcal{G}}^{(1)} \wedge d\hat{t}^1 \tag{5.58}$$

where

$$\widehat{\mathcal{G}}^{(0)} = \mathcal{G}^{(1)} + \mathcal{G}^{(0)} \wedge A_1 + \mathcal{G}^{(1)} \wedge A_1 \wedge A_2 \tag{5.59}$$

and

$$\widehat{\mathcal{G}}^{(1)} = \mathcal{G}^{(0)} - \mathcal{G}^{(1)} \wedge A_2 \tag{5.60}$$

5.4   Defects and T-duality on lens space

5.4.1   Fourier-Mukai kernel of $SU(2)$ WZW model and lens space T-duality

The construction of the previous section can be applied fibrewise to torus fibration and can be expected to relate pairs of torus fibrations $\pi : E \to M$ and $\tilde{\pi} : \tilde{E} \to M$. It should be noted that

in this case the fibre product $E \times_M \tilde{E}$ appearing e.g. in [32] can be identified with the subset of elements of the product space $E \times \tilde{E}$ of pairs $(e, \tilde{e})$ with $\pi(e) = \tilde{\pi}(\tilde{e})$. This submanifold is the world volume Q of the relevant bibrane.

We exemplify [163] the situation in the case of the T-duality relating conformal sigma-models with a lens space as a target space and the WZW theory based on the compact connected Lie group SU(2) at level k. As we explained in section 2.2.3, the relevant lens space is a quotient of the group manifold SU(2) by the right action of a cyclic subgroup $\mathbb{Z}_k$ of a maximal torus of SU(2). Recall that in the Euler coordinates for SU(2) (2.66), this corresponds to the identification $\varphi \sim \varphi + \frac{4\pi}{k}$.

We use another parametrisation [94] of the group manifold SU(2) in terms of Pauli matrices,

$$g = \mathrm{e}^{\mathrm{i}\phi\frac{\sigma_3}{2}} \mathrm{e}^{\mathrm{i}\theta\frac{\sigma_2}{2}} \mathrm{e}^{\mathrm{i}(\xi-\phi)\frac{\sigma_3}{2}} \ , \tag{5.61}$$

in which the metric takes form

$$\mathrm{d}s^2 = (\mathrm{d}\xi - (1 - \cos\theta)\,\mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \tag{5.62}$$

with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\xi \in [0, 4\pi]$.

Identifying ξ as the fibre coordinate exhibits the structure of the group manifold SU(2) as an S^1 bundle over S^2 of monopole charge 1, the Hopf bundle. Considering the bundle with charge m amounts to the substitution $\xi \to \frac{\xi}{m}$. Due to the orbifold description of the lens space SU(2)/$\mathbb{Z}_k$, the latter can be considered as an S^1-bundle over S^2 with Chern class k and thus admits a parametrization as in (5.61), but with $\xi \sim \xi + \frac{4\pi}{k}$. It is convenient to reparameterize the lens space bundle coordinate $\tilde{\tilde{\xi}}$ as $\tilde{\tilde{\xi}} = \xi'/k$.

To the S^1-bundle description of the lens space and the group manifold, we can apply the standard geometric T-duality construction [32] for torus fibrations. It involves a correspondence space $E \times_M \tilde{E}$, where in our case $E := \mathrm{SU}(2)$, $\tilde{E}$ is the lens space and the base manifold is $M := S^2$. It leads to the following relations for the first Chern classes of the S^1-bundles on M and the three-forms H and $\tilde{H}$ on E and $\tilde{E}$, respectively:

$$F = c_1(E) = \tilde{\pi}_* \tilde{H} \quad \text{and} \quad \tilde{F} = c_1(\tilde{E}) = \pi_* H \ , \tag{5.63}$$

where π_* is integration on the S^1-fibre. It is observed in [32] that the pullbacks $\pi^* F$ and $\tilde\pi^* \tilde F$ are exact on E and $\tilde E$ respectively, and therefore can be written as

$$\pi^* F = \mathrm{d}A \quad \text{and} \quad \tilde\pi^* \tilde F = \mathrm{d}\tilde A, \tag{5.64}$$

where $A \in \Omega^1(E)$ and $\tilde A \in \Omega^1(\tilde E)$ are global one-forms on E and $\tilde E$, respectively, which are assumed to be normalized such that

$$\pi_* A = 1 = \tilde\pi_* \tilde A \tag{5.65}$$

It is shown in Section 3 of [32] that there exists a three-form Ω on the base manifold M that obeys the two relations

$$H = A \wedge \pi^* \tilde F - \pi^* \Omega \quad \text{and} \quad \tilde H = \tilde\pi^* F \wedge \tilde A - \tilde\pi^* \Omega \ . \tag{5.66}$$

One then introduces a two-form ω on the correspondence space $E \times_M \tilde E$ by

$$\omega := \tilde p^* \tilde A \wedge p^* A \tag{5.67}$$

where p and $\tilde p$ are the projections $E \times_M \tilde E \to E$ and $E \times_M \tilde E \to \tilde E$ respectively. They obey the relation $\pi p = \tilde\pi \tilde p$.

It follows from (5.66) and (5.64) and commutativity $p^* \pi^* = \tilde p^* \tilde\pi^*$, that

$$\mathrm{d}\omega = -\tilde p^* \tilde H + p^* H \tag{5.68}$$

This two-form also enters [32] in the following isomorphism of twisted cohomologies

$$T_* : p_* \circ e^\omega \circ \tilde p^* : H^\bullet(\tilde E, \tilde H) \to H^{\bullet+1}(E, H) \ . \tag{5.69}$$

Let us comment on the important role of the equation (5.68) in the isomorphism (5.69). It is shown in [32] that thanks to this equation H-twisted cohomologies mapped to $\tilde H$-twisted cohomologies. On the other hand, this equation coincides with equation (3.24), which was derived in [78] from the requirement of a well-defined worldsheet action. This coincidence can be seen as additional evidence for the relation between defects and kernels of Fourier-Mukai transforms we propose in this thesis.

In the case when $E = \mathrm{SU}(2)$ and $\tilde{E}$ is a lens space, this yields

$$
\begin{aligned}
H &= \tfrac{k}{16\pi^2} \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}\xi \\
F &= \tfrac{1}{4\pi} \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \\
A &= \tfrac{1}{4\pi} \left(\mathrm{d}\xi - (1 - \cos\theta) \, \mathrm{d}\phi \right)
\end{aligned}
\tag{5.70}
$$

and

$$
\begin{aligned}
\tilde{H} &= \tfrac{1}{16\pi^2} \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}\xi' \\
\tilde{A} &= \tfrac{1}{4\pi} \left(\mathrm{d}\xi' - k(1 - \cos\theta) \, \mathrm{d}\phi \right) \\
\tilde{F} &= \tfrac{k}{4\pi} \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta
\end{aligned}
\tag{5.71}
$$

and thus surpressing the projectors p and $\tilde{p}$ for brevity in calculations in explicit coordinates

$$
\tilde{A} \wedge A = \frac{1}{16\pi^2} \left(\mathrm{d}a \wedge \mathrm{d}\xi + (1 - \cos\theta) \mathrm{d}\phi \wedge \mathrm{d}a \right)
\tag{5.72}
$$

where a is defined by the equation

$$
\tilde{\xi} = \frac{\xi'}{k} = \xi + \frac{a}{k} \, .
\tag{5.73}
$$

5.4.2   Defect operators on bulk fields

In this section, we describe defects by their action on bulk fields. In the case of rational conformal field theories, it is known (see Proposition 2.8 of [69]) that this action characterizes a defect uniquely.

The bulk partition function for the rational conformal field theory associated to a lens space is

$$
Z(q) = \sum_{j=0}^{k/2} \sum_{n \in \mathbb{Z}} \chi_j^{SU(2)}(q) \chi_{jn}^{PF}(\bar{q}) \psi_{-n}^{U(1)}(\bar{q}) \, .
\tag{5.74}
$$

To derive conformal defects between $\mathrm{SU}(2)_k$ and the lens space $\mathrm{SU}(2)/\mathbb{Z}_k$ we need the following endomorphisms of a direct sum of Fock spaces for left movers and right movers, respectively:

$$
\begin{aligned}
P_{r\pm}^{U(1)} &= \exp\left[\pm \sum_{n=1}^{\infty} \frac{\alpha_{-n}^0 \alpha_n^1}{n} \right] \sum_{l \in \mathbb{Z}} \left| \frac{r + 2kl}{\sqrt{2k}} \right\rangle_0 \otimes {}_1\!\left\langle \pm \frac{r + 2kl}{\sqrt{2k}} \right| \\
\bar{P}_{r'\pm}^{U(1)} &= \exp\left[\pm \sum_{n=1}^{\infty} \frac{\tilde{\alpha}_{-n}^0 \tilde{\alpha}_n^1}{n} \right] \sum_{l' \in \mathbb{Z}} \overline{\left| \pm \frac{r' + 2kl'}{\sqrt{2k}} \right\rangle_0} \otimes {}_1\!\overline{\left\langle \frac{r' + 2kl'}{\sqrt{2k}} \right|} \, ,
\end{aligned}
\tag{5.75}
\tag{5.76}
$$

where the subscripts 0 and 1 distinguish free boson theories on the two sides of the defect. The bra- and ket-states are highest weight states in Fock spaces. They obey the following conservation equations for the U(1)-currents

$$J_0^3 \pm J_1^3 = 0 \,, \quad \bar{J}_0^3 \pm \bar{J}_1^3 = 0 \,, \tag{5.77}$$

where e.g. the first equation is a short hand for the intertwining property

$$P_\pm^{U(1)} J_1^3 = \mp J_0^3 P_\pm^{U(1)} \ .$$

Similarly, we consider for the parafermion theories $\mathcal{A}_0^{PF(k)} \times \mathcal{A}_1^{PF(k)}$ the following two operators

$$P_{[j,n]}^{PF} = \sum_N |j,n,N\rangle_0 \otimes {}_1\langle j,n,N| \,, \tag{5.78}$$

$$\bar{P}_{[j,n]}^{PF} = \sum_M \overline{|j,n,M\rangle}_0 \otimes {}_1\overline{\langle j,n,M|} \,, \tag{5.79}$$

where the sums over M and N are over orthonormal bases of the parafermion state spaces. Here $j \in \{0, \frac{1}{2}, 1, \dots \frac{k}{2}\}$ and $n \in \mathbb{Z}/2k\mathbb{Z}$ satisfy the constraint $2j + n = 0 \bmod 2$. The pairs (j,n) and $(k/2 - j, k + n)$ have to be identified.

Our starting point are symmetry preserving defects in the SU(2)-theory. The corresponding operators on bulk fields can be expressed in terms of the modular matrix S of SU(2) and the identity operators on irreducible highest weight modules of the corresponding untwisted affine Lie algebra,

$$P_j^{SU(2)} = \sum_N |j,N\rangle_0 \otimes {}_1\langle j,N| \,, \tag{5.80}$$

$$\bar{P}_j^{SU(2)} = \sum_M \overline{|j,M\rangle}_0 \otimes {}_1\overline{\langle j,M|} \,, \tag{5.81}$$

where the sums over M and N are over orthonormal bases of the SU(2) state spaces. These endomorphisms preserve, of course, all SU(2) symmetries,

$$J_0^a \quad + \quad J_1^a = 0 \,, \tag{5.82}$$

$$\bar{J}_0^a \quad + \quad \bar{J}_1^a = 0 \,, \quad (a = 1, 2, 3) \,. \tag{5.83}$$

The action of a symmetry preserving defect on bulk fields is given in terms of these endomorphisms by [138]:

$$X_a = \sum_j \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \bar{P}_j^{SU(2)} \,. \tag{5.84}$$

Since in the situation at hand no field identification fixed points occur, we can apply the procedure described in [123, 160] to derive a new family of defects separating $\mathrm{SU}(2)_k$ and the lens space $\mathrm{SU}(2)/\mathbb{Z}_k$. Performing a T-duality in (5.84) yields

$$Y_a^{AB} = \sum_j \sum_n \frac{S_{aj}}{S_{0j}} \; P_j^{SU(2)} \bar{P}_{j,n}^{PF} \bar{P}_{n-}^{U(1)} \,. \tag{5.85}$$

The defects (5.85) preserve all left moving currents, but only the right moving current corresponding to the maximal torus,

$$J_0^a \;\; + \;\; J_1^a = 0 \,, \quad (a = 1, 2, 3) \tag{5.86}$$

$$\bar{J}_0^3 \;\; - \;\; \bar{J}_1^3 = 0 \,. \tag{5.87}$$

As a consequence of these equations, the defects (5.85) transform A-type branes on $\mathrm{SU}(2)_k$ to B-type brane on $\mathrm{SU}(2)/\mathbb{Z}_k$.

A third family of defects is obtained by summing over the images of (5.84) under the action of $\mathbb{Z}_k$, with a prefactor determined by the Cardy condition:

$$Y_a^{AA} = \sqrt{k} \sum_j \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \left(\bar{P}_{j,0}^{PF} \bar{P}_{0+}^{U(1)} + \bar{P}_{j,k}^{PF} \bar{P}_{k+}^{U(1)} \right) \,. \tag{5.88}$$

The defects (5.88) satisfy the conservation equations

$$J_0^a \;\; + \;\; J_1^a = 0 \,, \quad (a = 1, 2, 3) \tag{5.89}$$

$$\bar{J}_0^3 \;\; + \;\; \bar{J}_1^3 = 0 \,, \,. \tag{5.90}$$

and transform A-type branes on $\mathrm{SU}(2)_k$ to A-type branes on the lens space $\mathrm{SU}(2)/\mathbb{Z}_k$.

Performing a T-duality on the defects (5.88), one derives another family of defects on $\mathrm{SU}(2)_k$ that map an A-type brane on $\mathrm{SU}(2)_k$ to a B-type brane on $\mathrm{SU}(2)_k$:

$$X_a^{AB} = \sqrt{k} \sum_j \frac{S_{aj}}{S_{0j}} P_j^{SU(2)} \left(\bar{P}_{j,0}^{PF} \bar{P}_{0-}^{U(1)} + \bar{P}_{j,k}^{PF} \bar{P}_{k-}^{U(1)} \right) \,. \tag{5.91}$$

The defects (5.91) satisfy the conservation equations

$$J_0^a \;\; + \;\; J_1^a = 0 \,, \quad (a = 1, 2, 3) \tag{5.92}$$

$$\bar{J}_0^3 \;\; - \;\; \bar{J}_1^3 = 0 \,, \,. \tag{5.93}$$

Summing over images and performing T-duality in the left moving sector of (5.88) yields a fifth family of defects that map B-type branes on $SU(2)_k$ to A-type branes on $SU(2)/\mathbb{Z}_k$:

$$Y_a^{BA} = k \sum_j \frac{S_{aj}}{S_{0j}} \left(P_{j,0}^{PF} P_{0-}^{U(1)} + P_{j,k}^{PF} P_{k-}^{U(1)} \right) \left(\bar{P}_{j,0}^{PF} \bar{P}_{0+}^{U(1)} + \bar{P}_{j,k}^{PF} \bar{P}_{k+}^{U(1)} \right) . \tag{5.94}$$

The defects (5.94) satisfy the conservation equations:

$$J_0^3 \quad - \quad J_1^3 = 0 \tag{5.95}$$

$$\bar{J}_0^3 \quad + \quad \bar{J}_1^3 = 0 \, , . \tag{5.96}$$

5.4.3 Geometry of defects

We finally determine the geometry of the family of defects (5.85) relating $SU(2)$ and the lens space $SU(2)/\mathbb{Z}_k$. To this end, we parametrize bulk fields in terms of Euler angles $\vec{\theta}$ using the representation function $\mathcal{D}_{mm'}^j$ of the spin j representation (2.34):

$$|\vec{\theta}\rangle := \sum_{j,m,m'} \sqrt{2j+1} \mathcal{D}_{mm'}^j(\vec{\theta}) |j, m, m'\rangle . \tag{5.97}$$

We are thus interested in the overlap $\langle \vec{\theta}_0 | Y_a^{AB} | \vec{\theta}_1 \rangle$ as a function of two sets of Euler angles. As in the calculation in [29], the definition of the lens spaces as right quotients implies that only terms of the defect operator (5.85) with $n = 0, k$ contribute to the overlap; in the large k limit also the term with $n = k$ can be ignored. Therefore, we arrive in the limit of large level k at the function

$$\langle \vec{\theta}_0 | Y_a^{AB} | \vec{\theta}_1 \rangle \sim \sum_{j \in \mathbb{Z}} \frac{k}{\pi} \sin[(2j+1)\hat{\psi}] \, \mathcal{D}_{00}^j(g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1)) , \tag{5.98}$$

where the angle $\hat{\psi}$ is given in terms of a by $\hat{\psi} := \frac{(2a+1)\pi}{k+2}$. Using (2.84) and repeating the same steps as in section 2.2.4 we obtain

$$\langle \vec{\theta}_0 | Y_a^{AB} | \vec{\theta}_1 \rangle \sim \frac{\Theta(\cos \delta - \cos 2\hat{\psi})}{\sqrt{\cos \delta - \cos 2\hat{\psi}}} \quad . \tag{5.99}$$

Here Θ is the Heavyside step function and δ is the second Euler angle of the product element $g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1)$. As we explained in section 2.2.2 equation (5.99) implies that the "difference"

$g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1)$ takes its values in a subset consisting of products of an element in a fixed conjugacy class C with an element $L \in \mathrm{U}(1)$:

$$g_0^{-1}(\vec{\theta}_0)g_1(\vec{\theta}_1) \in CL \,. \tag{5.100}$$

We next determine the two-form ω satisfying equation $(H_1 - H_2)|_{\mathrm{bibrane}} = \mathrm{d}\omega$ that is part of the bibrane-data. Its value in the element $xfx^{-1}L$ with f a fixed element of the conjugacy class C and $x \in G$ and $L \in \mathrm{U}(1)$ arbitrary, can be derived from the Polyakov-Wiegmann identity (1.279). We compute for $g_0^{-1}g_1 \in CL$ the difference

$$
\begin{aligned}
\omega^{WZ}(g_0) - \omega^{WZ}(g_1) &= \omega^{WZ}(g_0) - \omega^{WZ}(g_0 CL) \\[2mm]
&= \omega^{WZ}(g_0) - [\omega^{WZ}(g_0) + \omega^{WZ}(CL) - \mathrm{d}\mathrm{Tr}\left(g_0^{-1}\mathrm{d}g_0 \mathrm{d}(CL)\,(CL)^{-1}\right)] \\[2mm]
&= -\omega^{WZ}(C) + \mathrm{d}\mathrm{Tr}\left(C^{-1}\mathrm{d}C\, \mathrm{d}LL^{-1}\right) + \mathrm{d}\mathrm{Tr}\left(g_0^{-1}\mathrm{d}g_0\, \mathrm{d}(CL)(CL)^{-1}\right) \,.
\end{aligned}
$$

As a consequence, the two-form

$$\omega := \frac{k}{8\pi^2}\mathrm{Tr}(C^{-1}\mathrm{d}C\, \mathrm{d}LL^{-1} + g_0^{-1}\,\mathrm{d}g_0\, \mathrm{d}(CL)(CL)^{-1}) - \omega_f(x) \,, \tag{5.101}$$

where the two form $\omega_f(x)$ is defined in (2.6):

$$\omega_f(x) = \frac{k}{8\pi^2}\mathrm{Tr}(x^{-1}\mathrm{d}xfx^{-1}\,\mathrm{d}xf^{-1}) \tag{5.102}$$

and obeys (2.4), has the desired property $\frac{k}{8\pi^2}\omega^{WZ}(g_0) - \frac{k}{8\pi^2}\omega^{WZ}(g_1) = \mathrm{d}\omega$. The coefficient fixed by the requirement $\int_{SU(2)} \frac{k}{8\pi^2}\omega^{WZ}(g) = k$ to make contact with the geometrical consideration.

Asymptotically, for large k, the situation simplifies in the case when $f \approx e$, and the bibrane worldvolume, i.e. the correspondence space, consists of all pairs of the form (g_0, g_0L), with $g_0 \in \mathrm{SU}(2)$ and $L \in \mathrm{U}(1)$. The corresponding two-form takes the form

$$\omega = \frac{k}{8\pi^2}\mathrm{Tr}(g_0^{-1}\,\mathrm{d}g_0\, \mathrm{d}LL^{-1}) \tag{5.103}$$

In this case, the defect acts as an isomorphism on bulk fields, and we thus expect a relation to T-duality. Indeed, we find in the parametrization (5.61)

$$(g^{-1}\mathrm{d}g)_{11} = -(g^{-1}\mathrm{d}g)_{22} = \mathrm{i}\frac{\mathrm{d}\xi}{2} - \mathrm{i}\mathrm{d}\phi\frac{1-\cos\theta}{2} \,. \tag{5.104}$$

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Writing $L = \mathrm{e}^{\mathrm{i}\frac{a\sigma_3}{2k}}$, we see that the two-form (5.103) coincides with the two-form (5.72) from the geometric approach. This nicely demonstrates how geometric structure familiar from Fourier-Mukai transformations is encoded in the algebraic data describing defects.

5.5 Defects between vectorially and axially gauged WZW models

In this section we construct topological defects mapping the axially gauged $\frac{G}{U(1)}_{\text{axial}}$ WZW model to the vectorially gauged $\frac{G}{U(1)}_{\text{vectorial}}$ WZW model for a general group G. For the case $G = SU(2)$ we analyze the corresponding operators acting in the Hilbert space of parafermions and find that for the level k parafermions there are $k+1$ such topological defects, labeled by the integrable spin $j = 0, \ldots, \frac{k}{2}$. This is another example of the case of a non trivial null space for the defect. The object is to realize these defects in the Lagrangian approach as a line separating axially and vectorially gauged WZW models. This problem is solved in this section. First we present the geometrical ansatz for the defects (formula (5.112) below) and check that it leads to the action that glues axially and vectorially gauged models. Then we study in detail the defect given by $j = 0$ and show that it coincides with the defect with the flux (5.38), studied in the previous section, and implements $\mathbb{Z}_k$ orbifolding together with the T-duality. In the rest of the section we construct defects as operators in the Hilbert space of the parafermions.

5.5.1 Geometry and flux of the defects gluing axially-vectorially gauged models

The action of the gauged WZW model is studied in section 1.5.5

$$S^{G/H}(g, A) = S^{\mathrm{WZW}} + S^{\mathrm{gauge}}, \tag{5.105}$$

where

$$S^{\mathrm{WZW}}(g) = \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}(\partial_+ g \partial_- g^{-1})dx^+ dx^- + \frac{k}{4\pi}\int_B \frac{1}{3}\mathrm{tr}(g^{-1}dg)^3 \tag{5.106}$$

$$\equiv \frac{k}{4\pi} \left[\int_\Sigma dx^+ dx^- L^{\text{kin}} + \int_B \omega^{\text{WZ}} \right],$$

$$S^{\text{gauge}} = \frac{k}{2\pi} \int_\Sigma L_v^{\text{gauge}} dx^+ dx^- , \qquad (5.107)$$

$$L_v^{\text{gauge}}(g, A) = \text{tr}[-g^{-1}\partial_+ g A_- + \partial_- g g^{-1} A_+ + A_- g^{-1} A_+ g - A_+ A_-] . \qquad (5.108)$$

Here H is subgroup of G, $g \in G$ and B is a 3-manifold such that $\partial B = \Sigma$ and A is a gauge field taking values in the H Lie algebra.

Using the Polyakov-Wiegmann identities (1.278) and (1.279) it is possible to verify that the action (5.105) is invariant under the gauge transformation:

$$g \to hgh^{-1} , \qquad A \to hAh^{-1} + dhh^{-1} \qquad (5.109)$$

for $h : \Sigma \to H$. This is a vectorially gauged model.

For the case of $H = U(1)$ considered here there exists the system is axially gauge invariant under the transformations

$$g \to hgh , \qquad A \to A + dhh^{-1} \qquad (5.110)$$

for $h : \Sigma \to U(1)$. In the axially gauged model the gauge field dependent term is

$$L_a^{\text{gauge}}(g, A) = \text{tr}[g^{-1}\partial_+ g A_- + \partial_- g g^{-1} A_+ - A_- g^{-1} A_+ g - A_+ A_-] . \qquad (5.111)$$

As we explained in section 3.2 to write the defect action with WZW terms we should specify a bibrane with a two-form satisfying the condition (3.24).

We suggest the following ansatz:

$$(g_1, g_2) = (C_\mu p, L_1 p L_2) \qquad (5.112)$$

Here $p \in G$, $L_1 \in U(1)$, $L_2 \in U(1)$ and C_μ is a conjugacy class

$$C_\mu = le^{2i\pi\mu/k}l^{-1}, \;\; l \in G \qquad (5.113)$$

where $\mu \equiv \boldsymbol{\mu} \cdot \boldsymbol{H}$ is a highest weight representation integrable at level k, taking value in the Cartan subalgebra of the G Lie algebra. This condition is a consequence of global issues [78]. Note that under the full gauge transformation

$$g_1 \mapsto h_1 g_1 h_1^{-1} \quad \text{and} \quad g_2 \mapsto h_2 g_2 h_2 \qquad (5.114)$$

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the parameters in (5.112) transform as

$$C_\mu \mapsto h_1 C_\mu h_1^{-1} \tag{5.115}$$

$$p \mapsto h_1 p h_1^{-1}$$

$$L_1 \mapsto L_1 h_1^{-1} h_2$$

$$L_2 \mapsto L_2 h_1 h_2$$

Using the Polyakov-Wiegamann identity (1.279) one can check that the condition (3.24) is satisfied with the following two-form

$$\varpi(C_\mu, p, L_1, L_2) = \omega_\mu(C_\mu) - \text{Tr}(C_\mu^{-1} dC_\mu dp p^{-1}) + \text{Tr}(p^{-1} dp d L_2 L_2^{-1}) + \tag{5.116}$$

$$+ \text{Tr}(L_1^{-1} dL_1 dp p^{-1}) + \text{Tr}(L_1^{-1} dL_1 p d L_2 L_2^{-1} p^{-1}) - \text{Tr}(L_1^{-1} dL_1 L_2^{-1} dL_2)$$

where $\omega_\mu(C_\mu) = \text{Tr}(l^{-1} dl e^{2i\pi\mu/k} l^{-1} dl e^{-2i\pi\mu/k})$ is defined in(2.6). Now the full action can be written as

$$S^{\text{A-V}} = S^{\text{kin-def}} + S^{\text{gauge-def}} + S^{\text{top-def}} \tag{5.117}$$

here

$$S^{\text{kin-def}} = \frac{k}{4\pi} \int_{\Sigma_1} dx^+ dx^- L^{\text{kin}}(g_1) + \frac{k}{4\pi} \int_{\Sigma_2} dx^+ dx^- L^{\text{kin}}(g_2) \tag{5.118}$$

and

$$S^{\text{gauge-def}} = \frac{k}{2\pi} \int_{\Sigma_1} L_v^{\text{gauge}}(g_1, A_1) dx^+ dx^- + \frac{k}{2\pi} \int_{\Sigma_2} L_a^{\text{gauge}}(g_2, A_2) dx^+ dx^- \tag{5.119}$$

$$S^{\text{top-def}} = \frac{k}{4\pi} \int_{B_1} \omega^{\text{WZ}}(g_1) + \frac{k}{4\pi} \int_{B_2} \omega^{\text{WZ}}(g_2) - \frac{k}{4\pi} \int_{D} \varpi(g_1, g_2) \tag{5.120}$$

with $\varpi(g_1, g_2)$ given by (5.116).

It is cumbersome but straightforward to check that the action (5.117) is invariant the gauge transformations:

$$g_1 \mapsto h_1 g_1 h_1^{-1}, \qquad A_1 \mapsto A_1 + dh_1 h_1^{-1} \tag{5.121}$$

$$g_2 \mapsto h_2 g_2 h_2, \qquad A_2 \mapsto A_2 + dh_2 h_2^{-1}$$

where $h_1 : \Sigma_1 \to U(1)$ and $h_2 : \Sigma_2 \to U(1)$.

5.5.2 Duality defect for the parafermion disc $SU(2)/U(1)$

Specialize now to the case of $G = SU(2)$ [17].

We write the group elements using the Euler coordinates:

$$g = e^{i\chi \frac{\sigma_3}{2}} e^{i\theta\sigma_1} e^{i\varphi \frac{\sigma_3}{2}} = e^{i(\tilde\phi+\phi)\frac{\sigma_3}{2}} e^{i\theta\sigma_1} e^{i(\tilde\phi-\phi)\frac{\sigma_3}{2}} \tag{5.122}$$

The ranges of the variables are $0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \chi \leq 4\pi, \quad -\pi \leq \phi, \tilde\phi \leq \pi$.

The axially gauged model $\frac{SU(2)}{U(1)}_{\text{axial}}$ is derived by the gauging of the $U(1)$ symmetry corresponding to shifting of $\tilde\phi$ and has the target space M_A with the following metric and dilaton field [96, 123]:

$$ds^2 = k(d\theta^2 + \tan^2\theta d\phi^2) \tag{5.123}$$

$$e^\Phi = \frac{g_s}{\cos\theta}$$

$$\phi \sim \phi + 2\pi$$

Using the T-duality rules of section 5.2 one can see that T-dual background to the axially gauged model is

$$\tilde{ds}^2 = k\left(d\tilde\theta^2 + \frac{d\tilde\phi^2}{\tan^2\tilde\theta}\right) \tag{5.124}$$

$$e^{\tilde\Phi} = \frac{g_s}{\sqrt{k}\sin\tilde\theta}$$

$$\tilde\phi \sim \tilde\phi + \frac{2\pi}{k}$$

Vectorially gauged model $\frac{SU(2)}{U(1)}_{\text{vec}}$ is derived by the gauging of the $U(1)$ symmetry corresponding to the shifting of ϕ and has the target space M_V with the metric and the dilaton:

$$\tilde{ds}^2 = k\left(d\tilde\theta^2 + \frac{d\tilde\phi^2}{\tan^2\tilde\theta}\right) \tag{5.125}$$

$$e^{\tilde\Phi} = \frac{g_s}{\sin\tilde\theta}$$

$$\tilde\phi \sim \tilde\phi + 2\pi$$

Comparing (5.124) and (5.125) one can see that the background T-dual to the axially gauged model is the Z_k orbifold of the vectorially gauged model.

According to the results of section 5.2 the world-volume of the T-duality defect D_A^T between backgrounds (5.123) and (5.124) is the submanifold $\theta = \tilde{\theta}$ of the product $M_V \times M_A$ with the flux $F = d\phi \wedge d\tilde{\phi}$. The defects between backgrounds (5.123) and (5.125) D_{V-A} has the same world volume but the flux is $F = kd\phi \wedge d\tilde{\phi}$. Consider the defects given by equation (5.112). The conjugacy class takes the form $C_j = le^{\frac{2\pi ij\sigma_3}{k}}l^{-1}$, $j = 0, \frac{1}{2} \ldots \frac{k}{2}$, (since we are working in the specific case of $G = SU(2)$, the general subscript μ was changed to j, which is standard for this group) and therefore we have a family of the defects labelled by j. Now we show that the T-duality defect above, D_{V-A}, corresponds to $j = 0$.

Let us examine this defect in more detail. Parameterizing $L_1 = e^{i\alpha_1\sigma_3/2}$ and $L_2 = e^{i\alpha_2\sigma_3/2}$ and writing p using the Euler coordinates, we obtain for this special defect:

$$(g_1, g_2) = \left(e^{i(\tilde{\kappa}+\kappa)\frac{\sigma_3}{2}} e^{i\theta\sigma_1} e^{i(\tilde{\kappa}-\kappa)\frac{\sigma_3}{2}}, \quad e^{i(\tilde{\kappa}+\kappa+\alpha_1)\frac{\sigma_3}{2}} e^{i\theta\sigma_1} e^{i(\tilde{\kappa}-\kappa+\alpha_2)\frac{\sigma_3}{2}} \right) \tag{5.126}$$

From (5.126) it can be seen that this defect satisfies the condition $\theta = \tilde{\theta}$. To project down this defect to the product space $M_V \times M_A$ we impose gauge fixing conditions $\kappa = 0$ for the first vectorially gauged model and

$$(\tilde{\kappa} + \kappa + \alpha_1) + (\tilde{\kappa} - \kappa + \alpha_2) = 0 \tag{5.127}$$

for the axially gauged model. From (5.127) one obtains:

$$\tilde{\kappa} = -\frac{\alpha_1 + \alpha_2}{2} \tag{5.128}$$

Therefore the angles ϕ and $\tilde{\phi}$ of the target spaces are related to the defect parameters by equations:

$$\tilde{\phi} = \tilde{\kappa} = -\frac{\alpha_1 + \alpha_2}{2} \tag{5.129}$$

$$\phi = \frac{\alpha_1 - \alpha_2}{2} \tag{5.130}$$

Let us evaluate the two-form (5.116). For $j = 0$ it simplifies to:

$$\varpi(p, L_1, L_2) = \text{Tr}(p^{-1}dpdL_2L_2^{-1}) + \text{Tr}(L_1^{-1}dL_1dpp^{-1}) + \tag{5.131}$$

$$\text{Tr}(L_1^{-1}dL_1pdL_2L_2^{-1}p^{-1}) - \text{Tr}(L_1^{-1}dL_1L_2^{-1}dL_2)$$

This implies

$$\text{Tr}(p^{-1}dpdL_2L_2^{-1}) = -(d\tilde{\kappa}\cos^2\theta - d\kappa\sin^2\theta)d\alpha_2 \tag{5.132}$$

$$\text{Tr}(L_1^{-1}dL_1dpp^{-1}) = -d\alpha_1(d\tilde{\kappa}\cos^2\theta + d\kappa\sin^2\theta)$$

$$\text{Tr}(L_1^{-1}dL_1pdL_2L_2^{-1}p^{-1}) = -d\alpha_1 d\alpha_2(\cos^2\theta - \frac{1}{2})$$

$$-\text{Tr}(L_1^{-1}dL_1L_2^{-1}dL_2) = \frac{d\alpha_1 d\alpha_2}{2}$$

Using that $\kappa = 0$ and (5.128), (5.129) and (5.130) one obtains that the θ dependent terms drop and we end up with

$$\frac{k}{4\pi}\varpi(p, L_1, L_2) = \frac{k}{4\pi}d\alpha_1 d\alpha_2 = \frac{k}{2\pi}d\tilde{\phi}d\phi \tag{5.133}$$

This is the flux on the defect D_{V-A} and as demonstrated in sec. 2, this defect is topological.

Geometry for a generic defect can be concluded noting that the bibrane (5.112) geometrically (but not the flux and symmetries) is folded version of the symmetry-breaking type II brane considered in section 2.5.2. Using the arguments explained there, one can conclude that a generic defect has a geometry given by the inequality (2.157):

$$\cos 2(\theta - \tilde{\theta}) \geq \cos\frac{4\pi j}{k} \tag{5.134}$$

5.5.3 Axial-vectorial defects as operators in the parafermion Hilbert space

It has been shown that the backgrounds (5.123) and (5.125) correspond to the parafermion theory, and therefore the defects above can be realized as operators in the parafermions Hilbert space.

To construct the corresponding operator one should start with the Cardy defect in the parafermion theory [138]:

$$X_{\hat{j},\hat{n}} = \sum_{j,n} \frac{S_{(\hat{j},\hat{n});(j,n)}^{\text{PF}}}{S_{(0,0);(j,n)}^{\text{PF}}} P_{j,n}^{\text{PF}} \bar{P}_{j,n}^{\text{PF}} \tag{5.135}$$

Here $S_{(\hat{j},\hat{n});(j,n)}^{\text{PF}}$ is the parafermion matrix of the modular transformation (1.333)

$$S_{(\hat{j},\hat{n});(j,n)}^{\text{PF}} = \sqrt{\frac{2}{k}}S_{\hat{j}j}^{\text{SU}(2)}e^{\frac{i\pi n\hat{n}}{k}} \tag{5.136}$$

$P_{j,n}^{\mathrm{PF}}$ and $\bar{P}_{j,n}^{\mathrm{PF}}$ are projectors defined in (5.78) and (5.79). We need to construct a defect mapping A- branes to B- branes. This can be done along the lines used in [123] for the parafermion B-branes construction. Recall that the $\mathbb{Z}_k$ orbifold of the parafermion theory at level k is T-dual to the original theory. To get a defect mapping A- branes to B- branes one should sum over $\mathbb{Z}_k$ images of $X_{\hat{j},\hat{n}}$ and perform T-duality. In order to circumvent the fixed point problem, we consider the case of odd k † . Summing over images leaves in (5.135) only the $n = 0$ term and T-duality exchanges $\bar{P}_{j,n}^{\mathrm{PF}}$ with its B-type version, which can be derived in the following way. Using the decomposition of $SU(2)_k$ as a product of parafermion and scalar theories (1.331) one can write

$$\bar{P}_j^{\mathrm{SU(2)}} = \sum_r \bar{P}_{j,r}^{\mathrm{PF}} \bar{P}_{r+}^{\mathrm{U(1)}} \tag{5.137}$$

where the projector $\bar{P}_j^{\mathrm{SU(2)}}$ for $SU(2)_k$ is defined in (5.81), for parafermions $\bar{P}_{j,r}^{\mathrm{PF}}$ in (5.79), and for $U(1)_k$ scalar $\bar{P}_{r+}^{\mathrm{U(1)}}$ in (5.76).

To define the T-dual projector $\overline{BP}_{j,n}^{\mathrm{PF}}$ we rotate the $SU(2)$ projector $\bar{P}_j^{\mathrm{SU(2)}}$ (5.81) with operator $e^{i\pi \bar{J}_0^1}$, satisfying

$$e^{i\pi \bar{J}_0^1} \bar{J}_0^3 e^{-i\pi \bar{J}_0^1} = -\bar{J}_0^3 \tag{5.138}$$

and afterwards decompose it again as a product of the parafermion and scalar theories:

$$1 \otimes e^{i\pi \bar{J}_0^1} \bar{P}_j^{\mathrm{SU(2)}} = \sum_r \overline{BP}_{j,r}^{\mathrm{PF}} \bar{P}_{r-}^{\mathrm{U(1)}} \tag{5.139}$$

where $\bar{P}_{r-}^{\mathrm{U(1)}}$ is defined in (5.76).

Combining the orbifolding and the T duality procedures results is:

$$Y_{\hat{j}}^{AB} = \sqrt{k} \sum_j \frac{S_{\hat{j},j}^{\mathrm{SU(2)}}}{S_{0,j}^{\mathrm{SU(2)}}} P_{j,0}^{\mathrm{PF}} \overline{BP}_{j,0}^{\mathrm{PF}} \tag{5.140}$$

Using the arguments of section 2.5.3, one can show that in the large k limit Y_j^{AB} has the geometry given with the overlap (2.166):

---

†In the case of an even k, the primary field $\frac{k}{4}$ has the non-trivial stabilizator $\mathbb{Z}_2$, which requires the fixed point resolution procedure. As a consequence the formulae for branes and defects derived in this way get modified. See for details [123].

$$\langle\theta,\phi|Y_{\hat{j}}^{AB}|\tilde{\theta},\tilde{\phi}\rangle \sim \tag{5.141}$$

$$\frac{k}{\pi^2}\int_{|2\theta-2\tilde{\theta}|}^{2\theta+2\tilde{\theta}}\frac{\Theta(\cos\gamma-\cos 2\hat{\psi})}{\sqrt{\cos\gamma-\cos 2\hat{\psi}}}\frac{\sin\gamma d\gamma}{\sqrt{[\cos\gamma-\cos 2(\theta+\tilde{\theta})][\cos 2(\theta-\tilde{\theta})-\cos\gamma]}}$$

where $\hat{\psi}=\frac{(2\hat{j}+1)\pi}{k+2}$ and Θ is the Heavyside step function. Eq. (5.141) shows that the world-volume of the defect should satisfy the inequality

$$\cos 2(\theta-\tilde{\theta}) \geq \cos 2\hat{\psi} \tag{5.142}$$

which in the large k limit coincides with the inequality (5.134), defining the geometry of a generic defect.

Note that in the defect Y_0^{AB}, the relation of the elements of the matrix of the modular transformation drops, and it is a sum of projectors, projecting down to the $n=0$ subspace and performing T-duality, thus mapping the $A_{j,n}$ Cardy branes to the B_j branes constructed in [123]. For generic $\hat{j}$ one derives a linear combination of the B_l branes with coefficients given by the fusion numbers $N_{\hat{j}j}^l$.

5.6 Fermionic T-duality

In this section we show how do defects generate T-duality on fermionic coordinates. We show here that the fermionic T-duality is implemented by the defect, given by the fermionic analogue of the Poincaré line bundle, which we call Super-Poincaré line bundle. This defect is invertible.

Then we define the super Fourier-Mukai transform, as in the bosonic case, as an integral with an appropriate kernel given by the exponent of the flux of a super Poincare line bundle.

5.6.1 Pseudodifferential forms integration

Pseudodifferential forms [25, 26, 117], defined on a supermanifold of p bosonic and q fermionic coordinates, are of the form

$$f = \sum_{v,u} f_{v,u}(x, d\theta)\theta^v dx^u \tag{5.143}$$

Where: $v = v_1, ..., v_q$; $u = u_1, ..., u_p$; $v_i, u_i \in 0, 1$; $x = x_1, ...x_p$; $d\theta = d\theta_1, ...d\theta_q$; $\theta^v = \theta_1^{v_1} \cdot ... \cdot \theta_q^{v_q}$; $dx^u = dx_1^{u_1} \cdot ... \cdot dx_p^{u_p}$, and the sum is over all possible values of u and v. Such an object can be integrated over the bundle on which it is defined. The integration is defined as

$$\int_{\mathcal{B}} f = \int_{B} f_{1,1,...,1} \tag{5.144}$$

Where $\mathcal{B}$ is the cotangent bundle of the supermanifold and B is its underlying bundle, with just the bosonic coordinates. The $d\theta$s are coordinates along the bundle, and unlike the case of the fibrewise integration presented above, they are bosonic. For that reason one needs f to be sufficiently rapidly decreasing in them in order for the integral to converge. As will be demonstrated bellow, this is indeed the case for the super Fourier-Mukai transform.

5.6.2 Review of the fermionic T-duality

Consider the action (5.19) for the case when one has fermionic as well as bosonic variables, and G_{ij} and B_{ij} are graded-symmetric and graded -antisymmetric tensors respectively. Suppose that G_{ij} and B_{ij} do not depend on the fermionic variable θ^1 [24]. Separating the variable θ^1 one has

$$S = \int dx^+ dx^- (B_{11}\partial\theta^1\bar{\partial}\theta^1 + E_{1N}\partial\theta^1\bar{\partial}X^N + E_{M1}\partial X^M\bar{\partial}\theta^1 + E_{MN}\partial X^M\bar{\partial}X^N) \tag{5.145}$$

Replacing derivatives of θ^1 by fermionic vector $(A, \bar{A})$ and introducing a Lagrange multiplier field $\tilde{\theta}^1$ one gets

$$S = \int dx^+ dx^- (B_{11}A\bar{A} + E_{1N}A\bar{\partial}X^N + E_{M1}\partial X^M\bar{A} + E_{MN}\partial X^M\bar{\partial}X^N + \tilde{\theta}^1(\partial\bar{A} - \bar{\partial}A)) \tag{5.146}$$

Integrating out $\tilde{\theta}^1$ imposes that

$$A = \partial \theta^1 \quad \text{and} \quad \bar{A} = \bar{\partial} \theta^1. \tag{5.147}$$

Integrating out $(A, \bar{A})$ results in:

$$\bar{A} = \frac{1}{B_{11}} \left((-)^{s_M} E_{1M} \bar{\partial} X^M + \bar{\partial} \tilde{\theta}^1 \right) \quad \text{and} \quad A = -\frac{1}{B_{11}} \left(E_{M1} \partial X^M - \partial \tilde{\theta}^1 \right) \tag{5.148}$$

Inserting (5.148) in (5.146) one obtains fermionic T-dual background:

$$\tilde{B}_{11} = -\frac{1}{B_{11}} \tag{5.149}$$

$$\tilde{E}_{1M} = \frac{E_{1M}}{B_{11}}$$

$$\tilde{E}_{M1} = \frac{E_{M1}}{B_{11}}$$

$$\tilde{E}_{MN} = E_{MN} - \frac{E_{1N} E_{M1}}{B_{11}}$$

or in the components:

$$\tilde{B}_{11} = -\frac{1}{B_{11}} \tag{5.150}$$

$$\tilde{G}_{1M} = \frac{G_{1M}}{B_{11}}$$

$$\tilde{B}_{1M} = \frac{B_{1M}}{B_{11}}$$

$$\tilde{G}_{MN} = G_{MN} - \frac{1}{B_{11}} (G_{1N} B_{M1} + B_{1N} G_{M1})$$

$$\tilde{B}_{MN} = B_{MN} - \frac{1}{B_{11}} (G_{1N} G_{M1} + B_{1N} B_{M1})$$

Equating (5.147) and (5.148) one gets:

$$\partial \tilde{\theta}^1 = B_{11} \partial \theta^1 + E_{M1} \partial X^M \quad \text{and} \quad \bar{\partial} \tilde{\theta}^1 = B_{11} \bar{\partial} \theta^1 - (-)^{s_M} E_{1M} \bar{\partial} X^M \tag{5.151}$$

The rest of the coordinates remains unchanged.

5.6.3 Defects implementing the fermionic T-duality and the Super Poincaré line bundle

We now consider the action with defect, with target spaces related by the equations (5.149), and the defect given again by the correspondence space

$$X^N = \tilde{X}^N, \qquad N = 2 \dots \dim M \tag{5.152}$$

and connection

$$A = \theta^1 d\tilde{\theta}^1 \tag{5.153}$$

with curvature

$$F = d\theta^1 \wedge d\tilde{\theta}^1. \tag{5.154}$$

We will call this super line bundle by analogy with the bosonic case a Super-Poincaré bundle. Now the defect equations of motion take the form:

$$E_{j1}\partial X^j - (-)^{s_j} E_{1j}\bar{\partial}X^j - \partial_\tau \tilde{\theta}^1 = 0 \tag{5.155}$$

$$E_{jN}\partial X^j - (-)^{s_j s_N} E_{Nj}\bar{\partial}X^j - \tilde{E}_{jN}\partial\tilde{X}^j + (-)^{s_j s_N}\tilde{E}_{Nj}\bar{\partial}\tilde{X}^j = 0, \quad N = 2\ldots \dim M \tag{5.156}$$

$$\tilde{E}_{j1}\partial\tilde{X}^j - (-)^{s_j}\tilde{E}_{1j}\bar{\partial}\tilde{X}^j + \partial_\tau \theta^1 = 0 \tag{5.157}$$

Additionally as before we have:

$$\partial X^N + \bar{\partial}X^N = \partial\tilde{X}^N + \bar{\partial}\tilde{X}^N, \quad N = 2\ldots \dim M \tag{5.158}$$

Solving (5.155), (5.156), (5.157), (5.158) we obtain

$$\bar{\partial}\tilde{X}^N = \bar{\partial}X^N, \qquad N = 2\ldots \dim M \tag{5.159}$$

$$\partial\tilde{X}^N = \partial X^N, \qquad N = 2\ldots \dim M$$

$$\partial\tilde{\theta}^1 = B_{11}\partial\theta^1 + E_{M1}\partial X^M$$

$$\bar{\partial}\tilde{\theta}^1 = B_{11}\bar{\partial}\theta^1 - (-)^{s_M} E_{1M}\bar{\partial}X^M$$

The details of the calculation can be found in [57]. The relations (5.159) coincide with the equations (5.151). Therefore the defect given by the Super-Poincare bundle on the super-correspondence space induces the fermionic T-duality.

One can check that equations (5.150) and (5.159) imply:

$$T = G_{ij}\partial X^i \partial X^j = \tilde{T} = \tilde{G}_{ij}\partial\tilde{X}^i \partial\tilde{X}^j \tag{5.160}$$

and

$$\bar{T} = G_{ij}\bar{\partial}X^i\bar{\partial}X^j = \tilde{\bar{T}} = \tilde{G}_{ij}\bar{\partial}\tilde{X}^i\bar{\partial}\tilde{X}^j \tag{5.161}$$

which means that the defect is topological.

All this again can be generalized to the T-dualizing of several coordinates. Suppose we T-dualize the first n coordinates, indexed by Greek letters.

The transformed background has the form

$$\tilde{E} = \begin{pmatrix} -E_{\alpha\beta}^{-1} & E_{\alpha\beta}^{-1}E_{\beta N} \\ E_{M\alpha}E_{\alpha\beta}^{-1} & E_{MN} - E_{\beta N}E_{M\alpha}E_{\alpha\beta}^{-1} \end{pmatrix} \tag{5.162}$$

Now we should consider the defect with the worldvolume

$$X^N = \tilde{X}^N, \qquad N = n+1\ldots \dim M \tag{5.163}$$

and connection

$$A = \sum_{\alpha=1}^{n} \theta^\alpha d\tilde{\theta}^\alpha. \tag{5.164}$$

It has the curvature

$$F = \sum_{\alpha=1}^{n} d\theta^\alpha \wedge d\tilde{\theta}^\alpha. \tag{5.165}$$

In the same way as above we can show that for M and $\tilde{M}$ related by equations (5.162) this defect is topological and implies the defect equations of motion:

$$\bar{\partial}\tilde{X}^N = \bar{\partial}X^N, \qquad N = n+1\ldots \dim M \tag{5.166}$$

$$\partial\tilde{X}^N = \partial X^N, \qquad N = n+1\ldots \dim M$$

$$\partial\tilde{\theta}^\alpha = E_{\beta\alpha}\partial\theta^\beta + E_{M\alpha}\partial X^M$$

$$\bar{\partial}\tilde{\theta}^\alpha = E_{\alpha\beta}\bar{\partial}\theta^\beta - (-)^{s_M}E_{\alpha M}\bar{\partial}X^N$$

We have obtained again T-duality relations for several T-dualized fermionic coordinates.

5.6.4 Super Fourier-Mukai transform

We now elaborate the Fourier-Mukai transform for fermionic T-duality. It has the form:

$$e^{-\hat{B}}\hat{\mathcal{G}} = \int d\eta e^{-B}\mathcal{G}e^{\eta\tilde{\eta}} \tag{5.167}$$

with $\mathcal{G}$ and B as in (5.50), where we set $\eta = d\theta^1$. As we explained η is a bosonic variable, so we have a usual integration over η. From (5.150) one obtains:

$$\hat{B} - B = -\frac{1}{2B_{11}}\tilde{\eta}^2 - \frac{1}{2}B_{11}\eta^2 \tag{5.168}$$

$$-\frac{1}{2B_{11}}(G_{1N}G_{M1} + B_{1N}B_{M1})dX^M dX^N + \frac{B_{1M}}{B_{11}}\tilde{\eta}dX^M - B_{1M}\eta dX^M$$

Suppose that $\mathcal{G}$ does not depend on η. Using the formula for the Gaussian integral

$$\int dx e^{-\frac{1}{2}ax^2 + Jx} = \frac{\sqrt{2\pi}}{\sqrt{a}}e^{\frac{J^2}{2a}} \tag{5.169}$$

we obtain that the terms in (5.167) containing B_{1M} and the first quadratic term are canceled and, we end up with

$$\hat{\mathcal{G}} = \frac{\sqrt{2\pi}}{\sqrt{B_{11}}}\mathcal{G}e^{-\frac{1}{2B_{11}}G_{1N}G_{M1}dX^M dX^N} \tag{5.170}$$

Note that G_{1N} and B_{1N} have parity $(-)^{s_N+1}$. Hence if dX^M and dX^N are differentials of the bosonic coordiantes, the product $G_{1N}G_{M1}$ contains fermionic coordinates and drops if we consider the lowest $\theta = 0$ components, in agreement with the observation [24] that the fermionic T-duality does not modify D-brane dimensionality. Note that the lowest $\theta = 0$ components of (5.170) coincide with the homogeneous part of the transformation of the Ramond-Ramond forms in [24].

Using the transformations rules (5.162) equation (5.170) can be generalized to the case of the T-dualization of several fermionic variables θ^α. Keeping in mind that eventually we are going to project to the $\theta = 0$ component we can set $G_{\alpha\beta} = 0$, since $G_{\alpha\beta} = \eta^{ab}E_a^\alpha E_b^\beta$, and taking into account that a and b are bosonic and α and β are fermionic, one sees that E_a^α and E_b^β are odd. With this simplification the Fourier-Mukai transform for $\mathcal{G}$ independent on θ^α can be computed to yield:

$$\hat{\mathcal{G}} = \frac{\sqrt{2\pi}}{\sqrt{\det||B_{\alpha\beta}||}}\mathcal{G}e^{-\frac{1}{2}B_{\alpha\beta}^{-1}G_{\alpha N}G_{M\beta}dX^M dX^N} \tag{5.171}$$

The lowest component of (5.171) again coincides with the homogeneous part of the transformation of Ramond-Ramond forms in [24] for the fermionic T-dualization of the n coordinates.

5.7 Non abelian T-duality

5.7.1 Review of non-abelian T-duality

Non-abelian T-duality was developed in early nineties [9, 30, 53, 97, 120, 121, 134].

Here we recall and collect some facts on non-abelian T-duality for isometry groups acting without isotropy [97, 110]. Suppose we have a target space with an isometry group G, with generators T^a, structure constants f^a_{bc}, and coordinates θ^a, and in some coordinates the metric and the NS two-form take the form

$$ds^2 = G_{\mu\nu}(Y)dY^\mu dY^\nu + 2G_{\mu a}(Y)\Omega^a_k dY^\mu d\theta^k + +G_{ab}(Y)\Omega^a_m \Omega^b_k d\theta^m d\theta^k \tag{5.172}$$

$$B = \frac{1}{2}B_{\mu\nu}(Y)dY^\mu \wedge dY^\nu + B_{\mu a}(Y)\Omega^a_k dY^\mu \wedge d\theta^k + +\frac{1}{2}B_{ab}(Y)\Omega^a_m \Omega^b_k d\theta^m \wedge d\theta^k \tag{5.173}$$

where Ω^a_k are components of the right-invariant Maurer-Cartan forms L^a:

$$dgg^{-1} = L^a T_a = \Omega^a_k T_a d\theta^k \tag{5.174}$$

The background fields depend on group coordinates θ^a only through the Maurer-Cartan forms. Also as it is clear from the notations they can depend on some spectator coordinates Y. Since $d(dgg^{-1}) = dgg^{-1} \wedge dgg^{-1}$, L^a and Ω^a_k satisfy the Maurer-Cartan relations

$$dL^a = \frac{1}{2}f^a_{bc}L^b L^c \tag{5.175}$$

and

$$\partial_i \Omega^c_j - \partial_j \Omega^c_i = f^c_{ab}\Omega^a_i \Omega^b_j \tag{5.176}$$

The corresponding Lagrangian density is

$$L = Q_{\mu\nu}\partial Y^\mu \bar{\partial} Y^\nu + Q_{\mu a}\Omega^a_k \partial Y^\mu \bar{\partial}\theta^k + Q_{a\mu}\Omega^a_k \partial\theta^k \bar{\partial} Y^\mu + Q_{ab}\Omega^a_m \Omega^b_k \partial\theta^m \bar{\partial}\theta^k \tag{5.177}$$

where

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}\,, \ Q_{\mu a} = G_{\mu a} + B_{\mu a} \tag{5.178}$$

$$Q_{a\mu} = G_{a\mu} + B_{a\mu}\,, \ Q_{ab} = G_{ab} + B_{ab}\,.$$

To find the dual action one can use the Buscher method and write the Lagrangian (5.177) in the form

$$L = Q_{\mu\nu}\partial Y^\mu \bar{\partial} Y^\nu + Q_{\mu a}\partial Y^\mu \bar{A}^a + Q_{a\mu}A^a \bar{\partial} Y^\mu + \tag{5.179}$$

$$Q_{ab}A^a \bar{A}^b - x^a(\partial \bar{A}^a - \bar{\partial} A^a - f_{bc}^a A^b \bar{A}^c)$$

The equations of motion of the Lagrangian multiplier x^a force the field strength

$$F_{+-}^a = \partial \bar{A}^a - \bar{\partial} A^a - f_{bc}^a A^b \bar{A}^c \tag{5.180}$$

to vanish. The solution to these constraints is

$$A^a = \Omega_k^a \partial \theta^k \quad \text{and} \quad \bar{A}^a = \Omega_k^a \bar{\partial} \theta^k. \tag{5.181}$$

Putting this solution into (5.179) yields the original action (5.177). On the other hand integrating out gauge fields A^a one obtains:

$$M_{ba}^{-1}(Q_{\mu b}\partial Y^\mu + \partial x^b) = -A^a \tag{5.182}$$

$$M_{ab}^{-1}(\bar{\partial} x^b - Q_{b\mu}\bar{\partial} Y^\mu) = \bar{A}^a$$

where

$$M_{ab} = Q_{ab} + x^c f_{ab}^c \tag{5.183}$$

Inserting expressions (5.182) back in (5.179) we find the dual action:

$$\hat{L} = \hat{E}_{\mu\nu}\partial Y^\mu \bar{\partial} Y^\nu + \hat{E}_{\mu a}\partial Y^\mu \bar{\partial} x^a + \hat{E}_{a\mu}\partial x^a \bar{\partial} Y^\mu + \hat{E}_{ab}\partial x^a \bar{\partial} x^b \tag{5.184}$$

where

$$\hat{E}_{\mu\nu} = Q_{\mu\nu} - Q_{\mu a}M_{ab}^{-1}Q_{b\nu} \tag{5.185}$$

$$\hat{E}_{\mu a} = Q_{\mu b}M_{ba}^{-1}$$

$$\hat{E}_{a\mu} = -Q_{b\mu}M_{ab}^{-1}$$

$$\hat{E}_{ab} = M_{ab}^{-1}$$

Equating (5.181) and (5.182), one gets the duality relations of non-abelian T-duality [30,121]

$$M_{ba}^{-1}(Q_{\mu b}\partial Y^\mu + \partial x^b) = -\Omega_k^a \partial \theta^k \tag{5.186}$$

$$M_{ab}^{-1}(\bar{\partial}x^b - Q_{b\mu}\bar{\partial}Y^\mu) = \Omega_k^a\bar{\partial}\theta^k \tag{5.187}$$

Separating in (5.185) symmetric and antisymmetric parts we derive metric and NS form of the dual theory:

$$\hat{G}_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}(Q_{\mu a}M_{ab}^{-1}Q_{b\nu} + Q_{\nu a}M_{ab}^{-1}Q_{b\mu}) \tag{5.188}$$

$$\hat{G}_{\mu a} = \frac{1}{2}(Q_{\mu b}M_{ba}^{-1} - Q_{b\mu}M_{ab}^{-1}) \tag{5.189}$$

$$\hat{G}_{ab} = \frac{1}{2}(M_{ab}^{-1} + M_{ba}^{-1}) \tag{5.190}$$

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{2}(Q_{\mu a}M_{ab}^{-1}Q_{b\nu} - Q_{\nu a}M_{ab}^{-1}Q_{b\mu}) \tag{5.191}$$

$$\hat{B}_{\mu a} = \frac{1}{2}(Q_{\mu b}M_{ba}^{-1} + Q_{b\mu}M_{ab}^{-1}) \tag{5.192}$$

$$\hat{B}_{ab} = \frac{1}{2}(M_{ab}^{-1} - M_{ba}^{-1}) \tag{5.193}$$

and for dilaton

$$\tilde{\Phi} = \Phi - \frac{1}{2}\log(\det M) \tag{5.194}$$

Let us recall the $SU(2)$ Principal Chiral Model [67, 68, 175]

$$S(g) = \int k\text{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g) \tag{5.195}$$

where $g \in SU(2)$. The metric in the Euler coordinates is

$$ds^2 = k(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi) \tag{5.196}$$

and there is no NS two-form. To obtain the dual background one should compute M_{ab}^{-1} matrix.

Denoting the dual coordinates x^a, $a = 1, 2, 3$, one has here

$$M_{ab} = k\delta_{ab} + \epsilon_{abc}x_c \tag{5.197}$$

and

$$M_{ab}^{-1} = \frac{1}{k^2 + r^2}\left(k\delta_{ab} + \frac{x_a x_b}{k} - \epsilon_{abc}x_c\right) \tag{5.198}$$

Separating symmetric and antisymmetric parts and denoting $r^2 = x^a x^a$ one derives

$$\hat{G}_{ab} = \frac{1}{k^2 + r^2}\left(k\delta_{ab} + \frac{x_a x_b}{k}\right) \tag{5.199}$$

$$\hat{B}_{ab} = -\frac{1}{k^2 + r^2}\epsilon_{abc}x_c \tag{5.200}$$

$$\hat{\Phi} = -\frac{1}{2}\log(k^3 + kr^2) \tag{5.201}$$

and hence one has

$$\hat{ds}^2 = \frac{dr^2}{k} + \frac{kr^2}{k^2 + r^2}ds^2(S^2) \tag{5.202}$$

$$\hat{B} = -\frac{r^3}{k^2 + r^2}\text{Vol}(S^2) \tag{5.203}$$

5.7.2   Defects generating non-abelian T-duality

Consider the action (5.2) with a defect as in the situation above, where M_1 is the target space with the coordinates (Y^μ, θ^k) and has metric and NS 2-form given by (5.172) and (5.173), M_2 is the space with the coordinates (Y^μ, x^a) and with metric and 2-form given by (5.188)-(5.193), and Q is the correspondence space, with the coordinates (Y^μ, θ^k, x^a), the connection

$$A = -x^a L^a = -x^a \Omega_k^a d\theta^k \tag{5.204}$$

and the curvature

$$F = dA = -(dx^a L^a + \frac{1}{2}x^a f_{bc}^a L^b L^c) \tag{5.205}$$

To derive (5.205) we used the Maurer-Cartan relation (5.175). By other words we take as L_1 in (5.2) the L given by (5.177), and as L_2 the $\hat{L}$ given by (5.184).

The conditions (5.204) and (5.205) define a line bundle $\mathcal{P}^{\text{NA}}$ over Q, with the curvature (5.205), which can be called non-abelian Poincarè line bundle. In this case the action (5.2) yields the following equations of motion on the defect line:

$$Q_{\mu a}\partial Y^\mu + Q_{ba}\Omega_m^b \partial\theta^m - Q_{a\mu}\bar{\partial}Y^\mu - Q_{ab}\Omega_m^b \bar{\partial}\theta^m = -x^c\Omega_m^b f_{ba}^c \partial_\tau \theta^m - \partial_\tau x^a \tag{5.206}$$

$$\hat{E}_{\mu a}\partial Y^\mu + \hat{E}_{ba}\partial x^b - \hat{E}_{a\mu}\bar{\partial}Y^\mu - \hat{E}_{ab}\bar{\partial}x^b = -\Omega_k^a \partial_\tau \theta^k. \tag{5.207}$$

$$Q_{\mu\alpha}\partial Y^\mu + Q_{a\alpha}\Omega_k^a\partial\theta^k - Q_{\alpha\mu}\bar{\partial}Y^\mu - \tag{5.208}$$

$$Q_{\alpha a}\Omega_k^a\bar{\partial}\theta^k - \hat{E}_{\mu\alpha}\partial Y^\mu - \hat{E}_{a\alpha}\partial x^a + \hat{E}_{\alpha\mu}\bar{\partial}Y^\mu + \hat{E}_{\alpha a}\bar{\partial}x^a = 0$$

In the first line we used the second of the Maurer-Cartan relations (5.176).

Solving equations (5.206)-(5.208) we obtain the duality relations of non-abelian T-duality (5.186) and (5.187)

$$Q_{\mu a}\partial Y^\mu + M_{ba}\Omega_m^b\partial\theta^m = -\partial x^a \tag{5.209}$$

$$Q_{a\mu}\bar{\partial}Y^\mu + M_{ab}\Omega_m^b\bar{\partial}\theta^m = \bar{\partial}x^a \tag{5.210}$$

$$M_{ba}^{-1}(Q_{\mu b}\partial Y^\mu + \partial x^b) = -\Omega_m^a\partial\theta^m \tag{5.211}$$

$$M_{ab}^{-1}(\bar{\partial}x^b - Q_{b\mu}\bar{\partial}Y^\mu) = \Omega_m^a\bar{\partial}\theta^m \tag{5.212}$$

Using expressions (5.185) and the duality relations (5.209) and (5.210) we obtain

$$T = \hat{T} \qquad \text{and} \qquad \bar{T} = \hat{\bar{T}} \tag{5.213}$$

where

$$T = G_{\mu\nu}\partial Y^\mu\partial Y^\nu + 2G_{\mu a}\Omega_k^a\partial Y^\mu\partial\theta^k + G_{ab}\Omega_m^a\Omega_k^b\partial\theta^m\partial\theta^k \tag{5.214}$$

$$\bar{T} = G_{\mu\nu}\bar{\partial}Y^\mu\bar{\partial}Y^\nu + 2G_{\mu a}\Omega_k^a\bar{\partial}Y^\mu\bar{\partial}\theta^k + G_{ab}\Omega_m^a\Omega_k^b\bar{\partial}\theta^m\bar{\partial}\theta^k \tag{5.215}$$

$$\hat{T} = \hat{G}_{\mu\nu}\partial Y^\mu\partial Y^\nu + 2\hat{G}_{\mu a}\partial Y^\mu\partial x^a + \hat{G}_{ab}\partial x^a\partial x^b \tag{5.216}$$

$$\hat{\bar{T}} = \hat{G}_{\mu\nu}\bar{\partial}Y^\mu\bar{\partial}Y^\nu + 2\hat{G}_{\mu a}\bar{\partial}Y^\mu\bar{\partial}x^a + \hat{G}_{ab}\bar{\partial}x^a\bar{\partial}x^b \tag{5.217}$$

what means that the defect is topological.

5.7.3 Non-abelian T-duality Fourier-Mukai transform of the Ramond-Ramond fields

Taking into account that the curvature of the defect generating the non-abelian T-duality is given by the formula (5.205), the Fourier-Mukai transform of the RR fields takes the form:

$$\widehat{\mathcal{G}} = \int_G \mathcal{G} \wedge e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2} x^a f^a_{bc} L^b \wedge L^c} \tag{5.218}$$

Here we apply this formula to the case of background considered in $[109, 175]$,

namely:

$$ds^2 = ds^2(M_7) + k(Y)ds^2(S^3) \tag{5.219}$$

Here M_7 is a seven-dimensional manifold, Y are coordinates on M_7, $k(Y)$ is a function of Y. One can have also B field on M_7. The second term is actually the $SU(2)$ principal chiral model, considered in section 5.7.1. Therefore, using formulae (5.202) and (5.203) the dual model takes the form:

$$\widehat{ds}^2 = ds^2_{M_7}(Y) + \frac{dr^2}{k} + \frac{kr^2}{k^2 + r^2} ds^2(S^2) \tag{5.220}$$

and

$$\hat{B} = B - \frac{r^3}{k^2 + r^2} \mathrm{Vol}(S^2) \tag{5.221}$$

Consider the following RR forms:

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(1)}_a \wedge L^a + \frac{1}{2}\mathcal{G}^{(2)}_{ab} \wedge L^a \wedge L^b + \mathcal{G}^{(3)} \wedge L^1 \wedge L^2 \wedge L^3 \tag{5.222}$$

Here $\mathcal{G}^{(0)}$, $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ are forms on M_7.

Denote the forms in the exponent of (5.218) as

$$A^{(2,0)} = \hat{B} - B \tag{5.223}$$

$$A^{(1,1)} = -dx^a \wedge L^a \tag{5.224}$$

$$A^{(0,2)} = -\frac{1}{2}x^a f^a_{bc} L^b \wedge L^c \tag{5.225}$$

In this notations we indicate by the first number the degree of the form in dx^a, and by the second in L^a. Expanding the exponent and remembering that one can have at most third degree terms in the both kinds of 1-forms we get:

$$e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2} x^a f^a_{bc} L^b \wedge L^c} = 1 + A^{(2,0)} + A^{(1,1)} + \tag{5.226}$$

$$A^{(0,2)} + \frac{1}{2}A^{(1,1)} \wedge A^{(1,1)} + A^{(2,0)} \wedge A^{(1,1)} +$$

$$A^{(1,1)} \wedge A^{(0,2)} + A^{(2,0)} \wedge A^{(0,2)} +$$

$$\frac{1}{6}A^{(1,1)} \wedge A^{(1,1)} \wedge A^{(1,1)} + A^{(2,0)} \wedge A^{(1,1)} \wedge A^{(0,2)}$$

Using the rules of the fiberwise integration we obtain that the dual of the first term comes from the all third order terms in L^a appearing in the expansion of the exponent:

$$\widehat{\mathcal{G}^{(0)}} = \mathcal{G}^{(0)} \wedge \omega^{(3)} \tag{5.227}$$

where

$$\omega^{(3)} = \int_G \frac{1}{6}A^{(1,1)} \wedge A^{(1,1)} \wedge A^{(1,1)} + A^{(2,0)} \wedge A^{(1,1)} \wedge A^{(0,2)} + A^{(0,2)} \wedge A^{(1,1)} \tag{5.228}$$

One can explicitly compute that

$$\frac{1}{6}A^{(1,1)} \wedge A^{(1,1)} \wedge A^{(1,1)} = dx^1 \wedge dx^1 \wedge dx^2 \wedge \mathrm{vol}(SU(2)) \tag{5.229}$$

where we introduced $\mathrm{vol}(SU(2)) = L^1 \wedge L^2 \wedge L^3$,

$$A^{(0,2)} \wedge A^{(1,1)} = x^a dx^a \wedge \mathrm{vol}(SU(2)) = rdr \wedge \mathrm{vol}(SU(2)) \tag{5.230}$$

$$A^{(2,0)} \wedge A^{(1,1)} \wedge A^{(0,2)} = -\frac{r^4 dr}{k^2 + r^2} \wedge \mathrm{Vol}(S^2) \wedge \mathrm{vol}(SU(2)) \tag{5.231}$$

To derive (5.231) we used the expressions (5.221) and (5.223) for $A^{(2,0)}$. Collecting all and using that $dx^1 \wedge dx^2 \wedge dx^3 = r^2 dr \wedge \mathrm{vol}(S^2)$ we obtain

$$\widehat{\mathcal{G}^{(0)}} = \mathcal{G}^{(0)} \wedge \left(\frac{r^2 k^2 dr}{k^2 + r^2} \wedge \mathrm{vol}(S^2) + rdr \right) \tag{5.232}$$

Similarly collecting all the second order terms in L^a in the expansion of the exponent one obtains the dual of the second term:

$$\widehat{\mathcal{G}_a^{(1)} \wedge L^a} = \int_G \frac{1}{2}\mathcal{G}_a^{(1)} \wedge L^a \wedge A^{(1,1)} \wedge A^{(1,1)} + \tag{5.233}$$

$$\int_G \mathcal{G}_a^{(1)} \wedge L^a \wedge A^{(0,2)} + \int_G \mathcal{G}_a^{(1)} \wedge L^a \wedge A^{(2,0)} \wedge A^{(0,2)}$$

$$= -\frac{1}{2}\epsilon_{abc}\mathcal{G}_a^{(1)} \wedge dx^b \wedge dx^c - \mathcal{G}_a^{(1)}x^a - A^{(2,0)} \wedge \mathcal{G}_a^{(1)}x^a$$

Picking up the first order terms in L^a gives us the dual of the third term:

$$\widehat{\mathcal{G}_{ab}^{(2)} \wedge L^a} \wedge L^b = \int_G \mathcal{G}_{ab}^{(2)} \wedge L^a \wedge L^b \wedge A^{(1,1)} + \int_G \mathcal{G}_{ab}^{(2)} \wedge L^a \wedge L^b \wedge A^{(1,1)} \wedge A^{(2,0)} \tag{5.234}$$

$$= -\epsilon_{abc}\mathcal{G}_{ab}^{(2)} \wedge dx^c + \epsilon_{abc}\mathcal{G}_{ab}^{(2)}x^c \wedge \frac{r^2 dr}{k^2 + r^2} \wedge \mathrm{vol}(S^2)$$

And finally the dual of the last term is given by the terms not containing L^a at all:

$$\int_G \mathcal{G}^{(3)} \wedge L^1 \wedge L^2 \wedge L^3 \wedge e^{\mathcal{F}} = \mathcal{G}^{(3)} + \mathcal{G}^{(3)} \wedge (\hat{B} - B) \qquad (5.235)$$

Rearranging the terms in order of dx^a we can write for the non-abelian T-dual of $\mathcal{G}$:

$$\widehat{\mathcal{G}} = \widehat{\mathcal{G}^{(0)}} + \widehat{\mathcal{G}^{(1)}} + \widehat{\mathcal{G}^{(2)}} + \widehat{\mathcal{G}^{(3)}} \qquad (5.236)$$

where

$$\widehat{\mathcal{G}^{(0)}} = -\mathcal{G}_a^{(1)} x^a + \mathcal{G}^{(3)} \qquad (5.237)$$

$$\widehat{\mathcal{G}^{(1)}} = \mathcal{G}^{(0)} \wedge r\,dr - \frac{1}{2}\epsilon_{abc}\mathcal{G}_{ab}^{(2)} \wedge dx^c \qquad (5.238)$$

$$\widehat{\mathcal{G}^{(2)}} = -\frac{1}{2}\epsilon_{abc}\mathcal{G}_a^{(1)} \wedge dx^b \wedge dx^c - (\hat{B} - B) \wedge \mathcal{G}_a^{(1)} x^a + \mathcal{G}^{(3)} \wedge (\hat{B} - B) \qquad (5.239)$$

$$\widehat{\mathcal{G}^{(3)}} = \mathcal{G}^{(0)} \wedge \frac{r^2 k^2 dr}{k^2 + r^2} \wedge \text{Vol}(S^2) + \frac{1}{2}\epsilon_{abc}\mathcal{G}_{ab}^{(2)} x^c \wedge \frac{r^2 dr}{k^2 + r^2} \wedge \text{vol}(S^2) \qquad (5.240)$$

As we have explained before, since the gauge invariant flux on the defect, which appears in the exponent of (5.218), satisfies the relation (5.52) , and the exterior differentiation commutes with the fiberwise integration, the dual fields satisfy the relation:

$$(d - \hat{H}) \wedge \widehat{\mathcal{G}} = \int_G e^{\hat{B} - B - dx^a \wedge L^a - \frac{1}{2}x^a f_{bc}^a L^b \wedge L^c} \wedge (d - H) \wedge \mathcal{G} \qquad (5.241)$$

The relation (5.241) guarantees that the hatted forms satisfy the supergravity Bianchi identity given that so do the original forms $\mathcal{G}$. In the mentioned papers, the non-abelian T-duality transformation of the RR fields was performed for backgrounds (5.219), using the approach based on equation (5.49), with the RR fields having the form:

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(3)} \wedge L^1 \wedge L^2 \wedge L^3 \qquad (5.242)$$

The results obtained in these works are in agreement with the formulae (5.237)-(5.240) for this case.

Chapter 6

Defects in the Liouville and Toda field theories

6.1 Liouville field theory

Let us review basic facts on the Liouville field theory (see e.g. [178]). Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab}\partial_a\varphi\partial_b\varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi}R\varphi\,, \tag{6.1}$$

where R is associated curvature. This theory is conformal invariant if the coupling constant b is related with the background charge Q as

$$Q = b + \frac{1}{b}\,. \tag{6.2}$$

The symmetry algebra of this conformal field theory is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m} \tag{6.3}$$

with the central charge

$$c_L = 1 + 6Q^2\,. \tag{6.4}$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q - \alpha)\,. \tag{6.5}$$

The spectrum of the Liouville theory is believed [33, 34, 40] to be of the following form

$$\mathcal{H} = \int_0^\infty dP \; R_{\frac{Q}{2}+iP} \otimes R_{\frac{Q}{2}+iP} \,, \tag{6.6}$$

where R_α is the highest weight representation with respect to Virasoro algebra. Characters of the representations $R_{\frac{Q}{2}+iP}$ are

$$\chi_P(\tau) = \frac{q^{P^2}}{\eta(\tau)} \,, \tag{6.7}$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^\infty (1 - q^n) \,. \tag{6.8}$$

Modular transformation of (6.7) is well-known:

$$\chi_P(-\frac{1}{\tau}) = \sqrt{2} \int \chi_{P'}(\tau) e^{4i\pi PP'} dP' \,. \tag{6.9}$$

Degenerate representations appear at

$$\alpha_{m,n} = \frac{1-m}{2b} + \frac{1-n}{2}b \tag{6.10}$$

and have conformal dimensions

$$\Delta_{m,n} = Q^2/4 - (m/b + nb)^2/4 \,, \tag{6.11}$$

where m, n are positive integers. At general b there is only one null-vector at the level mn. Hence the degenerate character reads:

$$\chi_{m,n}(\tau) = \frac{q^{-(m/b+nb)^2} - q^{-(m/b-nb)^2}}{\eta(\tau)} \,. \tag{6.12}$$

Modular transformation of (6.12) is worked out in [192]

$$\chi_{m,n}(-\frac{1}{\tau}) = 2\sqrt{2} \int \chi_P(\tau) \sinh(2\pi mP/b) \sinh(2\pi nbP) dP \,. \tag{6.13}$$

Given that the identity field is specified by $(m, n) = (1, 1)$ one finds the vacuum component of the matrix of modular transformation:

$$S_{0\alpha} = -2\sqrt{2} \sin \pi b^{-1}(2\alpha - Q) \sin \pi b(2\alpha - Q) \,. \tag{6.14}$$

To present formula (1.115) in the Liouville field theory we need two-point function

$$\langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle = \frac{S(\alpha)}{(z_1 - z_2)^{2\Delta_\alpha}(\bar{z}_1 - \bar{z}_2)^{2\Delta_\alpha}}. \tag{6.15}$$

Let us for this purpose recall some facts on the values of the correlation functions in the Liouville field theory in the Coulomb gas approach.

1. The three-point functions satisfying the relation $\alpha_1 + \alpha_2 + \alpha_3 = Q$ are set to 1. This rule actually sets normalization of the fields, since from here we receive that

$$\langle V_\alpha(z_1, \bar{z}_1) V_{Q-\alpha}(z_2, \bar{z}_2) \rangle = \frac{1}{(z_1 - z_2)^{2\Delta_\alpha}(\bar{z}_1 - \bar{z}_2)^{2\Delta_\alpha}}. \tag{6.16}$$

The fields V_α and $V_{Q-\alpha}$ have the same conformal dimensions and represent the same primary field, i.e. they are proportional to each other, and it follows from (6.15) and (6.16) that

$$V_\alpha = S(\alpha) V_{Q-\alpha} \tag{6.17}$$

2. The three-point functions $\mathcal{C}(\alpha_1, \alpha_2, \alpha_3)$ for the values of α_i satisfying the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = Q - nb, \tag{6.18}$$

are given by the Coulomb gas or screening integrals computed in [47]

$$I_n(\alpha_1, \alpha_2, \alpha_3) = \left(b^4 \gamma(b^2) \pi \mu\right)^n \frac{\prod_{j=1}^n \gamma(-jb^2)}{\prod_{k=0}^{n-1}[\gamma(2\alpha_1 b + kb^2)\gamma(2\alpha_2 b + kb^2)\gamma(2\alpha_3 b + kb^2)]}, \tag{6.19}$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

The structure constants derived as the Coulomb gas integrals are denoted by $\mathcal{C}$ to distinguish from their values derived from the DOZZ formula.

The structure constant are related to the three-point functions by the relation:

$$C_{\alpha_1, \alpha_2}^{\alpha_3} = C(\alpha_1, \alpha_2, Q - \alpha_3). \tag{6.20}$$

Thus we derive:

$$\mathcal{C}_{-b/2, \alpha}^{\alpha - b/2} = 1, \tag{6.21}$$

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and

$$\mathcal{C}_{-b/2,\alpha}^{\alpha+b/2} = \frac{\pi\mu b^4 \gamma(b^2)}{\gamma(2\alpha b)\gamma(b^2 - 2\alpha b + 2)}\,. \tag{6.22}$$

Now one can obtain the two-point function $S(\alpha)$ by the following trick [60]. Consider the auxiliary three-point function

$$\langle V_\alpha(x_1)V_{\alpha+b/2}(x_2)V_{-b/2}(z)\rangle\,. \tag{6.23}$$

Using the OPE

$$V_{-b/2}V_\alpha = \mathcal{C}_{-b/2,\alpha}^{\alpha-b/2}\left[V_{\alpha-b/2}\right] + \mathcal{C}_{-b/2,\alpha}^{\alpha+b/2}\left[V_{\alpha+b/2}\right]\,, \tag{6.24}$$

one receives that in the limit $z \to x_1$ the three-point function (6.23) takes the form:

$$\mathcal{C}_{-b/2,\alpha}^{\alpha+b/2}S(\alpha + b/2)\,, \tag{6.25}$$

whereas in the limit $z \to x_2$, it is

$$\mathcal{C}_{-b/2,\alpha}^{\alpha-b/2}S(\alpha)\,. \tag{6.26}$$

Equating (6.25) and (6.26) we get that the two-point function $S(\alpha)$ satisfies the condition:

$$\frac{S(\alpha)}{S(\alpha + b/2)} = \mathcal{C}_{-b/2,\alpha}^{\alpha+b/2}\,. \tag{6.27}$$

Solving (6.27) one derives:

$$S(\alpha) = \frac{(\pi\mu\gamma(b^2))^{b^{-1}(Q-2\alpha)}}{b^2}\frac{\Gamma(1 - b(Q - 2\alpha))\Gamma(-b^{-1}(Q - 2\alpha))}{\Gamma(b(Q - 2\alpha))\Gamma(1 + b^{-1}(Q - 2\alpha))}\,. \tag{6.28}$$

We have all the necessary ingredients to compute classifying algebra: two-point function $S(\alpha)$ and vacuum component of the matrix of the modular transformation. Before to continue let us recall that both of them can be conveniently written using ZZ function [192]:

$$W(\alpha) = -\frac{2^{3/4}(\pi\mu\gamma(b^2))^{-\frac{(Q-2\alpha)}{2b}}\pi(Q - 2\alpha)}{\Gamma(1 - b(Q - 2\alpha))\Gamma(1 - b^{-1}(Q - 2\alpha))}\,. \tag{6.29}$$

It can be easily shown that

$$\frac{W(Q - \alpha)}{W(\alpha)} = S(\alpha)\,, \tag{6.30}$$

and

$$W(Q - \alpha)W(\alpha) = S_{0\alpha}\,. \tag{6.31}$$

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Recalling (1.117), F_α takes the form:

$$F_\alpha = \frac{S_{00}}{W(Q-\alpha)W(\alpha)}. \tag{6.32}$$

Combining (6.30) and (6.32) we obtain coefficients ξ_α for the Liouville field theory:

$$\xi_\alpha^L = \sqrt{S(\alpha)F(\alpha)} = \frac{\sqrt{S_{00}}}{W(\alpha)}. \tag{6.33}$$

Eq. (1.115) implies:

$$C_{\alpha_1,\alpha_2}^{\alpha_3} F_{\alpha_3,0} \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \end{bmatrix} = W(0)\frac{W(\alpha_3)}{W(\alpha_1)W(\alpha_2)}. \tag{6.34}$$

Let us compare (6.34) with the calculations in literature. First of all recall the calculations in [60] for one of the momenta taking the degenerate value $\alpha_1 = -\frac{b}{2}$. The fusing matrix can be computed using that conformal blocks with the degenerate primary $-\frac{b}{2}$ satisfy the second order differential equation, which can be solved by the hypergeometric functions. The fusion matrix is given by the transformation properties of the hypergeometric functions. The fusion matrix derived in this way we denote by F^* to distinguish from the values of the fusion matrix derived from the Ponsot-Teschner formula. The corresponding values of F^* are [60, 180]:

$$F_{\alpha-b/2,0}^* \begin{bmatrix} -b/2 & -b/2 \\ \alpha & \alpha \end{bmatrix} = \frac{\Gamma(2\alpha b - b^2)\Gamma(-1-2b^2)}{\Gamma(2\alpha b - 2b^2 - 1)\Gamma(-b^2)}, \tag{6.35}$$

$$F_{\alpha+b/2,0}^* \begin{bmatrix} -b/2 & -b/2 \\ \alpha & \alpha \end{bmatrix} = \frac{\Gamma(2+b^2-2\alpha b)\Gamma(-1-2b^2)}{\Gamma(1-2\alpha b)\Gamma(-b^2)}. \tag{6.36}$$

Using ZZ function $W(\alpha)$ (6.29) one can compactly rewrite (6.35), (6.36) as:

$$F_{\alpha-b/2,0}^* \begin{bmatrix} -b/2 & -b/2 \\ \alpha & \alpha \end{bmatrix} = \frac{W(0)}{W(-\frac{b}{2})}\frac{W(\alpha-b/2)}{W(\alpha)}, \tag{6.37}$$

$$F_{\alpha+b/2,0}^* \begin{bmatrix} -b/2 & -b/2 \\ \alpha & \alpha \end{bmatrix} = \frac{W(0)}{W(-\frac{b}{2})}\frac{W(Q-\alpha-b/2)}{W(Q-\alpha)}. \tag{6.38}$$

Combining (6.21), (6.27), (6.30), (6.37), (6.38) we obtain

$$
C^{\alpha-b/2}_{-b/2,\alpha} F^{*}_{\alpha-b/2,0}
\begin{bmatrix}
-b/2 & -b/2 \\
\alpha & \alpha
\end{bmatrix}
= \frac{W(0)}{W(-\frac{b}{2})} \frac{W(\alpha - b/2)}{W(\alpha)} ,
\tag{6.39}
$$

$$
C^{\alpha+b/2}_{-b/2,\alpha} F^{*}_{\alpha+b/2,0}
\begin{bmatrix}
-b/2 & -b/2 \\
\alpha & \alpha
\end{bmatrix}
= \frac{W(0)}{W(-\frac{b}{2})} \frac{W(\alpha + b/2)}{W(\alpha)} ,
\tag{6.40}
$$

in agreement with (6.34).

Next we compute the left hand side of (6.34) using DOZZ formula for structure constants [46, 191] and the explicit expression for the fusing matrix found in [145]. It is instructive at the beginning to repeat the steps leading from (1.111) to (1.115) for the Liouville theory using the DOZZ formula. Using the relation between three-point functions and OPE structure constant (6.20) the associativity condition of the OPE in the Liouville field theory takes the form:

$$
C(\alpha_4, \alpha_3, \alpha_s) C(Q - \alpha_s, \alpha_2, \alpha_1) F_{\alpha_s,\alpha_t}
\begin{bmatrix}
\alpha_3 & \alpha_2 \\
\alpha_4 & \alpha_1
\end{bmatrix}
=
\tag{6.41}
$$

$$
= C(\alpha_4, \alpha_t, \alpha_1) C(Q - \alpha_t, \alpha_3, \alpha_2) F_{\alpha_t,\alpha_s}
\begin{bmatrix}
\alpha_1 & \alpha_2 \\
\alpha_4 & \alpha_3
\end{bmatrix}.
$$

Consider the limit $\alpha_t \to 0$ in (6.41).

From the DOZZ formula:

$$
C(\alpha_1, \alpha_2, \alpha_3) = \lambda^{(Q - \sum_{i=1}^{3} \alpha_i)/b} \times
\tag{6.42}
$$

$$
\frac{\Upsilon_b(b) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)},
$$

where

$$
\lambda = \pi \mu \gamma(b^2) b^{2 - 2b^2}
\tag{6.43}
$$

one can obtain [178]

$$
C(\alpha_2, \epsilon, \alpha_1) \simeq \frac{2\epsilon S(\alpha_1)}{(\alpha_2 - \alpha_1 + \epsilon)(\alpha_1 - \alpha_2 + \epsilon)} + \frac{2\epsilon}{(Q - \alpha_2 - \alpha_1 + \epsilon)(\alpha_1 + \alpha_2 - Q + \epsilon)}.
\tag{6.44}
$$

The functions $\Upsilon_b(\alpha)$ and their properties leading to (6.44) are described in appendix .1.

Using the reflection property

$$C(\alpha_3, \alpha_2, \alpha_1) = S(\alpha_3) C(Q - \alpha_3, \alpha_2, \alpha_1), \tag{6.45}$$

one receives in this limit, setting also $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_3$

$$C^2(\alpha_2, \alpha_1, \alpha_s) = \frac{4 S(\alpha_1) S(\alpha_2) S(\alpha_s)}{S(0)} \frac{F_{0,\alpha_s} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}}{\lim_{\epsilon \to 0} \epsilon^2 F_{\alpha_s,\epsilon} \begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}}. \tag{6.46}$$

It was shown in [179] that the limit

$$F''_{\alpha,0} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \equiv \lim_{\beta \to 0} \beta^2 F_{\alpha,\beta} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \tag{6.47}$$

exists and satisfies the equation:

$$F''_{\alpha,0} \begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix} F_{0,\alpha} \begin{bmatrix} \alpha_2 & \alpha_1 \\ \alpha_2 & \alpha_1 \end{bmatrix} = \frac{F_{\alpha_2} F_{\alpha_1}}{F_\alpha}. \tag{6.48}$$

Putting (6.48) in (6.46) one finally gets:

$$C(\alpha_1, \alpha_2, \alpha_s) F''_{\alpha_s,0} \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \end{bmatrix} = 2W(0) \frac{W(Q - \alpha_s)}{W(\alpha_1) W(\alpha_2)}. \tag{6.49}$$

and

$$C(\alpha_1, \alpha_2, \alpha_s) = 2W(Q - \alpha_1) \frac{W(Q - \alpha_2)}{W(Q) W(\alpha_s)} F_{0,\alpha_s} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}. \tag{6.50}$$

Here a sign factor could appear, but below we show that actually (6.49) and (6.50) hold without it. Recalling the relation (6.20) and (6.45) we obtain (6.34). The emergence of the factor 2 will be explained below. This derivation also explains that the double pole in the fusing matrix $F_{\alpha,0} \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \end{bmatrix}$ is related to the simple pole in the DOZZ formula.

One can compute the limit (6.47) also directly.

Recall that the boundary three-point function is given by [146]

$$C_{Q-\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = C_{\beta_3|\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1} = \frac{g_{\beta_3}^{\sigma_3\sigma_1}}{g_{\beta_2}^{\sigma_3\sigma_2}g_{\beta_1}^{\sigma_2\sigma_1}}F_{\sigma_2\beta_3}\begin{bmatrix}\beta_2 & \beta_1 \\ \sigma_3 & \sigma_1\end{bmatrix},\tag{6.51}$$

where

$$g_\beta^{\sigma_3\sigma_1} = \lambda^{\beta/2b}\frac{\Gamma_b(Q)\Gamma_b(Q-2\beta)\Gamma_b(2\sigma_1)\Gamma_b(2Q-2\sigma_3)}{\Gamma_b(2Q-\beta-\sigma_1-\sigma_3)\Gamma_b(\sigma_1+\sigma_3-\beta)\Gamma_b(Q-\beta+\sigma_1-\sigma_3)\Gamma_b(Q-\beta+\sigma_3-\sigma_1)}.\tag{6.52}$$

The function $\Gamma_b(x)$ is described in appendix .1.

Therefore the fusing matrix can be expressed as

$$F_{\sigma_2\beta_3}\begin{bmatrix}\beta_2 & \beta_1 \\ \sigma_3 & \sigma_1\end{bmatrix} = \frac{g_{\beta_2}^{\sigma_3\sigma_2}g_{\beta_1}^{\sigma_2\sigma_1}}{g_{\beta_3}^{\sigma_3\sigma_1}}C_{Q-\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}.\tag{6.53}$$

On the other side $C_{Q-\beta_3\beta_2\beta_1}^{\sigma_3\sigma_2\sigma_1}$ has a pole with residue 1 if $\beta_1 + \beta_2 - \beta_3 = 0$. Therefore using the invariance of the fusing matrix w.r.t. to the inversions $\alpha_i \to Q - \alpha_i$ one can write for the corresponding residue of the fusion matrix

$$F'_{\sigma_2,0}\begin{bmatrix}\beta_1 & \beta_1 \\ \sigma_1 & \sigma_1\end{bmatrix} = F'_{\sigma_2,Q}\begin{bmatrix}Q-\beta_1 & \beta_1 \\ \sigma_1 & \sigma_1\end{bmatrix} = \frac{g_{Q-\beta_1}^{\sigma_1\sigma_2}g_{\beta_1}^{\sigma_2\sigma_1}}{g_Q^{\sigma_1\sigma_1}}.\tag{6.54}$$

Using the explicit expressions (6.52) for $g_{\beta_1}^{\sigma_2\sigma_1}$, the DOZZ formula (6.42) for structure constants and the properties of the functions $\Gamma_b(x)$, $\Upsilon_b(x)$ reviewed in appendix .1, it is easy to compute that

$$g_{Q-\beta_1}^{\sigma_1\sigma_2}g_{\beta_1}^{\sigma_2\sigma_1} = 2^{1/4}\frac{2\pi W(Q-\sigma_1)W(Q-\sigma_2)}{W(\beta_1)}\frac{1}{C(\sigma_1,\sigma_2,\beta_1)}.\tag{6.55}$$

Using the properties of the functions $\Gamma_b(x)$, reviewed in appendix .1, one can compute the limit

$$\lim_{\beta_3\to Q}\frac{1}{g_{\beta_3}^{\sigma_1\sigma_1}},\tag{6.56}$$

and obtain that it has simple pole with the residue

$$\frac{2^{-1/4}W(0)}{\pi W(\sigma_1)W(Q-\sigma_1)}.\tag{6.57}$$

Combining (6.55) and (6.57) we again derive (6.49).

Some comments are in order at this point:

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1. This derivation shows that the fusing matrix element $F_{\sigma_2,0} \begin{bmatrix} \beta_1 & \beta_1 \\ \sigma_1 & \sigma_1 \end{bmatrix}$ indeed has double pole: one degree comes from the pole of the three-point function $C_{0,Q-\beta_1,\beta_1}^{\sigma_3\sigma_2\sigma_1}$ and the second from the pole of the $\frac{1}{g_Q^{\sigma_1\sigma_1}}$.

2. We have shown that (6.34) or (6.49) indeed always holds with the understanding that in the case of the singular behavior one should take the coefficients of the leading singularities.

3. Note that (6.49) evidently satisfies the reflection property (6.45) since the fusing matrix is invariant under the inversions $\alpha \to Q - \alpha$.

4. Let us explain the emergence of the factor 2 in (6.49). We have seen that the formula (6.49), derived by using the DOZZ formula for structure constant and Ponsot-Teschner (PT) formula for the fusing matrix has additional factor 2 compared to formulas (6.34), (6.39), (6.40) using the values of the structure constant derived as the Coulomb gas integrals and fusing matrix computed via the differential equations for the conformal blocks. The derivation of the formula (6.49) via the limiting procedure (6.46)-(6.49) indicates that the factor 2 originates from the coefficient 2 in formula (6.44). Point is that as the formula (6.44) shows, the two-point functions, derived from the DOZZ formula as residue of the pole in the limit $\alpha_3 \to 0$, are twice the two-point functions (6.15) and (6.16), derived in the Coulomb gas approach. Thus the states in the theory reconstructed from the DOZZ formula have twice the normalization of the fields used in the calculations leading to (6.34), (6.39), (6.40). This is the reason for emergence of the factor 2 in (6.49).

5. One can ask, what happens if one tries to compute the left hand side of formulae (6.39), (6.40) from the DOZZ and PT formulae. First of all let us recall that, as noted in [191], when the momenta α_i satisfy the relation (6.18), the DOZZ formula has a pole with the residue equals to the Coulomb gas integrals (6.19):

$$\text{res}_{\alpha_1+\alpha_2+\alpha_3=Q} C(\alpha_1, \alpha_2.\alpha_3) = 1, \tag{6.58}$$

and

$$\text{res}_{\alpha_1+\alpha_2+\alpha_3=Q-nb}C(\alpha_1,\alpha_2.\alpha_3) = I_n(\alpha_1,\alpha_2,\alpha_3). \tag{6.59}$$

But strictly speaking this is true only for the non-degenerate values of the momenta. For the degenerate values (6.10), as we see from the DOZZ formula, it may happen that additionally to the first vanishing term in the denominator, we have two more vanishing terms, one in the denominator and another one in the numerator. This makes the limiting procedure ambiguous and can bring to the values of the residue twice as the Coulomb gas results.

Consider the values of the momenta appearing in formulae (6.39), (6.40): $\alpha_1 = \alpha$, $\alpha_2 = -\frac{b}{2}$, $\alpha_3 = Q - \alpha \pm \frac{b}{2}$. For these α_i the DOZZ formula develops pole, and the matrix F'' defined in (6.47), vanishes. If now we set $\alpha_2 = -\frac{b}{2} + \delta$ and consider the limit $\delta \to 0$, we obtain

$$\lim_{\delta\to0} \delta \, C\left(\alpha, -\frac{b}{2}+\delta, Q-\alpha\mp\frac{b}{2}\right) = 2\mathcal{C}^{\alpha\pm b/2}_{-b/2,\alpha}, \tag{6.60}$$

$$\lim_{\delta\to0}\frac{1}{\delta}F''_{\alpha\pm b/2,0}\begin{bmatrix} -b/2+\delta & -b/2+\delta \\ \alpha & \alpha \end{bmatrix} = F^*_{\alpha\pm b/2,0}\begin{bmatrix} -b/2 & -b/2 \\ \alpha & \alpha \end{bmatrix}. \tag{6.61}$$

On the other hand it was suggested in [105] a limiting procedure reproducing the Coulomb gas values:

$$\lim_{\delta\to0}\left[\lim_{\epsilon\to0} \epsilon \, C\left(\alpha, -\frac{b}{2}+\delta, Q-\alpha\mp\frac{b}{2}-\delta+\epsilon\right)\right] = \mathcal{C}^{\alpha\pm b/2}_{-b/2,\alpha}. \tag{6.62}$$

But this procedure brings to the factor 2 in the fusion matrix:

$$\lim_{\delta\to0}\left[\lim_{\epsilon\to0}\frac{1}{\epsilon}F''_{\alpha\pm b/2+\delta-\epsilon,0}\begin{bmatrix} -b/2+\delta & -b/2+\delta \\ \alpha & \alpha \end{bmatrix}\right] = 2F^*_{\alpha\pm b/2,0}\begin{bmatrix} -b/2 & -b/2 \\ \alpha & \alpha \end{bmatrix}. \tag{6.63}$$

In any case we are in agreement (6.49).

Having demonstrated that (6.34) holds in the Liouville field theory we can use the the boundary bootstrap technique developed in sections 1.2.3, 1.2.5, 1.3 to derive D-branes, defects and permutation branes in the Liouville field theory in the simple and elegant way. To use equation (1.156) in non-rational theory we should take care to have a finite number of terms in the left hand side. This can be achieved taking $j = -\frac{b}{2}$.

Setting $j = -\frac{b}{2}$, $i = \alpha$, and $k = \alpha \pm b/2$, the equation (1.156) takes the form:

$$\Psi(\alpha)\Psi(-b/2) = \Psi(\alpha - b/2) + \Psi(\alpha + b/2). \tag{6.64}$$

The solution of the equation (6.64) is

$$\Psi_{m,n}(\alpha) = \frac{\sin(\pi m b^{-1}(2\alpha - Q))\sin(\pi n b(2\alpha - Q))}{\sin(\pi m b^{-1} Q)\sin(\pi n b Q)} = \frac{S_{m,n\,\alpha}}{S_{m,n\,0}}, \tag{6.65}$$

Using equations (1.155), (1.171) and (1.188), and recalling (6.33) we obtain one-point functions for ordinary branes

$$\tilde{U}_{m,n}(\alpha) = \Psi_{m,n}(\alpha)\frac{W(0)}{W(\alpha)}, \tag{6.66}$$

permutation branes on N-fold product

$$U^N_{\mathcal{P}m,n}(\alpha) = \Psi_{m,n}(\alpha)\left(\frac{W(0)}{W(\alpha)}\right)^N. \tag{6.67}$$

and defect two-point functions

$$\frac{D_{m,n}(\alpha)}{D_{m,n}(0)} = \Psi_{m,n}(\alpha)\left(\frac{W(0)}{W(\alpha)}\right)^2. \tag{6.68}$$

Using (1.148) one derives boundary state coefficients for ordinary branes:

$$B_{m,n}(\alpha) = \frac{S_{m,n\,\alpha}}{W(\alpha)}, \tag{6.69}$$

and permutation branes on N-fold product

$$B^N_{\mathcal{P}m,n}(\alpha) = \frac{S_{m,n\,\alpha}}{W^N(\alpha)}, \tag{6.70}$$

and for defects eq.(1.185) and (6.68) imply

$$D_{m,n}(\alpha) = \frac{S_{m,n\,\alpha}}{W^2(\alpha)} \tag{6.71}$$

$$\mathcal{D}_{m,n}(\alpha) = \frac{S_{m,n\,\alpha}}{S_{0\alpha}} \,. \tag{6.72}$$

To find continuous family of the brane and defects we will use the strategy developed in [60]. Consider at the beginning ordinary branes. The idea is that the boundary one-point function $\tilde{U}\left(-\frac{b}{2}\right)$ of the degenerate field $V_{-b/2}$ can be considered as a function $A(\mu_B)$ of the boundary cosmological constant μ_B. Setting in (1.154) $\tilde{U}\left(-\frac{b}{2}\right) = A$ and $\tilde{U}(\alpha) = \frac{\Lambda(\alpha)}{W(\alpha)}$ we obtain

$$\frac{W(-b/2)}{W(0)} A\Lambda(\alpha) = \Lambda(\alpha - b/2) + \Lambda(\alpha + b/2) \tag{6.73}$$

The solution of the eq.(6.73) is

$$\Lambda_s(\alpha) = -2^{1/2} \cosh(2\pi s(2\alpha - Q)) \,, \tag{6.74}$$

with

$$2\cosh 2\pi b s = A \frac{W(-b/2)}{W(0)} \tag{6.75}$$

Note that function (6.74) coincide with the matrix of the modular transformation (6.9). This leads to the FZZ (Fateev-Zamolodchikov-Zamolodchikov) boundary states:

$$B_s(\alpha) = \frac{\Lambda_s(\alpha)}{W(\alpha)} \,, \tag{6.76}$$

Similarly for permutation branes on N-fold product treating boundary one-point function $U_{\mathcal{P}}^N\left(-\frac{b}{2}\right)$ of the degenerate field $V_{-b/2}$ as a function $A_{\mathcal{P}}$ of the permutation branes boundary cosmological constant $\lambda_{\mathcal{P}}$ and setting $U_{\mathcal{P}}^N(\alpha) = \frac{\Lambda(\alpha)}{W(\alpha)^N}$ in (1.170) we obtain

$$\left(\frac{W(-b/2)}{W(0)}\right)^N A_{\mathcal{P}}\Lambda(\alpha) = \Lambda(\alpha - b/2) + \Lambda(\alpha + b/2) \tag{6.77}$$

Eq. (6.77) has the same solution (6.74) but with s related to $A_{\mathcal{P}}$ via

$$2\cosh 2\pi b s = A_{\mathcal{P}} \left(\frac{W(-b/2)}{W(0)}\right)^N \,. \tag{6.78}$$

Therefore the continuous family of the permutations brane has the form

$$B_{\mathcal{P}s}^N(\alpha) = \frac{\Lambda_s(\alpha)}{W^N(\alpha)} \,, \tag{6.79}$$

Finally considering the defect two-point function of the degenerate field $V_{-b/2}$ as a function of the defect cosmological constant κ and setting $D(-b/2)/D(0) = A_d$ and $D(\alpha) = \frac{\Lambda(\alpha)}{W^2(\alpha)}$ in (1.187) we obtain:

$$\left(\frac{W(-b/2)}{W(0)}\right)^2 A_d\Lambda(\alpha) = \Lambda(\alpha - b/2) + \Lambda(\alpha + b/2) \tag{6.80}$$

The eq.(6.80) is again solved by function (6.74) but with s and A_d related by

$$2\cosh 2\pi bs = A_d \left(\frac{W(-b/2)}{W(0)}\right)^2. \tag{6.81}$$

Therefore the continuous family of defects has two-point functions

$$D_s(\alpha) = -\frac{2^{1/2}\cosh(2\pi s(2\alpha - Q))}{W^2(\alpha)}. \tag{6.82}$$

and eigenvalues

$$\mathcal{D}_s(\alpha) = \frac{\cosh(2\pi s(2\alpha - Q))}{2\sin \pi b^{-1}(2\alpha - Q)\sin \pi b(2\alpha - Q)}. \tag{6.83}$$

6.2   Toda field theory

Recall some facts on Toda field theory [59]. The action of the $sl(n)$ conformal Toda field theory on a two-dimensional surface with metric g_{ab} and associated to it scalar curvature R has the form

$$\mathcal{A} = \int \left(\frac{1}{8\pi}g_{ab}(\partial_a\varphi\partial_b\varphi) + \mu\sum_{k=1}^{n-1} e^{b(e_k\varphi)} + \frac{(Q,\varphi)}{4\pi}R\right)\sqrt{g}d^2x. \tag{6.84}$$

Here φ is the two-dimensional $(n-1)$ component scalar field $\varphi = (\varphi_1\ldots\varphi_{n-1})$:

$$\varphi = \sum_i^{n-1} \varphi_i e_i, \tag{6.85}$$

where vectors e_k are the simple roots of the Lie algebra $sl(n)$, b is the dimensionless coupling constant, μ is the scale parameter called the cosmological constant and (e_k, φ) denotes the scalar product.

If the background charge Q is related with the parameter b as

$$Q = \left(b + \frac{1}{b}\right)\rho, \tag{6.86}$$

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where ρ is the Weyl vector , then the theory is conformally invariant. The Weyl vector is

$$\rho = \frac{1}{2}\sum_{e>0} e = \sum_{i}^{n-1}\omega_i\,, \tag{6.87}$$

where ω_i are fundamental weights, such that $(\omega_i, e_j) = \delta_{ij}$.

Conformal Toda field theory possesses higher-spin symmetry: there are $n-1$ holomorphic currents $W^k(z)$ with the spins $k = 2, 3, \ldots n$. The currents $W^k(z)$ form closed W_n algebra, which contains as subalgebra the Virasoro algebra with the central charge

$$c = n - 1 + 12Q^2 = (n-1)(1 + n(n+1)(b+b^{-1})^2)\,. \tag{6.88}$$

Primary fields of conformal Toda field theory are the exponential field parameterized by a $(n-1)$ component vector parameter α, $\alpha = \sum_{i}^{n-1}\alpha_i\omega_i$,

$$V_\alpha = e^{(\alpha,\varphi)}\,. \tag{6.89}$$

They have the simple OPE with the currents $W^k(z)$. Namely,

$$W^k(\xi)V_\alpha(z,\bar{z}) = \frac{w^{(k)}(\alpha)V_\alpha(z,\bar{z})}{(\xi - z)^k}\,. \tag{6.90}$$

The quantum numbers $w^{(k)}(\alpha)$ possess the symmetry under the action of the Weyl group $\mathcal{W}$ of the algebra $sl(n)$:

$$w^{(k)}(\alpha) = w^{(k)}(Q + \hat{s}(\alpha - Q)), \quad \hat{s} \in \mathcal{W}\,. \tag{6.91}$$

In particular

$$w^{(2)}(\alpha) = \Delta(\alpha) = \frac{(\alpha, 2Q - \alpha)}{2} \tag{6.92}$$

is the conformal dimension of the field V_α. Eq. (6.91) implies that the fields related via the action of the Weyl group should coincide up to a multiplicative factor. Indeed we have [61]

$$R_{\hat{s}}(\alpha)V_{Q+\hat{s}(\alpha-Q)} = V_\alpha\,, \tag{6.93}$$

where $R_{\hat{s}}(\alpha)$ is the reflection amplitude

$$R_{\hat{s}}(\alpha) = \frac{A(Q + \hat{s}(\alpha - Q))}{A(\alpha)}\,, \tag{6.94}$$

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$$A(\alpha) = (\pi\mu\gamma(b^2))^{\frac{(\alpha-Q,\rho)}{b}} \frac{2\pi b\sqrt{\Xi}}{\prod_{e>0}\Gamma(1-b(\alpha-Q,e))\Gamma(-b^{-1}(\alpha-Q,e))}\,, \tag{6.95}$$

where

$$\Xi = i^{n-1}\sqrt{\det C}\frac{1}{|\mathcal{W}|}\,, \tag{6.96}$$

and C is the Cartan matrix. Two-point functions in Toda field theory are

$$\langle V_\alpha(z_1,\bar{z}_1)V_{\alpha^*}(z_2,\bar{z}_2)\rangle = \frac{R(\alpha)}{(z_1-z_2)^{4\Delta_\alpha}(\bar{z}_1-\bar{z}_2)^{4\Delta_\alpha}}\,, \tag{6.97}$$

where $R(\alpha)$ is the maximal reflection amplitude defined as

$$R(\alpha) = \frac{A(2Q-\alpha)}{A(\alpha)}\,, \tag{6.98}$$

and α^* is defined by

$$(\alpha, e_k) = (\alpha^*, e_{n-k})\,. \tag{6.99}$$

The representations which appear in the spectrum of $sl(n)$ Toda field theory have momenta

$$\alpha \in Q + i\sum_{i}^{n-1} p_i\omega_i\,, \tag{6.100}$$

where p_i are real.

To describe degenerate representations it is useful to write α as

$$\alpha = Q + \nu\,. \tag{6.101}$$

Degenerate representations appear at the momentum ν satisfying the condition

$$-(\nu, e) = rb + \frac{s}{b}\,, \tag{6.102}$$

where e is a root and $r, s \in \mathbb{Z}_+$. Without loss of generality we can classify semi-degenerate representations by a collection of simple roots $\mathcal{I}$ for which the equation is satisfied:

$$-(\nu, e_i) = rb + \frac{s}{b} \qquad i \in \mathcal{I}\,. \tag{6.103}$$

Fully degenerate representations appear when $\mathcal{I}$ consists of all the simple roots. It is easy to show that for fully degenerate representations α takes the form:

$$\alpha_{R_1|R_2} = -b\lambda_1 - \frac{1}{b}\lambda_2\,, \tag{6.104}$$

where λ_1 and λ_2 are the highest weights correspondning to irreducible representations R_1 and R_2 of $sl(n)$.

The identity representation, as in the Liouville case before, belongs to the set of the fully degenerate representations.

To characterize generic semi-degenerate representations we need more notations. Denote by $\Delta_{\mathcal{I}}$ subsystem of roots which are linear combinations of the simple roots in $\mathcal{I}$, and by $\rho_{\mathcal{I}}$ restricted Weyl vector as half sum of the positive roots in $\Delta_{\mathcal{I}}$. For semi-degenerate representations ν takes the form

$$\nu_{\tilde{\nu},R_1,R_2} = \tilde{\nu} - (\rho_{\mathcal{I}} + \lambda_1)b - (\rho_{\mathcal{I}} + \lambda_2)/b, \tag{6.105}$$

where $\tilde{\nu}$ is continuous component of the ν in the direction orthogonal to simple roots in $\mathcal{I}$, and λ_1 and λ_2 are the highest weights correspomding to irreducible representations R_1 and R_2 of the Lie algebra built from $\Delta_{\mathcal{I}}$. The elements of the matrix of the modular transformation have been computed in [48] and given by the following expressions:

$$S_{\beta\alpha} = \Xi \sum_{\omega \in W} \epsilon(\omega) e^{2\pi i(\omega(\beta-Q),\alpha-Q)}, \tag{6.106}$$

$$S_{R_1|R_2,\alpha} = \chi_{R_1}(e^{2\pi i b(Q-\alpha)})\chi_{R_2}(e^{2\pi i b^{-1}(Q-\alpha)})S_{0\alpha}, \tag{6.107}$$

$$S_{0\alpha} = \Xi \prod_{e>0} 4\sin(\pi b(\alpha-Q,e))\sin(-\frac{\pi}{b}(\alpha-Q,e)), \tag{6.108}$$

$$S_{\tilde{\nu}R_1|R_2,\alpha} = \Xi \sum_{\tilde{\omega} \in W/W_{\mathcal{I}}} \epsilon(\omega) e^{2\pi i(\tilde{\omega}(\tilde{\mu}),\alpha-Q)}\chi_{R_1}(e^{2\pi i b\tilde{\omega}^{-1}(Q-\alpha)}) \times \tag{6.109}$$
$$\chi_{R_2}(e^{2\pi i b^{-1}\tilde{\omega}^{-1}(Q-\alpha)}) \prod_{e\in\Delta_{\mathcal{I}}^+} 4\sin(\pi b(\alpha-Q,\tilde{\omega}(e)))\sin(-\frac{\pi}{b}(\alpha-Q,\tilde{\omega}(e))).$$

$\chi_R(e^x)$ are the Weyl characters:

$$\chi_R(e^x) = \frac{\sum_{\omega \in W} \epsilon(\omega) e^{(\omega(\rho+\lambda),x)}}{\sum_{\omega \in W} \epsilon(\omega) e^{(\omega(\rho),x)}} \tag{6.110}$$

and Ξ is defined by (6.96).

Note that as in the Liouville field theory in the Toda field theory holds the relation as well

$$A(\alpha)A(2Q - \alpha) = S_{0\alpha}.\tag{6.111}$$

Recalling (1.117), we are ready to compute the coefficients ξ_α and η_α in the Toda field theory:

$$\xi_\alpha^T = \epsilon_\alpha \sqrt{\frac{A(2Q - \alpha)}{A(\alpha)} \frac{S_{00}}{A(\alpha)A(2Q - \alpha)}} = \epsilon_\alpha \frac{\sqrt{S_{00}}}{A(\alpha)},\tag{6.112}$$

$$\eta_\alpha^T = \epsilon_\alpha \sqrt{\frac{A(2Q - \alpha)}{A(\alpha)} \frac{A(\alpha)A(2Q - \alpha)}{S_{00}}} = \epsilon_\alpha \frac{A(2Q - \alpha)}{\sqrt{S_{00}}}.\tag{6.113}$$

Here ϵ_α denotes a possible sign factor.

Therefore one has in the Toda field theory

$$C_{\alpha_1,\alpha_2,\mu\bar\mu}^{\alpha_3} = \frac{\epsilon_{\alpha_1}\epsilon_{\alpha_2}}{\epsilon_0\epsilon_{\alpha_3}} \frac{A(2Q - \alpha_1)A(2Q - \alpha_2)}{A(2Q)A(2Q - \alpha_3)} F_{0,\alpha_3} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2^* \end{bmatrix}_{00}^{\bar\mu\mu}.\tag{6.114}$$

Here μ and $\bar\mu$ label multiplicity of the representation α_3 appearing in the fusion of α_1 and α_2. Eq. (6.114) implies:

$$\sum_{\mu\bar\mu} C_{\alpha_1,\alpha_2,\mu\bar\mu}^{\alpha_3} F_{\alpha_3,0} \begin{bmatrix} \alpha_1^* & \alpha_1 \\ \alpha_2 & \alpha_2 \end{bmatrix}_{\bar\mu\mu}^{00} = \frac{\epsilon_{\alpha_1}\epsilon_{\alpha_2}}{\epsilon_0\epsilon_{\alpha_3}} \frac{A(0)A(\alpha_3)}{A(\alpha_1)A(\alpha_2)} N_{\alpha_1\alpha_2}^{\alpha_3}.\tag{6.115}$$

Some comments are in order at this point.

1. Presently we have no closed expressions for fusing matrices and structure constants in the Toda field theory, and cannot verify the expression (6.114) fully as we have done in the Liouville field theory. But in the absence of these expressions, the formula (6.114) can help to draw many conclusions on different aspects of the Toda field theory.

2. Actually we can use equation (6.115) only for α_1, α_2 and α_3 possessing finite fusion multiplicity. This is always true for important for us case of the degenerate representations.

3. In the Toda field theory one has also analogue of the relations (6.20) and (6.45) in the Liouville field theory. In the Toda field theory they read:

$$C_{\alpha_1,\alpha_2}^{\alpha_3} = C(\alpha_1, \alpha_2, 2Q - \alpha_3)\tag{6.116}$$

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and

$$C(\alpha_3^*, \alpha_2, \alpha_1) = R(\alpha_3)C(2Q - \alpha_3, \alpha_2, \alpha_1). \tag{6.117}$$

It is easy to see that the relation (6.114) is in agreement with (6.116) and (6.117), observing that:

a) the fusing matrix is invariant under the Weyl reflections of the primaries, since they do not change the conformal dimensions, and therefore it is invariant under the replacement $\alpha_i^* \to 2Q - \alpha_i$ of any of its variables, and

b) using the definition (6.99) one can prove that the function $A(\alpha)$ is the same for α and α^*

$$A(\alpha) = A(\alpha^*). \tag{6.118}$$

We assume that possible sign factors satisfy $\epsilon_\alpha = \epsilon_{\alpha^*} = \epsilon_{2Q-\alpha}$.

4. It was computed in [62] that for $sl(3)$ Toda field theory

$$C_{-b\omega_1,\alpha}^{\alpha-bh} F_{\alpha-bh,0} \begin{bmatrix} \alpha^* & \alpha \\ -b\omega_1 & -b\omega_1 \end{bmatrix} = -\frac{\Gamma(-2 - 3b^2)}{\Gamma(-b^2)} \frac{\pi\mu}{\gamma(-b^2)} \frac{A(\alpha - bh)}{A(\alpha)}, \tag{6.119}$$

where $h \in H_{\omega_1}$ and $H_{\omega_1} = \{\omega_1, \omega_2 - \omega_1, -\omega_1\}$.

It is easy to show that for $sl(3)$ Toda field theory

$$-\frac{\Gamma(-2 - 3b^2)}{\Gamma(-b^2)} \frac{\pi\mu}{\gamma(-b^2)} = \frac{A(0)}{A(-b\omega_1)}. \tag{6.120}$$

Recalling that for this case there are no multiplicities we have perfect agreement with (6.115). We also see that for this case (6.115) satisfied without any sign factor.

5. All calculations leading to (6.114), (6.115) and (6.119) are performed in the Coulomb gas approach. Calculations using exact expressions for the structure constants and fusing matrix, still unknown in the Toda field theory, may bring to the modifications similar to what we encountered in the Liouville field theory.

The degenerate fields have in their OPE with general primary V_α only finite number of primaries $V_{\alpha'}$

$$V_{-b\lambda_1 - \frac{1}{b}\lambda_2} V_\alpha = \sum_{s,p} C^{\alpha'_{sp}}_{-b\lambda_1 - \frac{1}{b}\lambda_2, \alpha} V_{\alpha'_{sp}}, \tag{6.121}$$

where $\alpha'_{sp} = \alpha - bh_s^{\lambda_1} - b^{-1} h_p^{\lambda_2}$. $h_s^{\lambda_1}$ are weights of the representation λ_1:

$$h_s^{\lambda_1} = \lambda_1 - \sum_1^{n-1} s_i e_i, \tag{6.122}$$

where s_i are some non-negative integers.

Given the relation (6.115) we can write down the Cardy-Lewellen equations (1.151) for Toda field theory when one of the primaries, say j, taken the degenerate one, using general formalism developed in section 2.

Eq. (1.156) in Toda field theory takes the form:

$$\Psi(\alpha)\Psi(-b\omega_k) = \sum_s \Psi(\alpha - bh_s^{\omega_k}). \tag{6.123}$$

The solution of the equation (6.123) is given as in the rational conformal field theory by the relation of elements of the matrix of the modular transformation:

$$\Psi_{\lambda_1|\lambda_2}(\alpha) = \frac{S_{R_1|R_2,\alpha}}{S_{R_1|R_2,0}}. \tag{6.124}$$

Continuing as in the previous sections we obtain discrete family of the boundary state coefficients for ordinary branes, permutation branes and defects:

$$B_{R_1|R_2}(\alpha) = \frac{S_{R_1|R_2,\alpha}}{A(\alpha)} \epsilon_\alpha, \tag{6.125}$$

$$B^N_{\mathcal{P}\, R_1|R_2}(\alpha) = \frac{S_{R_1|R_2,\alpha}}{A^N(\alpha)} \epsilon_\alpha^N, \tag{6.126}$$

$$\mathcal{D}_{R_1|R_2}(\alpha) = \frac{S_{R_1|R_2,\alpha}}{S_{0\alpha}}. \tag{6.127}$$

The continuous family of branes and defects as explained in the previous section can be obtained via solutions of the equation;

$$\Lambda(\alpha)A\frac{A(-b\omega_k)}{A(0)} = \sum_s \Lambda(\alpha - bh_s^{\omega_k}). \tag{6.128}$$

The equation (6.128) as before can be solved by the elements of the matrix of modular transformation corresponding to non-degenerate and semi-degenerate representations:

$$\Lambda_\beta(\alpha) = S_{\beta\alpha}\,, \tag{6.129}$$

$$\Lambda_{\tilde\mu R_1 | R_2}(\alpha) = S_{\tilde\mu R_1 | R_2, \alpha}\,. \tag{6.130}$$

Dividing (6.129) and (6.130) by $A(\alpha)/\epsilon_\alpha$, $A^N(\alpha)/\epsilon_\alpha^N$ and $S_{0\alpha}$, we obtain ordinary branes, permutation branes and defects correspondingly.

6.3 Classical Liouville theory with defects

6.3.1 Review of Liouville solution

Let us recall some facts on classical Liouville theory.

The action of the Liouville theory is

$$S = \frac{1}{2\pi i} \int \left(\partial\phi \bar\partial\phi + \mu\pi e^{2b\phi} \right) d^2 z\,. \tag{6.131}$$

Here we use a complex coordinate $z = \tau + i\sigma$, and $d^2 z \equiv dz \wedge d\bar z$ is the volume form.

The field $\phi(z, \bar z)$ satisfies the classical Liouville equation of motion

$$\partial\bar\partial\phi = \pi\mu b e^{2b\phi}\,. \tag{6.132}$$

The general solution to (6.132), also derived below, was given by Liouville in terms of two arbitrary functions $A(z)$ and $B(\bar z)$ [119]

$$\phi = \frac{1}{2b} \log \left(\frac{1}{\pi\mu b^2} \frac{\partial A(z) \bar\partial B(\bar z)}{(A(z) + B(\bar z))^2} \right)\,. \tag{6.133}$$

The solution (6.133) is invariant if one transforms A and B simultaneously by the following constant Möbius transformations:

$$A \to \frac{\zeta A + \beta}{\gamma A + \delta}, \quad B \to \frac{\zeta B - \beta}{-\gamma B + \delta}, \quad \zeta\delta - \beta\gamma = 1\,. \tag{6.134}$$

Classical expressions for left and right components of the energy-momentum tensor are

$$T = -(\partial\phi)^2 + b^{-1}\partial^2\phi \,, \tag{6.135}$$

$$\bar{T} = -(\bar{\partial}\phi)^2 + b^{-1}\bar{\partial}^2\phi \,. \tag{6.136}$$

Substituting (6.133) in (6.135) and (6.136) we get, that components of the energy-momentum tensor are given by the Schwarzian derivatives of $A(z)$ and $B(\bar{z})$:

$$T = \{A; z\} = \frac{1}{2b^2}\left[\frac{A'''}{A'} - \frac{3}{2}\frac{(A'')^2}{(A')^2}\right]\,, \tag{6.137}$$

$$\bar{T} = \{B; \bar{z}\} = \frac{1}{2b^2}\left[\frac{B'''}{B'} - \frac{3}{2}\frac{(B'')^2}{(B')^2}\right]\,. \tag{6.138}$$

The Schwarzian derivative is invariant under arbitrary constant Möbius transformation:

$$\left\{\frac{\zeta F + \beta}{\gamma F + \delta}; z\right\} = \{F; z\}, \quad \zeta\delta - \beta\gamma = 1 \,. \tag{6.139}$$

Solutions of the Liouville equation (6.132) can be described also via linear combination of some holomorphic and anti-holomorphic functions. Let us introduce the function $V = e^{-b\phi}$. One can write the Liouville equation (6.132) as equation for V

$$V\partial\bar{\partial}V - \partial V\bar{\partial}V = -\pi\mu b^2 \,. \tag{6.140}$$

Also the left and right components of the energy-momentum tensor (6.135) and (6.136) can be written via V

$$\partial^2 V = -b^2 VT \,, \tag{6.141}$$

$$\bar{\partial}^2 V = -b^2 V\bar{T} \,. \tag{6.142}$$

It is straightforward to check that the general solution of eq. (6.140) is given by linear combination of two holomorphic $a_i(z)$, $i = 1, 2$, and two anti-holomorphic functions $b_i(\bar{z})$, $i = 1, 2$:

$$V = \sqrt{\pi\mu b^2}\left(a_1(z)b_1(\bar{z}) - a_2(z)b_2(\bar{z})\right)\,, \tag{6.143}$$

satisfying the condition

$$(a_1 a_2' - a_1' a_2)(b_1 b_2' - b_1' b_2) = 1 \,. \tag{6.144}$$

Usually the fields $a_i(z)$ and $b_i(\bar{z})$, $i = 1, 2$ are normalized to have the unit Wronskian:

$$a_1 a_2' - a_1' a_2 = 1 \tag{6.145}$$

and

$$b_1 b_2' - b_1' b_2 = 1 . \tag{6.146}$$

It is easy to see that the left and right components of the energy-momentum tensor can be expressed via a_i and b_i in the very simple form:

$$T = -\frac{1}{b^2} \frac{\partial^2 a_1}{a_1} = -\frac{1}{b^2} \frac{\partial^2 a_2}{a_2} \tag{6.147}$$

and

$$\bar{T} = -\frac{1}{b^2} \frac{\bar{\partial}^2 b_1}{b_1} = -\frac{1}{b^2} \frac{\bar{\partial}^2 b_2}{b_2} . \tag{6.148}$$

The solutions (6.133) and (6.143) can be related in the following way. One can solve the unit Wronskian conditions (6.145) and (6.146) via a holomorphic $A(z)$ and an anti-holomorphic function $B(\bar{z})$

$$a_1 = \frac{1}{\sqrt{\partial A}} \quad \text{and} \quad a_2 = \frac{A}{\sqrt{\partial A}} \tag{6.149}$$

and

$$b_1 = \frac{B}{\sqrt{\bar{\partial} B}} \quad \text{and} \quad b_2 = -\frac{1}{\sqrt{\bar{\partial} B}} . \tag{6.150}$$

Inserting (6.149) and (6.150) in (6.143) we get (6.133). Note that the Möbius transformations of A and B (6.134) become linear $SL(2, \mathbb{C})$ transformations of a_i and b_i:

$$\tilde{a}_1 = \delta a_1 + \gamma a_2 , \tag{6.151}$$

$$\tilde{a}_2 = \beta a_1 + \zeta a_2$$

and

$$\tilde{b}_1 = \zeta b_1 + \beta b_2 , \tag{6.152}$$

$$\tilde{b}_2 = \gamma b_1 + \delta b_2 .$$

It is straightforward to check that indeed (6.143) is invariant under (6.151) and (6.152), and both of them keep the unit Wronskian condition.

One can also check, that both component of the energy-momentum tensor (6.147) and (6.148) are invariant under these transformations as well.

We finish this section with a remark which will be important in the parts on the light asymptotic limit. In that parts we will consider an analytic continuation $\mu \to -\mu$. At this point the solution (6.143) is convenient to write as:

$$V = \sqrt{-\pi\mu b^2}\left(a_1(z)b_1(\bar{z}) + a_2(z)b_2(\bar{z})\right). \tag{6.153}$$

It is easy to check that (6.153) also solves the Liouville equation, given that a_i and b_i, $i = 1, 2$ obey the condition (6.144).

6.3.2 Lagrangian of the Liouville theory with defect

Recently in [3] the action of the Liouville theory with topological defects was suggested:

$$S^{\text{top-def}} = \frac{1}{2\pi i}\int_{\Sigma_1}\left(\partial\phi_1\bar{\partial}\phi_1 + \mu\pi e^{2b\phi_1}\right)d^2z + \frac{1}{2\pi i}\int_{\Sigma_2}\left(\partial\phi_2\bar{\partial}\phi_2 + \mu\pi e^{2b\phi_2}\right)d^2z \tag{6.154}$$

$$+ \int_{\partial\Sigma_1}\left[-\frac{1}{2\pi}\phi_2\partial_\tau\phi_1 + \frac{1}{2\pi}\Lambda\partial_\tau(\phi_1 - \phi_2) + \frac{\mu}{2}e^{(\phi_1+\phi_2-\Lambda)b} - \frac{1}{\pi b^2}e^{\Lambda b}\left(\cosh(\phi_1 - \phi_2)b - \kappa\right)\right]\frac{d\tau}{i}.$$

Here Σ_1 is the upper half-plane $\sigma = \text{Im}z \geq 0$ and Σ_2 is the lower half-plane $\sigma = \text{Im}z \leq 0$. The defect is located along their common boundary, which is the real axis $\sigma = 0$ parametrized by $\tau = \text{Re}z$. Note that $\Lambda(\tau)$ here is an additional field associated with the defect itself. The action (6.154) yields the following defect equations of motion at $\sigma = 0$:

$$\frac{1}{2\pi}(\partial - \bar{\partial})\phi_1 + \frac{1}{2\pi}\partial_\tau\phi_2 - \frac{1}{2\pi}\partial_\tau\Lambda + \frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b} - \frac{1}{\pi b}e^{\Lambda b}\sinh(\phi_1 - \phi_2)b = 0, \tag{6.155}$$

$$-\frac{1}{2\pi}(\partial - \bar{\partial})\phi_2 - \frac{1}{2\pi}\partial_\tau\phi_1 + \frac{1}{2\pi}\partial_\tau\Lambda + \frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b} + \frac{1}{\pi b}e^{\Lambda b}\sinh(\phi_1 - \phi_2)b = 0, \tag{6.156}$$

$$\frac{1}{2\pi}\partial_\tau(\phi_1 - \phi_2) - \frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b} - \frac{1}{\pi b}e^{\Lambda b}\left(\cosh(\phi_1 - \phi_2)b - \kappa\right) = 0. \tag{6.157}$$

The last equation is derived calculating variation by the Λ.

Using that $\partial_\tau = \partial + \bar{\partial}$ and forming various linear combinations of equations (6.155)-(6.157) we can bring them to the form:

$$\bar{\partial}(\phi_1 - \phi_2) = \pi \mu b e^{b(\phi_1 + \phi_2)} e^{-\Lambda b} , \qquad (6.158)$$

$$\partial(\phi_1 - \phi_2) = \frac{2}{b} e^{\Lambda b} \left(\cosh(\phi_1 - \phi_2) b - \kappa \right) . \qquad (6.159)$$

$$\partial(\phi_1 + \phi_2) - \partial_\tau \Lambda = \frac{2}{b} e^{\Lambda b} \sinh(b(\phi_1 - \phi_2)) . \qquad (6.160)$$

It is shown in [3] that requiring the defect equations of motion to hold for every σ brings additionally to the condition, that Λ is restriction to the real axis of a holomorphic field

$$\bar{\partial}\Lambda = 0 . \qquad (6.161)$$

This condition allows to rewrite (6.160) in the form

$$\partial(\phi_1 + \phi_2 - \Lambda) = \frac{2}{b} e^{\Lambda b} \sinh(b(\phi_1 - \phi_2)) . \qquad (6.162)$$

It is checked in [3] that the system of the defect equations of motion (6.158)-(6.162) guarantees that both components of the energy-momentum tensor are continuous across the defects and therefore describes topological defects:

$$-(\partial\phi_1)^2 + b^{-1}\partial^2\phi_1 = -(\partial\phi_2)^2 + b^{-1}\partial^2\phi_2 , \qquad (6.163)$$

$$-(\bar{\partial}\phi_1)^2 + b^{-1}\bar{\partial}^2\phi_1 = -(\bar{\partial}\phi_2)^2 + b^{-1}\bar{\partial}^2\phi_2 . \qquad (6.164)$$

Another interesting consequence of the defect equations of motion, found in [3], is existence together with the holomorphic field $\Lambda(z)$ of an anti-holomorphic field Ξ:

$$\partial\Xi = 0 , \qquad (6.165)$$

where

$$\Xi = e^{-b(\phi_1 + \phi_2)} e^{b\Lambda} \left(\cosh b(\phi_1 - \phi_2) - \kappa \right) . \qquad (6.166)$$

or alternatively

$$\Xi = \frac{b}{2} e^{-b(\phi_1 + \phi_2)} \partial(\phi_1 - \phi_2) . \qquad (6.167)$$

Now we will present the general solution for defect equations of motion (6.158)-(6.162).

We will follow essentially the same strategy which was used in [92] for analyzing the boundary Liouville problem. On the one hand since the defect is topological both components of the energy-momentum tensor are equal being computed in terms of ϕ_1 or ϕ_2. On the other hand each component of the energy-momentum tensor is given by the Schwarzian derivative, which is invariant under the Möbius transformation. This naturally leads to the following soluton:

$$\phi_1 = \frac{1}{2b} \log \left(\frac{1}{\pi \mu b^2} \frac{\partial A \bar{\partial} B}{(A + B)^2} \right) , \tag{6.168}$$

$$\phi_2 = \frac{1}{2b} \log \left(\frac{1}{\pi \mu b^2} \frac{\partial C \bar{\partial} D}{(C + D)^2} \right) , \tag{6.169}$$

where

$$C = \frac{\zeta A + \beta}{\gamma A + \delta} \quad \text{and} \quad D = \frac{\zeta' B + \beta'}{\gamma' B + \delta'} . \tag{6.170}$$

Remembering that ϕ_2 is invariant under the simultaneous Möbius transformation (6.134) of C and D, we can set $B = D$. Therefore without loosing generality we can look for a solution in the form:

$$\phi_1 = \frac{1}{2b} \log \left(\frac{1}{\pi \mu b^2} \frac{\partial A \bar{\partial} B}{(A + B)^2} \right) , \tag{6.171}$$

$$\phi_2 = \frac{1}{2b} \log \left(\frac{1}{\pi \mu b^2} \frac{\partial C \bar{\partial} B}{(C + B)^2} \right) , \tag{6.172}$$

where

$$C = \frac{\zeta A + \beta}{\gamma A + \delta} . \tag{6.173}$$

Substituting (6.171) and (6.172) in (6.158) we find that it is satisfied with

$$e^{-\Lambda b} = \frac{A - C}{\sqrt{\partial A \partial C}} . \tag{6.174}$$

Since A and C are holomorphic functions, Λ is holomorphic as well, as it is stated in (6.161).

It is straightforward to check that (6.162) is satisfied as well with ϕ_1, ϕ_2 and Λ given by (6.171), (6.172) and (6.174) respectively. And finally inserting (6.171), (6.172) and (6.174) in (6.159) we see that it is also fulfilled with

$$\kappa = \frac{\zeta + \delta}{2} . \tag{6.175}$$

Inserting (6.171), (6.172) in (6.167) one can check that

$$\Xi = \frac{\pi\mu b^2}{2} \frac{\gamma B^2 + B(\zeta - \delta) - \beta}{\bar{\partial} B} \,.$$ (6.176)

Remembering that B is anti-holomorphic we see that Ξ is anti-holomorphic as well.

We can write the solution of the defect equations of motion also using solution of the Liouville equation in the form (6.143). Recalling that the Möbius transformations of the functions A and B become linear $SL(2,\mathbb{C})$ transformations of the functions a_i and b_i, which leave the components of the energy-momentum tensor (6.147) and (6.148) invariant, we can write the solution (6.171)-(6.173) in the form:

$$e^{-b\phi_1} = \sqrt{\pi\mu b^2} \left(a_1(z)b_1(\bar{z}) - a_2(z)b_2(\bar{z}) \right),$$ (6.177)

$$e^{-b\phi_2} = \sqrt{\pi\mu b^2} \left(c_1(z)b_1(\bar{z}) - c_2(z)b_2(\bar{z}) \right),$$ (6.178)

where denoting $\vec{a} = (a_1, a_2)$, $\vec{c} = (c_1, c_2)$, and $D = \begin{pmatrix} \delta & \gamma \\ \beta & \zeta \end{pmatrix}$, one has

$$\vec{c} = D\vec{a}$$ (6.179)

and

$$2\kappa = \mathrm{Tr}\, D \,.$$ (6.180)

At this point we would like to make the following remark. Let us consider the identity defect. It has $A = C$, and $\kappa = 1$. Setting $A = C$ in (6.174) we obtain $e^{-\Lambda b} = 0$. This result can be derived also directly setting $\phi_1 = \phi_2$ in (6.158). Therefore the identity defect does not belong to the family of defects described by the action (6.154) and can be derived from them only in the limit $\Lambda \to \infty$. This can be understood recalling that defects described by (6.154) have a two-dimensional world-volume in a sense that the values of $\phi_1(\tau)$ and $\phi_2(\tau)$ at an arbitrary point τ on the defect line are not constrained and the point $(\phi_1(\tau), \phi_2(\tau))$ can take values in the whole plane $\mathbb{R}^2$. Contrary to this, the identity defect has a one-dimensional world-volume, since the point $(\phi_1(\tau), \phi_2(\tau))$ takes values on one-dimensional diagonal $\phi_1 = \phi_2$.

6.4 Heavy asymptotic limit

Let us consider the action (6.131) for the rescaled variable $\varphi = 2b\phi$

$$S = \frac{1}{8\pi i b^2} \int \left(\partial\varphi\bar{\partial}\varphi + 4\lambda e^\varphi \right) d^2z \,, \tag{6.181}$$

where $\lambda = \pi\mu b^2$.

This form shows that b^2 plays in the Liouville theory the role of the Planck constant, and one can study semiclassical limit taking the limit $b \to 0$, in such a way that λ is kept fixed.

Let us consider correlation functions in the path integral formalism:

$$\left\langle V_{\alpha_1}(z_1, \bar{z}_1) \cdots V_{\alpha_n}(z_n, \bar{z}_n) \right\rangle = \int \mathcal{D}\varphi \, e^{-S} \prod_{i=1}^n \exp\left(\frac{\alpha_i\varphi(z_i, \bar{z}_i)}{b} \right) \,. \tag{6.182}$$

We would like to calculate this integral in the semiclassical limit $b \to 0$ using the method of steepest descent, and we should decide how α_i scales with b. Since S scales like b^{-2}, for operators to affect the saddle point, we should take $\alpha_i = \eta_i/b$, with η_i fixed. The conformal weights $\Delta_\alpha = \eta(1-\eta)/b^2$ scale like b^{-2} as well. This is the heavy asymptotic limit. Another choice of the operator scaling will be discussed in the next subsection.

We see from (6.182) that in the semiclasscial limit the correlation function is given by $e^{-S_{\mathrm{cl}}}$ where, at least naively, in a sense which will be clarified below, S_{cl} is the action

$$S = \frac{1}{8\pi i b^2} \int \left(\partial\varphi\bar{\partial}\varphi + 4\lambda e^\varphi \right) d^2z + \sum_{i=1}^n \frac{\eta_i}{b^2}\varphi(z_i, \bar{z}_i) \,, \tag{6.183}$$

evaluated on the solution of its equation of motion:

$$\partial\bar{\partial}\varphi = 2\lambda e^\varphi - 4\pi \sum_{i=1}^n \eta_i \delta^2(z - z_i) \,. \tag{6.184}$$

Assuming that in the vicinity of the insertion point z_i, one can ignore the exponential term we get that in the neighborhood of the point z_i φ has the following behavior

$$\varphi(z, \bar{z}) = -4\eta_i \log|z - z_i| + X_i \quad \text{as} \quad z \to z_i \,. \tag{6.185}$$

One can insert this solution back into the equation of motion to check, if indeed the exponential term is subleading. We find, that this happens when

$$\mathrm{Re}\,\eta_i < \frac{1}{2} \,. \tag{6.186}$$

This constraint is known as Seiberg bound [170]. It is the semiclassial version of the quantum condition (6.17) stating that V_α and $V_{Q-\alpha}$ represent the same quantum operator. Either α or $Q - \alpha$ always obey the Seiberg bound.

Remembering that in the Liouville theory we have also background charge at infinity, conditions (6.185) should be complemented by the behavior at the infinity:

$$\varphi(z, \bar{z}) = -2 \log |z|^2 \quad \text{as} \quad |z| \to \infty. \tag{6.187}$$

Since the energy-momentum tensor in the presence of primary fields acquires a quadratic singularity, functions a_j, $j = 1, 2$, should solve the equation

$$\partial^2 a_j + b^2 T a_j = 0, \tag{6.188}$$

where

$$b^2 T = \sum_{i=1}^{n} \left(\frac{\eta_i (1 - \eta_i)}{(z - z_i)^2} + \frac{c_i}{(z - z_i)} \right) \tag{6.189}$$

and c_i are the so called accessory parameters.

If one tries naively to evaluate the action (6.183) on a solution obeying (6.185), we find that it diverges. Therefore we should consider a regularized action. It was constructed in [191]:

$$\begin{aligned} b^2 S^{\text{reg}} &= \frac{1}{8\pi i} \int_{D - \cup_i d_i} \left(\partial \varphi \bar{\partial} \varphi + 4 \lambda e^\varphi \right) d^2 z + \frac{1}{2\pi} \oint_{\partial D} \varphi d\theta + 2 \log R \\ &\quad - \sum_{i=1}^{n} \left(\frac{\eta_i}{2\pi} \oint_{\partial d_i} \varphi d\theta_i + 2 \eta_i^2 \log \epsilon_i \right). \end{aligned} \tag{6.190}$$

Here D is a disc of radius R, d_i is a disc of radius ϵ_i around z_i. It was shown in [191] that the action (6.190) satisfies the equation

$$\frac{\partial}{\partial \eta_i} b^2 S^{\text{reg}} = -X_i, \tag{6.191}$$

where X_i is defined by the boundary condition (6.185).

The Polyakov conjecture proved in [193] states, that the action (6.190) also obeys the relation:

$$\frac{\partial}{\partial z_i} b^2 S^{\text{reg}} = -c_i. \tag{6.192}$$

Let us write down regularized version of the action with a defect.

First of all let us write it in the terms of $\lambda = \pi\mu b^2$, $\varphi_1 = 2b\phi_1$, $\varphi_2 = 2b\phi_2$, and $\tilde{\Lambda} = 2b\Lambda$:

$$b^2 S^{\text{top−def}} = \frac{1}{8\pi i}\int_{\Sigma_1}\left(\partial\varphi_1\bar{\partial}\varphi_1 + 4\lambda e^{\varphi_1}\right)d^2z + \frac{1}{8\pi i}\int_{\Sigma_2}\left(\partial\varphi_2\bar{\partial}\varphi_2 + 4\lambda e^{\varphi_2}\right)d^2z \quad (6.193)$$

$$+ \int_{\partial\Sigma_1}\left[-\frac{1}{8\pi}\varphi_2\partial_\tau\varphi_1 + \frac{1}{8\pi}\tilde{\Lambda}\partial_\tau(\varphi_1 - \varphi_2) + \right.$$

$$\left.\frac{\lambda}{2\pi}e^{(\varphi_1+\varphi_2-\tilde{\Lambda})/2} - \frac{1}{\pi}e^{\tilde{\Lambda}/2}\left(\cosh\left(\frac{\varphi_1-\varphi_2}{2}\right) - \kappa\right)\right]\frac{d\tau}{i}\ .$$

Since we consider here only insertion of the bulk field, and do not consider insertion of the defect or boundary fields, the regularized action takes the form:

$$b^2 S^{\text{top−def}} = \frac{1}{8\pi i}\int_{\Sigma_1^R - \cup_i d_i}\left(\partial\varphi_1\bar{\partial}\varphi_1 + 4\lambda e^{\varphi_1}\right)d^2z \quad (6.194)$$

$$- \sum_{i=1}^n\left(\frac{\eta_i}{2\pi}\oint_{\partial d_i}\varphi_1 d\theta_i + 2\eta_i^2\log\epsilon_i\right) + \frac{1}{2\pi}\int_{s_{R1}}\varphi_1 d\theta + \log R$$

$$+ \frac{1}{8\pi i}\int_{\Sigma_2^R - \cup_j d_j}\left(\partial\varphi_2\bar{\partial}\varphi_2 + 4\lambda e^{\varphi_2}\right)d^2z$$

$$- \sum_{j=n+1}^{n+m}\left(\frac{\eta_j}{2\pi}\oint_{\partial d_j}\varphi_2 d\theta_j + 2\eta_j^2\log\epsilon_j\right) + \frac{1}{2\pi}\int_{s_{R2}}\varphi_2 d\theta + \log R +$$

$$\int_{-R}^R\left[-\frac{1}{8\pi}\varphi_2\partial_\tau\varphi_1 + \frac{1}{8\pi}\tilde{\Lambda}\partial_\tau(\varphi_1 - \varphi_2) + \right.$$

$$\left.\frac{\lambda}{2\pi}e^{(\varphi_1+\varphi_2-\tilde{\Lambda})/2} - \frac{1}{\pi}e^{\tilde{\Lambda}/2}\left(\cosh\left(\frac{\varphi_1-\varphi_2}{2}\right) - \kappa\right)\right]\frac{d\tau}{i}\ ,$$

where Σ_i^R is a half-disc of the radius R and s_{Ri} is a semicircle of the radius R in the half-plane Σ_i, $i = 1, 2$.

6.5   Defects in the heavy asymptotic limit

6.5.1   Heavy asymptotic limit of the correlation functions

In this section we consider the heavy asymptotic limit of two-point functions in the presence of defects (6.82):

$$\langle V_\alpha(z_1,\bar{z}_1)X_s V_\alpha(z_2,\bar{z}_2)\rangle = -\frac{1}{W^2(\alpha)}\frac{2^{1/2}i\cosh(2\pi s(2\alpha - Q))}{(z_1-z_2)^{2\Delta_\alpha}(\bar{z}_1-\bar{z}_2)^{2\Delta_\alpha}} \quad (6.195)$$

Now we should compute the inverse ZZ function (6.29) and the factor $\cosh(2\pi s(2\alpha - Q))$ in the limit $b \to 0$, setting $\alpha = \frac{\eta}{b}$, and $s = \frac{\sigma}{b}$. In the heavy asymptotic limit we should keep only

terms having the form $\sim e^{1/b^2}$.

To understand semiclassical origin of the denominator in (6.83) we find very useful to consider in the spirit of [105] analytic continuation of the Liouville theory with complex η and complex saddle points.

Taking the η satisfying the Seiberg bound (6.186) $\mathrm{Re}\,\eta < \frac{1}{2}$, using properties of Γ functions collected in appendix .2, we obtain

$$W^{-1}_{\alpha=\frac{\eta}{b}} \to C(b,\eta)\lambda^{\frac{1-2\eta}{2b^2}} \frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} \exp\left(\frac{2\eta-1}{b^2}\left[\log(1-2\eta)-1\right]\right) . \tag{6.196}$$

where

$$C(b,\eta) = -\frac{2^{-3/4}e^{-3i\pi/2}b^3\Gamma(2\eta)}{(2\eta-1)^2} \tag{6.197}$$

$$= \exp\left(-\frac{3}{4}\log 2 - \frac{i\pi}{2} + \log\Gamma(2\eta) - 2\log(1-2\eta) + 3\log b\right)$$

We see that all the terms in (6.197) are negligible compare to terms growing like $\sim e^{1/b^2}$ in the limit $b \to 0$, and therefore $C(b,\eta)$ can be omitted. The importance of the term $\frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)}$ is explained in [105]. It was shown that this term in the semiclassical interpretation arises as a sum over some "instanton" like sectors. As a preparation to this point we will expand this term in two ways as suggested in [105]. Denoting $y = e^{i\pi(2\eta-1)/b^2}$ one can write

$$\frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} = \frac{2i}{y-y^{-1}} = 2i\sum_{k=0}^{\infty} y^{-(2k+1)} = -2i\sum_{k=0}^{\infty} y^{2k+1} . \tag{6.198}$$

One expansion is valid for $|y| > 1$ and one for $|y| < 1$. So either way, there is a set T of integers with

$$\frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} = \pm 2i\sum_{M\in T} e^{2i\pi(M\mp 1/2)(2\eta-1)/b^2} , \tag{6.199}$$

where T consists of nonnegative integers if $\mathrm{Im}(2\eta-1)/b^2 > 0$ and of nonpositive ones if $\mathrm{Im}(2\eta-1)/b^2 < 0$.

Setting $\alpha = \frac{\eta}{b}$ and $s = \frac{\sigma}{b}$ we easily obtain:

$$\cosh 2\pi s(2\alpha - Q) \to e^{\frac{2\pi}{b^2}|\sigma|(1-2\eta)} . \tag{6.200}$$

Now we are position to write down the limiting form of the defects correlation functions.

Inserting (6.196), (6.200) in (6.195) we can write in the heavy asymptotic limit

$$\langle V_\alpha(z_1, \bar{z}_1) X V_\alpha(z_2, \bar{z}_2) \rangle \sim (z_1 - z_2)^{-2\eta(1-\eta)/b^2} (\bar{z}_1 - \bar{z}_2)^{-2\eta(1-\eta)/b^2} \tag{6.201}$$

$$\times \quad \lambda^{\frac{1-2\eta}{b^2}} \frac{1}{\sin^2 \pi \left(\frac{2\eta-1}{b^2}\right)} \exp\left(\frac{4\eta-2}{b^2}\left[\log(1-2\eta)-1\right]\right) e^{\frac{2\pi}{b^2}|\sigma|(1-2\eta)}.$$

Using also (6.199) we get

$$\langle V_\alpha(z_1, \bar{z}_1) X V_\alpha(z_2, \bar{z}_2) \rangle \sim \sum_{M_1, M_2 \in T} \exp\left(-S_{M_1, M_2}^{\mathrm{def}}\right), \tag{6.202}$$

where

$$b^2 S_{M_1, M_2}^{\mathrm{def}} = -2i\pi(M_1 + M_2 \mp 1)(2\eta - 1) + 4\eta(1-\eta)\log|z_1 - z_2| \tag{6.203}$$

$$-(1-2\eta)\log\lambda - (4\eta - 2)\log(1-2\eta) + (4\eta - 2) - 2\pi|\sigma|(1-2\eta).$$

It is instructive to compare the heavy asymptotic limit of the defect two-point function with the corresponding limit of usual two-point function, computed in [105]

$$\langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle \sim |z_1 - z_2|^{-4\eta(1-\eta)/b^2} \tag{6.204}$$

$$\times \quad \lambda^{(1-2\eta)/b^2} \frac{1}{\sin \pi(2\eta-1)/b^2} \exp\left(\frac{4\eta-2}{b^2}[\log(1-2\eta)-1]\right).$$

The relation of (6.201) to (6.204) naturally gives the heavy asymptotic limit of the eigenvalues $\mathcal{D}_s(\alpha)$ of the defect operator:

$$\mathcal{D}_s(\alpha) = \frac{\langle V_\alpha(z_1, \bar{z}_1) X V_\alpha(z_2, \bar{z}_2) \rangle}{\langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle} \rightarrow \frac{e^{\frac{2\pi}{b^2}|\sigma|(1-2\eta)}}{\sin \pi \left(\frac{2\eta-1}{b^2}\right)}. \tag{6.205}$$

Surely (6.205) can be also easily derived directly from (6.83) in the heavy asymptotic limit.

6.5.2 Evaluation of the action for classical solutions

According to general prescription of the semiclassical heavy asymptotic limit, we should find solutions of the Liouville equation, satisfying the defect equations of motion and possessing the logarithmic singularities (6.185) at points z_1 and z_2. The form of the solution of the defect equations of motion (6.171) and (6.172) implies that we should find functions $A(z)$, $C(z)$ and $B(\bar{z})$ in such a way that ϕ_1 has a logarithmic singularity at the point z_1 and ϕ_2 has a logarithmic

singularity at the point z_2. Since the energy-momentum tensor is continuous across a defect this implies that we should find solutions possessing two singular points. Two-point solutions are well known (see for example [105]) and we can build from them the Ansatz satisfying the defect equations of motion.

To build the solution with the required singularities one should take a function $A(z)$ which is smooth and holomorphic away from z_1 and z_2. Let us take $A(z)$ as

$$A(z) = e^{2\nu_1}(z - z_1)^{2\eta - 1}(z - z_2)^{1 - 2\eta} . \tag{6.206}$$

One has also

$$a_1 = \frac{1}{\sqrt{\partial A}} = \frac{e^{-\nu_1}}{\sqrt{(z_1 - z_2)(2\eta - 1)}} (z - z_1)^{1 - \eta}(z - z_2)^{\eta} , \tag{6.207}$$

$$a_2 = \frac{A}{\sqrt{\partial A}} = \frac{e^{\nu_1}}{\sqrt{(z_1 - z_2)(2\eta - 1)}} (z - z_1)^{\eta}(z - z_2)^{1 - \eta} . \tag{6.208}$$

Inserting (6.207) or (6.208) in (6.147) we obtain the energy-momentum tensor

$$b^2 T = \frac{\eta(1 - \eta)}{(z - z_1)^2} + \frac{\eta(1 - \eta)}{(z - z_2)^2} - \frac{2\eta(1 - \eta)}{z_1 - z_2} \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) , \tag{6.209}$$

indeed possessing two singular points (6.189), with accessory parameters

$$c_2 = -c_1 = \frac{2\eta(1 - \eta)}{z_1 - z_2} . \tag{6.210}$$

The anti-holomorphic part is:

$$B(\bar{z}) = -(\bar{z} - \bar{z}_1)^{1 - 2\eta}(\bar{z} - \bar{z}_2)^{2\eta - 1} , \tag{6.211}$$

$$b_1 = \frac{B}{\sqrt{\bar{\partial} B}} = \frac{1}{\sqrt{(\bar{z}_1 - \bar{z}_2)(2\eta - 1)}} (\bar{z} - \bar{z}_1)^{1 - \eta}(\bar{z} - \bar{z}_2)^{\eta} , \tag{6.212}$$

$$b_2 = -\frac{1}{\sqrt{\bar{\partial} B}} = \frac{1}{\sqrt{(\bar{z}_1 - \bar{z}_2)(2\eta - 1)}} (\bar{z} - \bar{z}_1)^{\eta}(\bar{z} - \bar{z}_2)^{1 - \eta} . \tag{6.213}$$

Let us take the holomorphic part for ϕ_2 as

$$C(z) = e^{2\nu_2}(z - z_1)^{2\eta - 1}(z - z_2)^{1 - 2\eta} = e^{2(\nu_2 - \nu_1)} A(z) , \tag{6.214}$$

and the antiholomorphic part again given by (6.211). Using (6.175) one gets

$$\kappa = \cosh(\nu_2 - \nu_1) . \tag{6.215}$$

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Inserting (6.206), (6.214) and (6.211) in (6.171) and (6.172) we obtain:

$$e^{-\varphi_1} = \frac{\lambda}{(2\eta - 1)^2 |z_1 - z_2|^2} \left(e^{\nu_1} |z - z_1|^{2\eta} |z - z_2|^{2-2\eta} \right. \tag{6.216}$$
$$\left. - e^{-\nu_1} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right)^2,$$

$$e^{-\varphi_2} = \frac{\lambda}{(2\eta - 1)^2 |z_1 - z_2|^2} \left(e^{\nu_2} |z - z_1|^{2\eta} |z - z_2|^{2-2\eta} \right. \tag{6.217}$$
$$\left. - e^{-\nu_2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right)^2.$$

It is easy to see that φ_1 and φ_2 given by (6.216) and (6.217) have the required singularity (6.185) around z_1 and z_2 respectively. In fact each of the functions φ_1 or φ_2 given by (6.216) and (6.217) coincides with the solution describing a saddle point for a two-point function considered in [105]. But in [105] this solution was considered on a full plane with the same parameter ν everywhere, whereas here each of them is considered on a corresponding half-plane, namely in (6.216) z belongs to the upper half-plane Σ_1, and in (6.217) z belongs to the lower half-plane Σ_2, and we should also remember that, $z_1 \in \Sigma_1$ and $z_2 \in \Sigma_2$. The defect is created by the choice of different parameters ν_1 and ν_2, $\nu_1 \neq \nu_2$.

From (6.216) and (6.217) we obtain

$$\varphi_1 = 4i\pi N_1 - \log \lambda + 2\log(1 - 2\eta) \tag{6.218}$$
$$-2\log \left(\frac{e^{\nu_1} |z - z_1|^{2\eta} |z - z_2|^{2-2\eta}}{|z_1 - z_2|} - \frac{e^{-\nu_1} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta}}{|z_1 - z_2|} \right),$$

$$\varphi_2 = 4i\pi N_2 - \log \lambda + 2\log(1 - 2\eta) \tag{6.219}$$
$$-2\log \left(-\frac{e^{\nu_2} |z - z_1|^{2\eta} |z - z_2|^{2-2\eta}}{|z_1 - z_2|} + \frac{e^{-\nu_2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta}}{|z_1 - z_2|} \right).$$

Here N_1 and N_2 are integer. The possibility to add the term $4i\pi N_j$, $j = 1, 2$, results from the invariance of the bulk (6.132) and defect (6.158)-(6.162) Liouville equations of motion under the transformation $\phi_j \to \phi_j + 2\pi i N_j/b$, or multiplying by $2b$, under $\varphi_j \to \varphi_j + 4\pi i N_j$, $j = 1, 2$. Note that the bulk Liouville equation (6.132) is invariant under the symmetry $\varphi_j \to \varphi_j + 2\pi i N_j$, and it is broken to the $\varphi_j \to \varphi_j + 4\pi i N_j$ by the exponential terms of the defect action (6.154).

To evaluate the action on solutions (6.216), (6.217), we will use the strategy used in [191]. Namely we will write the system of differential equations which this action should satisfy. The first equation is (6.191), which given that $\eta_1 = \eta_2 = \eta$, reads

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \eta} = -X_1 - X_2 \,. \tag{6.220}$$

where X_i is defined in (6.185). The leading terms of φ_1 around z_1 are

$$\varphi_1 \to -4\eta \log|z - z_1| + X_1 \,, \tag{6.221}$$

where

$$X_1 = 4\pi i N_1 - \log\lambda + 2\log(1 - 2\eta) - (2 - 4\eta)\log|z_1 - z_2| - 2\nu_1 \,. \tag{6.222}$$

The leading terms of φ_2 around z_2 similarly are

$$\varphi_2 \to -4\eta \log|z - z_2| + X_2 \,, \tag{6.223}$$

where

$$X_2 = 4\pi i N_2 - \log\lambda + 2\log(1 - 2\eta) - (2 - 4\eta)\log|z_1 - z_2| + 2\nu_2 \,. \tag{6.224}$$

Inserting (6.222) and (6.224) in (6.220) one obtains

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \eta} = -2\pi i \left(2N_1 + 2N_2\right) + 2\log\lambda - 4\log(1 - 2\eta) + (4 - 8\eta)\log|z_1 - z_2| + 2(\nu_1 - \nu_2) \,. \tag{6.225}$$

We would like to emphasize yet another difference from the calculation of the heavy asymptotic limit of the two-point function in [105]. In the case of the usual two-point function the integers N_1 and N_2 are equal since we have one continuous function ϕ. Here they can be different since we have two different functions φ_1 and φ_2.

The action with defect (6.194) implies also

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \kappa} = \frac{1}{i\pi} \int_{\partial \Sigma_1} e^{\Lambda b} d\tau \,. \tag{6.226}$$

Inserting (6.206) and (6.214) in eq. (6.174) one obtains

$$e^{\Lambda b} = \frac{1}{2\sinh(\nu_1 - \nu_2)} \frac{(2\eta - 1)(z_1 - z_2)}{(z - z_1)(z - z_2)} \,. \tag{6.227}$$

Using that

$$\frac{1}{i} \int_{\partial \Sigma_1} \frac{dz}{(z - z_1)(z - z_2)} = \frac{2\pi}{(z_1 - z_2)}, \tag{6.228}$$

we obtain

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \kappa} = \frac{2\eta - 1}{\sinh(\nu_1 - \nu_2)}. \tag{6.229}$$

Integrating equations (6.225) and (6.229) we obtain:

$$b^2 S_{N_1,N_2}^{\text{def}} = -2i\pi(2N_1 + 2N_2)\eta + 4\eta(1 - \eta)\log|z_1 - z_2| \tag{6.230}$$

$$+2\eta \log \lambda - (4\eta - 2)\log(1 - 2\eta) + 4\eta - (\nu_1 - \nu_2)(1 - 2\eta) + C,$$

where C is a constant. To derive the penultimate term we should remember the relation (6.215). To fix the constant term we can directly compute the action (6.194) for the Ansatz (6.218)-(6.219) with $\eta = 0$:

$$\varphi_1 = 4i\pi N_1 - \log \lambda - \log \left(\frac{e^{\nu_1}}{|z_1 - z_2|}|z - z_2|^2 - \frac{e^{-\nu_1}}{|z_1 - z_2|}|z - z_1|^2 \right)^2, \tag{6.231}$$

$$\varphi_2 = 4i\pi N_2 - \log \lambda - \log \left(\frac{e^{\nu_2}}{|z_1 - z_2|}|z - z_2|^2 - \frac{e^{-\nu_2}}{|z_1 - z_2|}|z - z_1|^2 \right)^2. \tag{6.232}$$

Evaluation of the action (6.194) on the Ansatz (6.231), (6.232) is done in [144]. The result is

$$b^2 S_0 = 2i\pi(N_1 + N_2) - \log \lambda - 2 - (\nu_1 - \nu_2). \tag{6.233}$$

Comparing (6.233) with (6.230) fixes the constant C:

$$C = 2i\pi(N_1 + N_2) - \log \lambda - 2. \tag{6.234}$$

Inserting this value of C in (6.230) we indeed obtain (6.203) if we set

$$N_1 = M_1, \tag{6.235}$$

$$N_2 = M_2 \mp 1, \tag{6.236}$$

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and

$$2\pi\sigma = \nu_1 - \nu_2 \,. \tag{6.237}$$

Some comments are in order at this point:

1. The action (6.230) satisfies the Polyakov relation (6.192) with the accessory parameters defined in (6.210):

$$b^2 \frac{\partial S_{\rm cl}^{\rm def}}{\partial z_i} = (-)^{i+1} \frac{2\eta(1-\eta)}{z_1 - z_2} \,, \qquad i = 1, 2 \,. \tag{6.238}$$

2. In eq. (6.199) M takes nonnegative integer values if $\text{Im}(2\eta - 1)/b^2 > 0$, and nonpositive integer values if $\text{Im}(2\eta-1)/b^2 < 0$. Therefore N_1 also runs over nonnegative or nonpositive integer values depending on the sign of $\text{Im}(2\eta-1)/b^2$, and N_2 takes values $\{1, 2, \ldots\}$, when $\text{Im}(2\eta - 1)/b^2 > 0$ and N_2 takes values $\{-1, -2, \ldots\}$, when $\text{Im}(2\eta - 1)/b^2 < 0$. The fact that for the different values of the parameter η we should take contribution of the different set of the saddle points is known as the Stokes phenomena, and was studied in detail for two- and three-point correlation functions of the Liouville field theory in [105]. Recall that it is caused by the fact that the sum (6.199) converges for the different values of M depending on the sign of $\text{Im}(2\eta-1)/b^2$. The values of parameters at which the jump of the set of the contributing saddle point occurs define a (anti-) Stokes line. As we explained in introduction the Stokes or anti-Stokes lines arise when at some values of parameters of the system imaginary or real parts of actions evaluated at the different saddle points coincide [105, 126, 188]. From (6.203) or (6.230) we see that $\text{Re}\, S_{N_1, N_2}^{\rm def}$ are the same for all N_1 and N_2 if $\text{Im}(2\eta - 1) = 0$. The line $\text{Im}(2\eta - 1) = 0$ is the anti-Stokes line at which indeed we observe jump in the set of the contributing saddle points. The jump is caused by the fact that at this line the magnitudes of the amplitudes of all saddle points coincide.

3. The discussion above of the differences between calculation of two-point function with and without defect suggests nice interpretation of the defect operator. We have seen that there exist two sources of discontinuity giving rise to the corresponding terms in the defect operators. The heavy asymptotic limit of $D(\alpha)$ (6.205) has an exponential in

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the numerator and sinus in the denominator. The exponential term in the numerator as we have seen originates from the discontinuity created by the choice of the different parameters ν_1 and ν_2. The correspondence between the N_i and M_i parameters makes clear that the different logarithmic branch solutions, given by N_1 and N_2, are responsible for the quadratic term $\sin^2 \pi \left(\frac{2\eta-1}{b^2} \right)$ term in the (6.201). On the other hand, as we have mentioned before, in the heavy asymptotic limit calculation of the usual two-point function one has $N_1 = N_2$, and it reflects the presence of the term $\sin \pi \left(\frac{2\eta-1}{b^2} \right)$ in the denominator of (6.204) in the first degree. Therefore the denominator $\sin \pi \left(\frac{2\eta-1}{b^2} \right)$ in $D(\alpha)$ reflects the possibility of the choice of different logarithmic branches with $N_1 \neq N_2$ in the solution of the defect equations of motion. The final quantum expression (6.83) results from the quantum corrections restoring $b \leftrightarrow b^{-1}$ duality of the Liouville theory.

Let us analyze in the heavy asymptotic limit also the relation (6.78) between parameter s and $A(b)$

$$2 \cosh 2\pi b s = A(b) \left(\frac{W(-b/2)}{W(0)} \right)^2 . \tag{6.239}$$

It is easy to compute that

$$\lim_{b \to 0} \frac{W(-b/2)}{W(0)} = -\frac{2}{\sqrt{\lambda}} . \tag{6.240}$$

Setting that $s = \frac{\sigma}{b}$, we get

$$\cosh 2\pi \sigma = \frac{2A(0)}{\lambda} . \tag{6.241}$$

This implies that parameter κ is proportional to $A(0)$:

$$\kappa = \frac{2A(0)}{\lambda} . \tag{6.242}$$

Note that as in the light asymptotic limit as well as in the heavy asymptotic limit we get the same relation between σ and κ

$$\kappa = \cosh 2\pi \sigma . \tag{6.243}$$

Chapter 7

Branes in (2,2,2,2) Gepner model

7.1 Simple current extension: brief review

Let us briefly remind the meaning of the simple current extension by simple currents of integral conformal weight [28, 70–72, 169], [73]. A primary J is called a simple current if, fused with any other primary λ, it yields just a single field $J\lambda$. Simple current extension means the combination of two operations:

- Projection. We keep only fields which obey $Q_J(\lambda) = 0$ where

$$Q_J(\lambda) = \Delta_\lambda + \Delta_J - \Delta_{J\lambda} \pmod{Z} \tag{7.1}$$

- Extension. We extend the chiral algebra by including the simple current J. This means that we organize the fields surviving the projection into orbits derived as a result of fusion with the simple current J.

Before writing the torus partition function we should discuss the important issue of fixed point resolution. If all the primaries form orbits of the same length, equal to the order $|G|$ of the full group G generated by the simple currents, or in other words have the same number of images under the repeated fusion with the simple current, the characters could be labelled by the primaries chosen, one from each orbit, called orbit representatives, and have the form:

$$\tilde{\chi}_{\hat{\lambda}} = \sum_{J \in G} \chi_{J\lambda} \tag{7.2}$$

The unitary matrix representing modular transformations on the extended theory is:

$$\tilde{S}_{\hat{a},\hat{b}} = |G| S_{ab} \tag{7.3}$$

where with hatted variables we denoted the orbit representatives. However it may happen that some of the primaries have a non-trivial stabilizer $\mathcal{S}_\lambda$, i.e. be fixed under the action of currents of a subgroup $\mathcal{S}_\lambda \in G$. In this case the freely acting group is the factor $G_a = G/\mathcal{S}_a$ and the orbit length is given by

$$|G_a| = \frac{|G|}{|\mathcal{S}_a|} \tag{7.4}$$

and therefore varies from orbit to orbit. The simple formula (7.3) for the modular transformation matrix does not work anymore. It turns out that in order to construct a unitary matrix representation of the modular transformation in this case one needs to resolve the primaries with non-trivial stabilizer, i.e. one should consider together with the orbit $\hat{a}$ additional $|\mathcal{S}_a|$ orbits*. Labelling the additional orbits by i we find the characters:

$$\tilde{\chi}_{\hat{\lambda},i} = m_{\hat{\lambda},i} \sum_{J \in G/\mathcal{S}_\lambda} \chi_{J\lambda} = \frac{m_{\hat{\lambda},i}}{|\mathcal{S}_\lambda|} \sum_{J \in G} \chi_{J\lambda} \tag{7.5}$$

where $m_{i,a}$ are usually equal to 1, but we keep them explicitly so as to keep track of the different resolved orbits.

The diagonal modular invariant torus partition function of the extended theory reads

$$Z_{\text{ext}} = \sum_{\hat{\lambda},i} |\tilde{\chi}_{\lambda,i}|^2 = \sum_{\text{orbits } Q(\lambda)=0} |\mathcal{S}_\lambda| \cdot |\sum_{J \in G/\mathcal{S}_\lambda} \chi_{J\lambda}|^2 \tag{7.6}$$

where we used that

$$|\mathcal{S}_a| = \sum_i (m_{a,i})^2 \tag{7.7}$$

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*Actually each primary should be resolved by the order of the subgroup $\mathcal{U}_a$ of the stabilizer, called untwisted stabilizer [70], on which a certain alternating $U(1)$-valued bihomomorphism, or discrete torsion, on the stabilizer $\mathcal{S}_a$ vanishes. It is well-known that discrete torsions are classified by the second $U(1)$-valued cohomology group $H^2(\mathcal{S}_a, U(1))$ [182], and since in Gepner models with diagonal (or charge conjugation) torus partition function – the situation of our interest below – the stabilizers are all isomorphic to the Z_2 group, for which $H^2(Z_2, U(1)) = 0$, one finds that the untwisted stabilizer coincides with stabilizer.

The unitary matrix representation of the modular transformation S on the characters (7.5), was constructed in [169], [70]. The following ansatz was suggested

$$\tilde{S}_{(a,i),(b,j)} = m_{a,i} m_{b,j} \frac{|G_a||G_b|}{|G|} S_{a,b} + \Gamma_{(a,i),(b,j)} \tag{7.8}$$

where $\Gamma_{(a,i),(b,j)}$ satisfies the equation

$$\sum_j \Gamma_{(a,i),(b,j)} m_{b,j} = 0 \tag{7.9}$$

and it is therefore different from zero only between fixed points. It was found in [169] that unitarity requires $\Gamma_{(a,i),(b,j)}$ to satisfy the condition:

$$\sum_{\text{orbits } Q(b)=0,j} \Gamma_{(a,i),(b,j)} \Gamma^*_{(c,k),(b,j)} = \delta_{ac}(\delta_{ik} - \frac{m_{a,i} m_{a,k}}{|S_a|}) \quad . \tag{7.10}$$

Using the matrix (7.8) one can compute the fusion rule coefficients using Verlinde formula and the annulus partition functions for the Cardy states. After some algebra we arrive at the expression:

$$
\begin{aligned}
A_{(a,i),(d,e)} &= \sum_{\text{orbits } Q(c)=0} \sum_{J \in G} \frac{m_{a,i} m_{d,e} \mathcal{N}^d_{Ja,c}}{|\mathcal{S}_a||\mathcal{S}_d|} \sum_{K \in G_c} \chi_{Kc} \\
&+ \sum_{\text{orbits } Q(c)=0} \sum_{(\text{orbits } Q(b)=0,j)} \frac{\Gamma_{(a,i),(b,j)} S_{c,b} \Gamma^*_{(b,j),(d,e)}}{S_{0,b}} \sum_{K \in G_c} \chi_{Kc}
\end{aligned}
\tag{7.11}
$$

Given that the resolving matrix $\Gamma_{(a,i),(b,j)}$ are different from zero only between fixed points we observe that formula (7.11) simplifies if one of the states is not fixed. When a is not fixed and d fixed (7.11) simplifies to

$$A_{(a),(d,e)} = \sum_{\text{orbits } Q(c)=0} \sum_{J \in G} \frac{m_{d,e} \mathcal{N}^d_{Ja,c}}{|\mathcal{S}_d|} \sum_{K \in G_c} \chi_{Kc} \tag{7.12}$$

When neither a nor d are fixed (7.11) further simplifies to

$$A_{ad} = \sum_{\text{orbits } Q(c)=0} \sum_{J \in G} \mathcal{N}^d_{Ja,c} \sum_{K \in G_c} \chi_{Kc} \tag{7.13}$$

For later application to Gepner models let us discuss the matrix $\Gamma_{(a,i),(b,j)}$ and the second term in (7.11) in the case when all the fixed points have a stabilizer isomorphic to Z_2. In this case

equations (7.9) and (7.10) can be satisfied by taking $\Gamma_{(a,i),(b,j)}$ in the form:

$$\Gamma_{(a,\psi),(b,\psi')} = \frac{|G_a||G_b|}{|G|} \hat{S}_{ab} \psi\psi' \delta_{af}\delta_{bf} \tag{7.14}$$

where ψ is the resolving index which takes two values $\pm$, and $\hat{S}_{ab}$ is a unitary matrix. Plugging (7.14) in (7.11) for the the second term one can write:

$$\frac{1}{|\mathcal{S}_a||\mathcal{S}_d|}\psi\psi'' \sum_{\text{Orbits } Q(c)=0} \sum_b \sum_{J\in G} \frac{\hat{S}_{Ja,b}S_{c,b}\hat{S}^*_{b,d}}{S_{0,b}} \left(\sum_{K\in G_c} \chi_{Kc} \right) \delta_{af}\delta_{bf}\delta_{df} \tag{7.15}$$

We also show in [162] that formulae (7.13) and (7.11) are actually equivalent to the formulae for the A-type annulus partition functions derived in [150] and [36].

7.2   Gepner models: generalities

Let us remind the basic facts about Gepner models [84]. The starting point of a Gepner model is the tensor product theory

$$\mathcal{C}^{\text{s-t}}_{k_1,\cdots,k_n} = \mathcal{C}^{\text{s-t}} \otimes \mathcal{C}_{k_1} \otimes \cdots \otimes \mathcal{C}_{k_n}, \tag{7.16}$$

where $\mathcal{C}^{\text{s-t}}$ is the D dimensional flat space-time part, and $\mathcal{C}_k$ is one of the $N=2$ minimal models, whose central charges $c_k = \frac{3k}{k+2}$ satisfy the relation

$$\sum_{i=1}^n c_{k_i} + \frac{3}{2}(D-2) = 12 \tag{7.17}$$

$N=2$ minimal models can be described as cosets $SU(2)_k \times U(1)_4/U(1)_{2k+4}$. Accordingly the primaries of $\mathcal{C}_k$ are labelled by three integers (l,m,s) with ranges $l \in (0,\cdots k)$, $m \in (-k-1,\cdots,k+2)$, $s \in (-1,0,1,2)$, subject to the selection rule $l+m+s \in 2Z$ and the field identification $(l,m,s) \equiv (k-l,m+k+2,s+2)$. Primaries with even values of s belong to the NS sector, while primaries with odd values of s belong to the R sector. The conformal dimension and charge of the primary (l,m,s) are given by:

$$h^l_{m,s} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \quad (\text{mod } 1) \tag{7.18}$$

$$q^l_{m,s} = \frac{m}{k+2} - \frac{s}{2} \quad (\text{mod } 2) \tag{7.19}$$

The exact dimensions and charges can be read off (7.18) and (7.19) using field identifications to bring (l, m, s) into the standard range

$$l \in (0, \cdots k), \quad |m - s| \leq l, \quad s \in (-1, 0, 1, 2) \tag{7.20}$$

The characters are given by

$$\chi_{m,s}^{l\,(k)}(z, \tau) = \sum_{j=0}^{k-1} c_{m+4j-s}^{l\,(k)}(\tau) \Theta_{2m+(4j-s)(k+2), 2k(k+2)}(z, \tau) \tag{7.21}$$

where

$$\Theta_{M,N}(z, \tau) = \theta \begin{bmatrix} \frac{M}{2N} \\ 0 \end{bmatrix} (z, 2N\tau) = \sum_{n \in Z} e^{2\pi i \tau N \left(n + \frac{M}{2N}\right)^2} e^{2i\pi z \left(n + \frac{M}{2N}\right)} \tag{7.22}$$

that obviously satisfy the identity

$$\Theta_{M+2N,N} = \Theta_{M,N} \tag{7.23}$$

and $c_m^{l\,(k)}$ are the characters of the parafermionic field theory at level k, satisfying identities:

$$c_m^{l\,(k)} = c_{-m}^{l\,(k)} = c_{m+2k}^{l\,(k)} = c_{k\pm m}^{k-l\,(k)} \tag{7.24}$$

The fusion coefficients are

$$\mathcal{N}_{m_1 m_2 m_3 s_1 s_2 s_3}^{N=2\ l_1 l_2 l_3} = \tag{7.25}$$

$$\mathcal{N}_{l_1 l_2}^{SU(2)\ l_3} \delta_{m_1 + m_2 - m_3} \delta_{s_1 + s_2 - s_3} + \mathcal{N}_{l_1 l_2}^{SU(2)\ k - l_3} \delta_{m_1 + m_2 - (m_3 + k + 2)} \delta_{s_1 + s_2 - (s_3 + 2)}$$

The space-time part can be described in terms of the $SO(D-2)_1$ algebra. $SO(2n)_1$ algebras have four primaries $\lambda = (o, v, s, c)$, with conformal dimensions

$$h_o = 0, \quad h_v = \frac{1}{2}, \quad h_s = h_c = \frac{n}{8} \tag{7.26}$$

charges

$$q_o = 0, \quad q_v = 1, \quad q_s = \frac{n}{2}, \quad q_c = \frac{n}{2} - 1 \tag{7.27}$$

and characters:

$$\chi_O^{SO(2n)} = \frac{1}{2\eta^n}(\theta_3^n + \theta_4^n) \tag{7.28}$$

$$\chi_V^{SO(2n)} = \frac{1}{2\eta^n}(\theta_3^n - \theta_4^n)$$

$$\chi_s^{SO(2n)} = \frac{1}{2\eta^n}(\theta_2^n + i^{-n}\theta_1^n)$$

$$\chi_c^{SO(2n)} = \frac{1}{2\eta^n}(\theta_2^n - i^{-n}\theta_1^n)$$

O and V primaries belong to the NS sector, while S and C belong to the R sector. For future use, let us write down also the fusion rules of the $SO(2n)_1$ algebras.

n odd	o	v	s	c
o	o	v	s	c
v	v	o	c	s
s	s	c	v	o
c	c	s	o	v

$$(7.29)$$

n even	o	v	s	c
o	o	v	s	c
v	v	o	c	s
s	s	c	o	v
c	c	s	v	o

$$(7.30)$$

The primaries of the product theory (7.16) can be labelled by the following collection of indices

$$(\lambda, \vec{l}, \vec{m}, \vec{s})) = (\lambda, l_1, m_1, s_1, \cdots, l_n, m_n, s_n) \tag{7.31}$$

The Gepner model is the simple current extension of the product $\mathcal{C}^{\text{s-t}}_{k_1, \cdots, k_n}$, with the following simple currents:

- supersymmetry current: $S_{\text{tot}} = (s, (0, 1, 1), \cdots (0, 1, 1))$

- alignment currents: $V_i = (v, \cdots (0, 0, 2) \cdots)$, with $(0, 0, 2)$ at the ith position.

Let us summarize the results of applying the formalism reviewed in the previous section to Gepner models [28], [36], [73]. In Gepner models the simple current projection or, in the

original Gepner's language, β-projection with respect to the supersymmetry current S_{tot} reads

$$Q_{(\omega,\vec{l},\vec{m},\vec{s}))} = q_\omega + \sum_{i=1}^{n} q_{m_i,s_i}^{l_i} = 1 \pmod{2Z} \tag{7.32}$$

and is nothing else than the famous GSO projection yielding space-time supersymmetry[†]. The projection with respect to the alignment current selects only primaries were all constituent primaries belong to the same sector, either NS either R and guarantees world-sheet supersymmetry.

To analyze the length of the orbits we should consider two cases:

1. all the levels k_i are odd

    In this case no fixed point occurs, all the primaries have trivial stabilizer, and the length of the S_{tot} current is $K = \text{lcm}\{4, 2k_i + 4\}$. All V_i currents always act freely and have length 2. But when all the k_i levels are odd, it turns out that the S_{tot} current has an overlap with the V_i currents, and to cover all orbit it is enough to sum over only $n-1$ of the n V_i currents. As a result, the orbit length in this case is $2^{n-1}K$.

2. one has $r \neq 0$ even levels k_i

Let us place the even levels in the first r positions. In this case for a generic primary the orbit length of the supersymmetry current is again $K = \text{lcm}\{4, 2k_i + 4\}$. But for the primaries with all l_i at the first r positions equal $\frac{k_i}{2}$:

$$l_i = \frac{k_i}{2} \quad i = 1, \ldots, r \tag{7.33}$$

due to the previously discussed field identification, which for them reads :

$$(\frac{k_1}{2}, m_1, s_1, \cdots, \frac{k_r}{2}, m_r, s_r, l_{r+1}, m_{r+1}, s_{r+1} \cdots, l_n, m_n, s_n) \equiv \tag{7.34}$$
$$(\frac{k_1}{2}, m_1 + k_1 + 2, s_1 + 2, \cdots, \frac{k_r}{2}, m_r + k_r + 2, s_r + 2, l_{r+1}, m_{r+1}, s_{r+1} \cdots, l_n, m_n, s_n)$$

there is a non-trivial stabilizer:

$$\mathcal{S}_{\vec{m},\vec{s}}^{\frac{k_1}{2},\cdots\frac{k_r}{2},l_{r+1},\cdots,l_n} = Z_2. \tag{7.35}$$

[†]Actually direct application of the formula (7.1) brings to shift 1 with respect to (7.32), but as explained in [36] and [73] the shift is absorbed by the superghost part, or alternatively by the bosonic string map.

We see that the stabilizer depends only on the values of l_i's $i = 1, \ldots, r$ and one can write:

$$|\mathcal{S}^{l_1, \cdots l_r}| = 1 + \delta_{l_1 \frac{k_1}{2}} \cdots \delta_{l_r \frac{k_r}{2}} \tag{7.36}$$

Therefore here we have two kinds of orbits, long orbits with length $2^n K$ for generic primary, and short orbits with length $2^{n-1} K$ for primaries of type (7.33). As we explained the short orbits should be resolved and acquire an additional label ψ taking two values, which we choose to be a sign $\psi = \pm$.

7.3   The (2,2,2,2) Gepner model

From now on we will specialize to the case of the $(2, 2, 2, 2)$ Gepner model, that corresponds to a compactification down to six dimensions. The flat part is described by an $SO(4)_1$ algebra. In order to write down the characters of the model, first of all we note that using the fusion rules (7.30) one can check that the subgroup generated by the currents S^2_{tot} and $V_i V_j$ has trivial action on the space-time part. The length of the S^2_{tot} current is $\frac{K}{2} = 4$. Using (7.32) we find it convenient to choose the primaries in the form $\{v, (l_1, m_1, s) \cdots, (l_n, m_n, s_n)\}$, with prescribed space-time part v, and neutral internal part, i.e.

$$\sum_{i=1}^{4} q^{l_i}_{m_i, s_i} = 0 \ \ (\text{mod} \ \ 2Z) \tag{7.37}$$

Now one can express the Gepner extension characters $\chi^{G \, \vec{l}}_{(\vec{m}, \vec{s})}$ in the form

$$\chi^{G \, \vec{l}}_{(\vec{m}, \vec{s})} = \frac{1}{|\mathcal{S}^{\vec{l}}_{\vec{m}, \vec{s}}|} (\mathcal{X}_v - \mathcal{X}_c + \mathcal{X}_o - \mathcal{X}_s) \tag{7.38}$$

where

$$\mathcal{X}_v = \frac{\chi^{SO(4)}_v}{\eta^4} A(m_1, s_1, m_2, s_2, m_3, s_3, m_4, s_4) \tag{7.39}$$

$$\mathcal{X}_c = \frac{\chi^{SO(4)}_c}{\eta^4} A(m_1 + 1, s_1 + 1, m_2 + 1, s_2 + 1, m_3 + 1, s_3 + 1, m_4 + 1, s_4 + 1)$$

$$\mathcal{X}_o = \frac{\chi^{SO(4)}_o}{\eta^4} A(m_1, s_1 + 2, m_2, s_2, m_3, s_3, m_4, s_4)$$

$$\mathcal{X}_s = \frac{\chi^{SO(4)}_s}{\eta^4} A(m_1 + 1, s_1 + 3, m_2 + 1, s_2 + 1, m_3 + 1, s_3 + 1, m_4 + 1, s_4 + 1)$$

with

$$A(m_i, s_i) = \sum_{\nu_0=0}^{3} \sum_{\nu_1=0,2} \sum_{\nu_2=0,2} \sum_{\nu_3=0,2} \chi_{m_1+2\nu_0,\, s_1+\nu_1+\nu_2+\nu_3+2\nu_0}^{l_1(2)}(z_1) \cdot \tag{7.40}$$

$$\chi_{m_2+2\nu_0,\, s_2+\nu_1+2\nu_0}^{l_2(2)}(z_2) \cdot \chi_{m_3+2\nu_0,\, s_3+\nu_2+2\nu_0}^{l_3(2)}(z_3) \cdot \chi_{m_4+2\nu_0,\, s_4+\nu_3+2\nu_0}^{l_4(2)}(z_4)$$

and, as explained above,

$$|\mathcal{S}_{\vec{m},\vec{s}}^{\vec{l}}| = 1 + \delta_{l_1 1} \delta_{l_2 1} \delta_{l_3 1} \delta_{l_4 1} \tag{7.41}$$

Using (7.21), (7.24) and (18) for the characters of the $k = 2$ minimal model one obtains the following simple expression

$$\chi_{m,s}^{l(2)}(z) = c_{m-s}^{l(2)}(\tau) \Theta_{4q,4}\left(\frac{z}{2}, \tau\right) \tag{7.42}$$

where $q = \frac{m}{4} - \frac{s}{2}$, and $c_m^{l\,(2)}$ are related to the Ising characters:

$$c_0^{0(2)} = \frac{1}{2\eta}\left(\sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}}\right) \tag{7.43}$$

$$c_0^{2(2)} = c_2^{0(2)} = \frac{1}{2\eta}\left(\sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}}\right) \tag{7.44}$$

$$c_1^{1(2)} = \frac{1}{\eta}\sqrt{\frac{\theta_2}{2\eta}} \tag{7.45}$$

Now let us compute $A(m_i, s_i)$. Repeatedly using theta functions product formulae from appendix .3, we have

$$A(m_i, s_i) = \Theta_{q_{\text{tot}},1}\left(\frac{z_{\text{tot}}}{8}, \tau\right) B(m_i, s_i) \tag{7.46}$$

where

$$z_{\text{tot}} = z_1 + z_2 + z_3 + z_4 \tag{7.47}$$

$$q_{\text{tot}} = q_1 + q_2 + q_3 + q_4 = \text{even} \tag{7.48}$$

and

$$B(m_i, s_i) = \tag{7.49}$$

$$\sum_{\nu_1, \nu_2, \nu_3 = 0, 2} c_{m_1-(s_1+\nu_1+\nu_2+\nu_3)}^{l_1(2)} c_{m_2-(s_2+\nu_1)}^{l_2(2)} c_{m_3-(s_3+\nu_2)}^{l_3(2)} c_{m_4-(s_4+\nu_3)}^{l_4(2)} \cdot$$

$$\sum_{a=0,2} \Theta_{(q_1-q_2+q_3-q_4)-\nu_2+a,2}(y_1, 2\tau) \cdot$$

$$\Theta_{(q_1-q_2-q_3+q_4)-\nu_3+a,2}(y_2, 2\tau) \cdot \Theta_{(q_1+q_2-q_3-q_4)-\nu_1+a,2}(y_3, 2\tau)$$

where

$$y_1 = \frac{z_1 - z_2 + z_3 - z_4}{4}, \quad y_2 = \frac{z_1 - z_2 - z_3 + z_4}{4}, \quad y_3 = \frac{z_1 + z_2 - z_3 - z_4}{4} \tag{7.50}$$

Note that $B(m_i, s_i) = B(m_i + 1, s_i + 1)$. Using (7.46), (7.28), (21), (23), this allows us to write for (7.38):

$$\chi_{(\vec{m},\vec{s})}^{G\,\vec{l}} = \tag{7.51}$$

$$\frac{1}{\eta^6 |\mathcal{S}_{\vec{m},\vec{s}}^{\vec{l}}|} \Bigg[\left(\theta_2^2(2\tau)\theta_3 \left(\frac{z_{\text{tot}}}{8}, 2\tau \right) - \theta_2(2\tau)\theta_3(2\tau)\theta_2 \left(\frac{z_{\text{tot}}}{8}, 2\tau \right) \right) B(m_i, s_i)$$

$$+ \left(\theta_3^2(2\tau)\theta_2 \left(\frac{z_{\text{tot}}}{8}, 2\tau \right) - \theta_2(2\tau)\theta_3(2\tau)\theta_3 \left(\frac{z_{\text{tot}}}{8}, 2\tau \right) \right) B(m_i, s_1 + 2, s_i)) \Bigg]$$

We see that whenever

$$z_1 + z_2 + z_3 + z_4 = 0 \tag{7.52}$$

the Gepner extension characters are supersymmetric. This plays a role in the study of magne-tized D-branes and in the computations of threshold connections [10].

From now on we put all $z_i = 0$. For this case the character (7.51) can be equivalently written as

$$\chi_{(\vec{m},\vec{s})}^{G\,\vec{l}} = \frac{\mathcal{J}}{2|\mathcal{S}_{\vec{m},\vec{s}}^{\vec{l}}|\eta^6} \left(\frac{B(m_i, s_i)}{\theta_3(0, 2\tau)} + \frac{B(m_i, s_1 + 2, s_i)}{\theta_2(0, 2\tau)} \right) \tag{7.53}$$

where $\mathcal{J} = \frac{1}{2}(\theta_3^4(0,\tau) - \theta_4^4(0,\tau) - \theta_2^4(0,\tau))$ is zero thanks to Jacobi *aequatio identica satis abstrusa*. Using (7.49) and taking into account that

$$\Theta_{\nu,2}(z,\tau) = \eta \chi_\nu^{\text{SO}(2)}\left(\frac{z}{2}, \tau \right) \tag{7.54}$$

as well as (7.43), (7.44), (7.45), we are now in a position to compute the characters for the various orbits.

To this end, we are going to present all the primaries of the model, or in other words to list all the orbit representatives. Surely one can pick up orbit representatives in many different ways. To be sure that we have not taken two primaries, belonging to the same orbit, one can resort to some kind of "gauge fixing". The gauge fixing chosen here, is the following.

1. We take the space-time part to be always v, as mentioned above.

2. we take $s_2 = s_3 = s_4 = 0$

3. we limit m_1 to the values 0 and 1.

4. to avoid taking primaries equivalent due to field identification, we always limit the values of the l_i to be 0 or 1.

The final picture is the following.

In this model we can divide primaries in 4 big groups.

The first group has $l_1 = l_2 = l_3 = l_4 = 0$, $s_1 = s_2 = s_3 = s_4 = 0$ and contains 16 primaries. We can divide them into three groups: K_1, K_2 and K_3. All primaries in the same group have the same conformal weights and characters. The results are presented in the tables below. It is understood that all the entries should be multiplied by $\frac{\mathcal{J}}{\eta^{12}}$.

K_1	$h_{K_1} = \frac{1}{2}$
$K_1 = (v)(0,0,0)(0,0,0)(0,0,0)(0,0,0)$	$\chi_{K_1}^G = \frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{16} + 3\frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}$

$$(7.55)$$

K_2	$h_{K_2} = 1$	
$K_{2a} = (v)(0,0,0)(0,0,0)(0,-2,0)(0,2,0)$		
$K_{2b} = (v)(0,0,0)(0,-2,0)(0,0,0)(0,2,0)$		
$K_{2c} = (v)(0,0,0)(0,2,0)(0,-2,0)(0,0,0)$		
$K_{2d} = (v)(0,0,0)(0,0,0)(0,2,0)(0,-2,0)$		
$K_{2e} = (v)(0,0,0)(0,-2,0)(0,2,0)(0,0,0)$		
$K_{2f} = (v)(0,0,0)(0,2,0)(0,0,0)(0,-2,0)$	$\chi_{K_2}^G = \frac{\theta_3^4(0,\tau)-\theta_4^4(0,\tau)}{16}$	(7.56)
$K_{2g} = (v)(0,0,0)(0,4,0)(0,2,0)(0,2,0)$		
$K_{2h} = (v)(0,0,0)(0,2,0)(0,4,0)(0,2,0)$		
$K_{2k} = (v)(0,0,0)(0,2,0)(0,2,0)(0,4,0)$		
$K_{2l} = (v)(0,0,0)(0,4,0)(0,-2,0)(0,-2,0)$		
$K_{2m} = (v)(0,0,0)(0,-2,0)(0,4,0)(0,-2,0)$		
$K_{2n} = (v)(0,0,0)(0,-2,0)(0,-2,0)(0,4,0)$		

K_3	$h_{K_3} = \frac{3}{2}$	
$K_{3a} = (v)(0,0,0)(0,0,0)(0,4,0)(0,4,0)$		
$K_{3b} = (v)(0,0,0)(0,4,0)(0,0,0)(0,4,0)$	$\chi_{K_3}^G = \frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{16} - \frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}$	(7.57)
$K_{3c} = (v)(0,0,0)(0,4,0)(0,4,0)(0,0,0)$		

The second group has $l_1 = l_2 = l_3 = l_4 = 0$, $s_1 = 2$, $s_2 = s_3 = s_4 = 0$ and also contains 16 primaries, which again can be divided into 3 subgroups, in such a way that all primaries inside each group have the same characters.

L_1	$h_{L_1} = 1$
$L_{1a} = (v)(0,0,2)(0,4,0)(0,4,0)(0,4,0)$	
$L_{1b} = (v)(0,0,2)(0,0,0)(0,0,0)(0,4,0)$	
$L_{1c} = (v)(0,0,2)(0,0,0)(0,4,0)(0,0,0)$	$\chi^G_{L_1} = \frac{\theta_3^4(0,\tau) - \theta_4^4(0,\tau)}{16}$
$L_{1d} = (v)(0,0,2)(0,4,0)(0,0,0)(0,0,0)$	

$$(7.58)$$

L_2	$h_{L_2} = \frac{1}{2}$
$L_{2a} = (v)(0,0,2)(0,4,0)(0,2,0)(0,-2,0)$	
$L_{2b} = (v)(0,0,2)(0,-2,0)(0,2,0)(0,4,0)$	
$L_{2c} = (v)(0,0,2)(0,-2,0)(0,4,0)(0,2,0)$	$\chi^G_{L_2} = \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{16} + \frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}$
$L_{2d} = (v)(0,0,2)(0,4,0)(0,-2,0)(0,2,0)$	
$L_{2e} = (v)(0,0,2)(0,2,0)(0,-2,0)(0,4,0)$	
$L_{2f} = (v)(0,0,2)(0,2,0)(0,4,0)(0,-2,0)$	

$$(7.59)$$

L_3	$h_{L_3} = \frac{3}{2}$
$L_{3a} = (v)(0,0,2)(0,0,0)(0,2,0)(0,2,0)$	
$L_{3b} = (v)(0,0,2)(0,2,0)(0,0,0)(0,2,0)$	
$L_{3c} = (v)(0,0,2)(0,2,0)(0,2,0)(0,0,0)$	$\chi^G_{L_3} = \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{16} - \frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}$
$L_{3d} = (v)(0,0,2)(0,0,0)(0,-2,0)(0,-2,0)$	
$L_{3e} = (v)(0,0,2)(0,-2,0)(0,-2,0)(0,0,0)$	
$L_{3f} = (v)(0,0,2)(0,-2,0)(0,0,0)(0,-2,0)$	

$$(7.60)$$

The third group containing 48 primaries with any two of l_i equal to 1, and other two of them to 0. This group consists of 6 subgroups:

$$l_1 = l_2 = 1 \quad l_3 = l_4 = 0 \tag{7.61}$$

$$l_1 = l_3 = 1 \quad l_2 = l_4 = 0$$

$$l_1 = l_4 = 1 \quad l_2 = l_3 = 0$$

$$l_2 = l_3 = 1 \quad l_1 = l_4 = 0$$

$$l_2 = l_4 = 1 \quad l_1 = l_3 = 0$$

$$l_3 = l_4 = 1 \quad l_1 = l_2 = 0$$

Each such a subgroup consists of 8 primaries and can be derived from, let's say, the first of them by permutations, so we will write down only one of them, the one with $l_1 = l_2 = 1$ and $l_3 = l_4 = 0$. We schematically denote the primaries in this group as $\Phi_{1a}^{1,1,\cdot,\cdot}$, indicating explicitly in the superscript which l_i are equal to 1.

Φ_1	$h_{\Phi_1} = \frac{3}{4}$
$\Phi_{1a}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,3,0)(0,2,0)(0,2,0)$	
$\Phi_{1b}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,-1,0)(0,0,0)(0,0,0)$	$\chi_{\Phi_1}^G = \frac{\theta_2^2(0,\tau)(\theta_3^2(0,\tau)+\theta_4^2(0,\tau))}{8}$

(7.62)

Φ_2	$h_{\Phi_2} = \frac{5}{4}$
$\Phi_{2a}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,3,0)(0,0,0)(0,4,0)$	
$\Phi_{2b}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,-1,0)(0,-2,0)(0,2,0)$	$\chi_{\Phi_2}^G = \frac{\theta_2^2(0,\tau)(\theta_3^2(0,\tau)-\theta_4^2(0,\tau))}{8}$

(7.63)

Φ_3	$h_{\Phi_3} = 1$
$\Phi_{3a}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,-3,0)(0,2,0)(0,0,0)$	
$\Phi_{3b}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,-3,0)(0,0,0)(0,2,0)$	$\chi_{\Phi_3}^G = \frac{\theta_3^4(0,\tau)-\theta_4^4(0,\tau)}{8}$

(7.64)

Φ_4	$h_{\Phi_4} = \frac{1}{2}$
$\Phi_{4a}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,1,0)(0,-2,0)(0,0,0)$	
$\Phi_{4a}^{1,1,\cdot,\cdot} = (v)(1,1,0)(1,1,0)(0,0,0)(0,-2,0)$	$\chi_{\Phi_4}^G = \frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{8}$

(7.65)

Finally we have a small group containing only 4 elements with $l_1 = l_2 = l_3 = l_4 = 1$, $s_1 = s_2 = s_3 = s_4 = 0$. All primaries in this group, as we explained in section 7.2, have a short orbit and should be resolved. After resolution we end up with 8 primaries. The $\pm$ in the notations refers to the resolution process.

R_1	$h_{R_1} = 1$
$R_{1a\pm} = (v)(1,1,0)(1,-1,0)(1,1,0)(1,-1,0)_\pm$	
$R_{1b\pm} = (v)(1,1,0)(1,-1,0)(1,-1,0)(1,1,0)_\pm$	$\chi^G_{R_1} = \frac{\theta_2^4(0,\tau)}{8}$
$R_{1c\pm} = (v)(1,1,0)(1,1,0)(1,-1,0)(1,-1,0)_\pm$	

$$\text{(7.66)}$$

R_2	$h_{R_2} = \frac{1}{2}$
$R_{2\pm} = (v)(1,1,0)(1,1,0)(1,-3,0)(1,1,0)_\pm$	$\chi^G_{R_2} = \frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{8}$

$$\text{(7.67)}$$

We see that before fixed points resolution we had 84 orbits: 31 orbits with conformal dimension 1, 12 orbits with conformal dimension $\frac{3}{4}$, 12 orbits with conformal dimension $\frac{5}{4}$, 20 orbits with conformal dimension $\frac{1}{2}$, 9 orbits with conformal dimension $\frac{3}{2}$. After the fixed points resolution we have 88 primaries: 34 orbits with conformal dimension 1, 12 orbits with conformal dimension $\frac{3}{4}$, 12 orbits with conformal dimension $\frac{5}{4}$, 21 orbits with conformal dimension $\frac{1}{2}$, 9 orbits with conformal dimension $\frac{3}{2}$ [190].

Collecting all the above results, we can write down the torus amplitude:

$$
Z = \left|\frac{\mathcal{J}}{\eta^{12}}\right|^2 \left[\left|\left(\frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{16} + 3\frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}\right)\right|^2\right.
$$

$$
16\left|\frac{\theta_3^4(0,\tau)-\theta_4^4(0,\tau)}{16}\right|^2 + 9\left|\frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{16} - \frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}\right|^2
$$

$$
+6\left|\frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{16} + \frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8}\right|^2 + 18\left|\frac{\theta_3^4(0,\tau)-\theta_4^4(0,\tau)}{8}\right|^2
$$

$$
+14\left|\frac{\theta_3^4(0,\tau)+\theta_4^4(0,\tau)}{8}\right|^2 + 12\left|\frac{\theta_2^2(0,\tau)(\theta_3^2(0,\tau)+\theta_4^2(0,\tau))}{8}\right|^2 +
$$

$$
12\left|\frac{\theta_2^2(0,\tau)(\theta_3^2(0,\tau)-\theta_4^2(0,\tau))}{8}\right|^2\left.\right]
$$

$$\text{(7.68)}$$

$$= \left| \frac{\mathcal{J}}{\eta^{12}} \right|^2 \left[\frac{1}{16} \Big(|\theta_3(0,\tau)|^4 + |\theta_4(0,\tau)|^4 + |\theta_2(0,\tau)|^4 \Big)^2 \right.$$
$$+ \frac{1}{4} \Big(|\theta_2(0,\tau)\theta_3(0,\tau)|^4 + |\theta_2(0,\tau)\theta_4(0,\tau)|^4 + |\theta_3(0,\tau)(\theta_4(0,\tau)|^4 \Big)$$
$$\left. + \frac{1}{2} \Big(|\theta_3(0,\tau)|^8 + |\theta_4(0,\tau)|^8 \Big) \right]$$

The partition function (7.68), as first noted in [51], coincides with the partition function of the T^4/Z_4 orbifold at the $SU(2)^4$ point, which we review in appendix .5.

Now we elaborate on the expression (7.11) for the annulus partition function for the $(2, 2, 2, 2,)$ Gepner model.

Let us denote the first Cardy state I:

$$I = (S_0, L_1, M_1, S_1, \cdots, L_4, M_4, S_4) \tag{7.69}$$

and the second J:

$$J = (\tilde{S}_0, \tilde{L}_1, \tilde{M}_1, \tilde{S}_1, \cdots, \tilde{L}_4, \tilde{M}_4, \tilde{S}_4) \tag{7.70}$$

Consider first the case when neither the first boundary state nor the second are fixed.

Now using (7.13) and the fusion coefficients (7.25) we can easily derive:

$$Z_{IJ} = \sum_{s_0} \sum_{l_i} |\mathcal{S}^{l_1 \cdots l_4}| \mathcal{N}_{v(S_0)\tilde{S}_0}^{SO(4)\, s_0} \prod_{i=1}^4 \mathcal{N}_{L_i \tilde{L}_i}^{SU(2)\, l_i} \chi_{M_1 - \tilde{M}_1 \cdots M_4 - \tilde{M}_4, \hat{s_0}, S_1 - \tilde{S}_1 \cdots S_4 - \tilde{S}_4}^{G \hat{l}_1 \cdots \hat{l}_4} \tag{7.71}$$

Actually the sum over J in (7.13) $\sum_{J \in G} \mathcal{N}_{Ja,c}^d$ is running over the orbit of the primary

$$(s_0, l_1, M_1 - \tilde{M}_1, S_1 - \tilde{S}_1 \cdots l_4, M_4 - \tilde{M}_4, S_4 - \tilde{S}_4) \tag{7.72}$$

while the sum over orbits in (7.13) runs over the specific representatives, for examples listed in the tables above. It means that generically in this sum only one term will survive, the specific representative of the orbit of the primary (7.72). If this primary has non-trivial stabilizer, due to field identification the sum over J will produce the representative twice. The fusion $v(S_0)$ in $\mathcal{N}_{v(S_0)\tilde{S}_0}^{SO(4)}$ is due to the bosonic string map [73]. Collecting all pieces we get (7.71). In practice in order to use formula (7.71) one needs to compute the primary (7.72) and then use the action of the simple current to find in the orbit which of the representatives listed in tables above it belongs to, and substitute its character.

Consider next the case when I is not fixed but J is. In this case elaborating on (7.12) we obtain:

$$Z_{IJ} = \frac{1}{|\mathcal{S}_J|} \sum_{s_0} \sum_{l_i} |\mathcal{S}^{l_1 \cdots l_4}| \mathcal{N}^{SO(4)\, s_0}_{v(S_0)\tilde{S}_0} \prod_{i=1}^{4} \mathcal{N}^{SU(2)\, l_i}_{L_i \tilde{L}_i} \chi^{G\, \hat{l}_1 \cdots \hat{l}_4}_{M_1 - \tilde{M}_1 \cdots M_4 - \tilde{M}_4, \hat{s}_0, S_1 - \tilde{S}_1 \cdots S_4 - \tilde{S}_4} \tag{7.73}$$

The last case is when both and I and J are fixed points. To elaborate on this case we need the matrices $S_{c,b}$ and $\hat{S}_{ab}$ in formula (7.15).

The matrix S_{ab} for Gepner models is the product of all the elementary S's and reads:

$$2^4 S^{SO(4)}_{s_0 s_0'} \prod_{i=1}^{4} S^{SU(2)_{k_i}}_{l_i l_i'} S^{U(1)_4}_{s_i s_i'} S^{U(1)_{k+2}}_{m_i m_i'} \tag{7.74}$$

The matrix $\hat{S}_{ab}$ was found in [73]. For the (2,2,2,2) model it has the form

$$-2^4 S^{SO(4)}_{s_0 s_0'} \prod_{i=1}^{4} S^{U(1)_4}_{s_i s_i'} S^{U(1)_{k+2}}_{m_i m_i'} \tag{7.75}$$

The numerical factors come from the field identification.

Plugging (7.74) and (7.75) in (7.15) one obtains:

$$Z_{I\psi J\psi'} = \frac{1}{|\mathcal{S}_I||\mathcal{S}_J|} \sum_{s_0} \sum_{l_i} \mathcal{N}^{SO(4)\, s_0}_{v(S_0)\tilde{S}_0} \cdot \tag{7.76}$$

$$\left(\prod_{i=1}^{4} \mathcal{N}^{SU(2)\, l_i}_{L_i \tilde{L}_i} + \psi\psi' \prod_{i=1}^{4} \sin\pi\frac{l_i+1}{2} \right) \chi^{G\, \hat{l}_1 \cdots \hat{l}_4}_{M_1 - \tilde{M}_1 \cdots M_4 - \tilde{M}_4, \hat{s}_0, S_1 - \tilde{S}_1 \cdots S_4 - \tilde{S}_4}$$

7.4 $D0$-branes on the T^4/Z_4 orbifold.

7.4.1   Fixed points

Defining complex coordinates $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ the Z_4 group action can be described as

$$z_1 \to e^{\frac{2i\pi k}{4}} z_1 \quad z_2 \to e^{-\frac{2i\pi k}{4}} z_2 \tag{7.77}$$

We can consider it as generated by the Z_2 subgroup acting as $z_1 \to -z_1$ and $z_2 \to -z_2$ and a Z_2' subgroup rotating by $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ the (x_1, x_2) and (x_3, x_4) planes: $z_1 \to iz_1$ and $z_2 \to -iz_2$.

The Z_2 group has 16 fixed points $(\pi Re_1, \pi Re_2, \pi Re_3, \pi Re_4)$, where $e_i = 0, 1$, out of which only 4 are also fixed under Z_2' : $D0_{1f} = (0,0,0,0)$, $D0_{2f} = (\pi R, \pi R, \pi R, \pi R)$, $D0_{3f} = (\pi R, \pi R, 0, 0)$, $D0_{4f} = (0, 0, \pi R, \pi R)$.

To begin with, let us calculate the annulus partition function for open strings having both ends at the same fixed point.

The partition function is given by

$$Z_{D0_f D0_f} = \tag{7.78}$$

$$\frac{1}{8} \sum_{k=0}^{3} \mathrm{Tr}(1 + (-)^F) g^k e^{-2\pi\tau L_0} = \frac{1}{4} \frac{\mathcal{J}}{\eta^{12}} Z_{\mathrm{windings}} + \frac{1}{8} \sum_{k=1}^{3} (4 \sin^2 \frac{\pi k}{4}) Z_{0,k}$$

where

$$Z_{\mathrm{windings}} = \tag{7.79}$$

$$\sum_{n_1,n_2,n_3,n_4} q^{n_1^2 + n_2^2 + n_3^2 + n_4^2} = \theta_3^4(0, 2\tau) = \frac{\theta_3^4(0, \tau) + \theta_4^4(0, \tau)}{4} + \frac{\theta_3^2(0, \tau)\theta_4^2(0, \tau)}{2}$$

and $Z_{0,k}$ can be found in (35), (36), (37) of appendix .5. Collecting all the pieces, we obtain:

$$Z_{D0_f D0_f} = \frac{\mathcal{J}}{\eta^{12}} \left(\frac{\theta_3^4(0, \tau) + \theta_4^4(0, \tau)}{16} + \frac{3\theta_3^2(0, \tau)\theta_4^2(0, \tau)}{8} \right) \tag{7.80}$$

We see that (7.80) coincides with (7.55):

$$Z_{D0_f D0_f} = \chi_{K_1} \tag{7.81}$$

In order to compute the partition function for strings with ends at different fixed point, we need to recall the partition function for a scalar X compactified at the self-dual radius $R = \frac{1}{\sqrt{2}}$ with Dirichlet boundary conditions placed at $2\pi R \xi_1$ and $2\pi R \xi_2$, so that

$$X = 2\pi R \xi_1 + (2R(\xi_2 - \xi_1) + 2nR)\sigma + \text{oscillators} \tag{7.82}$$

The partition function is easily calculated to be

$$Z_{x_1 x_2} = \frac{1}{\eta} q^{(\xi_2 - \xi_1)^2} \theta_3(2\tau(\xi_2 - \xi_1), 2\tau) \tag{7.83}$$

Using (7.83) we can then compute the annulus partition functions between different fixed points:

$$Z_{D0_{1f} D0_{2f}} = \frac{\mathcal{J}}{\eta^{12}} \left(\frac{\theta_3^4(0, \tau) + \theta_4^4(0, \tau)}{16} + \frac{\theta_3^2(0, \tau)\theta_4^2(0, \tau)}{8} \right) = \chi_{L_2} \tag{7.84}$$

$$Z_{D0_{1f}D0_{3f}} = \frac{\mathcal{J}}{\eta^{12}} \left(\frac{\theta_3^4(0,\tau) - \theta_4^4(0,\tau)}{16} + \frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{4} \right) \qquad (7.85)$$

It seems that (7.85) does not fall in the list of characters computed in section 7.3. We think

it means that the $D0_{f3}$ cannot be described by a Cardy state, and do not consider it any further

here.

7.4.2   Partially fixed points

Now we consider the case when the $D0$ branes lie at a point fixed only under Z_2. We have the

following list of such branes:

$$D0_1 = A_1 + A_1' : (0, \pi R, 0, 0) + (\pi R, 0, 0, 0) \qquad (7.86)$$

$$D0_2 = A_2 + A_2' : (\pi R, 0, 0, \pi R) + (0, \pi R, \pi R, 0)$$

$$D0_3 = A_3 + A_3' : (\pi R, \pi R, 0, \pi R) + (\pi R, \pi R, \pi R, 0)$$

$$D0_4 = A_4 + A_4' : (\pi R, 0, \pi R, \pi R) + (0, \pi R, \pi R, \pi R)$$

$$D0_5 = A_5 + A_5' : (0, 0, 0, \pi R) + (0, 0, \pi R, 0)$$

$$D0_6 = A_6 + A_6' : (\pi R, 0, \pi R, 0) + (0, \pi R, 0, \pi R)$$

The partition functions between branes (7.86) and fixed point branes are given by equation:

$$Z_{D0_i D0_f} = \text{Tr}_{A_i D0_f} \frac{(1 + (-)^F)}{2} \frac{(1 + g^2)}{2} e^{-2\pi\tau L_0} \qquad (7.87)$$

which taking into account (36) simplifies to

$$Z_{D0_i D0_f} = \text{Tr}_{A_i D0_f} \frac{(1 + (-)^F)}{4} e^{-2\pi\tau L_0} \qquad (7.88)$$

Using (7.83) we can easily compute all annulus partition functions of this type. The result

is presented in the following table:

Branes	$D0_{1f}$	$D0_{2f}$
$D0_{1f}$	$\frac{\theta_3^4+\theta_4^4}{16} + \frac{3\theta_3^2\theta_4^2}{8}$	$\frac{\theta_3^4+\theta_4^4}{16} + \frac{\theta_3^2\theta_4^2}{8}$
$D0_{2f}$	$\frac{\theta_3^4+\theta_4^4}{16} + \frac{\theta_3^2\theta_4^2}{8}$	$\frac{\theta_3^4+\theta_4^4}{16} + \frac{3\theta_3^2\theta_4^2}{8}$
$D0_1$	$\frac{\theta_2^2(\theta_3^2+\theta_4^2)}{8}$	$\frac{\theta_2^2(\theta_3^2-\theta_4^2)}{8}$
$D0_2$	$\frac{\theta_3^4-\theta_4^4}{8}$	$\frac{\theta_3^4-\theta_4^4}{8}$
$D0_3$	$\frac{\theta_2^2(\theta_3^2-\theta_4^2)}{8}$	$\frac{\theta_2^2(\theta_3^2+\theta_4^2)}{8}$
$D0_4$	$\frac{\theta_2^2(\theta_3^2-\theta_4^2)}{8}$	$\frac{\theta_2^2(\theta_3^2+\theta_4^2)}{8}$
$D0_5$	$\frac{\theta_2^2(\theta_3^2+\theta_4^2)}{8}$	$\frac{\theta_2^2(\theta_3^2-\theta_4^2)}{8}$
$D0_6$	$\frac{\theta_3^4-\theta_4^4}{8}$	$\frac{\theta_3^4-\theta_4^4}{8}$

$$(7.89)$$

where it is understood that all entries should be multiplied by $\frac{\mathcal{J}}{\eta^{12}} = \frac{1}{2\eta^{12}}(\theta_3^4 - \theta_4^4 - \theta_2^4)$.

Using the characters in section 7.3 one can present table (7.89) in the form

Branes	$D0_{1f}$	$D0_{2f}$
$D0_{1f}$	χ_{K_1}	χ_{L_2}
$D0_{2f}$	χ_{L_2}	χ_{K_1}
$D0_1$	χ_{Φ_1}	χ_{Φ_2}
$D0_2$	χ_{R_1}	χ_{R_1}
$D0_3$	χ_{Φ_2}	χ_{Φ_1}
$D0_4$	χ_{Φ_2}	χ_{Φ_1}
$D0_5$	χ_{Φ_1}	χ_{Φ_2}
$D0_6$	χ_{R_1}	χ_{R_1}

$$(7.90)$$

Table (7.90) already gives a hint for the candidate Cardy states, describing $D0$ branes located at fixed and partially fixed points.

To make things more precise we should compute also the partition functions between the different partially fixed branes (7.86). They have the form:

$$Z_{D0_i D0_j} = \mathrm{Tr}_{A_i A_j} \frac{(1+(-)^F)}{2}\frac{(1+g^2)}{2} e^{-2\pi\tau L_0} + \mathrm{Tr}_{A_i A_j'} \frac{(1+(-)^F)}{2}\frac{(1+g^2)}{2} e^{-2\pi\tau L_0} \qquad (7.91)$$

which using (36) simplifies to

$$Z_{D0_i D0_j} = \mathrm{Tr}_{A_i A_j} \frac{(1+(-)^F)}{4} e^{-2\pi\tau L_0} + \mathrm{Tr}_{A_i A_j'} \frac{(1+(-)^F)}{4} e^{-2\pi\tau L_0} \qquad (7.92)$$

Again using (7.83) we can present the result in the following table:

Branes	$D0_1$	$D0_2$	$D0_3$	$D0_4$	$D0_5$	$D0_6$
$D0_1$	$\frac{\theta_3^4}{4} + \frac{\theta_3^2\theta_4^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_3^4}{4} - \frac{\theta_3^2\theta_4^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$
$D0_2$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4+\theta_4^4}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$
$D0_3$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4}{4} + \frac{\theta_3^2\theta_4^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_3^4}{4} - \frac{\theta_3^3\theta_4^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$
$D0_4$	$\frac{\theta_3^4}{4} - \frac{\theta_3^2\theta_4^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_3^4}{4} + \frac{\theta_3^2\theta_4^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$
$D0_5$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4}{4} - \frac{\theta_3^3\theta_4^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_3^4}{4} + \frac{\theta_3^2\theta_4^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$
$D0_6$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4-\theta_4^4}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_2^2\theta_3^2}{4}$	$\frac{\theta_3^4+\theta_4^4}{4}$

$$(7.93)$$

where, as before, it is understood that all entries should be multiplied by $\frac{\mathcal{J}}{\eta^{12}} = \frac{1}{2\eta^{12}}(\theta_3^4 - \theta_4^4 - \theta_2^4)$.

After some trial and error we can solve these conditions with the following Cardy states:

$$D0_{1f} = |K_1\rangle^{\mathrm{Cardy}} \qquad (7.94)$$

$$D0_{2f} = |L_{2a}\rangle^{\mathrm{Cardy}}$$

$$D0_1 = |\Phi_{1a}^{1,1,\cdot,\cdot}\rangle^{\mathrm{Cardy}}$$

$$D0_2 = |R_{1a+}\rangle^{\mathrm{Cardy}}$$

$$D0_3 = |\Phi_{2b}^{\cdot,\cdot,1,1}\rangle^{\mathrm{Cardy}}$$

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$$D0_4 = |\Phi_{2b}^{1,1,\cdot,\cdot}\rangle^{\text{Cardy}}$$

$$D0_5 = |\Phi_{1a}^{\cdot,\cdot,1,1}\rangle^{\text{Cardy}}$$

$$D0_6 = |R_{1a-}\rangle^{\text{Cardy}}$$

Using the formulae (7.71),(7.73),(7.76) we obtain for the annulus partition functions between the states (7.94) the following table :

Branes	$D0_1$	$D0_2$	$D0_3$	$D0_4$	$D0_5$
$D0_1$	Z_{11}	$\chi_{\Phi_2} + \chi_{\Phi_1}$	$2\chi_{R_1}$	$2(\chi_{K_3} + \chi_{L_1})$	$2\chi_{R_1}$
$D0_2$	$\chi_{\Phi_2} + \chi_{\Phi_1}$	$\chi_{K_1} + 3\chi_{K_3}$	$\chi_{\Phi_2} + \chi_{\Phi_1}$	$\chi_{\Phi_1} + \chi_{\Phi_2}$	$\chi_{\Phi_1} + \chi_{\Phi_2}$
$D0_3$	$2\chi_{R_1}$	$\chi_{\Phi_2} + \chi_{\Phi_1}$	Z_{11}	$2\chi_{R_1}$	$2(\chi_{K_3} + \chi_{L_1})$
$D0_4$	$2(\chi_{K_3} + \chi_{L_1})$	$\chi_{\Phi_1} + \chi_{\Phi_2}$	$2\chi_{R_1}$	Z_{11}	$2\chi_{R_1}$
$D0_5$	$2\chi_{R_1}$	$\chi_{\Phi_1} + \chi_{\Phi_2}$	$2(\chi_{K_3} + \chi_{L_1})$	$2\chi_{R_1}$	Z_{11}
$D0_6$	$\chi_{\Phi_2} + \chi_{\Phi_1}$	$4\chi_{L_1}$	$\chi_{\Phi_2} + \chi_{\Phi_1}$	$\chi_{\Phi_1} + \chi_{\Phi_2}$	$\chi_{\Phi_1} + \chi_{\Phi_2}$

where $Z_{11} = \chi_{K_1} + \chi_{K_3} + 2\chi_{L_1}$, which coincides with table (7.93).

Chapter 8

Conclusion

Let us briefly summarize our findings.

1. We constructed geometrical realization of the Cardy states in coset model. We have shown that D-branes in coset are pointwise products of the cojugacy classes. We found geometrical meaning of the field identification and selection rules in coset models.

2. We found geometrical realization of the parafermionic D-branes in WZW model. We have shown that they are given by the pointwise products of a conjugacy and U(1) subgroup.

3. We found non-maximally symmetric non-factorizable D-branes in product of WZW models.

4. We have found geometrical realization of permutation D-branes and defects on cosets.

5. We have shown that certain diagonal embedding of the parafermionic D-branes in product of WZW models provides D-branes in the Nappi-Witten cosmological model as well as in the Guadagnini-Martelini-Mintchev model.

6. We proved symplectomorphism between phase space of the WZW model with boundaries and defects and that of 3D Chern-Simons theory with Wilson lines on a manifold of the form $\Sigma \times R$, where Σ is a certain Riemann surface and R is time line.

7. We proved symplectomorphism between phase space of the gauged WZW model with boundaries and defects and that of 3D double Chern-Simons theory with Wilson lines on a manifold of the form $\Sigma \times R$, where Σ is a certain Riemann surface and R is time line.

8. We constructed topological defects implementing abelian, non-abelian and fermionic T-dualities. We have shown that they are given by the Poincaré bundle and its non-abelian and super cousins correspondingly.

9. We studied in detail defects implementing abelian T-duality between $SU(2)$ WZW model and lens space. We have paid also special attention to the Fourier-Mukai transform of the twisted cohomology groups generated by the gauge invariant flux of this defect.

10. We studied in detail defects implementing T-duality between axially and vectorially gauged WZW model.

11. We calculated Fourier-Mukai transform of the Ramond-Ramond fields under the non-abelian T-duality.

12. We checked that the fusion matrix of the Liouville field theory with an intermediate state set to the vacuum gives rise to the DOZZ structure constants.

13. We constructed topological defects in the Liouville and Toda field theories as intertwining operators using Cardy-Lewelenn cluster equation. We have shown that in the Liouville field theory defects are labelled by the degenerate and physical primaries. We proved that in the Toda field theory topological defects are labelled by the physical, semi-degenerate and degenerate primaries.

14. We studied Lagrangian of the Liouville theory with defects and demonstrated its agreement with the operator description in the semiclassical limit.

15. We found geometrical realization of some Cardy states in (2,2,2,2,) Gepner model, using its equivalence with the T^4/Z_4 orbifold.

.1 Special functions

The function $\Gamma_b(x)$

The function $\Gamma_b(x)$ is a close relative of the double Gamma function studied in [18, 172]. It can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left(\frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-t/b})} - \frac{(Q - 2x)^2}{8e^t} - \frac{Q - 2x}{t} \right). \tag{1}$$

Important properties of $\Gamma_b(x)$ are

1. Functional equation: $\Gamma_b(x + b) = \sqrt{2\pi} b^{bx - \frac{1}{2}} \Gamma^{-1}(bx) \Gamma_b(x)$.

2. Analyticity: $\Gamma_b(x)$ is meromorphic, poles: $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$.

3. Self-duality: $\Gamma_b(x) = \Gamma_{1/b}(x)$.

From the property 1 one can obtain the following relations:

$$\Gamma_b(Q) = \sqrt{2\pi b}\, \Gamma_{1/b} \left(\frac{1}{b} \right) \tag{2}$$

$$\Gamma_b(Q) = \sqrt{\frac{2\pi}{b}}\, \Gamma_{1/b}(b) \tag{3}$$

$$W(x) = 2^{-1/4} \frac{\Gamma_b(2x)}{\Gamma_b(2x - Q)} \lambda^{\frac{2x - Q}{2b}}, \tag{4}$$

and the behaviour of the $\Gamma_b(x)$ near $x = 0$:

$$\Gamma_b(x) \sim \frac{\Gamma_b(Q)}{2\pi x}. \tag{5}$$

The function $\Upsilon_b(x)$

The Υ_b may be defined in terms of Γ_b as follows

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q - x)}. \tag{6}$$

An integral representation convergent in the strip $0 < \text{Re}(x) < Q$ is

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2} - x)\frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right]. \tag{7}$$

Important properties of $\Upsilon_b(x)$ are

1. Functional equation: $\Upsilon_b(x+b) = b^{1-2bx} \frac{\Gamma(bx)}{\Gamma(1-bx)} \Upsilon_b(x)$.

2. Analyticity: $\Upsilon_b(x)$ is entire analytic, zeros: $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$, $x = Q + nb + mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$.

3. Self-duality: $\Upsilon_b(x) = \Upsilon_{1/b}(x)$.

4. $\Upsilon_b(x) = \Upsilon_b(Q - x)$

These properties imply:

$$\frac{\Upsilon_b(2x)}{\Upsilon_b(2x - Q)} = S(x)\lambda^{\frac{2x-Q}{b}} \tag{8}$$

and

$$\Upsilon_b(x) \sim x\Upsilon_b(b), \tag{9}$$

when $x \to 0$.

The function $S_b(x)$

The function $S_b(x)$ may be defined in terms of $\Gamma_b(x)$ as follows

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q - x)} . \tag{10}$$

An integral that represents $\log S_b(x)$ is

$$\log S_b(x) = \int_0^\infty \frac{dt}{t} \left(\frac{\sinh t(Q - 2x)}{2 \sinh bt \sinh b^{-1}t} - \frac{Q - 2x}{2t} \right) . \tag{11}$$

The most important properties are

1. Functional equation: $S_b(x + b) = 2 \sin \pi bx S_b(x)$.

2. Analiticity: $S_b(x)$ is meromorphic, poles: $x = -(nb + mb^{-1})$, $n, m \in \mathbb{Z}^{\geq 0}$, zeros $x = Q + (nb + mb^{-1})$, $n, m \in \mathbb{Z}^{\geq 0}$.

3. Self-duality: $S_b(x) = S_{1/b}(x)$.

4. Inversion relation: $S_b(x)S_b(Q - x) = 1$.

These properties imply:

$$\frac{S_b(2x)}{S_b(2x - Q)} = \sqrt{2}W(x)W(Q - x) \tag{12}$$

.2 Properties of Γ functions

The limiting behavior of the terms with Γ functions can be calculated using the approximation

$$\Gamma(x) \sim e^{x \log x - x + O(\log x)} . \tag{13}$$

for x with big positive real part.

For negative x using the formula

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x \sin \pi x} , \tag{14}$$

one can bring problem to the previous case.

We also need well-known behavior of the $\Gamma(x)$ function for x around zero:

$$\Gamma(x) \sim \frac{1}{x} . \tag{15}$$

.3 Theta functions identities

We start by reviewing some useful identities satisfied by Theta functions [125].

$$\theta \begin{bmatrix} \frac{a}{n_1} \\ 0 \end{bmatrix} (x_1, n_1\tau) \theta \begin{bmatrix} \frac{b}{n_2} \\ 0 \end{bmatrix} (x_2, n_2\tau) = \tag{16}$$

$$\sum_{\mu=0}^{n_1+n_2-1} \theta \begin{bmatrix} \frac{n_1\mu+a+b}{n_1+n_2} \\ 0 \end{bmatrix} (x_1 + x_2, (n_1 + n_2)\tau) \cdot$$

$$\theta \begin{bmatrix} \frac{n_1 n_2 \mu + n_2 a - n_1 b}{n_1 n_2 (n_1+n_2)} \\ 0 \end{bmatrix} (n_2 x_1 - n_1 x_2, n_1 n_2 (n_1 + n_2)\tau)$$

where

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \sum_{n \in Z} \exp(i\pi(n + a)^2 \tau + 2i\pi(n + a)(x + b)) \tag{17}$$

Using the identity

$$\sum_{\mu=0}^{n-1} \theta \begin{bmatrix} \frac{\mu+a}{n} \\ 0 \end{bmatrix} (nx, n^2\tau) = \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (x, \tau) \tag{18}$$

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we can exploit (16) for the case relevant to our analysis i.e. $n_1 = r_1 n$ and $n_2 = r_2 n$

$$\theta \begin{bmatrix} \frac{a}{r_1 n} \\ 0 \end{bmatrix} (x_1, r_1 n\tau) \theta \begin{bmatrix} \frac{b}{r_2 n} \\ 0 \end{bmatrix} (x_2, r_2 n\tau) = \tag{19}$$

$$\sum_{\mu=0}^{r_1+r_2-1} \theta \begin{bmatrix} \frac{r_1 \mu}{r_1+r_2} + \frac{a+b}{(r_1+r_2)n} \\ 0 \end{bmatrix} (x_1 + x_2, (r_1 + r_2)n\tau) \cdot$$

$$\theta \begin{bmatrix} \frac{\mu}{r_1+r_2} + \frac{r_2 a - r_1 b}{r_1 r_2 (r_1+r_2)n} \\ 0 \end{bmatrix} (r_2 x_1 - r_1 x_2, r_1 r_2 (r_1 + r_2)n\tau)$$

Let us explicitly write this formula for the most relevant for us case: $n_1 = n_2 = n$, $r_1 = r_2 = 1$

$$\theta \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (x_1, n\tau) \theta \begin{bmatrix} \frac{b}{n} \\ 0 \end{bmatrix} (x_2, n\tau) = \tag{20}$$

$$\sum_{\mu=0}^{1} \theta \begin{bmatrix} \frac{\mu}{2} + \frac{a+b}{2n} \\ 0 \end{bmatrix} (x_1 + x_2, 2n\tau) \theta \begin{bmatrix} \frac{\mu}{2} + \frac{a-b}{2n} \\ 0 \end{bmatrix} (x_1 - x_2, 2n\tau)$$

.4 Other relevant identities

Recall the identities:

$$\theta_3^2(\tau) - \theta_4^2(\tau) = 2\theta_2^2(2\tau) \tag{21}$$

$$\theta_3^2(\tau) + \theta_4^2(\tau) = 2\theta_3^2(2\tau)$$

$$\theta_3(\tau)\theta_4(\tau) = \theta_4^2(2\tau)$$

$$\theta_2^2(\tau) = 2\theta_2(2\tau)\theta_3(2\tau)$$

From (21) we can derive another couple of useful identities:

$$\theta_3(2\tau)\theta_2^2(\tau) = \theta_2(2\tau)(\theta_3^2(\tau) + \theta_4^2(\tau)) \tag{22}$$

$$\theta_2(2\tau)\theta_2^2(\tau) = \theta_3(2\tau)(\theta_3^2(\tau) - \theta_4^2(\tau))$$

$$\Theta_{0,1}(z, \tau) = \theta_3(z, 2\tau) \tag{23}$$

$$\Theta_{1,1}(z, \tau) = \theta_2(z, 2\tau)$$

Let us also mention the following formulae.

$$\theta_1\left(\frac{1}{2},\tau\right) = \theta_2(0,\tau) \tag{24}$$

$$\theta_2\left(\frac{1}{2},\tau\right) = 0 \tag{25}$$

$$\theta_3\left(\frac{1}{2},\tau\right) = \theta_4(0,\tau) \tag{26}$$

$$\theta_4\left(\frac{1}{2},\tau\right) = \theta_3(0,\tau) \tag{27}$$

$$\theta_1\left(\frac{1}{4},\tau\right) = \theta_2\left(\frac{1}{4},\tau\right) = \theta_1\left(\frac{3}{4},\tau\right) = -\theta_2\left(\frac{3}{4},\tau\right) \tag{28}$$

$$\theta_3\left(\frac{1}{4},\tau\right) = \theta_4\left(\frac{1}{4},\tau\right) = \theta_3\left(\frac{3}{4},\tau\right) = \theta_4\left(\frac{3}{4},\tau\right) \tag{29}$$

$$\frac{\theta_3^2(\frac{1}{4},\tau)}{\theta_1^2(\frac{1}{4},\tau)} = \frac{\theta_3(0,2\tau)}{\theta_2(0,2\tau)} = \frac{\theta_2^2(0,\tau)}{\theta_3^2(0,\tau) - \theta_4^2(0,\tau)} \tag{30}$$

.5 Partition function of the T^4/Z_4 orbifold

$$Z = \frac{1}{4}Z_{\text{lattice}}\left|\frac{\mathcal{J}}{\eta^{12}}\right|^2 + \sum_{r,s}{}' n_{r,s}|Z_{r,s}|^2 \tag{31}$$

where

$$Z_{\text{lattice}} = (|\chi_1^{\text{SU}(2)}|^2 + |\chi_2^{\text{SU}(2)}|^2)^4 = \frac{1}{4}\left(|\theta_3(0,\tau)|^4 + |\theta_4(0,\tau)|^4 + |\theta_2(0,\tau)|^4\right)^2 \tag{32}$$

and

$$Z_{r,s} = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{\theta^2\begin{bmatrix}\alpha\\\beta\end{bmatrix}(0,\tau)\,\theta\begin{bmatrix}\alpha+\frac{r}{4}\\\beta+\frac{s}{4}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\alpha-\frac{r}{4}\\\beta-\frac{s}{4}\end{bmatrix}(0,\tau)}{\eta^6 \quad \theta\begin{bmatrix}\frac{1}{2}+\frac{r}{4}\\\frac{1}{2}+\frac{s}{4}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\frac{1}{2}-\frac{r}{4}\\\frac{1}{2}-\frac{s}{4}\end{bmatrix}(0,\tau)} \tag{33}$$

Consider the Ramond part.

$$Z_{r,s}^R = \frac{\theta_2^2(0,\tau)}{\eta^6}\frac{\theta\begin{bmatrix}\frac{1}{2}+\frac{r}{4}\\0+\frac{s}{4}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\frac{1}{2}-\frac{r}{4}\\0-\frac{s}{4}\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{2}+\frac{r}{4}\\\frac{1}{2}+\frac{s}{4}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\frac{1}{2}-\frac{r}{4}\\\frac{1}{2}-\frac{s}{4}\end{bmatrix}(0,\tau)} \tag{34}$$

$$Z_{0,1}^R = -\theta_2^4(0,\tau)\frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{35}$$

$$Z_{0,2}^R = 0 \tag{36}$$

$$Z_{0,3}^R = -\theta_2^4(0,\tau)\frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{37}$$

$$Z_{2,0}^R = \theta_2^4(0,\tau)\frac{\theta_3^4(0,\tau)}{4\eta^{12}} \tag{38}$$

$$Z_{2,1}^R = \theta_2^4(0,\tau)\frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{39}$$

$$Z_{2,2}^R = \theta_2^4(0,\tau)\frac{\theta_4^4(0,\tau)}{4\eta^{12}} \tag{40}$$

$$Z_{2,3}^R = \theta_2^4(0,\tau)\frac{\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{41}$$

$$Z_{1,0}^R = \theta_2^4(0,\tau)\frac{\theta_3^4(0,\tau) + \theta_2^2(0,\tau)\theta_3^2(0,\tau)}{4\eta^{12}} \tag{42}$$

$$Z_{1,2}^R = -\theta_2^4(0,\tau)\frac{\theta_3^4(0,\tau) - \theta_2^2(0,\tau)\theta_3^2(0,\tau)}{4\eta^{12}} \tag{43}$$

$$Z_{3,0}^R = \theta_2^4(0,\tau)\frac{\theta_3^4(0,\tau) + \theta_2^2(0,\tau)\theta_3^2(0,\tau)}{4\eta^{12}} \tag{44}$$

$$Z_{3,2}^R = -\theta_2^4(0,\tau)\frac{\theta_3^4(0,\tau) - \theta_2^2(0,\tau)\theta_3^2(0,\tau)}{4\eta^{12}} \tag{45}$$

$$Z_{1,1}^R = \theta_2^4(0,\tau)\frac{\theta_4^4(0,\tau) + i\theta_2^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{46}$$

$$Z_{1,3}^R = \theta_2^4(0,\tau)\frac{\theta_4^4(0,\tau) - i\theta_2^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{47}$$

$$Z_{3,1}^R = \theta_2^4(0,\tau)\frac{\theta_4^4(0,\tau) - i\theta_2^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{48}$$

$$Z_{3,3}^R = \theta_2^4(0,\tau)\frac{\theta_4^4(0,\tau) + i\theta_2^2(0,\tau)\theta_4^2(0,\tau)}{4\eta^{12}} \tag{49}$$

The numbers $n_{r,s}$ are given by the following formulae: $n_{0,s} = 4\sin^4\frac{\pi s}{4}$, $n_{r,s} = n_{r,s+r}$, $n_{r,s} = n_{s,4-r}$.

Plugging all in (31) we get (7.68).

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