Group Theory

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CONTENTS

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I. GROUPS AND LIE GROUPS

In this section we briefly remind the general properties of groups and their representations. In physics groups are appeared as the groups of symmetries of the system. They gather certain transformations in space-time or integral degrees of freedom preserving the Hamiltonian or, in general, the action of the system. In classical mechanics, due to Noether theorem, any continues symmetry group provides the system with integrals of motion, which can be used further for reducing the number of degrees of freedom.

A. Groups and their structure

1. Groups

Remember that the *group* G is a set of elements g, endowed with an associative multiplication, with an identity element 1 and an inverse g^{-1} obeying $gg^{-1} = 1$.

If the product is commutative, i.e. $g_1 g_2 = g_2 g_1$ for any two elements, it is called *abelian*. For the abelian groups, the multiplication is often referred as a sum and the identity as 0.

If the group is called *finite* if it has a finite number of elements. That number is called the *order* of the group and denoted it by $|G|$. The infinite groups could be discrete and continuous.

A *subgroup* $H \subset G$ is a subset of elements which forms a group under the group product of G.

By definition, the *center* of G is the abelian subgroup consisting of elements $g \in G$ commuting with the whole group.

2. Homomorphisms

A group *homomorphism* is a map τ: $G \to G'$ between two groups which preserves the group structure, i.e. which satisfies

$$\tau(g_1 g_2) = \tau(g_1)\tau(g_2).$$

The invertible group homomorphism is called an *isomorphism*. If there exists a group isomorphism between G and G', the groups are called isomorphic, $G \cong G'$.

Finally, a group isomorphism from a group to itself, $G \to G$, is an *automorphism*.

3. Group actions

How to define a group? A common way of definition is the description of group action on a general set M. All one to one invertible maps between the elements of this set (bijections) naturally form a group, which is sometimes called the symmetric group of M, denoted by Sym(M). Note that if M is finite and contains n elements, this reproduces our previous definition of the *symmetric group*, or *permutation group* S_n.

A *group action* or *realization* of the group consists in associating with each its element g a transformation t_g of the point set M such that the product of the group elements maps to the product of corresponding transformations, i.e.

$$t_{gg'}x = t_g t_{g'}x \qquad \text{for any} \quad x \in M, \quad g, g' \in G. \tag{1.A.1}$$

In other words, the group action is a homomorphism from G into Sym(M). Clearly, t_1 is the trivial map. Often in the definition the $t_g x$ is replaced by the left group action: $gx = t_g x$.

The group action is called *faithful* when to distinct elements of the group correspond distinct transformations. Clearly, an action is faithful if the only identity transformation is the t_1.

It is easy to see that the *left translations* $t_a g = ag$ for $a \in G$ define a natural action on the group, $M = G$.

***Problem* 1.A.1:** Proof that the inverse right translations $\tau_a g = ga^{-1}$ and conjugations $\tau_a g = aga^{-1}$ define also group realizations.

Fix a point $x \in M$. Then its *orbit* in M is the set of all images, i.e.

$$Gx = \{gx \,|\, \forall g \in G\}.$$

Conversely, the *stabiliser*, or *little group* is the set of all group elements that leave x invariant,

$$G_x = \{g \in G \,|\, gx = x\}. \tag{1.A.2}$$

4. Equivalence relation and quotient sets

A given binary relation $\sim$ among the elements x of the set M is said to be an *equivalence* relation if is reflexive ($x \sim x$), symmetric (from $x_1 \sim x_2$ it follows that $x_2 \sim x_1$), and transitive (if $x_1 \sim x_2$ and $x_2 \sim x_3$ then $x_1 \sim x_3$).

For a given element x, the *equivalence class* denoted $[x]$, is defined as a subset of all elements, which are equivalent to x:

$$[x] = \{x' \in M \,|\, x' \sim x\}. \tag{1.A.3}$$

It is easy to see that two equivalence classes are either equal or disjoint, so the whole group is the union of all them.

The equivalence classes $[x]$ themself form another, smaller set, which is called a *quotient set* or *coset*.

5. Cosets

For given groups different equivalence relations can be defined. For example, Let H be a subgroup of G. Then two elements $g_1, g_2 \in G$ are equivalent with respect to the right multiplication on H, if there exists an element from $g \in H$ such that $g_2 = g_1 h$.

The equivalence class $[g]$, which is usually denoted by gH, consists of all elements gh, where $h \in H$ runs over all subgroup elements, and called a *left coset* of H in G. Similarly, a *right coset* $[g] = Hg$ of the subgroup H in G is defined. In other words,

$$gH = \{gh \mid \forall h \in H\}, \qquad Hg = \{hg \mid \forall h \in H\}. \tag{1.A.4}$$

Clearly the unit element belongs to the coset corresponding to the subgroup H, i.e. $[1] = H$. All cosets have the same number of elements equal to the order of the subgroup H. Clearly, the cosets gH and Hg do not form subgroups in G.

***Problem* 1.A.2:** Why the left, right multiplication define equivalence relations amon the elements of the group? Why they are not subgroups?

6. Conjugacy classes

Two group elements g_1 and g_2 are called *conjugate*, if there exists an element g such that

$$g_2 = g g_1 g^{-1}.$$

The conjugacy is another equivalence relation among the elements of the group.

***Problem* 1.A.3:** Check that the conjugacy is an equivalence relation.

Therefore, one can form equivalence classes, called conjugacy classes. More presizely, given any group element g, its *conjugacy class* $[g]$ consists on the set of all conjugate elements:

$$[g] = \{hgh^{-1} | h \in G\}. \tag{1.A.5}$$

For example, the conjugate class of the unity consists only from the unity itself, $[1] = 1$. For an abelian group, these classes consist on single element: $[g] = g$. For nonabelian groups, they contain different number of elements.

***Problem* 1.A.4:** Does the conjugacy relation respect the group operations?

7. Conjugacy and normal subgroup

Alternatively, for a given subgroup H and an element g from G, the set of elements $H_g = gHg^{-1}$ define a *conjugate subgroup* which is equivalent to H.

***Problem* 1.A.5:** Prove that H_g is a subgroup in G and that it is isomorphic to H.

A subgroup N is called *normal*, *invariant* or *self conjugate* if it is coincides with its conjugate with respect to any element of the group, $N_g = N$, or

$$gNg^{-1} = N \quad \text{for all } g \in G. \tag{1.A.6}$$

***Problem* 1.A.6:** Prove that for a given group action on the point set M, the group elements $g \in G$, which leave all points $x \in M$ invariant, $gx = x$, form a normal subgroup. Prove that the stabiliser G_x (1.A.2) is a normal subgroup.

***Problem* 1.A.7:** Show that the left, right cosets and the factor sets are not, in general, groups. Describe them for an abelian group.

A group which has no nontrivial normal subgroups (i.e. other than 1 and G itself) is called simple.

8. Quotient groups

Normal subgroups allow for the construction of the quotient groups, or factor groups. For general subgroup H, we define the *quotient set* as the set of all left cosets of H,

$$G/H = \{gH \mid \forall g \in G\}. \tag{1.A.7}$$

In general, this quotient is not a group because it does not inherit the group structure from G.

However, for the normal subgroup N, the quotient G/N inherits the group structure from the initial group G. It called a *quotient group* or *factor group*. The group operation is defined by

$$(gN)(g'N) = gg'N. \tag{1.A.8}$$

Indeed, the multiplication does not depend of the representative of the quotient set since, due to invariance of the subgroup N, we have:

$$(gn)(g'n') = gg'n'' \qquad \text{with} \quad n'' = \tau_g(n)n' \in N, \quad \tau_g(n) = g^{-1}ng \in N. \tag{1.A.9}$$

It is clear that the map τ_g from N.

9. Direct and semidirect products

We will now finally discuss ways to combine groups into bigger ones.

Given two groups, we can define a trivially combined group. The *direct product* of G_1 and G_2 groups is given by the set

$$G_1 \times G_2 = \{(g_1, g_2) \,|\, g_1 \in G_1, g_2 \in G_2\}. \tag{1.A.10}$$

The product is defined componentwise,

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2'). \tag{1.A.11}$$

In other words any element of the product group is expressed as a product of two elements from each group, the order is irrelevant, $g = g_1 g_2 = g_2 g_1$. So, both subgroups mutually commute in the direct product: $G_1 G_2 = G_2 G_1$. The unity element if the product of unities in each group.

The direct product notion can be generalized in order to allow shifted commutations among elements of the two ingredients $G_1 = N$ and $G_2 = H$.

To define the shifted product, we need in an exact realization τ_h of the H as automorphisms of the group N, i.e.

$$\tau_h(nn') = \tau_h(n)\tau_h(n'). \tag{1.A.12}$$

Then the *semidirect product* is defined as the set

$$G = N \rtimes H = \{(n, h) \,|\, n \in N, h \in H\} \tag{1.A.13}$$

endowed with the group product

$$(n, h)(n', h') = (n\tau_h n', hh'). \tag{1.A.14}$$

Let explain the constriction in a more transparent way. The group element in (1.A.13) can be written as

$$(n, h) = (n, 1)(1, h) = nh \tag{1.A.15}$$

and can be viewed as a definition of the product between the left n and right h. The realization τ_h determines now the same product taken in the reverse order by

$$hn := \tau_h(n)h \qquad \text{so that} \qquad \tau_h(n) = hnh^{-1}. \tag{1.A.16}$$

The last relation ensures that τ_h is an exact H-realization as an automorphism on N as was prescribed initially.

B. Structure of group representations

1. Representations

A *representation* (ρ, V) of a group G is a group action on a vector space V by invertible linear maps. So, the map $g \to \rho_g$ respects the group product. This implies that for any two group elements $g, g' \in G$ and any two vectors $v_{1,2} \in V$ we have

$$\rho_{g'g} = \rho_{g'}\rho_g, \qquad \rho_g(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \rho_g v_1 + \alpha_2 \rho_g v_2, \tag{1.B.1}$$

there $\alpha_{1,2}$ are any two numbers. We will consider the representations over the *complex* or *real* space.

A representation is *faithful* if it is a faithful group action. It separates the distinct group elements: $\rho_{g_1} = \rho_{g_2}$ implies $g_1 = g_2$.

***Problem* 1.B.8:** Prove that ρ_g is faithful if the unit matrix represents only the unit element of the group, $g = e$.

For instance, in any group there exists a simplest one-dimensional *trivial* representation given by the unity map,

$$\rho_g = \epsilon_g = 1, \qquad g \in G. \tag{1.B.2}$$

For the nontrivial groups it is not faithful and says nothing about the intrinsic structure of the group. From this viewpoint it is completely useless.

2. Equivalence

Two representations ρ and ρ' are *equivalent* if they are related by a similarity transformation, i.e. if there is an operator R such that

$$R\rho_g R^{-1} = \rho'_g \tag{1.B.3}$$

for all elements g. Two equivalent representation can be thought of as the same representation in different bases. We will normally regard equivalent representations as being equal.

3. Reducible and irreducible representations

An important question is whether we can break up a representation into smaller parts. This will be the case if there is a subspace of V which gets mapped to itself, because then the representation can be restricted to that subspace.

We say that a subspace $V' \subset V$ is an *invariant* subspace if $\rho_g V' \subset V'$ for all $g \in G$.

A representation ρ is called *reducible* if V contains a nontrivial invariant subspace, i.e. a subspace which does not coincide with the null and whole space V.

Otherwise, a representation ρ is called *irreducible* if V does not contain any nontrivial invariant subspace.

A representation is called *fully reducible* if V can be written as the direct sum of two nontrivial invariant subspaces, $V = V_1 \oplus V_2$. Recall from linear algebra that this means that every vector v can be uniquely written as $v = v_1 + v_2$, with $v_i \in V_i$. Then $\dim V = \dim V_1 + \dim V_2$.

Assume ρ is reducible, i.e. there is an invariant subspace V_1 so that $\rho_g V_1 = V_1$. If one chooses a basis of V such that the first n_1 basic vectors span V_1, the matrices of ρ take the block-upper-triangular form

$$\rho_g = \begin{pmatrix} \rho_g^{(1)} & \rho_g^{(12)} \\ 0 & \rho_g^{(2)} \end{pmatrix}$$

for all g. Here $\rho_g^{(i)}$ are $n_i \times n_i$ dimensional matrices, while $\rho_g^{(12)}$ is a $n_1 \times n_2$ matrix. If ρ is fully reducible, $\rho^{(12)} = 0$, i.e. it can be brought to block-diagonal form

$$\rho_g = \begin{pmatrix} \rho_g^{(1)} & 0 \\ 0 & \rho_g^{(2)} \end{pmatrix}.$$

4. Unitary representations

A *unitary representation* is given by unitary matrixes, i.e. all operators (1.B.1) are unitary. In order to emphasize the unitary, we denote the corresponding operators by U:

$$\rho_g = U_g, \qquad U_g U_g^+ = U_g^+ U_g = I. \tag{1.B.4}$$

Any reducible unitary representation is fully reducible.

Let V_1 be an invariant subspace of a unitary representation (U_g, V). Denote by

$$(v_1, v_2) = v_1^+ v_2$$

the corresponding Hermitian inner product. Then the orthogonal space

$$V_1^\perp = \{v_2 \in V \,|\, (v_2, v_1) = 0 \quad \forall v_1 \in V_1\}.$$

will be invariant too. Indeed, for any vector v_2 belonging to the space $V_1^\perp$, all vectors $U_g v_2$, where g is any group element, also will belong to $V_1^\perp$.

$$(U_g v_2, v_1) = (v_2, U_g^+ v_1) = (v_2, U_{g^{-1}} v_1) = 0. \tag{1.B.5}$$

Since both spaces form entire representation space, $V = V_1 \oplus V_1^\perp$, we conclude that the initial representation is fully reduced to (U_g, V_1) and $(U_g, V^\perp)$.

5. Dual and conjugate representations

For each complex representation (ρ, V) of the group G, define a *complex conjugate representation* $(\bar{\rho}, V)$ by the complex conjugate matrices:

$$\bar{\rho}_g = (\rho_g)^*, \qquad g \in G.$$

***Problem* 1.B.9:** Verify that $\bar{\rho}$ is a representation.

Similarly, from any representation one can construct a *dual representation* $(\tilde{\rho}, V)$, sometimes call also a *contragredient representation*, given by transposed matrixes:

$$\tilde{\rho}_g = (\rho_{g^{-1}})^{\tau}. \tag{1.B.6}$$

The dual to a complex conjugate representation gives a Hermitian congugate representations

$$\rho_g^+ = (\rho_{g^{-1}})^+, \qquad g \in G. \tag{1.B.7}$$

***Problem* 1.B.10:** For an irreducible representation, will be irreducible its dual and complex conjugate representations?

6. Product representations

Given the spaces V_1 and V_2, we can form the tensor product (or direct product) $V = V_1 \otimes V_2$. This space is constructed as follows: Take bases $\{e_i\}$ and $\{f_j\}$ are some basises in V_1 and V_2 respectively. A basis of the tensor product space V is then given by the set $v_{ij} = \{e_i \otimes f_j\}$. It is parametrized by the multi-index (i, j). Clearly, $\dim V = \dim V_1 \cdot \dim V_2$.

Given two operators acting on V_1 and V_2 , we can define their tensor product via their matrix elements. Let the matrix elements be $\rho^{(1)}_{ii'}$ and $\rho^{(2)}_{jj'}$. Then the matrix the product

$$\rho_{ij\,i'j'} = \rho^{(1)}_{ii'} \rho^{(2)}_{jj'}$$

is parameterized by the multi-index.

7. Schur lemma

Consider two irreducible representations $(\rho^{(1)}, V_1)$ and $(\rho^{(2)}, V_2)$ of the same group G.

Lemma 1. *(Schurs Lemma) A linear operator R: $V_1 \to V_2$ between the two irreducible representations, which satisfy*

$$R\rho_g^{(1)} = \rho_g^{(2)} R \tag{1.B.8}$$

for any group element g, is either an isomorphism, in which case the representations are equivalent, or zero.

The operator R is called an intertwiner. The relation (1.B.8) implies the commutativity of the following diagram:

$$\begin{tikzcd} v_1 \arrow[r, "R"] \arrow[d, "\rho_g^{(1)}"'] & v_2 \arrow[d, "\rho_g^{(2)}"] \\ v_1' \arrow[r, "R"] & v_2' \end{tikzcd}$$

Proof. Recall the definition of the kernel $\mathrm{Ker}R \subset V_1$ and the image $\mathrm{Im}R \subset V_2$ of a linear map R. The first is the null space of R, and the second collects all its images,

$$\mathrm{Ker}R = \{v_1 \in V_1 \,|\, Rv_1 = 0\}, \qquad \mathrm{Im}R = \{Rv_1 \,|\, v_1 \in V_1\}. \tag{1.B.9}$$

***Problem* 1.B.11:** Prove that the $\mathrm{Ker}R \subset V_1$ and $\mathrm{Im}R \subset V_2$ are invariant subspaces of the representations $\rho_g^{(1)}$ and $\rho_g^{(2)}$ respectively. *Hint:* use the intertwiner condition (1.B.10).

***Problem* 1.B.12:** Show that either $\mathrm{Ker}R = 0$ and $\mathrm{Im}R = V_2$ or $\mathrm{Ker}R = V_1$ and $\mathrm{Im}R = 0$. *Hint:* use the irreducibility condition.

The first case the last task corresponds the equivalents representations while the second case describes the zero map. This completes the proof of the lemma. □

Consider now the a single representation on the complex space. As a consequence of the Schur lemma, we have the following assertion which often is considered as a Schur lemma.

Corollary 1. *A linear operator R on the complex space V, which commutes with the group action,*

$$R\rho_g = \rho_g R, \tag{1.B.10}$$

is a multiple of the identity map:

$$R = \lambda I. \tag{1.B.11}$$

Proof. Since the identity map I also obeys the intertwining relation (1.B.10), the combined operator $R - \lambda I$ for any $\lambda \in \mathbb{C}$ is also an intertwiner obeying the Schur Lemma condition. But at the root point $\lambda = \lambda_0$ where $\det(R-\lambda_0 I) = 0$, the operator $R - \lambda I$ is not invertible. Hence, it must vanish providing the equation (1.B.11) for $\lambda = \lambda_0$. □

***Problem* 1.B.13:** Is the corollary fulfilled for the real representations?

8. Characters

For a n-dimensional representation (ρ, V) of the group G, define a function dependent on the group element g taking the trace of the representation matrix,

$$\chi_g^\rho = \mathrm{Tr}\ \rho_g = \sum_{i=1}^{n} (\rho_g)_{ii}. \tag{1.B.12}$$

The immediate consequence of this definition is that *a character value at the unity element is the dimension of the representation*,

$$\chi_e^\rho = \dim V = n. \tag{1.B.13}$$

Then applying the trace on both sides of the equivalence relation (1.B.3) and the cyclicity property of the trace, we conclude that *the equivalent representations have the same character*:

$$\chi_g^\rho = \chi_g^{\rho'} \qquad \text{if} \qquad \rho \equiv \rho'. \tag{1.B.14}$$

A character is a function on the conjugacy classes of the group. It has the same value on conjugate group elements:

$$\chi_g^\rho = \chi_{hgh^{-1}}^\rho, \qquad \forall g, h \in G. \tag{1.B.15}$$

***Problem* 1.B.14:** Prove above formula.

If the representation ρ is fully reduces to the representations $\rho^{(1)}$ and $\rho^{(2)}$, then *the characters also obey the sum rule:*

$$\chi_g = \chi_g^{(1)} + \chi_g^{(2)}.$$

Similarly, if the representation ρ is a product of $\rho^{(1)}$ and $\rho^{(2)}$, then *the characters obey the product rule:*

$$\chi_g = \chi_g^{(1)} \chi_g^{(2)}.$$

C. Some useful algebras

1. Quaternions

Consider the three two-dimensional matrices τ_1, τ_2 and τ_3 expressed via the Pauli matrices σ_i

$$\tau_i = \imath\sigma_i, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.C.1}$$

Together with the identity matrix $I = 1$, they form a closed algebra over the real numbers. They are three imaginary unities $\tau_i^2 = -1$ obeying the rule $\tau_1\tau_2 = -\tau_3$ and the other rules obtained by the cyclic permutations. In a single form,

$$\tau_i\tau_j = -\delta_{ij} - \epsilon_{ijk}\tau_k, \tag{1.C.2}$$

where ϵ_{ijk} is the Levi-Civita tensor.

The associative algebra generated by τ_i (1.C.2) is called the *algebra of quaternions* and denoted as $\mathbb{H}$.

Any element $q \in \mathbb{H}$ is presented as $q = q_0 + q_i\tau_i$. The conjugate is given by $q^+ = q_0 - q_i\tau_i$. Clearly, $\mathbb{H}$ is closed under the multiplication.

The norm of the quaternion is given by

$$|q| = \sqrt{\det qq^+}. \tag{1.C.3}$$

***Problem* 1.C.15:** Show that: 1) $qq^+ = |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$, 2) $|q_1q_2| = |q_1||q_2|$.

This implies that $q = 0$ if and only if its norm vanishes, and the inverse elements exist for all nonzero quaternions and are given by

$$q^{-1} = \frac{q^+}{|q|^2}. \tag{1.C.4}$$

These are peculiar features of the quaternions which distinguish them from other matrix algebras and allow them to consider as a noncommutative extension of the complex numbers.

Denote by $\mathbb{H}_1$ the quaternions with the unity norm, $|q| = 1$. This subset contains the unity and is closed under the multiplication and inverse, but, of course, is not invariant under the linear operations. Therefore, $\mathbb{H}_1$ is a group described by the three-dimensional sphere S^3.

***Problem* 1.C.16:** Prove that $\mathbb{H}_1 \equiv SU(2)$. *Hint*: use the representation (1.G.18).

2. Group algebras

Consider the finite group G and construct the finite-dimensional vector space on it by

$$a = \sum_{g\in G} a_g g. \tag{1.C.5}$$

Here the group elements are associated with the basic states of the linear space whose dimension coincides with the order of the group $k = |G|$. The a_g are the real or complex coefficients.

The corresponding algebra is called a *group algebra* of G and mentioned by $\mathbb{R}[G]$ (or by $\mathbb{C}[G]$ in the complex case).

The product of two elements induces a product among their coefficients by

$$ab = \sum_{g\in G} (a * b)_g g \tag{1.C.6}$$

with the *convolution product* between two coefficients

$$(a * b)_g = \sum_{h\in G} a_h b_{h^{-1}g}. \tag{1.C.7}$$

Above construction seems to be somehow an artificial one, and a natural question arises: why the group algebra is introduced? The answer becomes evident if we remember that many applications of the groups in quantum physics are related to the properties of their representations. The group algebra incorporates the common properties of all irreducible representations of the group.

D. Irreducible representation of finite groups

1. Equivalence to unitary representation

Recall that the unitary representations are given by unitary matrices, see Sec. I.B.4.

Lemma 2. *Any representation of finite group is equivalent to some unitary representation.*

Proof. Take a k-dimensional representation ρ_g and construct there the followng metrics

$$B = \sum_{g\in G} \rho_g^+ \rho_g. \tag{1.D.1}$$

First, this metrics is positive, $B > 0$. Indeed, for the square of any vector v from the representation space we have

$$v^+ B v = \sum_g (\rho_g v)^+ \rho_g v > 0,$$

Note that we have excluded the zero value since it implies a presence of the null vector v, $\rho_g v = 0$, which violates the invertibility of ρ_g. Hence, the metrics can be converted to the Euclidean form by coordinate transformation

$$I = X^+ B X \tag{1.D.2}$$

Second, it is easy to see that the constructed metrics is invariant:

$$\rho_g^+ B \rho_g = \sum_{g'\in G} (\rho_g \rho_{g'})^+ \rho_g \rho_{g'} = \sum_{g'\in G} \rho_{gg'}^+ \rho_{g'} = \sum_{g'\in G} \rho_{g'}^+ \rho_{g'} = B.$$

Then in new coordinate system, as easy to verify, the representation (1.B.4) will be unitary.

$$U_g = X \rho_g X^{-1} \tag{1.D.3}$$

The above equation provides the equivalence of both representations. □

2. Orthogonality of matrix elements

Let U_g and U'_g are nonequivalent irreducible representations of the finite group G on the spaces V and V' with dimension n, n' correspondingly.

Lemma 3. *The matrix elements of nonequivalent irreducible representations are orthogonal*

$$\sum_{g\in G} (U_g^*)_{ij} (U'_g)_{i'j'} = 0, \qquad 1 \le i, j \le n, \quad 1 \le i', j' \le n'. \tag{1.D.4}$$

Proof. Take an arbitrary rectangular matrix A of dimension $n' \times n$ and construct another matrix

$$R = R(A) = \sum_{g\in G} U_g^+ A U'_g. \tag{1.D.5}$$

Then the matrix R obeys the intertwiner condition for all group elements g,

$$U_g^+ R U'_g = R, \qquad \text{or} \qquad R U'_g = U_g R. \tag{1.D.6}$$

The Schur lemma implies that it must vanish: $R = 0$. Choosing A equal successively to the nn' matrices $E_{jj'}$ with a single nonzero element,

$$A_{jj'} = E(i, i')_{jj'} = \delta_{ij}\delta_{i'j'}, \tag{1.D.7}$$

we come to the desired equation (1.D.4). □

Lemma 4. *The matrix elements of an irreducible n-dimensional representation are orthogonal*

$$\sum_{g\in G}(U_g^*)_{ij}(U_g)_{kl} = \frac{|G|}{n}\delta_{ik}\delta_{jl}, \tag{1.D.8}$$

Proof. In that case, we must set $U'_g = U_g$ in the relations (1.D.5) and (1.D.6). According to the Schur lemma, the intertwining operator must be proportional to the identity with the coefficient A-dependent coefficient, $R(A) = \lambda(A)I$. Substituting $A = E(i,k)$ into

$$U^+ A U_g = \lambda(A) I$$

as in the previous case, and we at the following equation on (j,l)th matrix element:

$$\sum_{g\in G}(U_g^*)_{ij}(U_g)_{kl} = \lambda_{ik}\delta_{jl}. \tag{1.D.9}$$

Setting $j = l$, taking sum over j and using the unitarity condition (1.B.4), we get

$$\lambda_{ik} = \frac{|G|}{n}\delta_{ik},$$

which completes the proof. □

The two formulas (1.D.4), (1.D.8) can be unified into a singe one if we consider only nonequivalent irreducible representations labeling by Greek indexes:

$$\sum_{g\in G}(U_g^{\alpha *})_{ij}(U_g^{\beta})_{kl} = \frac{|G|}{n_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{jl}, \tag{1.D.10}$$

where n_α is the dimension of the representation U^α.

The above relation has many important consequences.

The dimension of the Euclidian space onder consideration coincides with the order of the finite group. As soon as the orthogonal vectors are linearly independent, we must have the inequality

$$\sum_\alpha n_\alpha^2 \leq |G|, \tag{1.D.11}$$

which puts a strong restriction on the amount of irreducible representations and their dimensions.

3. Orthogonality of characters

As we have mentioned before, the equivalent representations have the same character

$$\chi_\alpha(g) = \mathrm{Tr}\ U_g^\alpha = \sum_{i=1}^{n_\alpha}(U_g^\alpha)_{ii}. \tag{1.D.12}$$

For simplicity, rescale the Hermitian scalar product, considered above, as

$$\mathcal{M}(a,b) = \frac{1}{|G|}\sum_{g\in G} a_g^* b_g. \tag{1.D.13}$$

Then the matrix elements orthogonality relation (1.D.10) implies that the characters of nonequivalent irreducible representations are orthogonal, and their norm equals one in this metrics:

$$\mathcal{M}(\chi_\alpha, \chi_\beta) = \delta_{\alpha\beta}. \tag{1.D.14}$$

The established properties of characters have many interesting consequences.

First, since a character is a function on conjugacy classes of the group (see Sec. I.B.8), one can conclude that *the amount of all nonequivalent irreducible representations of a given group does not exceed the number of its conjugacy classes.*

Second, a character of any representation U_g expressed via irreducible characters

$$\chi(g) = \sum_{\alpha} m_\alpha \chi_\alpha(g),$$

where $m_\alpha = 0, 1, 2, \ldots$ are a municipalities of the irreducible representation U^α in U. So, the square of the character is represented as the sum of squares of interes:

$$\mathcal{M}(\chi, \chi) = \sum_{\alpha} m_\alpha^2. \tag{1.D.15}$$

We obtained in this way a criteria of irreducibility: *a representation is irreducible if and only if the norm of its character is unity.*

Consider the simplest nontrivial example of the abelian cyclic group Z_n generated by s single element a

$$Z_n = \{e = a^0, a, a^2, \ldots, a^{n-1}\}, \qquad a^n = e. \tag{1.D.16}$$

Of course, the irreducible representations are one-dimensional and coincide with the characters. Define n representations by

$$\chi_l^\alpha = \chi_{a^l}^\alpha = \exp\left(\frac{2\pi\imath\alpha l}{n}\right), \qquad \alpha = 0, 1, 2, \ldots, n-1. \tag{1.D.17}$$

***Problem* 1.D.17:** *a)* Why there is no other irreducible representation for the group Z_n? *b)* Check that the characters (1.D.17) are orthogonal.

4. Regular representation

Here we investigate a representation given on the group algebras (see Sec. I.C.2). The left group action on the group algebra (1.C.5) defined the *regular representation* of the group G. More precisely,

$$\rho_g^{\mathrm{reg}} a = ga = \sum_{g' \in G} a_{g'} g g' = \sum_{g' \in G} a_{g^{-1}g'} g'. \tag{1.D.18}$$

The dimension of the representation is the order of the group:

$$n_{\mathrm{reg}} = |G|.$$

Take a natural basis in the group algebra consisting from the group elements. Then the representation ρ_g^{reg} just permutes all the basic elements. The exsection is the $g = e$ case when it coincides with the unity map. In the first case, all diagonal elements of the representation matrix vanish, so that we have for the character of the regular representation very simple expression

$$\chi_g^{\mathrm{reg}} = |G| \delta_{eg}. \tag{1.D.19}$$

Decompose the regular representation into irreducible representations with multiplicity m_α^{reg} of the αth irreducible representation (here and in the following we the superscript marking the regular representation):

$$\chi_g = \sum_{\alpha} m_\alpha \chi_g^\alpha, \qquad \chi = \chi^{\mathrm{reg}}. \tag{1.D.20}$$

The orthogonality of characters (1.D.14)together with the character formula for regular representation (1.D.19) imply

$$m_\alpha = \mathcal{M}(\chi, \chi^\alpha) = \frac{\chi_e \chi_e^\alpha}{|G_i|} = n_\alpha. \tag{1.D.21}$$

We conclude that *the regular representation contains all irreducible representations with the multiplicities equal to their dimensions*. Substituting (1.D.21) into (1.D.15), we

$$|G| = \sum_{\alpha} n_{\alpha}^2. \tag{1.D.22}$$

For example, it include a sum of all group elements,

$$e_s = \frac{1}{|G|} \sum_{g \in G} g. \tag{1.D.23}$$

Since a multiplication on a particular element g defines a one-to-one map of a group into itself, we have

$$\rho_g e_s = g e_s = s g = e_s \qquad \text{so that} \qquad e_s^2 = e_s. \tag{1.D.24}$$

Thus, we have extracted a trivial representation from the regular one. It is easy to see that its multiplicity is one. Indeed if $ga = a$ for all group elements g, then all coefficients a_g in the decomposition (1.C.5) are the same.

5. Regular representation and conjugacy classes

Denote by G_i the elements belonging to the ith conjugate class of the group. Clearly, for any element G,

$$g G_i g^{-1} = G_i, \qquad \bigcup_{i=1}^{r} G_i = G$$

where r is the number of conjugate classes. Then the element

$$h_i = \sum_{g \in G_i} g \tag{1.D.25}$$

from the group algebra commutes with any element g of the group:

$$g h_i g^{-1} = \sum_{g' \in G_i} g g' g^{-1} = \sum_{g' \in G_i} g = h_i. \tag{1.D.26}$$

Clearly, the elements (1.D.25) belong to the center of the group algebra, Moreover, they form a basis in the center so that if $ga = ag$ for a given group algebra element $a = \sum_{g \in G} a_g g$ and all group elements g, then

$$a = \sum_{i=1}^{r} c_i(a) h_i \tag{1.D.27}$$

with some coefficients $c_i(a)$.

***Problem* 1.D.18:** Prove exactly the above statement.

Clearly, the product $h_i h_j$ belongs to the center too, so that one can decompose it on the cental basic elements (1.D.27) and set $c_{ijk} = c_k(h_i h_j)$. Thus *the elements h_i form a closed abelian algebra*:

$$h_i h_j = h_j h_i = \sum_{k=1}^{r} c_{ijk} h_k, \tag{1.D.28}$$

where c_{ijk} are some numbers. In general, they have a complicated form dependent on the group structure.

Clearly, the structure coefficients c_{ijk} are symmetric on i, j. Let $i = 1$ marks the conjugacy class of unity: $G_1 = \{e\}$. Then it is clear that $h_1 = e$ and

$$h_1 h_j = h_j, \qquad \text{so} \qquad c_{1jk} = c_{j1k} = \delta_{jk}. \tag{1.D.29}$$

Is the value of c_{ij1} also so simple? Note that if it does not vanish then there is an element $g \in G_i$ such that $g^{-1} \in G_j$. But in this case the inverse of any other element from G_i belongs to G_j and vice versa. Indeed, if two element conjugate, then their inverses are conjugate too:

$$g' = sgs^{-1} \quad \Rightarrow \quad g'^{-1} = sg^{-1}s^{-1}.$$

Therefore the class G_j is composed from the inverses of all elements from the class G_i, and the index j completely specified by i. We will mark this fact by setting $j = i'$. Both classes contain the same number of elements,

$$G_{i'} = G_i^{-1}, \qquad |G_i| = |G_{i'}| \tag{1.D.30}$$

It is easy to derive from the definition (1.D.25) the particular form of the algebraic relations (1.D.28):

$$h_i h_{i'} = |G_i| e + \sum_{k \neq 1} c_{ijk} h_k. \tag{1.D.31}$$

Finally, we get

$$c_{ij1} = |G_i| \delta_{i'j}. \tag{1.D.32}$$

6. Characters and conjugacy classes

According to the Schur lemma, on irreducible representations U_g^α, these elements are proportional to the identity matrix:

$$U_{h_i}^\alpha = \lambda_i^\alpha I.$$

It can be written in terms of the characters,

$$|G_i| \chi_i^\alpha = n_\alpha \lambda_i^\alpha,$$

where we have restricted the characters to the conjugate classes so that $\chi_i^\alpha = \chi_{g_i}^\alpha$, where g_i is some representative from the class G_i. So, we have

$$\lambda_i^\alpha = \frac{|G_i| \chi_i^\alpha}{n_\alpha}. \tag{1.D.33}$$

The algebraic relation (1.D.28) in terms of characters is:

$$|G_i||G_j| \chi_i^\alpha \chi_j^\alpha = \sum_{k=1}^{r} |G_k| c_{ijk} n_\alpha \chi_k^\alpha. \tag{1.D.34}$$

***Problem* 1.D.19:** Derive the relation (1.D.34).

Now we are going to strengthen the relation between the irreducible representations and conjugacy classes, established in Sec. I.D.3. In fact, *the number of all nonequivalent irreducible representations of a given group equals the number r of its conjugacy classes.*

It is enough to set the completeness of the restricted characters χ^α. The relation (1.D.34) is very useful for this. Indeed, applying the sum over α and noting that the term $n_\alpha \chi_k^\alpha$ produces the character of the regular representation which has a very simple form, we arrive at

$$\sum_\alpha \chi_i^\alpha \chi_j^\alpha = \sum_{k=1}^{r} \frac{|G_k|}{|G_i||G_j|} c_{ijk} \chi_k = \frac{|G|}{|G_i||G_j|} c_{ij1} = \frac{|G|}{|G_i|} \delta_{i'j}, \tag{1.D.35}$$

where we have applied the expression (1.D.32). This is not a desired relation since the scalar product on the left side is not Hermitian. So, we have to replace

$$\chi_i^\alpha \to (\chi_i^\alpha)^* = (\chi_{g_i}^\alpha)^* = \chi_{g_i^{-1}}^\alpha = \chi_{g_{i'}}^\alpha = \chi_{i'}^\alpha,$$

which provides a desired completeness relations for the characters,

$$\sum_\alpha (\chi_i^\alpha)^* \chi_j^\alpha = \frac{|G|}{|G_i|} \delta_{ij}. \tag{1.D.36}$$

E. Symmetric group

Recall that the symmetric group S_n is formed by all permutations of n elements.

1. Two-line representation

Any element $s \in S_n$ can be presented in two-line notation

$$s = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ s_1 & s_2 & \dots & s_{n-1} & s_n \end{pmatrix}. \tag{1.E.1}$$

Here s_i is the image of the ith element. The examples are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 6 & 2 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix} \tag{1.E.2}$$

The product of two permutations s and p in this notation is

$$sp = \begin{pmatrix} 1 & \dots & n \\ s_1 & \dots & s_n \end{pmatrix} \begin{pmatrix} 1 & \dots & n \\ p_1 & \dots & p_n \end{pmatrix} = \begin{pmatrix} p_1 & \dots & p_n \\ s_{p_1} & \dots & s_{p_n} \end{pmatrix} \begin{pmatrix} 1 & \dots & n \\ p_1 & \dots & p_n \end{pmatrix} = \begin{pmatrix} 1 & \dots & n \\ s_{p_1} & \dots & s_{p_n} \end{pmatrix} \tag{1.E.3}$$

For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \tag{1.E.4}$$

The inverse element corresponds to the switch of two rows. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

***Problem* 1.E.20:** Show the conjugation rule

$$sps^{-1} = \begin{pmatrix} s_1 & \dots & s_n \\ s_{p_1} & \dots & s_{p_n} \end{pmatrix}. \tag{1.E.5}$$

Hint. Use the product rule (1.E.3).

2. Disjoint cycle representation

Alternatively, one can present this element in one-line notation by constructing the chain of images starting from the first element (we set $s(i) := s_i$)

$$1 \to s(1) \to s(s(1)) \to \dots \to 1.$$

Clearly, all its elements besides the first one are distinct. Denote by $k+1$ the length of this chain. It defines the cyclic permutation from the symmetric group $S_k \subset S_n$

$$c_1 = (1, s_1, \dots s_1^k) = \begin{pmatrix} 1 & s(1) & \dots & s^k(1) \\ s(1) & s(s(1)) & \dots & 1 \end{pmatrix}. \tag{1.E.6}$$

where by $s^k(i)$ we denote the k-fold nested action of the permutation on the ith element. If $k < n$ then by selecting one of the remaining elements i (where $i \neq s^k(1)$ for all k), one can construct another cycle

$$c_2 = (i, s(i), \dots s^l(i)).$$

Evidently, both cycles contain nonintersecting sets and mutually commute as group elements

$$c_1 c_2 = c_2 c_1.$$

Continuing this process inductively until all n elements are included into the game, we decompose the permutation into the product of commutative cycles,

$$s = c_1 c_2 \dots c_\kappa.$$

For example, the first permutation (1.E.4) is expressed in one-line form as

$$(1,4,3)(2) = (2)(1,4,3) = (1,4,3).$$

***Problem* 1.E.21:** Find the one-line representation for the last two elements in (1.E.4).

***Problem* 1.E.22:** For the cycle $c = (a_1, a_2 \dots, a_k)$ the inverse is $c^{-1} = (a_k, \dots, a_2, a_1)$.

Note that the cycle made up from a single element is unity: $(i) = e$ and can be omitted from the cycle decomposition.

3. Conjugacy classes

Given two elements $s \in S_n$ and $p \in S_k$ with $k \le n$, it is easy to verify that

$$s(p_1, \dots, p_k)s^{-1} = (s_{p_1}, \dots, s_{p_k}). \tag{1.E.7}$$

***Problem* 1.E.23:** Prove the formula (1.E.5) using the conjugation rule in two-line notation (1.E.3).

Therefore, the conjugacy map does not change the cycle structures of permutations. From the other side, it is easy to check that two elements with the same cycle structure are conjugate to each other. For example, the elements $s_1 = (123)(45)$ and $s_2 = (253)(14)$

$$s_2 = p s_1 p^{-1}, \qquad p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}.$$

From the other side, neither s_1 nor s_2 is not conjugate to the element $(12)(45)$.

Proposition 1. *The two elements are conjugate if and only if they consist of the same number of disjoint cycles of the same lengths.*

Therefore, a conjugacy class of the symmetric group S_n is uniquely characterized by the partitions of n,

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_k, \qquad 1 \le \lambda_k \le \ldots \le \lambda_1 \le n, \tag{1.E.8}$$

where λ_i is the length of the ith cycle.
For example in case of S_3 group, there are three partitions

$$3, \qquad 2+1, \qquad 1+1+1,$$

which describes the $r = 3$ conjugacy classed formed by all following elements with distinct values of i_1, i_2 and i_3:

$$(i_1, i_2, i_3), \qquad (i_1, i_2), \qquad e.$$

Note that one can select only two nonequal permutations for the fist class of thee-length cycles. The standard choice is $(1,2,3)$ and $(1,3,2)$, the others are obtained from them by cyclic shifts, which do not change them. So, this class consists of two elements. There are three elements of the second type, $(1,2)$, $(1,3)$ and $(2,3)$. The third class contains a single unit element.

***Problem* 1.E.24:** Check that the conjugacy class of n-cycles in S_n contains $(n-1)! = n!/n$ elements.

There are five possible partitions of 4:

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1. \tag{1.E.9}$$

***Problem* 1.E.25:** *a)* Describe the corresponding conjugacy classes of the symmetric group S_4. *b)* Write down the partitions of 5 and corresponding conjugacy classes for S_5.

The group S_4 consists of 24 permutations. The first class above consists of $3! = 6$ elements. The second class is determined by a three-cycle. As we have seen, for a given set of three numbers there are only two such permutations, so we have $8 = 4 \cdot 2$ permutations there. The third class consists two 2-cycle permutations, $(i_1, i_2)(i_3, i_4)$. Clearly the first cycle completely determines the second one, an we have $3 = \frac{1}{2}\binom{4}{2}$ elements here. The additional $\frac{1}{2}$ factor here identifies a given permutation with $(i_3, i_4)(i_1, i_2)$. This factor is absent in the fourth conjugacy class having $6 = \binom{4}{2}$ members. The last one is the unity class from a single member.

Let us count the number of group elements, $|G_i|$ in ith conjugacy class for general symmetric group $G = S_n$. Suppose the conjugacy class consists of n_1 1-cycles, n_2 2-cycles, etc. Then the number of different permutations in the conjugacy class is

$$|G_i| = \frac{n!}{2^{n_2}3^{n_3}\dots n_1!n_2!n_3!\dots}, \qquad \text{where} \qquad n_1 + 2n_2 + 3n_3 + \dots = n. \tag{1.E.10}$$

The factors k^{n_k} in the denominator is due to a cyclic order doesn't matter within a cycle: $(1, 2, 3, 4)$ is the same as $(2, 3, 4, 1)$. The factors $n_k!$ come from the fact that order of elements doesn't matter between cycles of the same length.

4. Irreducible representations in group algebra of S_3

Let us investigate the group algebra of the simplest symmetric groups.

Note that for S_2 it consists of two elements e and $(1, 2)$ with the constrain $s^2 = e$. Two distinct one dimensional representations are the trivial and antisymmetric ones given, respectively, by the totally symmetric e_s (1.D.23) and antisymmetric e_a elements:

$$e_s = e_1 = \frac{e + (1,2)}{2}, \qquad e_a = e_2 = \frac{e - (1,2)}{2}$$

It is easy to check that this basis satisfies

$$e_i e_j = \delta_{ij} e_i \tag{1.E.11}$$

for $i, j = a, s$.

The S_3 case is more complicate. Of course, like in any symmetric group, there are symmetric and antisymmetric elements here,

$$e_s = \frac{e + (1,2) + (1,3) + (2,3) + (1,2,3) + (1,3,2)}{6}, \qquad e_a = \frac{e - (1,2) - (1,3) - (2,3) + (1,2,3) + (1,3,2)}{6}.$$

***Problem* 1.E.26:** Check that the above elements satisfy the product rule (1.E.11).

Note that line e_s, the element e_a projects out all elements into itself:

$$e_a g = g e_a = e_a, \qquad g \in G. \tag{1.E.12}$$

***Problem* 1.E.27:** Check the above relation for e_a (case of e_s it has been treated already (1.D.24)]).

The S_3 group has three nonequivalent irreducible representations according to its three conjugacy classes (see Sec. I.E.3). So, there is a one representation $U^{(3)}$, yet undiscovered. The equation (1.D.22) get immediately its dimension and multiplicity: $n_3 = m_3 = 2$. So, it must present twice in the regular representation.

Indeed, e_s and e_a does not span whole six-dimensional group algebra, so the corresponding projectors do not form a complete set:

$$e_t = e - e_a - e_s = \frac{2e - (1,2,3) - (1,3,2)}{3}.$$

It is easy to check that e_t together with e_s and e_a satisfy the 'orthogonality' condition (1.E.11). Obviously, e_t projects to the four-dimensional space of the group algebra.

In order to extract both subspaces, split the projector e_t into two parts, $e_t = e_3 + e_4$, with

$$e_3 = \frac{e + (1,2) - (1,3) - (1,2,3)}{3} \sim \frac{e - (1,3)}{2} \cdot \frac{e + (1,2)}{2}, \tag{1.E.13}$$

$$e_4 = \frac{e - (1,2) + (1,3) - (1,3,2)}{3} \sim \frac{e - (1,2)}{2} \cdot \frac{e + (1,3)}{2}. \tag{1.E.14}$$

The second equations in (1.E.13), (1.E.14) imply that

$$e_3e_4 = e_4e_3 = 0, \qquad e_3e_a = e_4e_a = e_se_3 = e_se_4 = 0, \qquad e_3e_s = e_4e_s = e_ae_3 = e_ae_4 = 0. \tag{1.E.15}$$

The first and second sets of equations are easy to check since subsequent symmetrization and antisymmetrisation procedures vanish. Remember that the elements e_a and e_s commute with whole group algebra (1.D.24), (1.E.12), so they can be moved between the two multiplies in (1.E.13) and (1.E.14). This proves the third set of equations above.

So, the four projectors e_i with $i = 1, 2, 3, 4$ are orthogonal and satisfy (1.E.11). The elements $e_{1,2}$ define the trivial and antisymmetric representation, while $e_{3,4}$ generate two equivalent two-dimensional representations of S_3.

***Problem* 1.E.28:** Show by direct calculation that the elements (1.E.13) and (1.E.14) are idempotent, i.e. $e_i^2 = e_i$.

5. Irreducible representations in group algebra of S_n

Now we come to the general symmetric group algebra and briefly describe how to extract irreducible representations from it. The procedure is an extension of the $n = 3$ case. The totally symmetric and antisymmetrization elements of the group algebra are now

$$e_s = \frac{1}{n!}\sum_{g\in S_n} g, \qquad e_a = \frac{1}{n!}\sum_{g\in S_n}\epsilon_g g = \frac{1}{n!}\sum_{s\in S_n}\epsilon_{s_1\ldots s_n} s, \tag{1.E.16}$$

where $\epsilon_g = \pm 1$ is the parity of the permutation represented by the Levi-Civita tensor in two-line representation (1.E.1).

6. Cayley's theorem

It turns out that the set of symmetric groups is big enough: it contains any finite group as a subgroup. More precisely, Cayleys theorem states that

Every finite group G of order n is isomorphic to a subgroup of the symmetric group S_n.

Let us numerate the elements of the group

$$G = \{g_1, g_2, \ldots, g_n\}$$

and fix some element $g = g_k$. Then the left (right) multiplication of the group elements on g just permutes in some way their indexes:

$$\{gg_1, gg_2, \ldots, gg_n\} = \{g_{s_1}, g_{s_2}, \ldots, g_{s_n}\}.$$

The permutation $s \in S_n$ depends on g. This follows from the fact that the multiplication on g is a one-to-one map on G, so gg_i and gg_j must differ for $i \neq j$. It can be also proven that the map respects the group product.

7. Transpositions and generators

The exchange between ith and jth elements with all others elements fixed is called a transposition. It is easy to see that every permutation can be written as a product of transpositions. In other words, they generate the whole group S_n. Moreover, any transposition can be written as a product of the $n-1$ adjacent transitions $\sigma_i = (i, i+1)$. Using (1.E.7), it is easy to verify, for example, te relation

$$(i, i+2) = \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

In extension of this relation is

$$(i, i+k) = s\sigma_{i+k-1}s^{-1} \qquad \text{with} \qquad s = \sigma_i\sigma_{i+1}\ldots\sigma_{i+k-2}. \tag{1.E.17}$$

***Problem* 1.E.29:** Prove the equation (1.E.17). *Hint:* Apply the induction on k and use the inverse expression

$$s^{-1} = \sigma_{i+k-2}\ldots\sigma_{i+1}\sigma_i.$$

The $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ are independent and the only relations among them are:

$$\sigma_i{}^2 = 1, \tag{1.E.18a}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{with} \qquad j \neq i \pm 1, \tag{1.E.18b}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{1.E.18c}$$

***Problem* 1.E.30:** Show that the relation (1.E.18c) is equivalent to

$$(\sigma_i \sigma_{i+1})^3 = e.$$

***Problem* 1.E.31:** *a)* Show that $(1,2,3) = s_2 s_1$ and $(3,2,1) = s_1 s_2$

b) Write explicitly all relations among generators in S_3.

c) Show that S_3 is isomorphic to the symmetry group of the regular triangle D_3.

Note that the relations (1.E.18b) and (1.E.18c) solely define a *baraid group* with infinite number of elements. It represents braids in three-dimensional space with a multiplication defined as a composition of braids.

8. Alternating group

The representation of a permutation as a product of transpositions is not unique and can be modified using the relations (1.E.18a)–(1.E.18c). However, the same relations imply that the parity of their number must be the same. So, the number of transpositions needed to represent a given permutation is either always even or always odd.

Clearly, the even permutations form a subgroup $A_n \subset S_n$, which is called an alternating group. Moreover, it is an invariant subgroup, since for every $a \in A_n$ the element gag^{-1} is even too. The latter becomes evident if we express $g \in S_n$ as a transposition product .

A single transposition, say, σ_1, maps between even and odd parts of the symmetric group: $S_n = A_n \cup \sigma_1 A_n$. So, they have the same number of elements and the order of A_n is $n!/2$. For instance, the group A_3 is composed from the cyclic permutations isomorphic to Z_3.

Is the alternating group A_n simple, or it still contain a nontrivial invariant subalgebra? In fact, it is simple except for $n = 4$ case.

$$S_n/A_n = Z_2 = \{\pm 1\}. \tag{1.E.19}$$

Consider now the symmetric group S_3 containing six elements. It is isomorphic to the symmetry group of regular triangle D_3.

***Problem* 1.E.32:** Verify that

1) the three cyclic permutations e, $(2,3,1)$ and $(3,1,2)$ correspond to the triangle rotations on angle $\varphi = 0, \pm\frac{2\pi}{3}$ and form the cyclic subgroup Z_3;

2) the two-element exchanges $(1,2)$, $(1,3)$ and $(2,3)$ correspond to the reflections with respect to the three symmetry axes.

The cyclic permutations are even here, while the reflection are odd. In this particular case, the alternating group is formed by cyclic permutations, so

$$A_3 \equiv Z_3.$$

9. Groups S_4 and A_4

Consider now the group S_4 having the order 24. As a subalgebra, it contains the symmetry group of square D_4 with 8 elements Four permutations are odd and correspond to the rotations of the square on angles $\varphi = 0, \pm\frac{\pi}{2}, \pi$. Clearly, they form a cyclic group Z_4. Another four elements of D_4 are even and provide the reflections with respect to the symmetry axis.

***Problem* 1.E.33:** Construct explicitly the D_4 subgroup elements in the cycle representation and calculate their products.

Note that the D_4 subgroup is not invariant.

In contrast, the alternating subgroup A_4 is invariant. It contains only permutations with two transitions. There are three such elements acting on distinct points,

$$a = (1,2)(3,4), \qquad b = (1,3)(2,4), \qquad c = (1,4)(2,3). \tag{1.E.20}$$

The eight other elements transitions with one common point, which form three-length cycles. They are

$$(1,2,3), \qquad (2,3,4), \qquad (1,2,4), \qquad (1,3,4) \tag{1.E.21}$$

and their inverses. The order of A_4 is 12, so apart from the unity, there is no other element in this group.

There elements (1.E.20) commute with each other. Moreover, the product of two of them gives the third one:

$$ab = ba = c, \qquad a^2 = b^2 = e. \tag{1.E.22}$$

***Problem* 1.E.34:** Verify the first two equations above.

So, one can conclude that the elements a, b, c together with e form the group $Z_2 \times Z_2$. From the other side, this subgroup is invariant in S_4 and A_4 since according to the rule (1.E.7), any conjugation just permutes the three elements a, b, c. Therefore, the group A_4 is not simple.

What prevent to extent this construction to higher A_n? In fact, in order to ensure the invariance, we have to involve into the subset all elements

$$(ij)(kl) \qquad \text{with} \qquad 1 \le i < j < k < l \le n.$$

Then for $n > 4$, this set fails to be a subgroup, as is easy to see.

F. Finite reflection group

1. Reflection groups and root system

Consider the n-dimensional Euclidean space $\mathbb{R}^n$. Any vector $\alpha \in \mathbb{R}^n$ defines a hyperplane $(x, \alpha) = 0$, which is orthogonal to it and passes through the origin. The orthogonal reflection s_α with respect to this hyperplane as a mirror can be described by the following formula:

$$s_\alpha x = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha. \tag{1.F.1}$$

Indeed, it alters the direction of α, $s_\alpha \alpha = -\alpha$ while all points of the hyperplane $(x, \alpha) = 0$ are stable: $s_\alpha x = x$.

Clearly, the reflection belongs the orthogonal group: $s_\alpha \in O(n)$. It considers with the inverse map: $s_\alpha^2 = 1$. Moreover, since the $O(n)$ group preserves the scalar product, it maps the reflections in the following way:

$$s_{g\alpha} = g s_\alpha g^{-1}, \qquad \forall g \in O(n). \tag{1.F.2}$$

Actually, a reflection (1.F.1) is given by the direction of the vector α but not by its length. In order to set one-to-one correspondence between two sets, one must normalise the vectors somehow. A given set of hyperplanes generate a discrete group of isometries in $\mathbb{R}^n$, which, in general could be infinite. We consider the case then this group contains only finite number of elements.

Definition. A normalized root system $\mathcal{R}$ is a finite subset of vectors in $\mathbb{R}^n$ normalized by $\alpha^2 = 2$ such that $s_\alpha \beta \in \mathcal{R}$ for any $\alpha, \beta \in \mathcal{R}$.

In particular, since $s_\alpha \alpha = -\alpha$, the roots $\pm\alpha$ participate together in $\mathcal{R}$. Moreover, they define the same reflection: $s_\alpha = s_{-\alpha}$.

The second statement in the definition means that the root system remains invariant with respect to any reflection from this set. The reflection formula (1.F.1) is simplified for a normalized system:

$$s_\alpha x = x - (\alpha, x)\alpha. \tag{1.F.3}$$

Definition. The subgroup $\mathcal{W} = \mathcal{W}_\mathcal{R}$ of $O(n)$ generated by the reflections s_α for $\alpha \in \mathcal{R}$ is called the finite reflection group or the Weyl group associated with the normalized root system $\mathcal{R}$.

The abstract group of a reflection group is a Coxeter group, while conversely a reflection group can be seen as a linear representation of a Coxeter group. For finite reflection groups, this yields an exact correspondence: every finite Coxeter group admits a faithful representation as a finite reflection group of some Euclidean space. For infinite Coxeter groups, however, a Coxeter group may not admit a representation as a reflection group.

2. Examples of root systems

Consider the standard orthogonal basis $e_1, \dots, e_n$ in $\mathbb{R}^n$ with $(e_i, e_j) = \delta_{ij}$.

$A_{n-1} = sl(n)$. This normalized root system is formed by the vectors $\alpha_{ij} = e_i - e_j$:

$$\mathcal{R}_{A_{n-1}} = \{\pm(e_i - e_j) \,|\, 1 \le i < j \le n\}. \tag{1.F.4}$$

The associated Weyl group $\mathcal{W}_{\mathcal{R}}$ coincides with the symmetric group S_n of permutations of n elements considered in Sect. (I.E). Indeed, the reflection in the root α_{ij} interchanges the two basis vectors e_i and e_j and leaves the remaining ones fixed:

$$s_{\alpha_{ij}} x = x_j e_i + x_i e_j + \sum_{k \neq i,j} x_k e_k. \tag{1.F.5}$$

$C_n = sp(2n)$ or $B_n = so(2n+1)$ are characterized by same set forms a normalized root system defines as follows:

$$\{\pm(e_i - e_j), \pm(e_i + e_j), \pm\sqrt{2} e_i \,|\, 1 \le i < j \le n\}. \tag{1.F.6}$$

The associated Weyl group consists of permutations S_n and sign changes of n coordinates. The permutations do not commute with sign changes. So that the semidirect product of both groups is taken place here: $\mathcal{W}_{\mathcal{R}} = S_n \ltimes Z_2^n$.

$D_n = so(2n)$. It is described by the set

$$\{\pm(e_i + e_j), \pm(e_i - e_j) \,|\, 1 \le i < j \le n\}. \tag{1.F.7}$$

Its Weyl group consists of permutations S_n of permutations and simultaneous sign changes of even number of coordinates so that $\mathcal{W}_{\mathcal{R}} = S_n \ltimes Z_2^{n-1}$.

3. Weyl chambers

Removing the hyperplanes defined by the roots of $\mathcal{R}$ cuts up n-dimensional Euclidean space into a finite number of open regions, called Weyl chambers. It forms a convex polyhedral cone. Since each reflection maps the hyperplanes to each other, the chambers are permuted by the action of the Weyl group. In particular, the number of Weyl chambers equals the order of the Weyl group.

4. Positive and negative roots

Fix one Weyl chamber as a positive Weyl chamber V_+. It partitions the root system into into positive and negative roots:

$$\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-. \tag{1.F.8}$$

By definition positive roots have positive inner products with all vectors in V_+ while negative roots are minus positive roots. We shall write $\alpha > 0$ if $\alpha \in \mathcal{R}_+$. It is clear that

$$\mathcal{R}_+ = \{\alpha \in \mathcal{R} \,|\, (\lambda, \alpha) > 0 \quad \forall \lambda \in V_+\}, \tag{1.F.9}$$

$$V_+ = \{\lambda \in \mathbb{R}^n \,|\, (\lambda, \alpha) > 0 \quad \forall \alpha \in \mathcal{R}_+\}, \tag{1.F.10}$$

and so V_+ and $\mathcal{R}_+$ mutually determine each other.

Using positive root system, one can define a partial ordering among all vectors by setting $x \ge y$ if $x - y$ is a nonnegative superposition of positive roots:

$$x - y = \sum_{\alpha \in \mathcal{R}_+} c_\alpha \alpha, \qquad c_\alpha \ge 0. \tag{1.F.11}$$

5. Dihedral group

A regular polygon with n sides has $2n$ different symmetries: n rotational symmetries generated by the n reflection symmetries. Together they make up the dihedral group D_n. It is formed by rotations r_k on the angle $2\pi k/N$, forming an invariant cyclic subgroup Z_n, and the reflections s_k, where $0 \le k \le n-1$. The kth reflection mirrors the plane with respect to the axis making the angle $\pi k/N$ with the abscissa axis. Their composition is given by the following formulae:

$$r_k r_l = r_{k+l}, \quad r_k s_l = s_{k+l}, \quad s_k r_l = s_{k-l}, \quad s_k s_l = r_{k-l}. \tag{1.F.12}$$

The above indexes are taken modulus n. If n is odd, each axis of symmetry connects the midpoint of one side to the opposite vertex. If n is even, there are $n/2$ axes of symmetry connecting the midpoints of opposite sides and $n/2$ axes of symmetry connecting opposite vertices. Reflecting in one axis of symmetry followed by reflecting in another axis of symmetry produces a rotation through twice the angle between the axes.

The dihedral group is generated by the $r = r_1$ and $s = s_0$ subjecting to the rule

$$r^n = s^2 = (rs)^2 = 1. \tag{1.F.13}$$

It can be expressed as a semidirect product $D_n = Z_n \rtimes Z_2$ with Z_2 acting on Z_n by inversion.

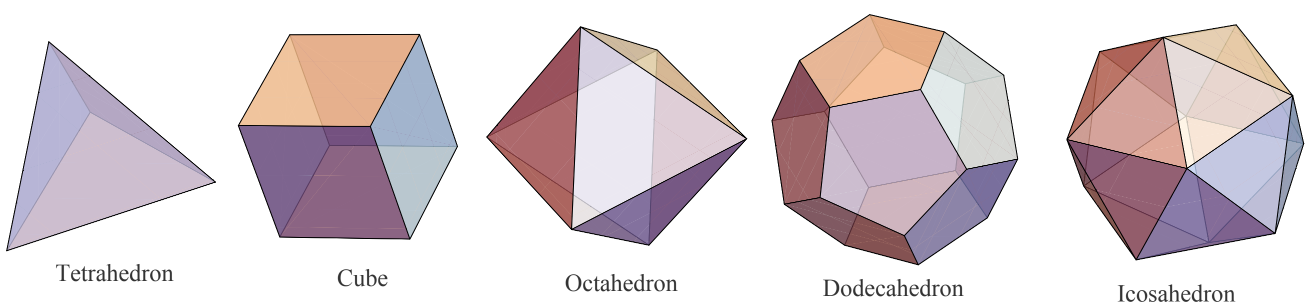
If we identify the complex plane $\mathbb{C}$ with the Euclidean plane via $z = x + iy$ then the set

$$\mathcal{R} = \{\sqrt{2}\exp(\pi\imath k/n) \,|\, k = 0, 1, \ldots, 2n-1\} \tag{1.F.14}$$

of renormalized $2n$-th roots of unity has Weyl group equal to the dihedral group D_n. Indeed, the composition of two reflections is a rotation over twice the angle between their mirrors.

6. Symmetry groups of regular polytopes

All symmetry groups of regular polytopes are finite Coxeter groups. Note that dual polytopes have the same symmetry group.



There are three series of regular polytopes in all dimensions. The symmetry group of a regular n-simplex is the symmetric group S_{n+1}, also known as the Coxeter group of type A_n. The symmetry group of the n-cube and its dual, the n-cross-polytope, is B_n, and is known as the hyperoctahedral group.

The exceptional regular polytopes in dimensions two, three, and four, correspond to other Coxeter groups. In two dimensions, the dihedral groups, which are the symmetry groups of regular polygons, form the series $I_2(p)$. In three dimensions, the symmetry group of the regular dodecahedron and its dual, the regular icosahedron, is H_3, known as the full icosahedral group. In four dimensions, there are three special regular polytopes, the 24-cell, the 120-cell, and the 600-cell. The first has symmetry group F_4, while the other two are dual and have symmetry group H_4.

The Coxeter groups of type D_n, E_6, E_7, and E_8 are the symmetry groups of certain semiregular polytopes.

G. Matrix groups

In this subsection we briefly recall the properties of the standard matrix groups. The matrix multiplication and inverse provides this set with the group structure.

1. Linear groups

We consider the complex and real $N \times N$ matrices $A = A_{ij}$. Any matrix defines a linear transformation in N dimensional space

The invertible matrices form a *general linear* group $GL(N)$. It is described by N^2 complex or real parameters given by the matrix elopements excluding $(N^2 - 1)$ dimensional surface $\det A \neq 0$ in the parameter region. The group unity is given by the identity matrix given by the Kronecker delta, $I_{ij} = \delta_{ij}$. The matrices λI with $\lambda \neq 0$ are isomorphic to $GL(1)$ and form the center of $GL(N)$. Therefore $GL(N)$ is not simple.

The *special linear group* $SL(N)$ is restricted by the area-preserving transformations given by the matrices with unit determinant,

$$\det A = 1. \tag{1.G.1}$$

Of course, this condition respects the matrix multiplication. The group manifold is a $(N^2 - 1)$ dimensional surface in the parameter space given by the matrix entries. Clearly, $SL(N)$ is an invariant subgroup in $GL(N)$.

***Problem* 1.G.35:** Show that $GL(N) = SL(L) \times GL(1)$ where the second group represents the center of general linear group.

So far we have not mentioned whether the space is real or complex. In it is necessary, the space type in mentioned in the definition of the group, like $GL(N, \mathbb{C})$ or $GL(N, \mathbb{R})$. Clearly, the real group is a subgroup in the complex group, and the latter requires twice more parameters. For example, $GL(N, \mathbb{R}) \subset GL(N, \mathbb{C})$, which is parametrized by N^2 real numbers.

2. Orthogonal groups

Consider the subgroup of $GL(N)$ formed by the orthogonal transformations. Such transformations form the *orthogonal group* $O(N)$ given by the matrices

$$AA^\tau = I \qquad \text{with} \qquad A^\tau_{ij} = A_{ji}, \tag{1.G.2}$$

where A^τ is the transposed matrix A. They leave invariant the Euclidean metrics $\sum_i x_i^2$.

The above equation means that the columns or rows form an orthogonal basis.

***Problem* 1.G.36:** Show that the dimension of $O(N)$ is $\frac{1}{2}(N-1)N$.

Taking the trace of both side of matrix equation (1.G.2), we get $\sum_{i,j} A_{ij}^2 = N$, so that $O(N)$ is a $\frac{1}{2}(N-1)N$ dimensional surface on the $(N^2 - 1)$ dimensional sphere. Therefore it is a compact group.

From the other side, the matrix relations (1.G.2) imply that $\det A = \pm 1$ so that the group splits on two disjoint parts. The $\det A = 1$ part forms a subgroup called a *special orthogonal group* and denoted as $SO(N)$.

***Problem* 1.G.37:** Check that the $SO(2)$ elements are parameterized by an angle, so that the group forms a cycle S^1,

$$A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{1.G.3}$$

***Problem* 1.G.38:** Prove that $SO(N)$ is a normal subgroup of $O(N)$.

The structure of the $O(N)$ differs for even and odd space dimensions Fix an orthogonal reflection R. It must obey the relations $\det R = -1$ and $R^2 = I$ and generates the Z_2 group. Clearly, R maps between the two parts of the group so that

$$O(N) = SO(N) \cup R\, SO(N). \tag{1.G.4}$$

. For odd values of N the simplest choice is $R = -I$ when R belongs to the center of the group. This strengthens the above relation to the direct product

$$O(N) = SO(N) \times Z_2. \tag{1.G.5}$$

For even values of N any reflection does not commute with the whole group so that the following, more general relation holds

$$Z_2 = O(N)/SO(N). \tag{1.G.6}$$

Note that the above properties of the orthogonal groups are valid for real and complex spaces since the orthogonality condition (1.G.2) is an analytic function on the matrix entries. So, although usually the real groups $O(N, \mathbb{R})$ and $SO(N, \mathbb{R})$ are considered, their complex counterparts $O(N, \mathbb{C})$ and $SO(N, \mathbb{C})$ also matter.

3. Pseudo-orthogonal groups

Consider the $(M+N)$-dimensional pseudo-Euclidean space with the Minkowski metrics described by the signature ϵ_i,

$$B_{ij} = \epsilon_i \delta_{ij}, \qquad \epsilon_1 = \cdots = \epsilon_M = 1, \qquad \epsilon_{M+1} = \cdots = \epsilon_{M+N} = -1. \tag{1.G.7}$$

The *pseudo-orthogonal group* $O(M, N)$ given by the linear maps which preserve Minkowski metrics:

$$ABA^\tau = B. \tag{1.G.8}$$

The above equation means that the columns or rows form an orthogonal basis in Minkowski space. In particular the row square is give by

$$A_{i1}^2 + \cdots + A_{iM}^2 - A_{i\,M+1}^2 - \cdots - A_{i\,M+N}^2 = 1. \tag{1.G.9}$$

The dimension of $O(M, N)$ coincides with the dimension of $SO(M+N)$ and equals $\frac{1}{2}(N+M-1)(N+M)$. It is a noncompact group for $M, N > 0$.

Consider now the simplest case of the two dimensional *Lorentz group* $O(1, 1)$.

***Problem* 1.G.39:** Prove that any $O(1, 1)$ element is expressed by the matrix

$$A = \begin{pmatrix} \epsilon_1 \cosh\phi & \epsilon_2 \sinh\phi \\ \epsilon_2 \sinh\phi & \epsilon_1 \cosh\phi \end{pmatrix} \qquad \text{with} \qquad \epsilon_i = \pm 1. \tag{1.G.10}$$

Consider the subgroup $O^+(1, 1)$, called *proper Lorentz group*, formed by the transformations

$$A_\phi = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \tag{1.G.11}$$

***Problem* 1.G.40:** 1) Verify that $A_{\phi_1} A_{\phi_2} = A_{\phi_1+\phi_2}$ so that the $O^+(1, 1)$ is isomorphic to $\mathbb{R}^1$. 2) Show that the $O^+(1, 1)$ is invariant subgroup and

$$O(1, 1)/O^+(1, 1) = Z_2 \times Z_2. \tag{1.G.12}$$

Denote by P and T the space inversion and time reversal operators:

$$P = \mathrm{diag}(1, -1), \qquad T = \mathrm{diag}(-1, 1). \tag{1.G.13}$$

They generate the discrete $Z_2 \times Z_2$ subgroup. From the representation (1.G.10) we see that any element $A \in O(1, 1)$ can be written in the four possible ways:

$$A = \{A_\phi,\, PA_\phi,\, TA_\phi,\, PTA_\phi\}. \tag{1.G.14}$$

Each of the four disjoint subsets is equivalent to $\mathbb{R}$. The first and last parts form the subgroup $SO(1, 1)$ with unit determinant.

The described picture is extended to the case of the Lorentz group in higher dimensions. It splits into four noncompact disjoint parts with the proper Lorentz group being a connected component of the group unity,

$$O(1, N) = G \cup P\,G \cup T\,G \cup PT\,G, \qquad G = O^+(1, N). \tag{1.G.15}$$

where the space inversion and time reversal operators are given now by

$$P = \mathrm{diag}(1, -1, \ldots, -1), \qquad T = \mathrm{diag}(-1, 1, \ldots, 1). \tag{1.G.16}$$

4. Unitary groups

Consider the subgroup of $GL(N,\mathbb{C})$ formed by the unitary transformations. Such transformations form the *unitary group* $U(N)$ given by the matrices

$$AA^{+} = I \qquad \text{with} \qquad A^{+}_{ij} = A^{*}_{ji}, \tag{1.G.17}$$

where A^{+} is the Hermitian conjugate of A. Since the above relation is not analytic, the unitary group is a real group.

***Problem* 1.G.41:** Show that the dimension of $U(N)$ is N^2.

The above implies that $\sum_{i,j}|A_{ij}|^2 = N$, which defines a sphere in the $2N^2$ dimensional real space. Hence, the $U(N)$ is a N^2 dimensional surface on the $(2N^2-1)$ dimensional sphere. Like the orthogonal group, it is compact.

Furthermore, the matrix relations (1.G.17) imply that $\det A = e^{i\phi}$ where ϕ is a real phase parameter.

The center of the unitary group consists of the matrices $e^{i\phi/N}I$ which, in fact, constitute the one dimensional group $U(1)$.

Fixing ϕ it to zero, which corresponds to the condition $\det A = 1$, we arrive at the subgroup called a *special unitary group* and denoted as $SU(N)$. Due to elimination of the phase, it has one less dimension equal to N^2-1. Clearly, $SU(N) = U(N)\cap SL(N,\mathbb{C})$.

The $SU(2)$ group forms a three-dimensional sphere S^3 in $\mathbb{R}^4$, which follows from the explicit matrix representation of its element,

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \text{with} \qquad |a|^2+|b|^2 = 1, \quad a,b\in\mathbb{C}. \tag{1.G.18}$$

***Problem* 1.G.42:** Prove the parametrization (1.G.18) for $A\in SU(2)$. *Hint*: Resolve explicitly the relations (1.G.1) and (1.G.17).

***Problem* 1.G.43:** Prove that $U(N) = SU(N)\times U(1)$.

The above task implies that $SU(N)$ is a normal subgroup in $U(N)$.

5. Symplectic groups

Consider the $2N$ dimensional phase space with coordinates and momenta. The canonical Poisson brackets are

$$\{q_i,p_j\} = \delta_{ij}, \qquad \{p_i,p_j\} = \{q_i,q_j\} = 0, \tag{1.G.19}$$

where $i = 1,\ldots,N$. Defining a single phase space variable with $x_i = q_i$ and $x_{i+N} = p_i$, we rewrite the canonical Poisson brackets in terms of the $2N\times 2N$ symplectic (antisymmetric) metrics Ω, given in block diagonal form

$$\{x_i,x_j\} = \Omega_{ij}, \qquad \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \Omega^{\tau} = -\Omega. \tag{1.G.20}$$

In classical mechanics, the canonical transformations of phase space variables x_i preserve the symplectic stricture, i.e. the Ω.

The linear canonical transformations form the *symplectic group* $Sp(2N)$. In other words, $Sp(2N)$ is defined by the matrices obeying

$$A\Omega A^{\tau} = \Omega, \qquad \mathrm{Det}A = 1. \tag{1.G.21}$$

***Problem* 1.G.44:** Show that $Sp(2N)$ is formed by the transformations which preserve the skew-symmetric bilinear form

$$(x,x') = \sum_{i=1}^{N}(q_ip'_i - q'_ip_i)$$

***Problem* 1.G.45:** Show that the dimension of $Sp(2N)$ is $(2N+1)N$.

***Problem* 1.G.46:** Show that the $Sp(2N)$ is not compact.

***Problem* 1.G.47:** Check that for $N = 1$, the second relation in (1.G.21) is automatically fulfilled. Check also the equivalence $Sp(2) \equiv SL(2)$.

In the literature the same notation is applied often for the so-called *unitary symplectic group*. It is the compact part of the complex symplectic group,

$$USp(N) = SU(2N) \cap Sp(2N, \mathbb{C}). \tag{1.G.22}$$

***Problem* 1.G.48:** Show that the $USp(N)$ group as the $Sp(N, \mathbb{R})$ is characterized by $N(2N + 1)$ real parameters.

$USp(N)$ is the subgroup of $GL(n, \mathbb{H})$ (invertible quaternionic matrices) that preserves the standard hermitian form on the N dimensional quaternionic space $\mathbb{H}^N$:

$$(x, y) = \sum_{i=1}^{N} x_i^+ y_i, \qquad x_i, y_i \in \mathbb{H}. \tag{1.G.23}$$

That is, $USp(N)$ is just the quaternionic unitary group, $U(N, \mathbb{H})$, so that it is sometimes called the hyperunitary group. Clearly, $USp(1) \equiv SU(2)$.

6. Equivalence of low dimensional groups

***Problem* 1.G.49:** Prove that $SL(2, \mathbb{C}) \equiv SO(1, 3) \times Z_2$, where the Z_2 group in the center of the $SL(2, \mathbb{C})$ formed by the elements $\{\pm I\}$.

***Problem* 1.G.50:** Prove the following relation between orthogonal and unitary groups

$$SO(4) \equiv \frac{SU(2) \times SU(2)}{Z_2}.$$

H. Lie groups

1. Smooth parametrization

More generally, a *Lie group* is a group that is also a N-dimensional smooth manifold, in which the group multiplication and inversion are smooth maps. In fact, it is enough to require the smoothness of the single map $G \times G$ given by $(g_1, g_2) \to g_1 g_2^{-1}$. The *real* and *complex* Lie groups are parameterized by the real and complex smooth manifolds correspondingly.

It is convenient to use a parametrization with zero describing the group unity,

$$g(a) = g(u^1, \dots, u^N), \qquad g(0) = 1, \tag{1.H.1}$$

so that a is either from $\mathbb{R}^N$ or $\mathbb{C}^N$. Then the product of any two group elements $g(u)$ and $g(v)$ is an element $g(w)$ of the group, where

$$g(u)g(v) = g(w) \qquad \text{so that} \qquad w = \varphi(u, v) \tag{1.H.2}$$

is a smooth function on the group parameters. Similarly, the inverse function $\psi(u)$ is defined by another smooth function,

$$g(u)^{-1} = g(\psi(v)). \tag{1.H.3}$$

Since $g(0) = 1$, they obey

$$\varphi(u, 0) = \varphi(0, u) = u, \qquad \psi(0) = 0. \tag{1.H.4}$$

The associativity of the product and the inverse definition impose the following restrictions on the functions:

$$\varphi(u, \varphi(v, w)) = \varphi(\varphi(u, v), w), \qquad \varphi(\psi(u), v) = \varphi(u, \psi(v)) = 0. \tag{1.H.5}$$

Smooth groups satisfying the above requirements are referred to as *Lie groups*.

2. Tangent space and structure constants

For sufficiently small values of the parameters $u, w \sim \varepsilon$ we may apply a Teylor expansion to both functions φ and ψ. We cut the series at the third order in ε:

$$\varphi^k(u,v) = u^k + v^k + a^k_{ij}u^iv^j + O(\varepsilon^3), \qquad \phi^k(u) = -u^k + a^k_{ij}u^iu^j + O(\varepsilon^3), \tag{1.H.6}$$

$$a^k_{ij} = \frac{1}{2}\frac{\partial^2}{\partial u^i \partial v^j}\varphi^k(0,0) = \frac{1}{2}\frac{\partial^2}{\partial u^i \partial u^j}\psi^k(0). \tag{1.H.7}$$

***Problem* 1.H.51:** Derive these relations. *Hint*: Apply the general Teylor expansion to the relations (1.H.5) and get the corresponding restrictions for its coefficients.

The tangent space at the identity element of the Lie group acquires a linear structure. In particular, the commutator of $g(u)$ and $g(v)$ can be defined for sufficiently small u and v. Using the Teylor series (1.H.6), we obtain up to the third order term

$$\varphi^k(u,v) - \varphi^k(v,u) = c^k_{ij}u^iv^j + O(\varepsilon^3) \qquad \text{with} \qquad c^k_{ij} = a^k_{ij} - a^k_{ji}, \tag{1.H.8}$$

The tensor c^k_{ij}, called *structure constants* of the Lie group, is antisymmetric on the lower indexes: $c^k_{ij} = -c^k_{ji}$. For the abelian group it vanished. The structure constants obey the Jacobi identity, which is a consequence of the associativity of the group product:

$$c^l_{is}c^s_{jk} + c^l_{js}c^s_{ki} + c^l_{ks}c^s_{ij} = 0. \tag{1.H.9}$$

***Problem* 1.H.52:** Prove the Jacobi identity (1.H.9). *Hint*: Apply twice the Teylor expansion of the product in (1.H.6) for the left and right side of the associativity equation in (1.H.5).

We have defined in this way a *Lie algebra* being a tangent space on the Lie group at the unity.

In fact, the structure constants can be obtained without using the linear structure of the tangent space.

***Problem* 1.H.53:** 1) Verify that the Teylor expansion of the adjoint action $g(w) = g(u)g(v)g(u)^{-1}$ is given by

$$w^k = v^k + c^k_{ij}u^iv^j + O(\varepsilon^3).$$

2) Verify that the Teylor expansion of the Lie group element $g(w) = g(u)g(v)g(u)^{-1}g(v)^{-1}$ is given by

$$w^k = c^k_{ij}u^iv^j + O(\varepsilon^3).$$

3. Adjoint representation

The adjoint representation of the group is defined on the tangent space at the group unity as a infinitesimal conjugate action near the unity [see problem 1.A.1]

$$\mathrm{Ad}_g x = \frac{d}{dt}\big(g\tilde{g}_0(t)g^{-1}\big)\Big|_{t=0} = gxg^{-1}, \qquad x = \dot{g}_0(0). \tag{1.H.10}$$

Literature

The definition of groups, representations, and their structure is presented in almost any book on the group theory. The description on the representation of finite groups, characters of their irreducible representations and orthogonality, regular representation etc. is well described in (Weyl, 1931), (Hamermesh, 1964), (Tung, 1985). The description of the symmetric group and its representation mainly based on (Hamermesh, 1964). For the description of the matrix groups, Lie groups and algebras, see (Dubrovin, 1992).

II. LIE ALGEBRAS

A. Structure and properties

1. Lie algebras from Lie groups

Actually, in section I.H we have studied the structure of the tangent space at the unity element of the Lie group. We have observed that this linear space with the same dimension N is endowed by the commutator (1.H.8) inherited from the group product. The commutator has been defined via the stricture constants of the Lie group obeying the Jacobi identity.

More precisely, consider any curve $g(u(t))$ in the Lie group G passing through the unity at $t = 0$ so that $u(0) = 0$. Then the tangent vector at the unity is formed by all such vectors:

$$\left.\frac{dg}{dt}\right|_{t=0} = \left.\frac{du}{dt}\right|_{t=0} \left.\frac{\partial g}{\partial u^i}\right|_{x=0} = \dot{u}(0)\partial_i g(0). \tag{2.A.1}$$

The elements of the tangent space can be presented in invariant form $u = \dot{u}^i x_i$ with $x_i = \partial_i$ being the basic vectors along the coordinate directions. The commutator for infinitesimal group elements (1.H.8) provides the tangent space with the following commutator

$$[\dot{u}, \dot{v}]^k = c^k_{ij}\dot{u}^i\dot{v}^j \qquad \text{so that} \qquad [x_i, x_j] = c^k_{ij}x_k. \tag{2.A.2}$$

Then the Jacobi identity on the structure constants (1.H.9) can be rewritten in term of the commutators,

$$[\dot{u}, [\dot{v}, \dot{w}]] + [\dot{v}, [\dot{w}, \dot{u}]] + [\dot{w}, [\dot{u}, \dot{v}]] = 0. \tag{2.A.3}$$

***Problem* 2.A.1:** Derive this identity using the relation (1.H.9).

2. Lie algebras, subalgebras and ideals

We have obtained all necessary ingredients for independent definition of the *Lie algebra* L. It is a vector space equipped with the commutator, which is bilinear antisymmetric operation obeying the Jacobi identity:

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0. \tag{2.A.4}$$

In physics we deal with the *complex* and *real* Lie algebras. They have common features, but the structure of the complex algebras is simper and more familiar.

The commutator of basic elements x_i defines the tensor c^k_{ij} called now the structure constants of the Lie algebra:

$$[x_i, x_j] = c^k_{ij}x_k. \tag{2.A.5}$$

Recall that the Jacobi identity puts a quadtatic relation (1.H.9) on them.

The subalgebra $H \subset L$ is a subspace in L closed with respect to the commutator: $[H, H] \subset H$.

The ideal or invariant subalgebra I is a subalgebra of L obeying $[I, L] \subset I$.

Let a Lie algebra space is represented as a direct sum of the linear spaces of two its subalgebras. If both subalgebras commute, $[L_1, L_2] = 0$, then it is said that L is a *direct sum* of the Lie algebras L_1 and L_2.

More generally, if L_1 is an ideal in L, i.e. $[L_1, L_2] = L_1$, then a Lie algebra L is called their *semidirect sum*, $L = L_1 \oplus L_2$.

3. Representations

A *representation* (ρ, V) of a Lie algebra L is a linear map $x \to \rho_x$ which associates to any element x the linear operator ρ_x acting on the vector space V obeying

$$\rho_{[x,y]} = \rho_x\rho_y - \rho_y\rho_x. \tag{2.A.6}$$

From any representation (ρ_g, V) of the Lie group G, one can obtain a representation of corresponding Lie algebra by taking the derivative at the group unity,

$$\rho_x = \left.\frac{d\rho_{g(t)}}{dt}\right|_{t=0} = \rho_{\dot{g}(0)}, \qquad x = \dot{g}(0) \in L. \tag{2.A.7}$$

***Problem* 2.A.2:** Prove that a valid representation of the Lie algebra is produced in this way, i.e. the relation (2.A.6). *Hint:* use one of the representations for the commutator in the group from the problem 1.H.53.

A representation is *faithful* if it separates the distinct elements of L.

the action of the Lie algebra by the commutator is a representation called an *adjoint representation* and mentioned by ad or by the hat:

$$\mathrm{ad}_x y = \hat{x} y = [x, y], \qquad x, y \in L. \tag{2.A.8}$$

***Problem* 2.A.3:** Verify that the adjoint action obeys (2.A.6), i.e.

$$\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]. \tag{2.A.9}$$

Hint: use the Jacobi identity.

Obviously, the adjoint representation of the Lie algebra can be obtained as an infinitesimal adjoint representation of the underlying Lie group (1.H.10)

$$\mathrm{ad}_x y = \frac{d}{dt} \mathrm{Ad}_{g(t)} x \big|_{t=0}, \tag{2.A.10}$$

where the smooth curve $g(t)$ passes though the origin, $g(0) = 1$.

4. Killing metrics

The trace of products of two elements in the adjoint representation is a metrics called a *Killing form* or a *Killing metrics*,

$$(x, y) = \mathrm{Tr} \hat{x} \hat{y}, \qquad x, y \in L. \tag{2.A.11}$$

Its matrix in the chosen basis is given by

$$g_{ij} = (x_i, x_j) = \mathrm{Tr}\, \hat{x}_i \hat{x}_j = c^l_{ik} c^k_{jl}. \tag{2.A.12}$$

***Problem* 2.A.4:** Derive the metric matrix (2.A.12).

The Killing form is *invariant*, i.e. obeys the relation

$$\hat{z}(x, y) = (\hat{z}x, y) + (x, \hat{z}y) = 0. \tag{2.A.13}$$

***Problem* 2.A.5:** Establish the invariance of the Killing metrics. *Hint*: present (2.A.13) as

$$\mathrm{Tr}(\mathrm{ad}_{[z,x]} \mathrm{ad}_y) + \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_{[z,y]}) = 0$$

and apply (2.A.9).

The invariance of the metrics g_{ij} is equivalent to the complete antisymmetry of the tensor $c_{ijk} = c^l_{ij} g_{lk}$. Indeed,

$$(\hat{x}_i x_j, x_k) = c^l_{ij} (x_l, x_k) = c^l_{ij} g_{lk} = c_{ijk}. \tag{2.A.14}$$

Then the identity (2.A.13) applied for the basic elements results in $c_{ijk} = -c_{ikj}$:

$$(\hat{x}_i x_j, x_k) + (x_j, \hat{x}_i x_k) = c^l_{ij} g_{lk} + c^l_{ik} g_{jl} = c_{ijk} + c_{ikj} = 0, \tag{2.A.15}$$

Together with $c_{ijk} = -c_{jik}$ this implies the total antisymmetry of the tensor c_{ijk}.

The invariance of the metrics implies that the orthogonal complement $I^\perp$ of any ideal $I \subset L$ is also an ideal. Indeed, using short schematic notations, we have

$$([L, I^\perp], I) = (I^\perp, [I, L]) = (I^\perp, I) = 0. \tag{2.A.16}$$

In particular, all null vectors of the Killing metrics make up an ideal, which we mention by I_0.

5. Solvable and nilpotent Lie algebras

We may construct the derived algebra $L^{(1)}$ of a Lie algebra L spanned by the commutators of the elements of L. Clearly, it forms an ideal. Formally we write

$$L^{(1)} = [L, L]. \tag{2.A.17}$$

Starting with a Lie algebra, we may form a whole series of derived algebras. If we write for the kth derived algebra

$$L^{(k+1)} = [L^{(k)}, L^{(k)}] \qquad \text{with} \qquad L^{(0)} = L. \tag{2.A.18}$$

The chain of the embedded algebras

$$L \supset L^{(1)} \supset L^{(2)} \supset \ldots \supset L^{(k)} \supset \ldots \tag{2.A.19}$$

is called the derived series of the Lie algebra L.

If the series ends up at some k with $L^{(k)} = 0$, the algebra L is said to be a *solvable* Lie algebra.

***Problem* 2.A.6:** Show that the Lie algebras formed by all upper triangular matrices are solvable.

Define another sequence of embedded ideals by the induction:

$$L^{k+1} = [L^k, L^k], \quad L^2 = L^{(1)} = [L, L], \tag{2.A.20}$$

$$L = L^1 \supset L^2 \supset \ldots \supset L^k \supset \ldots \tag{2.A.21}$$

It is called the descending central series or descending sequence of ideals. If the series terminates for some positive integer k with $L^k = 0$, then the Lie algebra L is called *nilpotent*.

A nilpotent Lie algebra is necessarily solvable, but a solvable Lie algebra need not be nilpotent.

***Problem* 2.A.7:** Show that the Lie algebra formed by all upper triangular matrices with equal diagonal elements is nilpotent.

6. Simple and semisimple Lie algebras

A Lie algebra is said to be *simple* if it contains no proper ideals. It is said to be *semisimple* if it does not contain nontrivial abelian ideals. A simple algebra is necessarily semisimple.

Theorem 1. *A Lie algebra is semisimple if and only if its Killing form is nondegenerate.*

Proof. Suppose a Lie algebra L is not semisimple, i.e. possesses a commutative ideal I which is spanned by the first elements of the chosen basis in L. We distinguish I by attaching primes to the basic states: $x_{i'}$. If we take the second metric index from I, the Killing metrics will vanish:

$$g_{ij'} = c^k_{il} c^l_{j'k} = c^k_{il'} c^{l'}_{j'k} = c^{k'}_{il'} c^{l'}_{j'k'} = 0. \tag{2.A.22}$$

The first two identities above fulfil since N is ideal so $[x_{j'}, x_k] = c^{l'}_{j'k} x_{l'}$. The last identity is due to the commutativity of the ideal I. Therefore, the matrix is degenerate.

The inverse statement is harder to prove and we only sketch the proof. Note that the null vectors of the Killing form make up an ideal $I_0 = L^{\perp}$. In fact, it is not a noncommutative ideal we are looking for.

□

Proposition 2. *For a simple complex Lie algebra any invariant metrics is a scalar multiple of the Killing form.*

Proof. Indeed, any linear combination of two invariant metrics is again an invariant metrics. Thus, one can take a combination with vanishing determinantL: $\det(g - \alpha g_1) = 0$ for some complex α. Then the invariant metrics $g' = g - \alpha g_1$ would have a singular vector x_0 with $(x_0, L) = 0$. But the null space of the invariant form forms an ideal I'_0 in the Lie algebra. This can be proven in the same way as was established for the Killing form. Since the algebra is simple, I'_0 must coincide with L. □

B. Matrix Lie algebras

1. General Lie algebra

The matrices operating in N-dimensional space form a linear space endowed with the natural commutator $[A, B] = AB - BA$. The corresponding Lie algebra is called a *general linear algebra* $gl(n)$. Clearly, they describe the tangent space at the identity point of $GL(N)$.

Choose as a basic elements the N^2 matrices $E_{i,j}$ with a single nonzero entry at the intersection of the i-th row and j-th column,

$$(E_{i,j})_{i'j'} = \delta_{ii'}\delta_{jj'}, \qquad 1 \le i, j \le N. \tag{2.B.1}$$

***Problem* 2.B.8:** Verify that $E_{i,j}$ form an algebra

$$E_{i,j}E_{k,l} = \delta_{jk}E_{i,l}. \tag{2.B.2}$$

Sometimes it is convenient to present these matrices in the Dirac bracket formalism where their algebra (2.B.2) becomes evident:

$$E_{i,j} = |i\rangle\langle j|. \tag{2.B.3}$$

The structure constants are defined by the commutator

$$[E_{i,j}, E_{k,l}] = \delta_{jk}E_{i,l} - \delta_{li}E_{k,j}. \tag{2.B.4}$$

Proposition 3. *The Killing form (2.A.12) for two matrices $X, Y \in gl(N)$ is reduced to the usual traces:*

$$(X, Y) = Tr\,\hat{X}\hat{Y} = 2N\,trXY - 2trX\,trY. \tag{2.B.5}$$

Proof. Since $\mathrm{tr}\,AE_{k,l} = A_{lk}$ for any matrix A, it is easy to see that

$$\mathrm{Tr}\,\hat{X}\hat{Y} = \sum_{k,l}\mathrm{tr}(E_{l,k}\hat{X}\hat{Y}E_{k,l}), \tag{2.B.6}$$

where the trace on the left part is taken in the adjoint representation but the trace on the right part is a usual matrix trace. Using the cyclicity of the trace, we further simplify the trace (2.B.6):

$$\begin{aligned}\mathrm{Tr}\,\hat{X}\hat{Y} &= 2\sum_{k,l}\mathrm{tr}(E_{l,k}XYE_{k,l} - E_{l,k}XE_{k,l}Y) = 2N\sum_{l}\mathrm{tr}\,E_{l,l}XY - 2\sum_{k,l}X_{kk}Y_{ll}\\ &= 2N\,\mathrm{tr}XY - 2\,\mathrm{tr}X\,\mathrm{tr}Y.\end{aligned} \tag{2.B.7}$$

This derivation becomes more transparent if we apply the Dirac form for the basic matrices (2.B.3). For instance,

$$\mathrm{tr}(E_{l,k}A) = \mathrm{tr}\big(|l\rangle\langle k|A\big) = \langle k|A|l\rangle = A_{kl}, \tag{2.B.8}$$

$$\mathrm{tr}(E_{l,k}AE_{i,j}B) = \langle k|A|i\rangle\langle j|B|l\rangle = A_{ki}B_{jl}. \tag{2.B.9}$$

□

Since the $\hat{I} = 0$, the identity matrix $I = \sum_i E_{i,i}$ is a null vector of the Killing metrics, $(I, A) = 0$ for any matrix A. Using (2.B.5), one can easily deduce the metrics in the above basis

$$g_{ij,kl} = (E_{i,j}, E_{k,l}) = 2N\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}. \tag{2.B.10}$$

2. Special Lie algebra

Consider now the tangent space of $SL(N)$ group. Substituting $A = I + \varepsilon X$ into the unity determinant condition (1.G.1), at the first order on ε we come to the zero trace condition,

$$\mathrm{tr}X = 0. \tag{2.B.11}$$

***Problem* 2.B.9:** Show that $\det(I + \varepsilon X) = 1 + \varepsilon X + O(\varepsilon^2)$.

Since the commutator trace vanish, the condition (2.B.11) remains invariant under Lie algebra operations giving rise to the *special Lie algebra* $sl(N)$.

***Problem* 2.B.10:** Show that $gl(N) = sl(N) \oplus gl(1)$, where $gl(1)$ is spanned by the identity matrix.

The Killing metrics simplifies for the $sl(N)$. Note one the basis of $sl(N)$ contains one element less than the $gl(N)$ basis which can be set up to be the identity matrix I. The latter, however, disappears in the Killing form, so that the Killing form of $sl(N)$ and $gl(N)$ coincide. From the other side, due to the condition (2.B.11), the metrics (2.B.5) is reduced to

$$(X, Y) = 2N\mathrm{tr}XY \quad \text{with} \quad X, Y \in sl(N). \tag{2.B.12}$$

Define now more conventional, orthogional basis for the $sl(N)$ Lie algebra. The $N(N-1)$ off-diagonal matrices $E_{i,j}$, $i \neq j$ remain unhanded, and we have set $N-1$ traceless diagonal matrices. The simplest choice are the elements $E_{i,i} - E_{i+1,i+1}$ which are not orthogonal with respect to the Killing metrics. The orthogonal diagonal basis is

$$E_k = \sum_{i=1}^{k} E_{k,k} - kE_{k+1,k+1}, \qquad k = 1, \ldots, N-1. \tag{2.B.13}$$

We write down explicitly all nonzero scalar products between the basic states,

$$(E_{i,j}, E_{j,i}) = 2N \quad \text{for} \quad i \neq j, \qquad (E_k, E_k) = 2Nk(k+1). \tag{2.B.14}$$

3. Orthogonal Lie algebra

Consider now the infinitesimal generators of the $SO(N)$ group. Substituting $A = I + \varepsilon X$ into the orthogonality condition (1.G.2), we see that the infinitesimal matrix must be antisymmetric at the first order,

$$X^\tau = -X \qquad \text{so that} \qquad X_{ij} = -X_{ji}. \tag{2.B.15}$$

***Problem* 2.B.11:** Show that the antisymmetric matrices form a linear space closed under the commutator, while the symmetric matrix space is not closed.

So, the antisymmetric matrices form an *orthogonal Lie algebra* $so(N)$. Evidently, they obey belong to $sl(N)$ because their diagonal elements vanish, $X_{ii} = 0$. They are described by the upper (or lower) diagonal elements of the matrix so that as for the $SO(N)$ group,

$$\dim so(N) = \frac{1}{2}N(N-1). \tag{2.B.16}$$

The simplest basis consists of the antisymmetric matrices

$$L_{i,j} = E_{i,j} - E_{j,i} \quad \text{with} \quad i < j. \tag{2.B.17}$$

which are subjected to the commutation rules

$$[L_{i,j}, L_{k,l}] = \delta_{jk}L_{i,l} + \delta_{il}L_{j,k} - \delta_{jl}L_{i,k} - \delta_{ik}L_{j,l}. \tag{2.B.18}$$

***Problem* 2.B.12:** Verify the commutation relations (2.B.18).

Since the matrix trace of the product of the basic states is

$$\mathrm{tr}\, L_{i,j}L_{k,l} = -2\delta_{ik}\delta_{jl}, \tag{2.B.19}$$

where $i < j$ and $k < l$, we have:

$$\mathrm{Tr}\,\hat{X}\hat{Y} = -\frac{1}{2}\sum_{k<l}\mathrm{tr}(L_{k,l}\hat{X}\hat{Y}L_{k,l}) = \frac{1}{2}\sum_{k,l}\mathrm{tr}(E_{l,k}\hat{X}\hat{Y}E_{k,l}) - \frac{1}{2}\sum_{k,l}\mathrm{tr}(E_{k,l}\hat{X}\hat{Y}E_{k,l}). \tag{2.B.20}$$

The first term on the right side is a half of the $sl(N)$ Killing form, i.e. is $N\mathrm{tr}\,XY$. The second term is calculated using the identity (2.B.9) and the skew-symmetry of X:

$$-\frac{1}{2}\sum_{k,l}\mathrm{tr}(E_{k,l}\hat{X}\hat{Y}E_{k,l}) = \sum_{k,l}\mathrm{tr}(E_{k,l}XE_{k,l}Y - E_{k,l}XYE_{k,l}) = \sum_{k,l}X_{lk}Y_{lk} - \sum_{k}\mathrm{tr}(XYE_{k,k}) \\ = -2\,\mathrm{tr}XY. \tag{2.B.21}$$

Inserting this into the expression (2.B.6), we get the Killing form for the orthogonal Lie algebra,

$$(X,Y) = (N-2)\mathrm{tr}\,XY. \tag{2.B.22}$$

Using the identity (2.B.19), we see that the Killing metrics is negative and diagonal in Lij,

$$g_{ij,kl} = (L_{i,j}, L_{k,l}) = -2(N-2)\delta_{ik}\delta_{jl}. \tag{2.B.23}$$

4. Unitary and special unitary Lie algebras

Consider now the infinitesimal generators of the $U(N)$ and $SU(N)$ groups. Substituting $A = I + \varepsilon X$ into the unitary condition (1.G.17), we see that the X mist be shew-Hermitian at the first order,

$$X^{+} = -X \qquad \text{or} \qquad X^{*}_{ij} = -X_{ji}. \tag{2.B.24}$$

***Problem* 2.B.13:** Show that the skew-Hermitian matrices form a linear space closed under the commutator.

So, the skew-Hermitian matrices form an *unitary Lie algebra* $u(N)$. Its subalgebra formed by traceless skew-Hermitian matrices is called a *special unitary Lie algebra* $su(N)$. Their dimensions, of course, are inherited from their groups:

$$\dim u(N) = N^2, \qquad \dim su(N) = N^2 - 1. \tag{2.B.25}$$

The simplest $u(N)$ basis consists of the matrices

$$\imath E_{i,i}, \qquad L_{i,j} = E_{i,j} - E_{j,i}, \qquad S_{i,j} = \imath(E_{i,j} + E_{j,i}) \qquad \text{where} \quad i < j. \tag{2.B.26}$$

The related $su(N)$ basis as for the $sl(N)$ case, is the same apart from the the diagonal elements which can be set either to the $\imath E_{i,i} - \imath E_{i+1,i+1}$ or to the $\imath E_k$ as is defined in (2.B.13):

$$E'_i = \imath E_i, \qquad L_{i,j} = E_{i,j} - E_{j,i}, \qquad S_{i,j} = \imath(E_{i,j} + E_{j,i}), \tag{2.B.27}$$

where the indexes i,j are restricted by $i < N$ and $i < j$.

***Problem* 2.B.14:** Derive the commutation relations among the basis elements (2.B.27).

Both algebras are real Lie algebras: $u(N) = u(N,\mathbb{R})$ and $su(N) = su(N,\mathbb{R})$. Indeed, the complex analytic function do nor survive under the Hermitian conjugate. The complexification of the unitary Lie algebras produce the linear algebras

$$u(N)_{\mathbb{C}} \equiv gl(N,\mathbb{C}), \qquad su(N)_{\mathbb{C}} \equiv sl(N,\mathbb{C}). \tag{2.B.28}$$

***Problem* 2.B.15:** Prove the above isomorphisms. *Hint:* compare the basses of the corresponding algebras.

The correspondence between the unitary and linear algebras (2.B.28) makes similar their algebraic properties. In particular, the decomposition $u(N) \equiv su(N) \oplus u(1)$ resembles the aforementioned splitting of the $gl(N)$.

The correspondence (2.B.28) means that the form of the Killing metrics for $u(N)$ or $su(N)$ coincides with the forms of the metrics for $gl(N)$ (2.B.5) and $sl(N)$ (2.B.12), respectively. The $sl(N)$ basis (2.B.27) is orthogonal with respect to the Killing metrics with the only nontrivial scalar products counting the lengths of the basic elements,

$$(E'_k, E'_k) = -2Nk(k+1), \qquad (L_{i,j}, L_{i,j}) = -4N, \qquad (S_{i,j}, S_{i,j}) = -4N, \tag{2.B.29}$$

where $i < j$ and $k < N$. So, the metrics is negatively defined.

Note that the basis (2.B.27) in the absence of the coefficient $\imath$ is an orthogonal basis of the $sl(N,\mathbb{R})$. The Killing metrics there has a positive signature on $\frac{1}{2}N(N+1)-1$ vectors and the negative signature on the remaining $\frac{1}{2}N(N-1)$ states.

5. Pseudoorthogonal Lie algebra

Recall that the pseudoorthogonal group $SO(M,N)$ preserves the generalized Minkowsli metrics B_{ij} characterized by the signature $\epsilon_i = \pm 1$ (1.G.7). The related Lie algebra $so(M,N)$ called a *pseudoorthogonal Lie algebra* is given by the matrices obeying

$$X^\tau = -BXB \qquad \text{or} \qquad X_{ji} = -\epsilon_i \epsilon_j X_{ij}. \tag{2.B.30}$$

***Problem* 2.B.16:** 1) Show that such matrices form a linear space closed under the commutator. 2) Derive the relation (2.B.30) from (1.G.8) for $A = I + \varepsilon X$.

Like the skew-symmetric matrices, they obey belong to $sl(M+N)$ because their diagonal elements vanish, $X_{ii} = 0$. They are described by the upper (or lower) diagonal elements of the matrix so that as for the $so(M+N)$ so that

$$\dim so(M,N) = \frac{1}{2}(M+N)(M+N-1).$$

The simplest basis consists of the antisymmetric matrices

$$L_{i,j} = \epsilon_i E_{i,j} - \epsilon_j E_{j,i} \quad \text{with} \quad i < j. \tag{2.B.31}$$

which are subjected to the commutation rules

$$[L_{i,j}, L_{k,l}] = B_{jk} L_{i,l} + B_{il} L_{j,k} - B_{jl} L_{i,k} - B_{ik} L_{j,l}. \tag{2.B.32}$$

***Problem* 2.B.17:** Show that the matrices L_{ij} satisfy the rule (2.B.30) and derive their commutation relations.

The Killing metrics is diagonal in the selected basis (2.B.31),

$$g_{ij,kl} = (L_{i,j}, L_{k,l}) = -2(N-2)\epsilon_i \epsilon_j \delta_{ik} \delta_{jl}. \tag{2.B.33}$$

***Problem* 2.B.18:** Derive the form of the Killings metrics (2.B.33) in the basic (2.B.31).

For the real $so(M,N,\mathbb{R})$ algebra, the metrics contains MN positive signs and $\frac{1}{2}M(M-1) + \frac{1}{2}N(N-1)$ negative signs.

Note that the complex Lie pseudoorthogonal Lie algebras make no sense since they the are isomorphic to the complex orthogonal algebra,

$$so(M,N,\mathbb{C}) \equiv so(M+N,\mathbb{C}).$$

6. Symplectic Lie algebra

Recall that the symplectic group $Sp(2N)$ is formed by the linear transformations of $2N$ dimensional space preserving the skew-symmetric symplectic metrics Ω_{ij}, defined by (1.G.20). The related Lie algebra $sp(2N)$ called a *symplectic Lie algebra* is given by the matrices obeying

$$X^\tau = \Omega X \Omega \qquad \text{or} \qquad X_{ji} = \epsilon_i \epsilon_j X_{ij}. \tag{2.B.34}$$

***Problem* 2.B.19:** 1) Show that such matrices form a linear space closed under the commutator. 2) Derive the relation (2.B.34) from (1.G.21) for $A = I + \varepsilon X$.

Applying the race to both sides of the first equation in (2.B.34) and using that $\Omega^2 = -I$, we come to the zero-trace condition, $\mathrm{tr} X = 0$, for the $sp(2N)$ algebra. The latter becomes clear also from the following representation of the general $sp(2N)$ matrix X in the block diagonal form with N-dimensional matrices F, U and V:

$$X = \begin{pmatrix} F & U \\ V & -F^\tau \end{pmatrix}, \qquad U = U^\tau, \quad V = V^\tau \tag{2.B.35}$$

***Problem* 2.B.20:** Prove the formula (2.B.35) for any $X \in sp(2N)$.

The dimension of the symplectic Lie algebra is given by the corresponding group dimension, so

$$\dim sp(2N) = N(2N+1).$$

The suitable basis is formed by the elements

$$D_{i,j} = E_{i,j} - E_{j+N,i+N}, \qquad P^+_{i,j} = E_{i,j+N} + E_{j,i+N}, \qquad P^-_{i,j} = E_{i+N,j} + E_{j+N,i} \tag{2.B.36}$$

provided that all indexes are restricted by $i, j \le N$.

C. Invariant vector fields

1. Lie algebra of vector fields

As an example of the infinite dimensional Lie algebra consider the space of vector fields ξ defined on a n-dimensional smooth manifold M

$$\xi = \xi^i \frac{\partial}{\partial u_i}. \tag{2.C.1}$$

They act on the functions $f(x) = f(u_1, \ldots, u_n)$,

$$\hat{\xi} f = \xi^i \partial_i f. \tag{2.C.2}$$

A vector field induces a *vector flow* along its direction (a one-parametric transformation family) as the solution of the system of the differential equations

$$\dot{x}(t, u_0) = \hat{\xi} x(t, u_0), \qquad u(0, u_0) = u_0. \tag{2.C.3}$$

The flow operators

$$\xi_t : \quad u_0 \to x(t, u_0). \tag{2.C.4}$$

form the abelian group with the exponential rule $\xi_{t_1+t_2} = \xi_{t_1}\xi_{t_2}$, so that the relation transformation sometimes called an *exponential map*. In fact, they act as true exponentials:

$$\xi_t u = \exp(t\hat{\xi})u = \sum_{i=0}^{\infty} \frac{t^n}{n!} \hat{\xi}^n u. \tag{2.C.5}$$

In particular, for $\xi = \partial_1$ this produces the map $(u_1, u_2, \ldots u_k) \to (u_1 + t, u_2, \ldots u_k)$.

***Problem* 2.C.21:** Show that the flow induced by the field $x\partial_y - x\partial_x$ results in $z \to e^{\imath t} z$ in complex coordinates.

Suppose now we have two vector fields, ξ and η on the same manifold.

***Problem* 2.C.22:** Proof that the product of two vector files, $\xi^i \eta^j$ is not a tensor.

Nevertheless, the commutator of two vector fields defined also a vector field,

$$(\hat{\xi}\hat{\eta} - \hat{\eta}\hat{\xi}) f = \xi^i \frac{\partial \eta^j}{\partial x_i} \partial_j f - \eta^i \frac{\partial \xi^j}{\partial x_i} \partial_j f = [\hat{\xi}, \hat{\eta}]^i \partial_j f \tag{2.C.6}$$

so that

$$[\xi, \eta]^i = \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j}. \tag{2.C.7}$$

The Jacobi equation follows immediately from the associativity of the operator product.

The commutator of two vector fields is expressed via the related infinitesimal one-parametric shifts,

$$\xi_t \eta_\tau - \eta_\tau \xi_t = [\xi, \eta] t\tau + o(t\tau). \tag{2.C.8}$$

Clearly, any smooth map of two manifolds, $f:\ M \to M'$ induces a map of the vector fields,

$$\xi^{i'} = \frac{\partial m^{i'}}{\partial m^i} \xi^i. \tag{2.C.9}$$

Then the commutator of two vector fields maps to the commutator of their images,

$$f[\xi, \eta] = [f\xi, f\eta]. \tag{2.C.10}$$

***Problem* 2.C.23:** Prove the above relation. *Hint*: The map f is similar to the coordinate change.

Let a Lie group G acts on the manifold M. The action $m \to gm$, where m is a manifold point and g is a group element, induces the map of the tangent vectors given by the Jacobian matrix dg. A vector field $\xi = \xi_m$ on M is called *G-invariant* if it is unchanged by the action of any group element, meaning

$$g\xi_m := (dg)\xi_m = \xi_{gm} \qquad \text{for all } g \in G \text{ and } m \in M. \tag{2.C.11}$$

2. Left invariant vector fields

Let now $M = G$ and the group acts on itself by the left multiplication $g(v) \to g(u)g(v)$. A vector field is called *left invariant* if it remains unchanged under the left multiplications:

$$\xi_{g(u)g(v)} = g(u)\xi_{g(v)} \tag{2.C.12}$$

for any group elements $g(u)$ and $g(v)$ dependent on smooth group parameters. More precisely, the group action on the vector is given by the Jacobian so that in the coordinate space we have

$$\xi^k(\varphi(u,v)) = \frac{\partial \varphi^k(u,v)}{\partial v^i}\xi^i(v) \tag{2.C.13}$$

Similarly, one can define a *right invariant* vector field.

The left invariant vector field induces a flow (2.C.3), (2.C.5) on the group manifold G. In particular, it maps the unity element to some g_t. Applying both sides of the equation (2.C.12) to the e, we obtain

$$\dot{g}_t = g_t \dot{g}_0, \qquad g_0 = e. \tag{2.C.14}$$

The solution can be written formally in the form

$$g_t = g_0 \exp(tx) \qquad \text{with} \qquad x = \dot{g}_0. \tag{2.C.15}$$

In more explicit form the both equations can be presented via the coordinates as follows:

$$\dot{u}_t^k = \frac{\partial \varphi^k(u_t,0)}{\partial v^i}x^i, \qquad u_t = \varphi(u_0, \exp(tx)), \tag{2.C.16}$$

where the briefly denote the second group parameter by a single exponential, $\exp(tx) = (e^{tx_1}, \ldots, e^{tx_N})$ with N being the dimension of the Lie group G. Note that in the derivation the associativity condition on φ is used (1.H.5).

We see that the flow generated by the left invariant vector field on a Lie group is given by the right multiplication on the exponential map.

***Example* 1:** Consider the case of the $GL(N)$ group. The right multiplication induces the vector field action $X \to = AX$ for all $A \in GL(N)$ and any fixed $X \in gl(N)$. The solution to the system

$$\dot{A}_t = A_t X \tag{2.C.17}$$

has the exponential form in complete analogy of the general case considered above,

$$A_t = A_0 \exp(tX). \tag{2.C.18}$$

Proposition 4. *The commutator $[\xi, \eta]$ of two left invariant vector fields is also left invariant. The algebra of left invariant vector fields of the Lie group is isomorphic to the corresponding Lie algebra.*

Proof. □

3. Exponential map

We have defined a Lie algebra as a tangent space at the group unity point. In this context, the Lie algebras are differentials of the Lie groups. But now we have realized that that the Lie algebra, in in its turn, can recover the Lie group, at least at some neighboring of the unity element. The corresponding map is given by the usual exponential function. Schematically the relation between the Lie group and Lie algebra can be described briefly by

$$L = dG|_{g=e} \qquad \text{and} \qquad G_c = \exp(L). \tag{2.C.19}$$

Note that in the second equation we denoted by $G_c \subset G$ the connected component of the unity, i.e. all group element which can be connected with the group unity e by a continuous curve in the parameter space. Clearly, G_c has the dimension of G.

***Problem* 2.C.24:** Prove that the connected components G_c is a normal subgroup in G.

Consider now the exponential maps for the standard matrix groups.

***Problem* 2.C.25:** Prove the identity for any matrix X

$$\det e^X = e^{\mathrm{tr} X}. \tag{2.C.20}$$

Hint: Map X by the matrix transformation SXS^{-1} to a triangular form.

***Example* 2:** The solution of the above problem implies that

$$\exp[gl(N)] = GL(N), \qquad \exp[sl(N)] = SL(N). \tag{2.C.21}$$

***Example* 3:** For the orthogonal and unitary algebras we have

$$\exp[so(N)] = SO(N), \qquad \exp[su(N)] = SU(N). \tag{2.C.22}$$

Let us focus on $su(N)$ consisting of skew-hermitian matrices X. Then

$$\exp(X)^+ = \exp(X^+) = \exp(-X) \qquad \text{so that} \qquad \exp(X)^+ \exp(X) = I. \tag{2.C.23}$$

Since the $SU(N)$ group is simple, it does not contain any normal subgroup, so the exponential map must produce the whole group.

***Example* 4:** For the Lorentz Lie algebra, the exponent gives rise to the proper Lorentz group,

$$\exp[so(1,3)] = O^+(1,3). \tag{2.C.24}$$

The algebra $so(1,3)$ consists of matrices obeying $X^\tau = -BXB$ where $B$is the Minkowski metrics. Since $B^2 = I$, we have

$$A^\tau = \exp(X)^\tau = \exp(X^\tau) = \exp(-BXB) = B\exp(-X)B = BA^{-1}B. \tag{2.C.25}$$

From the other side ...

D. Universal enveloping algebras

An important topic in Lie algebras studies and probably the main source of their appearance in applications is representation of the Lie algebra. Recall that the representation assigns to any element x of a Lie algebra a linear operator ρ_x. The space of linear operators is not only a Lie algebra, but also an associative algebra and so one can consider products $\rho_x \rho_y$. The main point to introduce the universal enveloping algebra is to study such products in various representations of a Lie algebra. It appears that certain properties are *universal for all representations*. The universal enveloping algebra is a way to grasp all such properties and only them.

1. Poincaré-Birkhoff-Witt theorem

Consider the associative algebra of polynomials in elements of the Lie algebra L where the commutator of any two elements $x, y \in L$ is expressed, like in any representation, via the product as $[x, y] = xy - yx$. This algebra is called a *universal enveloping algebra* of the Lie algebra $U(L)$. It can be considered as an analogue of the group algebra for the Lie algebras. Clearly, the dimension of $U(L)$ is infinite.

First, it contains the unity 1. Next, the first order homogeneous polynomials $a^i x_i$ are formed by the Lie algebra itself with the basic elements $x_i \in L$. The second order homogeneous polynomials $a^{ij} x_i x_j$ are not independent. Instead, only the restricted sum with $i \le j$ is independent since

$$x_j x_i = x_i x_j + c^k_{ji} x_k. \tag{2.D.1}$$

Continuing these steps by induction we arrive at the decomposition of any n-th order polynomial $u \in U(L)$ in terms of the basic monomials.

$$u = a + \sum_i a^i x_i + \sum_{i \le j} a^{ij} x_i x_j + \sum_{i \le j \le k} a^{ijk} x_i x_j x_k + \ldots + \sum_{i_1 \le \cdots \le i_n} a^{i_1 \ldots i_n} x_{i_1} x_{i_2} \ldots x_{i_n}. \tag{2.D.2}$$

This assertion called a *Poincaré-Birkhoff-Witt theorem*. Usually the basic monomials are written in the power product form

$$x_1^{n_1} x_2^{n_2} \dots x_N^{n_N}, \qquad \sum_i n_i = n \tag{2.D.3}$$

with the involved zero powers: $n_i = 0, 1, \dots$.

***Problem* 2.D.26:** Show that the symmetrized products form another basis of $U(L)$

$$x_{(i_1 \dots i_n)} = \frac{1}{n!} \sum_{s \in S_n} x_{i_{s_1}} x_{i_{s_2}} \dots x_{i_{s_n}}. \tag{2.D.4}$$

2. Invariant tensors

So far, we have constructed an invariant metrics in adjoint representation using the trace. This constriction can be generalised to covariant tensor of any rank n. Namely, it is easy to see that the tensor

$$g_{i_1 i_2 \dots i_n} = \operatorname{Tr} \hat{x}_{i_1} \hat{x}_{i_2} \dots \hat{x}_{i_n}. \tag{2.D.5}$$

is invariant, i.e. obeys the identity

$$c_{ii_1}^l g_{li_2 \dots i_n} + c_{ii_2}^l g_{i_1 l \dots i_n} + \dots + c_{ii_n}^l g_{i_1 i_2 \dots l} = 0 \tag{2.D.6}$$

***Problem* 2.D.27:** Prove the above identity. *Hint*: Show that the left side of (2.D.6) is equal to the trace of the commutator which certainly vanishes: $\operatorname{Tr} [\hat{x}_i, \hat{x}_{i_1} \hat{x}_{i_2} \dots \hat{x}_{i_n}] = 0$.

Moreover, since the adjoint action $\hat{x}_i$ one the basic states x_j is described by the matrix $C_j^{j'} = c_{ij}^{j'}$ the invariant tensor is expressed in the matrix product form,

$$g_{i_1 i_2 \dots i_n} = c_{i_1 j_1}^{i} c_{i_2 j_2}^{j_1} \dots c_{i_n i}^{j_{n-1}}. \tag{2.D.7}$$

In case of nondegenerate metrics, a similar-type equation holds for the contravariant tensor obtained by the rising the indexes with the inverse metrics

$$c_{il}^{i_1} g^{li_2 \dots i_n} + c_{il}^{i_2} g^{i_1 l \dots i_n} + \dots + c_{il}^{i_n} g^{i_1 i_2 \dots l} = 0, \qquad g^{i_1 \dots i_n} = g^{i_1 j_1} \dots g^{i_n j_n} g_{j_1 \dots j_n}. \tag{2.D.8}$$

For $n = 2$ the defined tensor is reduced to the Killing metrics which is symmetric. For higher ranks it is not so, but we can symmetrize it

$$g_{(i_1 i_2 \dots i_n)} = \operatorname{Tr} \hat{x}_{(i_1 \dots i_n)} = \frac{1}{n!} \sum_{s \in S_n} g_{i_{s_1} i_{s_2} \dots i_{s_n}}. \tag{2.D.9}$$

Obviously, the symmetrized tensor is also invariant.

The invariant tensors (2.D.5) are defined so far in the adjoint representation. One can construct in the same way the invariant tensors in arbitrary representation (ρ, V) of the Lie algebra,

$$g_{i_1 i_2 \dots i_n}^{\rho} = \operatorname{Tr}_V \rho_{x_{i_1}} \rho_{x_{i_2}} \dots \rho_{x_{i_n}}. \tag{2.D.10}$$

The additive action of the representation matrix on such tensor vanishes:

$$\rho_{i_1}^l g_{li_2 \dots i_n}^{\rho} + \rho_{i_2}^l g_{i_1 l \dots i_n}^{\rho} + \dots + \rho_{i_n}^l g_{i_1 i_2 \dots l}^{\rho} = 0 \qquad \text{with} \quad \rho = \rho_{x_i}. \tag{2.D.11}$$

***Problem* 2.D.28:** Prove the above identity.

3. Casimir invariants

Imagine that the Killing metrics g_{ij} (2.A.12) is *nondegenerate*. Let $g^{ij} = g^{-1}$ be the inverse metrics. Then the second order *Casimir operator* $C_2 \in U(L)$ commutes with the Lie algebra

$$C_2 = g^{ij} x_i x_j, \tag{2.D.12}$$

$$[C_2, x_k] = g^{ij}(c^l_{ik} x_l x_j + c^l_{jk} x_i x_l) = g^{ij} c^l_{ik}(x_l x_j + x_j x_l) = g^{ij} g^{lm} c_{ikm}(x_l x_j + x_j x_l) = 0 \tag{2.D.13}$$

Recall that the tensor $c_{ikm} = c^l_{ik} g_{lm}$ is antisymmetric. Therefore, the term $g^{ij} g^{lm} c_{ikm}$ is antisymmetric in $l \leftrightarrow j$ while the terms in parentheses is symmetric.

The higher order Casimir operators can be constructed by contracting the basic monomials with the contravariant invariant tensors of the same order.

Theorem 2. *The following n-the order Casimir element commutes with the Lie algebra,*

$$C_n = g^{i_1 i_2 \ldots i_n} x_{i_1} x_{i_2} \ldots x_{i_n}, \qquad [C_n, L] = 0. \tag{2.D.14}$$

Proof. The x monomials in the expression of C_n form the rank n covariant tensor under the adjoint action $\hat{x}_i$ of the Lie algebra. And they are contracted with the contravariant tensor which must produce a scalar.

In more detail, let us calculate the commutator with the basic elements x_i of L using the invariance condition contravariant tensor (2.D.8),

$$\begin{aligned} [x_i, C_n] &= g^{i_1 i_2 \ldots i_n}\left(c^l_{ii_1} x_l x_{i_2} \ldots x_{i_n} + c^l_{ii_2} x_{i_1} x_l \ldots x_{i_n} + \ldots + c^l_{ii_n} x_{i_1} x_{i_2} \ldots x_l\right) \\ &= \left(g^{l i_2 \ldots i_n} c^{i_1}_{il} + g^{i_1 l \ldots i_n} c^{i_2}_{il} + \ldots + g^{i_1 i_2 \ldots l} c^{i_n}_{il}\right) x_{i_1} x_{i_2} \ldots x_{i_n} = 0. \end{aligned} \tag{2.D.15}$$

□

Note that the first order Casimir element vanishes for nondegenerate metrics since $g_k = c^i_{ki} = c_{kij} g^{ij} = 0$.

It was established that the Casimir elements (2.D.14) generate the whole center of the universal enveloping algebra $U(L)$. Moreover, for the first few C_n are independent, the higher Casimir elements are polynomially expressed via them. For the simple Lie algebra, only the number of independent C_n coincides with the rank of the Lie algebra.

Finally we tone that the Casimir invariant can be contracted also in any representation from the corresponding invariants (2.D.10). However, all they would be expressed via the invariants in the adjoint representation (2.D.14).

***Problem* 2.D.29:** Construct an analogue of the (2.D.14) by means of the invariants (2.D.10) and prove that they commute with the Lie algebra. *Hint*: use the relation (2.D.11).

4. Exponents

The infinite powers series like the Teylor expansion do not belong to the universal enveloping algebra $U(L)$. However, we can consider them in certain context as a formal power series without focusing on their convergence which can be considered in concrete representations.

Among the infinite power series, a special role has the exponential functions on Lie algebra elements since they are convergent in almost all cases,

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x \in L. \tag{2.D.16}$$

From the other viewpoint, we have observed already that the exponential maps the Lie algebra to the Lie group.

Proposition 5. *For any two Lie algebra elements $x, y \in L$ we have the identity*

$$e^x y e^{-x} = e^{\hat{x}} y = y + [x, y] + \frac{1}{2}[x, [x, y]] + \ldots. \tag{2.D.17}$$

Proof. The above equation is verified on any order of power series on both sides. Using the expansion (2.D.16) it is easy to show that the identity (2.D.17) splits in n-th order to the following equation

$$\hat{x}^n y = \sum_{k=0}^{n} \binom{n}{k} x^k y (-x)^{n-k} \qquad \text{with} \qquad \binom{n}{k} = \frac{n!}{(k!(n-k)!} \tag{2.D.18}$$

being the Newton binomial coefficient. This formula is reminiscent of the binomial identity and it is easily proven by induction. □

***Problem* 2.D.30:** Prove by induction the above formula.

5. Coproduct

As we said before, the algebra $U(L)$ gathers the universal properties of all representations of the Lie algebra L. Including into the game the tensor product representations, we need to introduce the means to describe the universal tensor products, which can be applied to any representation. This mean is the coproduct.

The coproduct Δ provides the associative algebra homomorphism from $U(L)$ to the tensor product $U(L) \otimes U(L)$ first by setting it up to the element of Lie algebra by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \tag{2.D.19}$$

then by extending it by induction to the whole enveloping algebra by requiring the homomorphism,

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(ab) = \Delta(a)\Delta(b). \tag{2.D.20}$$

The relation (2.D.19) is interpreted as the additivity of the Lie algebra action.

Note that the algebra equipped by a coproduct (with some additional structures which we do not touch here) is called a *Hopf algebra*.

For higher order monomial the coproduct becomes more complicate. For example,

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}, \qquad x \in L. \tag{2.D.21}$$

***Problem* 2.D.31:** Prove by induction the above formula.

Any element $a \in U(L)$ has a general comultiplication form

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}. \tag{2.D.22}$$

However, the exponential of Lie algebra element possesses a quite simple and nics coproduct. Using the derived formula (2.D.21) and the homomorphism rule (2.D.20), we immediately get:

$$\Delta(e^x) = e^x \otimes e^x, \qquad x \in L. \tag{2.D.23}$$

This means that the group action on the product representation is multiplicative.

6. Baker-Campbell-Hausdorff formula

As soon as $\exp(x)$ is associated with the group element for $x \in L$, the product of such exponentials must be also the group element which must be presented as a single exponent:

$$\exp(z) = \exp(x)\exp(y), \qquad z \in L \quad \text{if} \quad x, y \in L. \tag{2.D.24}$$

In general, z is expressed as a infinite power series of commutators including two elements x and y. The corresponding formula is very complicate and called a *Baker-Campbell-Hausdorff formula*.

$$\begin{aligned} z = z(x,y) = \log[e^x e^y] =& F - \frac{F^2}{2} + \frac{F^3}{3} - \dots, \\ & F = e^x e^y - 1 = x + y + xy + \frac{1}{2}(x^2 + y^2) + \dots \end{aligned} \tag{2.D.25}$$

Up to up to the second order, we have $z = x + y + \frac{1}{2}[x, y]$. A nontrivial result is that

Lemma 5. *The element $z(x,y)$ belongs to the Lie algebra L.*

Proof. The roof is as nice as the lemma itself, and we we just outline the steps. First, it is easy to see that the coproduct acts on z as on the Lie algebra elements (2.D.19). Such elements are called *primitive*. Indeed,

$$\Delta(e^x e^y) = \Delta(e^x)\Delta(e^y) = e^x e^y \otimes e^x e^y = (e^x e^y \otimes 1)(1 \otimes e^x e^y). \tag{2.D.26}$$

Hence,

$$\Delta(z) = \log\left[(e^x e^y \otimes 1)(1 \otimes e^x e^y)\right] = \log(e^x e^y) \otimes 1 + 1 \otimes \log(e^x e^y) = z \otimes 1 + 1 \otimes z. \tag{2.D.27}$$

Therefore, the element z is primitive.

The next fact is that all primitive elements form $U(L)$ belong to the Lie algebra L. This can be shown in any order term in the decomposition of universal enveloping algebra element (2.D.2). For example, consider the second-order element u and suppose that it is primitive. Decompose it through the basic elements $u = \sum_{i,j} c^{ij} x_i x_j$. Without any restriction, one can suppose that the tensor c^{ij} is symmetric. Then it is easy to see that this condition implies that the following equation holds:

$$\sum_{i,j} c^{ij} x_i \otimes x_j = 0.$$

Since all terms in this decomposition are independent, the coefficients c^{ij} must vanish, so we get $u = 0$. The steps can be further extended to any $k > 2$ order.

□

***Problem* 2.D.32:** Calculate the Baker-Campbell-Hausdorff expansion up to the third order:

$$z = x + y + \frac{1}{2}[x,y] + \frac{1}{12}\big([x,[x,y]] + [y,[y,x]]\big). \tag{2.D.28}$$

Literature

The properties of Lie algebra, the matrix Lie algebras are described in (Dubrovin, 1992), (Georgi, 1999), (Wybourne, 1974). A universal enveloping Lie algebra, its structure and casimir invariants are well described in (Barut, 1986).

III. SEMISIMPLE LIE ALGEBRAS

A. Structure

1. Cartan subalgebra

We define first the notion of Cartan subalgebra. This is a maximal abelian subalgebra of g such that all its elements are diagonalisable (hence simultaneously diagonalisable) in the adjoint representation. That such an algebra exists is non trivial and must be established, but we shall admit it. This Cartan subalgebra is non unique, but one may prove that two distinct choices are related by an automorphism of the algebra.

Choose the basis h_i for the Cartan subalgebra H. Then they will commute also in the adjoint representation:

$$[h_i, h_j] = 0, \qquad [\mathrm{ad}_{h_i}, \mathrm{ad}_{h_j}] = 0, \qquad i = 1, 2, \ldots, l. \tag{3.A.1}$$

We may thus diagonalise simultaneously these ad_{h_i}. The subalgebra H produces the zero eigenvectors with vanishing eigenvalue. Complete them to make a basis by finding a set of eigenvectors e_α linearly independent of the h_i,

$$\mathrm{ad}_{h_i} e_\alpha = [h_i, e_\alpha] = \alpha_i e_\alpha \tag{3.A.2}$$

with the α_i not all vanishing, otherwise the subalgebra H would not be maximal.

In these expressions, the α_i are eigenvalues of the operators ad_{h_i}. Since we chose them Hermitian, their eigenvalues are real. By linearity, for an arbitrary element of $h \in H$ written as

$$\mathrm{ad}_h e_\alpha = [h, e_\alpha] = \alpha(h) e_\alpha, \qquad h = x^i h_i \tag{3.A.3}$$

and the eigenvalue of ad_h on e_α is $\alpha(h) = x^i \alpha_i$, which is a linear form on H. The linear forms on a vector space H is called the dual space H^*. One may thus consider the root α as a vector of the dual space of H, hence $\alpha \in H^*$, the root space. Note that $\alpha(h_i) = \alpha_i$.

2. Dual Cartan subspace

Since the h_j are diagonalisable, the total number of their eigenvectors must be equal to the dimension of the space, here the dimension of the adjoint representation, i.e. of the Lie algebra. We denote by Δ the set of all roots.

In the basis h_i, e_α of L, the Killing form takes a simple form

$$(h_i, e_\alpha) = 0, \qquad (e_\alpha, e_\beta) = 0 \quad \text{unless} \quad \alpha + \beta = 0. \tag{3.A.4}$$

Actually, both equations follows from the invariance of the Killing form, according to which the total H-eigenvalue of both arguments in the form must vanish. More precisely, for any $h \in H$ with $\alpha(h) \neq 0$ we have

$$\alpha(h)(h_i, e_\alpha) = (h_i, [h, e_\alpha]) = ([h_i, h], e_\alpha) = 0. \tag{3.A.5}$$

Choosing $\alpha(h) \neq 0$, we get the first relation. For the second relation becomes evident from the following identity:

$$(\alpha(h) + \beta(h))(e_\alpha, e_\beta) = ([h, e_\alpha], e_\beta) + (e_\alpha, [h, e_\beta]) = 0. \tag{3.A.6}$$

From the second relation in (3.A.4) and nondegeneracy of the Cartan metrics it follows that the root $\pm\alpha$ enters in the root system Δ together.

The restriction of this form to the Cartan subalgebra is non-degenerate, otherwise the null vector h_0 there, $(h_0, H) = 0$, would be a null vector in entire space, $(h_0, L) = 0$ due to (3.A.4). Therefore, the Killing form induces an isomorphism between H and H^*: to any root $\alpha \in H^*$ one associates the unique vector $h_\alpha \in H$ such that

$$(h_\alpha, h) = \alpha(h) \qquad \forall h \in H. \tag{3.A.7}$$

One has also a bilinear form on H^* inherited from the Killing form

$$(\alpha, \beta) := (h_\alpha, h_\beta) = \alpha(h_\beta) = \beta(h_\alpha). \tag{3.A.8}$$

Note also that due to the signature change by a coordinate map $x_i \to \imath x_i$, one can choose an orthonormal basis on the Cartan subalgebra: $(h_i, h_j) = \delta_{ij}$.

In is easy to show that roots span the whole space H^*, or, equivalently, the Cartan subalgebra elements h_α span the whole subalgebra. Indeed, suppose that the general combination $\sum_{\alpha\in\Delta} c_\alpha h_\alpha$ does not span H so that there is an element $h \in H$ orthogonal to any h_α. Then it commutes with any e_α since

$$[h, e_\alpha] = \alpha(h) e_\alpha = (h, h_\alpha) e_\alpha = 0. \tag{3.A.9}$$

Therefore, the nontrivial vector h belongs to the center of the Lie algebra, which contradicts with the semisimplicity condition.

3. Cartan-Weyl basis

Evidently, the entire algebra decomposes into the sum of the root spaces and Cartan subalgebra, which we consider here as a generalized root space for vanishing root:

$$L = \bigoplus_{\alpha\in\{\Delta,0\}} L_\alpha, \qquad L_0 = H. \tag{3.A.10}$$

The following commutation relation between the generalized root eigenspaces become apparent after applying the operator ad_h for any $h \in H$ on both sides and using (2.A.8) and (3.A.3),

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta}. \tag{3.A.11}$$

Take any two elements $e_\alpha \in L_\alpha$ and $e_{-\alpha} \in L_{-\alpha}$ such that $(e_\alpha, e_{-\alpha}) \neq 0$. Such an element must exist since otherwise, from (3.23), e_α would be a singular vector for the Cartan metrics. Normalise the scalar product by rescaling the root vectors to the canonical form,

$$(e_\alpha, e_{-\alpha}) = 1. \tag{3.A.12}$$

Owing to the invariance of the Cartan metrics (2.A.13), we have

$$([e_\alpha, e_{-\alpha}], h) = (e_\alpha, [e_{-\alpha}, h]) = \alpha(h). \tag{3.A.13}$$

Combining with (3.A.11) we get

$$[e_\alpha, e_{-\alpha}] = h_\alpha. \tag{3.A.14}$$

The most important step is to show that the eigenspaces L_α, which gather all root vectors of a given root α are actually one dimensional. Consider the subspace of the algebra spanned as

$$L'_\alpha = e_{-\alpha} \oplus L_\alpha \oplus L_{2\alpha} \oplus \ldots \oplus L_{k\alpha}, \tag{3.A.15}$$

where k is the largest multiple of the root α that can occur in the root system. The crucial observation id that this subspace is invariant with respect to the action of ad_{e_α}, $\mathrm{ad}_{e_{-\alpha}}$ and ad_{h_α}. In that case the commutation relation can be restricted to the subspace L'_α. Restricting it, and taking the trace on the subspace one obtains

$$0 = -1 + \dim L_\alpha + 2\dim L_{2\alpha} + \ldots + \dim L_{k\alpha}, \tag{3.A.16}$$

which fixes

$$\dim L_\alpha = 1, \qquad \dim L_{k\alpha} = 0 \quad \text{for} \quad k > 1. \tag{3.A.17}$$

We have used the vanishing of the commutator's trace and the definition (3.A.7) with $h = h_\alpha$. Eventually we reach to the following simple decomposition for the subspace (3.A.15)

$$L'_\alpha = e_{-\alpha} \oplus e_\alpha. \tag{3.A.18}$$

Note that the restriction of the metrics on the Cartan subspace can be madden a positive definite. Indeed, for any $h \in H$, the operator ad_h in the adjoint representation has a block diagonal form. Each pair of the root vectors $(e_\alpha, e_{-\alpha})$ forms a separate block, and the Cartan subalgebra forms a vanishing block. It is easy to derive in this way the restriction of the Cartan form on H:

$$(h_1, h_2) = \sum_{\alpha \in \Delta} \alpha(h_1)\alpha(h_2) \tag{3.A.19}$$

Substituting h_α and h_β instead of h_1 and h_2 one can rewrite above formula in terms of the products in the root space:

$$(\alpha, \beta) = \sum_{\gamma \in \Delta} (\alpha, \gamma)(\gamma, \beta). \tag{3.A.20}$$

We summarize the results in the following statements.

1. The root spaces L_α are nondegenerate.

2. For any root α, the $k\alpha$ is another root only when $k = \pm 1$.

3. The canonical basis is formes by the elements h_i and e_α with $i = 1, 2, \ldots, l$ and $\alpha \in \Delta$ obeying the commutation rules:

$$[h_i, h_j] = 0, \qquad [h_i, e_\alpha] = \alpha(h_i)e_\alpha, \qquad [e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root,} \\ h_\alpha & \text{if } \alpha + \beta = 0, \\ 0 & \text{otherwize.} \end{cases} \tag{3.A.21}$$

$$(h_i, h_j) = \delta_{ij}, \qquad (e_\alpha, e_\beta) = \delta_{\alpha+\beta,0}. \tag{3.A.22}$$

4. Roots and reflection groups

For a fixed root α, the root vectors $e_{\pm\alpha}$ together with the Cartan subalgebra element h_α form a $sl(2,\mathbb{C})$ subalgebra. It is easy to see from Eqs. (3.A.8), (3.A.21) that the normalized elements

$$s_\pm = \frac{\sqrt{2}}{(\alpha,\alpha)^{\frac{1}{2}}} e_{\pm\alpha} \qquad \text{and} \qquad s_z = \frac{1}{(\alpha,\alpha)} h_\alpha \tag{3.A.23}$$

form the standard ladder basis there:

$$[s_z, s_\pm] = \pm s_\pm, \qquad [s_+, s_-] = 2s_z. \tag{3.A.24}$$

This subalgebra operates on entire algebra L by means of the adjoint action. So, one can apply the representation theory of the $sl(2,\mathbb{C})$ algebra in order to study the properties of the root system in L.

First of all, the finite dimensionality of L demands that the eigenvalues of s_z must to be half-integers:

$$\mathrm{ad}_{s_z} e_\beta = \tfrac{1}{2} k e_\beta, \qquad k \in \mathbb{Z}. \tag{3.A.25}$$

Substituting the last equation (3.A.23) and the definitions (3.A.7), (3.A.8) and the formulas (3.A.21), we arrive at the following condition on the roots:

$$k = \langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}. \tag{3.A.26}$$

The $\langle \alpha, \beta \rangle$ is conversional notation for this integral, and it is neither symmetric nor linear in the first argument.

Since the roles of α and β are interchangeable the above equation implies that

$$\cos^2 \theta_{\alpha\beta} = \frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = \frac{k_1 k_2}{4}, \tag{3.A.27}$$

where $\theta_{\alpha\beta}$ is the angle between the roots α and β, and k_1 and k_2 are integers. The equation (3.A.27) is the sought-for requirement on the finiteness of the Lie algebra. It looks like a kind of quantization condition. It implies that the only angles between roots are those with $\cos\theta = 0, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1$. Excluding the trivial case of ± 1 corresponding to the trivial cases of to the same $\alpha = \beta$ or opposite $\alpha = -\beta$ roots, this yields the possible angle values

$$\theta_{\alpha\beta} = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}. \tag{3.A.28}$$

The possible square-length rations of any two roots are restricted to

$$\frac{(\alpha,\alpha)}{(\beta,\beta)} = \left(\frac{k_1}{k_2}\right)^2 = \frac{1}{3}, \frac{1}{2}, 1, 2, 3. \tag{3.A.29}$$

5. Long and short roots

For a particular root system either all roots have the same lengths, or there are *long roots* and *short roots*. In the latter case, their ratio is either 3 or 2. The existence of roots with three or more distinct lengths is forbidden since in that case the ratio 2/3 does not occur in (3.A.29).

All roots of a given length are conjugate under the Weyl group $\mathcal{W}$. Let α and β be roots of the same root length, $|\alpha| = |\beta|$. We may assume $(\alpha,\beta) > 0$. If they are distinct, then the value for the integrals (3.A.26) is fixes to one: $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = 1$. Therefore,

$$s_\beta \alpha = \alpha - \beta, \qquad s_\alpha \beta = \beta - \alpha, \tag{3.A.30}$$

which implies

$$s_\alpha s_\beta s_\alpha \beta = s_\alpha s_\beta (\beta - \alpha) = s_\alpha(-\alpha) = \alpha \tag{3.A.31}$$

so that the Weyl group element $w = s_\alpha s_\beta s_\alpha$ maps β and α.

Thus, we have established

Lemma 6. *For a given Lie algebra at most two lengths occur in the root system Δ. All roots of a same length are in the same orbit under the Weyl group action $\mathcal{W}$.*

6. Spin algebra action

More detailed look at the root vectors which participate in this $sl(2,\mathbb{C})$ representation will reveal the hidden structure of the root system. For simplicity consider the case then $\alpha+\beta$ is not a root providing that $[e_\alpha, e_\beta]=0$. This means that the e_β is the highest state for the adjoint action of the Lie algebra (3.A.23). The lower states are obtained by the successive action of the $e_{-\alpha}$:

$$f_k = \mathrm{ad}^k_{e_{-\alpha}} e_\beta. \tag{3.A.32}$$

Applying the induction step on k, it is easy to derive the following rising action on the defines states,

$$\mathrm{ad}_{e_\alpha} f_{k+1} = (k+1)\left(\alpha, \beta - \tfrac{1}{2}k\alpha\right) f_k. \tag{3.A.33}$$

It is clear that $f_{k+1}=0$ if and only if the coefficient in front of f_k vanishes, which leads to the expression obtained in (3.A.26) for k. Thus, we have the chain of root β, $\beta-\alpha$, $\beta-2\alpha$, …, $\beta-k\alpha$ with the constructed root vectors f_0, f_1, …, f_k, correspondingly. Amazingly, the last and the fist roots in this chain are related by the reflection in the root space H^* [see equation (1.F.1) in the section devoted to the reflection groups]:

$$\beta - k\alpha = s_\alpha \beta \qquad \text{with} \qquad k = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}. \tag{3.A.34}$$

Moreover, the entire chain remains invariant with respect to the α reflection. It is easy to see that

$$s_\alpha(\beta - j\alpha) = \beta - (k-j)\alpha, \qquad 0 \le j \le k, \tag{3.A.35}$$

i.e. the j-th root is mapped to the $(k-j)$-th one.

Now take any root γ. Applying the rising adjoint action of the vector e_α on e_γ, we will produce the string of roots in the same way until reaching the highest root $\beta=\gamma+p\alpha$ for some positive p. Then the above procedure is repeated till the lowest non-vanishing root

$$s_\alpha \beta = s_\alpha \gamma - p\alpha. \tag{3.A.36}$$

We argue in this way that the Weyl group $\mathcal{W}$ formed by all root reflections s_α leaves invariant the set of roots Δ of a semisimple Lie algebra L.

As a by product of the proof we established that for any two roots α and γ all intermediate elements in the chain between γ and $s_\alpha\gamma$ are also roots:

$$\{\gamma - k'\alpha,\ \gamma-(k'-1)\alpha,\ \ldots, \gamma\} \in \Delta, \qquad k' = \frac{2(\alpha,\gamma)}{(\alpha,\alpha)}. \tag{3.A.37}$$

We summarize the obtained results in the following statements ...

7. Positive roots

According to the previous studies, the root system Δ of the Lie algebra defines a finite reflection group $\mathcal{W}$. The normalised roots of Δ produce the root set $\mathcal{R}$ of the reflection group. This is not a one-to-one correspondence: two different Lie algebra may have s common normalized root set $\mathcal{R}$ due to differences in the lengths. From the other side, there are reflection groups which do not correspond to any semisimple Lie algebra. However, most of the properties of both systems are similar. In particular, the definition of Weyl chambers, positive and simple roots are automatically transferred to the Lie algebra case.

$$\theta_{\alpha\beta} = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}. \tag{3.A.38}$$

Namely, we say that the root α is positive if its first nonzero coefficient is nonzero: $\alpha(h_i)>0$ provided that $\alpha(h_j)>0$ for all $j<i$. The ordering is evidently not unique and depends on the choice of the orthogonal basis in H but its properties are commonly described. The entire set decouple into equal positive and negative parts:

$$\Delta = \Delta_+ \cup \Delta_-. \tag{3.A.39}$$

Clearly, if the root α is positive, then the $-\alpha$ is a negative root and vise versa.

One can introduce a well defined a lexicographic ordering between the roots in the following way: we say that $\alpha>\beta$ if $\alpha-\beta>0$.

8. Simple roots

A root α is called simple if it is positive and cannot be expressed as the sum of positive roots. We denote the set of simple roots by Π.

It is easily seen that every positive root $\alpha > 0$ can be expressed as a sum of the simple roots $\alpha_i \in \Pi$ with positive integral coefficients:

$$\alpha = \sum_i n_i \alpha_i, \qquad \alpha_i \in \Pi, \quad n_i \in \mathbb{Z}_+. \tag{3.A.40}$$

The positive roots space the entire dual Cartan space H^*. This fact follows from the partition of the roots into positive and negative parts (3.A.39). However, the basis formed by the positive roots is highly overcomplicated as we will show latter. The most important property of simple roots is that they provide a natural basis in the dual Cartan space so that any element $\lambda \in H^*$ decomposes into them as

$$\lambda = \sum_{i=1}^{l} c_i \alpha_i, \qquad \alpha_i \in \Pi, \quad l = \dim H. \tag{3.A.41}$$

To establish this property, it enough to prove that the simple roots are linearly independent.

First we note that the scalar product of two distinct simple roots is nonpositive:

$$(\alpha_i, \alpha_j) \le 0, \qquad \alpha_i, \alpha_j \in \Pi, \quad i \ne j. \tag{3.A.42}$$

In order to demonstrate this, let look at the chain of roots constructed in Eq. (3.A.37) between α_i and $s_{\alpha_j}\alpha_i$. If $(\alpha_i, \alpha_j) > 0$ then the penultimate root in that chain would be $\alpha_i - \alpha_j$ which contradicts with the simplicity of α_i.

Suppose now that a certain linear combination of the simple roots vanishes:

$$\lambda = \sum_i c_i \alpha_i = 0 \qquad \alpha_i \in \Pi \tag{3.A.43}$$

We split it into the sum with positive and negative coefficients, $\lambda = \lambda_+ - \lambda_-$, where $\lambda_\pm$ involves the terms with positive (negative) coefficients only:

$$\lambda_+ = \sum_{i_+} c_{i_+} \alpha_{i_+}, \qquad \lambda_- = -\sum_{i_-} c_{i_-} \alpha_{i_-}, \qquad c_i > 0. \tag{3.A.44}$$

Since the dual subspace of the Cartan subalgebra is Euclidean, we have: $(\lambda_+, \lambda_+) \ge 0$. From the other side, using the property (3.A.42), we obtain that

$$(\lambda_+, \lambda_-) = -\sum_{i_+, i_-} c_{i_+} c_{i_-} (\alpha_{i_+}, \alpha_{i_-}) \ge 0. \tag{3.A.45}$$

So, the requirement $\lambda = 0$ imposes the condition $\lambda_\pm = 0$. But a linear combination of positive roots with positive coefficients cannot vanish, since each term has its first nonzero element positive, The same argument works for negative roots too. As a consequence, all coefficients c_i must vanish, which completes the proof.

9. Cartan matrix

The Cartan matrix is defined via the scalar products of the simple roots in the following way:

$$c_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{3.A.46}$$

Note that this matrix is not a priori symmetric.

By definition, the diagonal elements equal two, $c_{ii} = 2$, while the off-diagonal entries are nonpositive integers due to the equations (3.A.26) *and* (3.A.42).

The quantised angles for the simple roots is more restrictive than for the general root established above [see equation (3.A.27)]. Rewrite the scalar product in terms of the vector lengths and the angle,

$$(\alpha_i, \alpha_j) = 2|\alpha_i||\alpha_j| \cos\theta_{ij} \qquad \text{with} \quad |\alpha_i| = \sqrt{(\alpha_i, \alpha_i)}, \quad \theta_{ij} = \theta_{\alpha_i \alpha_j}. \tag{3.A.47}$$

In contrast to the generic case, the acute angles between simple root are forbidden. Then from equation (3.A.46), we obtain the angle and the length ration of the simple roots,

$$\cos\theta_{ij} = -\tfrac{1}{2}\sqrt{c_{ij}c_{ji}}, \qquad \frac{|\alpha_i|}{|\alpha_j|} = \sqrt{\frac{c_{ij}}{c_{ji}}}. \tag{3.A.48}$$

The straight angle $\theta_{ij} = \pi$ is also excluded due to the positivity of both roots. Therefore, the off-diagonal entries of the matrix c_{ij} may take only the values 0, -1, -2 or -3. Thus, the only possible values of the $\cos\theta$ are 0, $-\frac{1}{2}$, $-\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{3}}{2}$. And the only possible angles between simple roots are $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$ and $\frac{5\pi}{6}$. The ratios of lengths are respectively equal to 1, $\sqrt{2}$ and $\sqrt{3}$.

A single algebra of rank $l = 1$ is the $sl(2,\mathbb{C})$ algebra, which will be called below A_1. Its Cartan matrix is one dimensional: $C_{A_1} = 2$. There are only three simple (not semisimple!) algebras of rank $l = 2$ with the Cartan matrices

$$C_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad C_{B_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \qquad C_{G_2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \tag{3.A.49}$$

B. Classification of simple Lie algebras

For a given semisimple Lie algebra L, we have constructed a set of simple roots Π in the dual space of the Cartan subalgebra. Amazingly, it turned out that having a simple root system, one can recover completely the structure of the algebra. In other words, Π it encodes the whole L so that

There is a one-to-one correspondence between the sets of simple roots and the semisimple Lie algebras.

It is easy to see that for decomposable simple-root systems, the simple roots in the two orthogonal subsystems commute (p and q are zero for all pairs), and the entire system of roots splits into two commuting subsets.

In general, if the set of roots may be split into two mutually orthogonal subsets,

$$\Pi = \Pi_1 \cup \Pi_2, \qquad (\alpha,\beta) = 0 \quad \text{if } \alpha \in \Pi_1,\ \beta \in \Pi_2. \tag{3.B.1}$$

Each subsystem Π_i, along with the Cartan generators associated with the subspace it spans, forms an invariant subalgebra. Recalling that any semi-simple algebra may be decomposed into the direct sum of simple subalgebras. It can be shown that the Lie algebra associated with such a *decomposable* root system is not simple and decomposes into a direct sum of two algebras generated by Π_1 and P_2. The opposite statement is also true. Therefore,

there is a one-to-one correspondence between the sets of indecomposable simple roots and the simple Lie algebras.

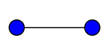
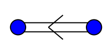
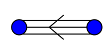
Therefore, the classification of the simple complex Lie algebras is reduced to the classification of the simple root systems which can not be split into two orthogonal subsets.

1. Dynkin diagram

So far, we have classified the indecomposable simple root sets in one and two dimensions $l = 1, 2$. For higher dimension of the root space, it becomes difficult to visualise the root system. Another representation is adopted, by encoding the Cartan matrix into a diagram in the following way.

A soon as the off-diagonal elements of the Cartan matrix take specific With each simple root is associated a vertex of the diagram. Two different vertices ($i \neq j$) are linked by an edge if they are not orthogonal to each other: $(\alpha_i, \alpha_j) \neq 0$. The edge is simple if $c_{ij} = c_{ji} = -1$, which corresponds to the angle of $2\pi/3$ and equal lengths. It is double if $c_{ij} = -2$ and $c_{ji} = -1$, producing the angle of $3\pi/4$ with a length ratio of $\sqrt{2}$. The triple link describes the entries $c_{ij} = -3$ and $c_{ji} = -1$. It gives rise to an angle of $5\pi/6$ with a length ratio of $\sqrt{3}$. Finally, the double and triple links carries an arrow from i to j indicating which root is the longest.

In this context, the Cartan matrices of the rank two algebras (3.A.49) are drawn as

A_2 :  B_2 :  G_2 : 

The described graphical description can be summarised as follows.

Definition. The *Dynkin diagram* associated with a complex semi-simple Lie algebra of rank l is a graph with l nodes, each node corresponding to one of the simple roots. The i-th and j-th node, $i \neq j$, are connected by the

$$n_{ij} = c_{ij}c_{ji} = 4\cos^2\theta_{ij} \tag{3.B.2}$$

edges. If two nodes are connected by more than one edge then an arrow is added in the direction of the shorter root. If two nodes are connected by only one edge, then no arrow is added.

Proposition 6. *The Dynkin diagram of a simple Lie algebra is a connected tree (without loops) with exactly $l-1$ links.*

Proof. If the diagram decomposes into two disconnected components, then the simple roots can be split into two orthogonal sets Π_1 and Π_2. The root system Δ also splits into two orthogonal subsystems Δ_1 and Δ_2 generated by the corresponding simple roots.

Consider the element from the dual Cartan algebra

$$\lambda = \sum_{i=1}^{l} \frac{\alpha_i}{|\alpha_i|}$$

Then

$$(\lambda, \lambda) = l + \sum_{i<j} \frac{2(\alpha_i, \alpha_j)}{|\alpha_i||\alpha_j|}, \quad \text{and thus} \quad \sum_{i<j} -\frac{2(\alpha_i, \alpha_j)}{|\alpha_i||\alpha_j|} < l \tag{3.B.3}$$

due to the positivity of the metrics. Supposing for instance that $|\alpha_i| \geq |\alpha_j|$ and using the Cartan matrix definition (3.A.46), we get that each term in the second sum is bounded from below by $-c_{ij}$:

$$-\frac{2(\alpha_i, \alpha_j)}{|\alpha_i||\alpha_j|} \geq -\frac{2(\alpha_i, \alpha_j)}{|\alpha_i|^2} = -c_{ij}. \tag{3.B.4}$$

Inserting both relations into the inequality (3.B.3), we come to the conclusion that possible values of off-diagonal elements are restricted by

$$\sum_{i<j} -c_{ij} \leq l - 1. \tag{3.B.5}$$

Since the only nonzero off-diagonal entries of c_{ij} being equal to $\{-1, -2, -3\}$, the total amount of links in the Dynkin diagram does not exceed $l-1$. Together with connectivity, it implies that the diagram is tree-type without internal cycles. □

Proposition 7. *In a diagram each vertex meets at most three lines.*

Proof. Suppose that the simple roots $\alpha_i, \alpha_2, \dots \alpha_k$, $k < l$, are connected to the root α_l. Since the diagram do not contain loops, the first k roots are orthogonal to the last one: $c_{il} = 0$ for any $i \leq k$. Denote by α_0 (not a simple root!) the vector λ spanned by the $k+1$ roots $\alpha_i, \alpha_2, \dots \alpha_k, \alpha_l$ and being orthogonal to α_i for $i \leq k$: (α_0, α_i). Then

$$\sum_{i=0}^{k} \cos^2 \theta_{il} = 1. \tag{3.B.6}$$

Since the above set is linearly independent then $(\alpha_i, \alpha_0) \neq 0$ so that

$$4 \sum_{i=1}^{k} \cos^2 \theta_{il} = \sum_{i=1}^{l-1} n_{il} < 4, \tag{3.B.7}$$

where n_{ij} is the number of links between i-th and j-th nodes of the diagram (3.B.2). □

Proposition 8. *The diagrams depicted on figure 1 can not be part of the Dynkin diagram.*

Theorem 3. *A finite dimensional complex simple Lie algebra is described by a Dynkin diagram from the list in figure 2.*

a)

b)

c)

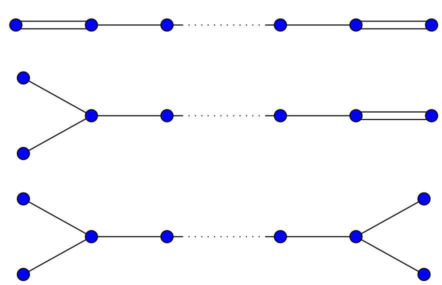


FIG. 1 Forbidden subdiagrams.

2. Chevalley generators

In this basis the Cartan matrix is appeared explicitly in the commutation relations. We redefine the orthogonal basis in the Cartan subalgebra taking instead a basis associated with the simple roots. The related lowering-rising vectors in L are normalised according to

$$h_i = h_{\alpha_i}, \qquad e_i = \frac{\sqrt{2}}{|\alpha_i|} e_{\alpha_i} \qquad f_i = \frac{\sqrt{2}}{|\alpha_i|} e_{-\alpha_i}, \qquad i = 1, \dots, l.$$

Their commutation relations read

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = c_{ij} e_j, \qquad [h_i, f_j] = -c_{ij} f_j, \qquad [e_i, f_j] = \delta_{ij} h_i \tag{3.B.8}$$

There are additional relations inside, separately, the lowering and the rising generators, which are called Serre relations:

$$\mathrm{ad}_{e_j}^{1-c_{ij}}(e_i) = 0, \qquad \mathrm{ad}_{f_j}^{1-c_{ij}}(f_i) = 0, \qquad i \neq j. \tag{3.B.9}$$

This proves that the whole algebra is indeed encoded in the Cartan matrix or Dynkin diagram. Note also the remarkable property that in that basis, all the structure constants are integers.

3. Weyl group

Consider the irreducible root system Δ. Let the vector ρ is the half sum of all positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{3.B.10}$$

Evidently, it is not a root. For the simplicity, let shorten the notation for the Weyl reflection with respect to the hyperplane defined by the simple root as follows

$$s_i := s_{\alpha_i}. \tag{3.B.11}$$

Lemma 7. *The reflection s_i permutes all positive roots except α_i itself. Its action on ρ is rather simple: $s_i \rho = \rho - \alpha_i$.*

Proof. Indeed, $s_i \alpha$ is also a root. $\alpha = \sum_i n_i \alpha_i$ with $n_i \geq 0$ Then the reflection with respect to any simple root α_i acts on it as □

Lemma 8. *Take a subset $\alpha_{i_1}, \alpha_{i_2} \dots, \alpha_{i_p}$ (not necessarily distinct), $p \leq l$ from the simple roots. If the root $\beta = s_{i_1} \dots s_{i_{p-1}} \alpha_{i_p}$ is negative ($\beta \in \Delta_-$) then for some $p' < p$ we will have*

$$s_{i_1} \dots s_{i_p} = s_{i_1} \dots s_{i_{p'}} s_{i_{p'+1}} \dots s_{i_{p-1}}. \tag{3.B.12}$$

Proof. □

Corollary 2. *Let $w = s_{i_1} \dots s_{i_p}$ be an expression for the Wey group element $w \in \mathcal{W}$ in terms of refections simple roots, with p as small as possible. Then the root $\beta = w \alpha_{i_p}$ is negative: $\beta \in \Delta_-$.*

For any root $\alpha \in \Delta$, there exists some Weyl group elements $w \in \mathcal{W}$ mapping it to a simple root, $w\alpha \in \Pi$. Recall now that at most two root lengths occur among all roots Δ.

Proposition 9. *For every root $\beta \in \Delta$ there exists a simple root α_j and a sequence of simple reflections $s_{i_1}, \ldots, s_{i_s}$ whose composition carries β to α_j.*

Proof. Suppose first that the root is positive: $\beta \in \Delta_+$. Define the *height* of a positive root $\beta = \sum_i k_i \alpha_i$ as the sum of the coefficients k_i. Clearly, the positive root is simple iff its height is one. From the other side, there exists a simple root α_i obeying $(\alpha_i, \beta) > 0$, otherwise

$$(\beta, \beta) = \sum_i k_i (\alpha_i, \beta) \leq 0, \tag{3.B.13}$$

which contradicts with the positivity of the Cartan metrics. The related reflection would reduce the height of β by the integral number $\langle \alpha_i, \beta \rangle \geq 1$. Clearly $s_i \beta$ would be a positive root since only the i-th coefficient of β changes. Applying this procedure recursively, we will arrive finally at a simple root α_j so that

$$s_{i_s} \ldots s_{i_2} s_i \beta = \alpha_j = s_j s_{i_s} \ldots s_{i_2} s_i (-\beta). \tag{3.B.14}$$

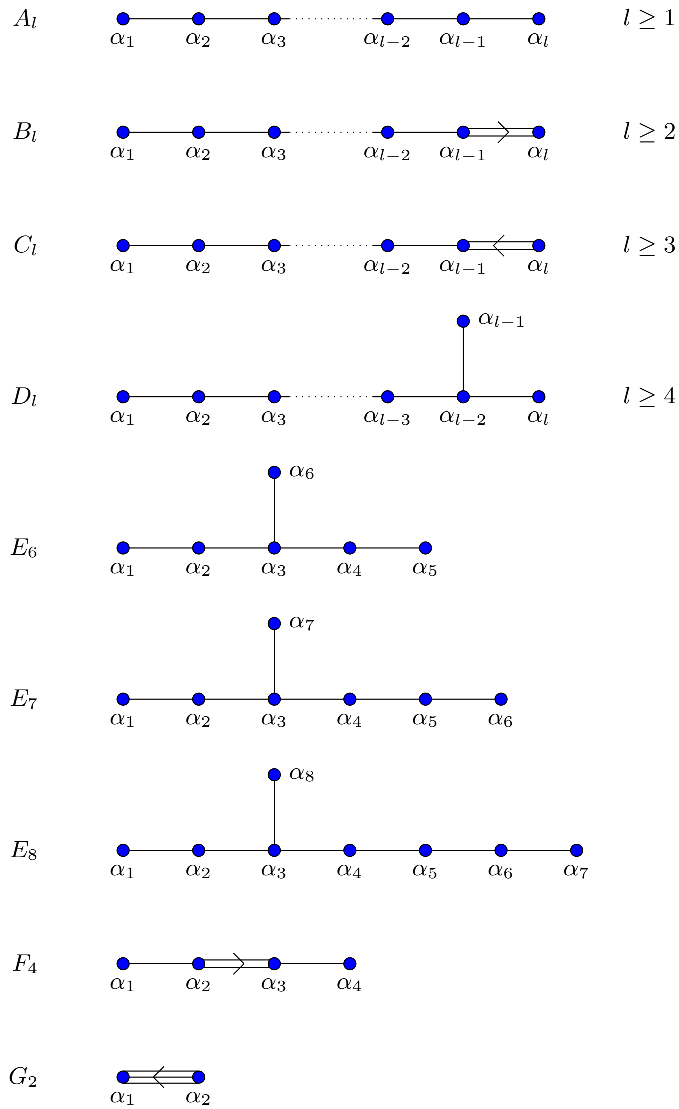


FIG. 2 Dynkin diagrams for the finite dimensional complex simple Lie algebras.

The second equation above assures the statement for a negative root $\beta \in \Delta_-$, where the additional simple reflection s_j is needed. □

Note that since the Weyl group is an orthogonal transformation, for any root and any element $w \in \mathcal{W}$ we have:

$$s_{w\alpha} = w s_\alpha w^{-1}. \tag{3.B.15}$$

As a consequence of the proven proposition, the simple reflections solely generate the whole Weyl group $\mathcal{W}$. A reduced expression for w is its factorization as a product of k simple reflections $w = s_1 s_2 \dots s_k$, where k is minimal. This minimal number is called the length of w.

4. Generators

On the simple roots, the action of generators (3.B.11) takes the following form:

$$s_i \alpha_j = \alpha_j - c_{ji}\alpha_i. \tag{3.B.16}$$

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \quad \text{if} \quad a_{ij} = 0 \tag{3.B.17}$$

$$(s_i s_j)^{m_{ij}} = 1 \quad \text{with} \quad m_{ij} = \frac{\pi}{\pi - \theta_{ij}}, \quad i \neq j \tag{3.B.18}$$

5. Maximal root

Recall that every root $\alpha \in \Delta$ is an integer combination of the l simple roots α_i with all coefficients non-negative (a positive root), or with all coefficients nonpositive (negative root). We write $\alpha \succ 0$ in the first case, and $\alpha \prec 0$ in the second case. This defines a partial order on the roots by

$$\alpha \succ \beta \quad \text{if and only if} \quad \alpha - \beta = \sum_{i=1}^{l} k_i \alpha_i \quad \text{with} \quad k_i \geq 0. \tag{3.B.19}$$

The order in partial which means that not all rotts can be compared in this way.

Lemma 9. *There is a unique maximal root γ in the root system Δ.*

Proof. Choose a $\gamma = \sum_i k_i \alpha_i$ so that it is maximal among all roots that it is comparable to. At least one of the $k_i > 0$. We claim that all the $k_i > 0$. Indeed, suppose not. This partitions the set of simple roots Π into Π_1, the subset of Π for which $k_i > 0$ and Π_2, the subset for which $k_i = 0$. Now the scalar product of any two distinct simple roots is nonpositive. In particular, $(\alpha_i, \alpha_j) \leq 0$ if $i \in \Pi_1$ and $j \in \Pi_2$ so that $(\gamma, \alpha_j) \leq 0$. The irreducibility of the simple root system Π implies that the straight inequality holds at lest for two pair of indexes which we set to be i, j again. This implies that $(\gamma, \alpha_j) < 0$. In its turn, this means that the reflected root $s_i\gamma$ is greater than γ

$$s_i \gamma = \gamma - \langle \alpha_i, \gamma \rangle \alpha_i \succ \gamma, \tag{3.B.20}$$

which contradicts with the maximality condition. So we have proved that the subset Π_2 is empty all the k_i are positive. Furthermore, this same argument shows that $(\gamma, \alpha_i) \geq 0$ for all simple roots α_i. We will now show that this maximal root is unique.

Suppose there were two maximal roots γ and γ'. Write $\gamma' = \sum_i k_i' \alpha_i$ with $k_i' > 0$. Then clearly $(\gamma, \gamma') > 0$ since $(\gamma, \alpha_i) \geq 0$ and the inequality holds at least for one i. Then since $s_\gamma \gamma'$ is a root it follows that all the members of the root chain from $s_\gamma \gamma'$ to γ' must be also roots including the element $\beta = \gamma' - \gamma$. But if β is a root, it is either positive or negative, contradicting the maximality of γ or γ', correspondingly. □

6. Dual Coxeter number

The coefficients k_i in the decomposition of the maximal root over the simple roots (3.B.21) are called the *marks*.

$$\gamma = \sum_{i=1}^{l} k_i \alpha_i, \qquad k_i \in \mathbb{N}. \tag{3.B.21}$$

The dual Coxeter number is defined as

$$g = \sum_{l=1}^{l} \frac{1}{2}(\alpha_i, \alpha_i) k_i + 1. \tag{3.B.22}$$

Literature

The properties of semisimple Lie algebra, the Dynkin classification is given in detail in (Goto, 1978), see also (Zuber, 2009). (Georgi, 1999), (Wybourne, 1974). A brief description with examples of the Cartan-Weyl basis is given in (Barut, 1986).

IV. QUANTUM GROUPS

A. Hopf algebras

1. Coalgebras and bialgebras

Consider a rela associative algebra $\mathcal{A}$, although all results remain true for the complex algebras too. At the starting point we denote by m: $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ the multiplication map whose associativity can be written as follows:

$$(ab)c = a(bc) \qquad \text{or} \qquad \begin{tikzcd} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[dl, "m\otimes 1"'] \arrow[dr, "1\otimes m"] & \\ \mathcal{A}\otimes\mathcal{A} \arrow[dr, "m"'] & & \mathcal{A}\otimes\mathcal{A} \arrow[dl, "m"] . \\ & \mathcal{A} & \end{tikzcd} \tag{4.A.1}$$

Clearly, a unit for the algebra defines a map $\eta : \mathbb{R} \to \mathcal{A}$ with

$$\eta(\alpha) = \alpha \cdot 1, \qquad \alpha \in \mathbb{R}. \tag{4.A.2}$$

with multiplicative property described by the following diagram

$$\eta(\alpha)a = a\eta(\alpha) = \alpha a \qquad \text{or} \qquad \begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[dr, "m"] & \\ \mathbb{R}\otimes\mathcal{A} \arrow[u, "\eta\otimes 1"] = & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[dr, "m"] & \\ \mathcal{A}\otimes\mathbb{R} \arrow[u, "1\otimes\eta"] = & \mathcal{A} \end{tikzcd} . \tag{4.A.3}$$

A *coalgebra* is a dual analog of this construction. It is a vector space endowed with the *comultiplication* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and *counity* $\epsilon : \mathcal{A} \to \mathbb{R}$. Both maps are linear and dual, respectively, to the multiplication and unity maps of the algebra.

The comultiplication obeys the coassociativity property, which can be described by the commutative diagram obtained from the diagram (4.A.1) by reversing the arrows:

$$\Delta(a_i) \otimes a^i = a_i \otimes \Delta(a^i) \qquad \text{or} \qquad \begin{tikzcd} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \\ \mathcal{A}\otimes\mathcal{A} \arrow[ur, "\Delta\otimes 1"] & & \mathcal{A}\otimes\mathcal{A} \arrow[ul, "1\otimes\Delta"'] \\ & \mathcal{A} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd} \tag{4.A.4}$$

Respectively, the counity obeys the counitarity property, described by inversion of the unity diagram:

$$\epsilon(a_i)a^i = a_i\epsilon(a^i) = a \qquad \text{or} \qquad \begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[d, "\epsilon\otimes 1"'] & \\ \mathbb{R}\otimes\mathcal{A} = \mathcal{A} \arrow[u, leftarrow, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[d, "1\otimes\epsilon"'] & \\ \mathcal{A}\otimes\mathbb{R} = \mathcal{A} \arrow[u, leftarrow, "\Delta"'] & \end{tikzcd} \tag{4.A.5}$$

A *bialgebra* is an algebra with coalgebra structure. It is endowed by the multiplication m, comultiplication Δ, unity η and counity ϵ maps, which satisfy all of the commutative diagrams (4.A.1), (4.A.4) (4.A.3), (4.A.5), provided that the comultiplication and counity are algebra homomorphisms. The last condition implies the following compatibility equations holding for all $a, b \in \mathcal{A}$:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \Delta(1) = 1\otimes 1, \qquad \epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(1) = 1. \tag{4.A.6}$$

2. Hopf algebras

Consider the space of linear maps $f : \mathcal{A} \to \mathcal{A}$ on the bialgebra $\mathcal{A}$

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b), \qquad \alpha, \beta \in \mathbb{R}, \quad a, b \in \mathcal{A}. \tag{4.A.7}$$

Define there the *convolution product* by the formula

$$f * f' = m(f\otimes f')\Delta \qquad \text{or} \qquad (f * f')(a) = f(a_i)f'(a^i). \tag{4.A.8}$$

***Problem* 4.A.1:** Prove that the product $*$ is associative.

***Problem* 4.A.2:** Prove that the composite map

$$I_* : \quad I_*(a) = \eta(\epsilon(a)) \tag{4.A.9}$$

is the unity in the star product:

$$I_* * f = f * I_* = f. \tag{4.A.10}$$

Hint: Use the coassosiativity and the counity definition.

Note that the convolution product differs from the operator product (or matrix product). Therefore, the identity elements I_* of the convolution map does not coincide with the identity transformation of the algebra $\mathcal{A}$: $I_* \neq I_{\mathcal{A}}$. As a linear operator, I_* is highly degenerate.

A *Hopf algebra* $\mathcal{A}$ is a bialgebra where the identity map $I_{\mathcal{A}}$ is invertible for the convolution product. The inverse $S = I_{\mathcal{A}}^{-1}$ is called an *antipode*.

***Problem* 4.A.3:** 1) Express the identity (4.A.10) as a commutativity of the following diagrams:

$$\begin{tikzcd} A \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & \mathbb{R} \arrow[r, "\eta"] & A \\ A\otimes A \arrow[rr, "1\otimes S"] & & A\otimes A \arrow[u, "m"'] \end{tikzcd}, \qquad \begin{tikzcd} A \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & \mathbb{R} \arrow[r, "\eta"] & A \\ A\otimes A \arrow[rr, "S\otimes 1"] & & A\otimes A \arrow[u, "m"'] \end{tikzcd}, \tag{4.A.11}$$

The defining property of the antipode can also be expressed via the identity

$$S(a_i)a^i = a_iS(a^i) = \epsilon(a)\cdot 1, \qquad \forall a \in \mathcal{A}. \tag{4.A.12}$$

$$S = S * (I_{\mathcal{A}} * I_*) = S* \tag{4.A.13}$$

***Example* 5:** A universal enveloping algebra $U(L)$ over a Lie algebra L, described in Sect. II.D in detail, is a Hopf algebra with the standard additive comultiplication (2.D.19). The counity vanishes on nontrivial monomials: $\epsilon(L^k) = 0$ for $k \geq 1$. The antipode is given by the extension of the inverse map $S(x) = -x$ for any $x \in L$.

***Example* 6:** Consider the real (or complex) group algebra $\mathbb{R}[G]$ over the group G, described in Sect. I.C.2 in detail. It is a Hopf algebra with comultiplication given by the standard group action $\Delta g = g \times g$ for any $g \in G$. The antipode on the group elements is just the inverse map: $S(g) = g^{-1}$, while the counity vanishes for all group entries excepts the unity: $\epsilon(g) = \delta_{g,e}$. These three operations are extended to whole group algebra $\mathbb{R}[G]$ by linearity.

***Problem* 4.A.4:** Prove exactly that the group algebra $\mathbb{R}[G]$ and Lie algebra L with the above defined comultiplication, counity and antipode satisfies the Hopf algebra conditions.

3. Action on representation products

The counity is just the trivial one-dimensional "vacuum" representation of the Hopf algebra,

$$a|0\rangle = \epsilon(a)|0\rangle. \tag{4.A.14}$$

The counitarity means that its product does not change a representation,

$$V_\rho \otimes V_0 \equiv V_\rho. \tag{4.A.15}$$

The antipode permits to create a conjugate representation (ρ^*, V) from any representation (ρ, V) using the matrix transpose or hermitian conjugate

$$\rho^*_a = \rho^\tau_{s(a)}, \qquad \rho^*_{ab} = \rho^*_a \rho^*_b. \tag{4.A.16}$$

The antipode property means that the product of a representation with its conjugate always contains a trivial represnetation,

$$\rho \otimes \rho^* = \epsilon \oplus \rho \oplus \rho^* \oplus \dots \tag{4.A.17}$$

4. Adjoint representation

The Hopf algebra $\mathcal{A}$ provides the tools for defining an *adjoint representation* of the algebra on itself. We define the adjoint action by

$$\hat{a}b = \mathrm{ad}_a b = a_i b S(a^i). \tag{4.A.18}$$

Lemma 10. *The action* (4.A.18) *obeys the representation rule.*

Proof. We must ensure that for any two elements a, b of the Hopf algebra we have

$$\mathrm{ad}_{ab} = \mathrm{ad}_a \mathrm{ad}_b. \tag{4.A.19}$$

Acting by the right part of the above rule to an arbitrary element c of the algebra $\mathcal{A}$, we reproduce this operator identity:

$$\hat{a}\hat{b}c = a_i b_j c S(b^j) S(a^i) = a_i b_j c S(a^i b^j) = \mathrm{ad}_{ab} c. \tag{4.A.20}$$

In the last equation we have used the homomorphism property of the coproduct,

$$\Delta(ab) = \Delta(a)\Delta(b) = a_i b_j \otimes a^i b^j. \tag{4.A.21}$$

□

B. Quantum universal enveloping algebras

1. q-deformed $U(sl_2)$ algebra

As was shown, the universal enveloping algebra $U(L)$ over any Lie algebra L can be endowed with a natural Hopf algebra structure. It appeared this Hopf algebra can be deformed in a nontrivial way for any semisimple Lie algebras. The deformation coefficient q can be associated with the Plank constant by $q = \exp(\hbar)$ so that in the classical limit $\hbar \to 0$ it reduces to the usual nondeformed case with $q = 1$.

Such deformations are called *quantum groups* or, more precisely, *quantized universal enveloping algebras* and denoted as $U_q(L)$.

First consider the simplest case of the complex Lie algebra $U_q(sl_2)$ which is defined by the three basic elements $e_\pm$ and h. We deform their commutation relations in the following way,

$$\begin{aligned} &[h, e_\pm] = \pm 2 e_\pm, \\ &[e_+, e_-] = [h]_q \qquad \text{with} \qquad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh \hbar x}{\sinh \hbar}. \end{aligned} \tag{4.B.1}$$

Next, deform the standard Lie algebra coproduct in the following way

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \qquad \Delta(e_\pm) = e_\pm \otimes q^{\frac{1}{2}h} + q^{-\frac{1}{2}h} \otimes e_\pm. \tag{4.B.2}$$

Note that in the last equation we admit the infinite power series which formally is out of the $U(sl_2)$. Identities involving such series are supposed true if they are fulfilled at any order. They are treated as formal power series without studding their convergence, although the exponentials converges in all realizable representations.

The infinite series can be avoided if we include the exponential generator k with its inverse instead of h.

$$k^{\pm 1} = q^{\pm\frac{1}{2}h} \tag{4.B.3}$$

The commutation relations (4.B.1) are written in terms of new generators as follows:

$$ke_\pm k^{-1} = q^{\pm 1} e_\pm, \qquad [e_+, e_-] = \frac{k^2 - k^{-2}}{q - q^{-1}} \tag{4.B.4}$$

The coproduct is given by

$$\Delta(k) = k \otimes k, \qquad \Delta(e_\pm) = e_\pm \otimes k + k^{-1} \otimes e_\pm. \tag{4.B.5}$$

The antipode is also deformed and defined by

$$S(e_\pm) = -q^{\pm 1} e_\pm, \qquad S(h) = -h \quad \text{or} \quad S(k^{\pm 1}) = k^{\mp 1}. \tag{4.B.6}$$

The county remains the same as in the Lie algebra.

$$\epsilon(e_\pm) = \epsilon(h) = 0, \qquad \text{so that} \qquad \epsilon(k) = k. \tag{4.B.7}$$

Lemma 11. *The generators* (4.B.1) *or* (4.B.5) *form a Hopf algebra* $U_q(sl_2)$ *with the coproduct* (4.B.2)*, antipode* (4.B.6) *and countity* (4.B.7)*.*

Proof. First check that the comultiplication is an homomorphism [see equations (4.A.6)]. We verify the most complicated commutator, the remaining commutators are easier to check. Thus, we have

$$\begin{aligned}[\Delta(e_+), \Delta(e_-)] &= [e_+, e_-] \otimes k^2 + k^{-2} \otimes [e_+, e_-] = \frac{(k^2 - k^{-2}) \otimes k^2 + k^{-2} \otimes (k^2 - k^{-2})}{q - q^{-1}} \\ &= \frac{k^2 \otimes k^2 - k^{-2} \otimes k^{-2}}{q - q^{-1}} = \frac{\Delta\left(k^2 - k^{-2}\right)}{q - q^{-1}} = \Delta\left([h]_q\right).\end{aligned} \tag{4.B.8}$$

Here the mixed commutators disappear. For example,

$$\left[e_+ \otimes k\,,\, k^{-1} \otimes e_-\right] = e_+ k^{-1} \otimes k e_- - k^{-1} e_+ \otimes e_- k = 0 \tag{4.B.9}$$

due to the first relation in the system (4.B.4).

Next let us check the first antipode property (4.A.12) for $a = e_\pm$:

$$S(a_i)a^i = S(e_\pm)k + S(k^{-1})e_\pm = -q^{\pm 1} e_\pm k + k e_\pm = 0. \tag{4.B.10}$$

The second relation can be verified in same way. The case of $a = k$ is evident.

The counitarity condition (4.A.5) is verified trivially. ◻

2. Deformation of simple Lie algebras

Now we will describe the quantum universal enveloping algebra $U_q(L)$ associated with general simple Lie algebra L. We construct the deformation of the Chevalley generators $h_i, e_{\pm i}$ of the algebra L defined in the section III.B.2. The deformation of their commutation relations (3.B.8) looks like

$$[h_i, h_j] = 0, \qquad [h_i, e_{\pm j}] = \pm c_{ij} e_{\pm j}, \qquad [e_{+i}, e_{-j}] = \delta_{ij} [h_i]_{q_i}. \tag{4.B.11}$$

They are quite simple extension of ones for the simplest case of $L = sl_2$. Remember that the integral numbers c_{ij} form the Cartan matrix of L provided by its Dynkin scheme. The numbers q_i depend on a single deformation parameter q,

$$q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)}. \tag{4.B.12}$$

We have, however the Serre relations (3.B.9) among positive and negative generators, which need to be deformed too. They can be written via the q-deformed adjoint action which we will do below. Currently, we give their explicit form here:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_{\pm i}^{1-a_{ij}-k} e_{\pm j} e_{\pm i}^{k} = 0, \qquad i \neq j. \tag{4.B.13}$$

The q-deformed binomial coefficients here are expressed in the same way as the usual binomial coefficient, except that the factorial function is replaced by the quantum factorial:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad [n]_q! = [n]_q[n-1]_q \dots [1]_q, \tag{4.B.14}$$

and the function $[x]_q$ is defined in (4.B.1).
[1]

The coproduct is defined by the equations

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \qquad \Delta(e_{\pm i}) = e_{\pm i} \otimes q_i^{\frac{1}{2}h_i} + q_i^{-\frac{1}{2}h_i} \otimes e_{\pm i}, \tag{4.B.15}$$

The counit, and antipode are described by the maps

$$\epsilon(h_i) = \epsilon(e_{\pm i}) = 0, \qquad S(h_i) = -h_i, \qquad S(e_{\pm i}) = -q_i^{\pm 1} e_{\pm i}. \tag{4.B.16}$$

Literature

The quantum groups and Hopf algebras are described in detail in the book (Majid, 2002).

V. CLIFFORD ALGEBRAS AND SPINORS

A. Spinless fermions

1. Fermionic oscillator algebra

We imagine n spinless electrons with the *creation* and *destruction* operators obeying the anticommutation rules

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \qquad \{a_i^+, a_j\} = \delta_{ij}, \tag{5.A.1}$$

where we have used the curved bracket for the anticommutator of two operators: $\{a, b\} = ab + ba$. It is well knows from the quantum field theory that the operators $a_i^{\pm}$ are associated with i-th fermion, and the operator $n_i = a_i^+ a_i$ describes the number of i-th particle.

The relations (5.A.1) are appeared in the quantum field theory as a *fermionic oscillator algebra*. Its anticommutative part generated by the annihilation operators a_i is called a *Grassman algebra*.

Consider a 2^N-dimensional representation of the fermionic oscillator algebra spanned by the vacuum state $|\Omega\rangle$, which annihilates all a_i and produces all other states under action of a_i^+:

$$a_i|\Omega\rangle = 0, \qquad |i_1 \dots i_p\rangle = a_{i_1}^+ \dots a_{i_p}^+ |\Omega\rangle \quad \text{with} \quad i_1 < \dots < i_p. \tag{5.A.2}$$

***Problem* 5.A.1:** Show that the state $|i_1 \dots i_p\rangle$ contains p fermions.

---

[1] quantum factorials $[n]_{q_i}!$ of the form utilised here were defined by Heine in 1846!

2. Irreducible representation

Proposition 10. *The representation* (5.A.2) *is a unique irreducible representation of the fermionic oscillator algebra.*

Proof. Use the anticommutation relations it is easy to conclude that the following subspace V_0 of the representation space V annihilates under the action of the creation operators:

$$V_0 = a_1^+ \dots a_N^+ V, \qquad a_i^+ V_0 = 0. \tag{5.A.3}$$

First we notice that V_0 does not vanish. Overwise, acting on it by the annihilation operator a_1 we conclude that the subspace $a_2^+ \dots a_N^+$ vanished too

$$0 = a_1 V_0 = (1 - a_1^+ a_1) a_2 \dots a_N^+ V = a_2^+ \dots a_N^+ V + (-1)^N a_1^+ \dots a_N^+ a_1 V = a_2^+ \dots a_N^+ V. \tag{5.A.4}$$

Continuing by the successive actions of $a_2, \dots a_N$ we come to the conclusion that $V = 0$.

Let the basic vectors, describing different vacua, $|\Omega_s\rangle$, span the space V_0. It is easy to see that the 2^N dimensional subspaces V_s generates by these vacua, remain invariant under the action of the whole algebra:

$$V_s = a_{i_1} \dots a_{i_p} |\Omega_s\rangle, \qquad a_i^\pm V_s \subset V_s. \tag{5.A.5}$$

The only possibility compatible with the irreducibility is the uniqueness of the vacuum state, producing the representation (5.A.2).

□

B. Clifford algebras

1. Euclidean and Minkowskian Dirac matrices

Let us briefly describe the construction of the spinorial representations for the orthogonal groups. They can not be obtained from tensor products of the defining representations. However, there is an explicit way to find the generators in this representations via the *Clifford algebra*. The latter is the algebra generated by the N elements $\gamma_0 = \gamma_1, \gamma_2, \dots, \gamma_N$, obeying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \begin{cases} 2\delta^{\mu\nu} & \text{Euclidean case} \\ 2\eta^{\mu\nu} & \text{Minkowskian case,} \end{cases}, \tag{5.B.1}$$

where we chose the Minkowski metrics as

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, \dots, -1). \tag{5.B.2}$$

Here the existence of the unity element on the right part is supposed so that, for instance, the following notations are equivalent: $\delta^{\mu\nu} = \delta^{\mu\nu} \cdot 1 = \delta^{\mu\nu} I$.

For complex Clifford algebra all metric signatures become equivalent after the phase rescaling $\gamma^\mu \to \imath\gamma^\mu$ of some generators. Here we will restrict the discussion to real Clifford algebras with Minkowskian and Euclidean signatures so that we adopt two notations for the first element since the index value $i = 0$ is the conventional choice for the time coordinate in Minkowskian space-time. Note that the Minkowskian and Euclidean Clifford algebras are mapped to each other by the change $\gamma^0 \to \imath\gamma^0$.

The extension to arbitrary signatures is possible.

***Problem* 5.B.2:** Show that the defining relation (5.B.1) remains invariant with respect to the orthogonal and pseudoorthogonal transformations for the Euclidean and Minkowskian Clifford algebras, respectively:

$$\gamma^\mu \to \sum_j A_{ij}\gamma^\nu, \qquad A \in O(N) \quad \text{or} \quad A \in O(1, N). \tag{5.B.3}$$

Define the anti-symmetrized products

$$\gamma^{\mu_1 \cdots \mu_p} = \gamma^{[\mu_1} \dots \gamma^{\mu_p]} = \frac{1}{p!} \sum_{s \in S_p} \epsilon_{s_1 \dots s_p} \gamma^{\mu_{s_1}} \dots \gamma^{\mu_{s_p}} \tag{5.B.4}$$

where ϵ is the rank-p Levi-Civita tensor. They form the $\binom{N}{p}$ dimensional space. All such products with $p = 1, \dots N$ together with the identity matrix span the entire Clifford algebra in 2^N dimensions.

Moreover, the elements ± 1, $\pm\gamma^\mu$, $\pm\gamma^{\mu\nu}$, ... form a finite group of order 2^{N+1}. Every representation of a finite group can be made unitary by a similarity transformation. So, we assume that all gamma matrices are unitary. Then it follows from the Clifford algebra (5.B.1) that all γ's are Hermitian, except γ^0, which is anti-Hermitian in Minkowski signature:

$$(\gamma^\mu)^+ = g^{\mu\mu}\gamma^\mu = \begin{cases} \gamma^\mu & \text{for Euclidean metrics } g^{\mu\nu} = \delta^{\mu\nu}, \\ \gamma^0\gamma^\mu\gamma^0 & \text{for Minkowski metrics } g^{\mu\nu} = \eta^{\mu\nu}. \end{cases} \tag{5.B.5}$$

It follows from the algebra that the product of odd number of gamma matrices are traceless,

$$\operatorname{tr}\gamma^{\mu_1}\dots\gamma^{\mu_p} = 0, \qquad p = 1, 3, 5, \dots, \quad p \le N-1. \tag{5.B.6}$$

***Problem* 5.B.3:** Proof the above trace identity. *Hint*: Take γ^ν which does not participate in the product and calculate the trace $\operatorname{tr}\gamma^\nu\gamma^{\mu_1}\dots\gamma^{\mu_p}\gamma^\nu$ first by using the cyclicity, second, by applying the anticommutation relations among γ^μ.

Taking trace to (5.B.1), we get

$$\operatorname{tr}\gamma^\mu\gamma^\nu = N g^{\mu\nu}, \tag{5.B.7}$$

where $g^{\mu\nu}$ is either Euclidean $\delta^{\mu\nu}$ or Minkowski $\eta^{\mu\nu}$ metrics dependent on the Clifford algebra choice. Moreover, using the cyclicity property, one can calculate a trace of the product of any even number of gamma matrices. It is expressed via the metrics tensor.

***Problem* 5.B.4:** Calculate trace of the product of four gamma matrices, $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\delta$.

2. Even and odd N

Consider the (rescaled) product of all Clifford algebra generators. For odd N, it commutes with all γ^μ while for even N it anticommutes with them:

$$\gamma^{N+1} = \lambda\gamma^0\gamma^1\dots\gamma^{N-1}, \qquad \begin{array}{ll} N = 2n+1: & [\gamma^{N+1}, \gamma^\mu] = 0 \\ N = 2n: & \{\gamma^{N+1}, \gamma^\mu\} = 0 \end{array} \qquad 0 \le \mu \le N-1, \tag{5.B.8}$$

$$\lambda = \begin{cases} \imath^{n-1} & \text{(Minkowskian)} \\ \imath^n & \text{(Euclidean)} \end{cases} \tag{5.B.9}$$

The phase factor λ is chosen to assure that

$$(\gamma^{N+1})^2 = 1, \qquad (\gamma^{N+1})^+ = \gamma^{N+1}. \tag{5.B.10}$$

***Problem* 5.B.5:** Verify the relations (5.B.10). *Hint:* Check the identities for the Euclidean case:

$$(\gamma^1\dots\gamma^N)^2 = (-1)^{\frac{N(N-1)}{2}} = (-1)^n.$$

According to Schur lemma, in an irreducible representation of odd-dimensional Clifford algebra, the element γ^{N+1} must be multiple of the identity. Taking into account (5.B.10), we conclude that the corresponding factor must be ± 1:

$$N = 2n+1, \text{ irreducible representation}: \qquad \gamma^{N+1} = \pm 1. \tag{5.B.11}$$

For even N, the introduced element defines a projectors

$$\gamma_\pm = \frac{1 \pm \gamma^{N+1}}{2}, \qquad \gamma_\pm^2 = \gamma_\pm, \quad \gamma_\pm\gamma_\mp = 0. \tag{5.B.12}$$

***Problem* 5.B.6:** Check
1) that $\gamma_\pm$ satisfy the above equations.
2) their relations with other Dirac matrices:

$$\gamma^\mu\gamma_\pm = \gamma_\mp\gamma^\mu.$$

3. Pauli matrix representation for even N

There is a simple real representation of the even-dimensional Clifford algebra in terms of the tensor products of the Pauli matrices, which, hence, represents also the Majorana fermions.

In Euclidean case, we have:

$$\begin{aligned}
\gamma^{2i} &= \underbrace{1\otimes\cdots\otimes 1}_{i}\otimes\sigma_1\otimes\underbrace{\sigma_3\otimes\cdots\otimes\sigma_3}_{n-i-1} = \sigma_1^{(i+1)}\prod_{j=i+2}^{n}\sigma_3^{(j)},\\
\gamma^{2i+1} &= \underbrace{1\otimes\cdots\otimes 1}_{i}\otimes\sigma_2\otimes\underbrace{\sigma_3\otimes\cdots\otimes\sigma_3}_{n-i-1} = \sigma_2^{(i+1)}\prod_{j=i+2}^{n}\sigma_3^{(j)},
\end{aligned} \tag{5.B.13}$$

where the matrix $\sigma^{(j)}$ acts on the j-th space of the tensor product and $n = \frac{N}{2}$ in accordance with (5.B.8).

In Minkowski signature, the matrix γ^0 differs by a factor of $\imath$ producing

$$\gamma^0 = \imath\sigma_1\otimes\sigma_3\otimes\ldots\otimes\sigma_3 \tag{5.B.14}$$

and the other γ^k for $k = 1,\ldots,N-1$ remain unchanged.

The chilarily gamma matrix is expressed as

$$\gamma^{N+1} = \begin{cases}\imath^n\gamma^0\ldots\gamma^{N-1} & \text{in Euclidean case}\\ \imath^{n-1}\gamma^0\ldots\gamma^{N-1} & \text{in Minkowski case}\end{cases} = \sigma_3\otimes\ldots\otimes\sigma_3. \tag{5.B.15}$$

In the Euclidean case all matrices γ^i are Hermitian. For Minkowski metrics, all are Hermitian besides the γ^0, which is anti-Herminian.

***Problem* 5.B.7:** Check:
1) Euclidean Clifford algebra anticommutation rules for the representation (5.B.13).
2) Minkowski anticommutation rules when γ^0 is replaced by (5.B.14).

***Example* 7:** As an example, consider the simplest case of $N = 2$. The Clifford algebra representation (5.B.13) is described by the first two Pauli matrices: $\gamma^0 = \sigma_1$ and $\gamma^1 = \sigma_2$, whereas the chirality gamma matrix is given by

$$\gamma^3 = \imath\sigma_1\sigma_2 = -\sigma_3.$$

All three γ-matrices anticommute together, and the Clifford algebra coincides with the Pauli matrix algebra.

***Example* 8:** Next, we consider the $N = 4$ case with Minkowski metrics, where we have the following representation:

$$\gamma^0 = \imath\sigma_1\otimes\sigma_3, \qquad \gamma^1 = \sigma_2\otimes\sigma_3, \qquad \gamma^2 = 1\otimes\sigma_1, \qquad \gamma^3 = 1\otimes\sigma_2. \tag{5.B.16}$$

The chirality gamma matrix is given by

$$\gamma^5 = \imath\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma_3\otimes\sigma_3.$$

4. Spinless fermion representation for even N

In even dimension, $N = 2n$, the Euclidean Clifford algebra generators has the following representation in terms of the fermion operators $a_i^\pm$ with $i = 0,\ldots,n-1$, obeying the Fermi-Dirac statistics (5.A.1),

$$\gamma^{2j} = a_j + a_j^+, \qquad \gamma^{2j+1} = \imath(a_j - a_j^+). \tag{5.B.17}$$

The inverse relations are

$$a_j^\pm = \frac{\gamma^{2j}\pm\imath\gamma^{2j+1}}{2}. \tag{5.B.18}$$

***Problem* 5.B.8:** Verify that:
1) the operators γ^i obey the anticommutation rules (5.B.1);
2) the fermion number and parity operators are given by

$$n_i = a_i^+ a_i = \frac{1 - \imath\gamma^{2j}\gamma^{2j+1}}{2}, \tag{5.B.19}$$
$$p_i = (-1)^{n_i} = \imath\gamma^{2j}\gamma^{2j+1}. \tag{5.B.20}$$

The using the representations (5.B.18) and (5.B.13), one can express the fermion operators via the Pauli matrices. The corresponding map is known as a *Jordan-Wigner transformation* and demonstrates the nonlocality of fermion creation-annihilation operators:

$$a_{i-1}^{\pm} = \sigma_{\pm}^{(i)} \prod_{j=i+1}^{n} \sigma_3^{(j)}, \qquad \sigma_{\pm} = \frac{\sigma_1 \pm \imath\sigma_2}{2}. \tag{5.B.21}$$

For Minkowskian Clifford algebra the formula for the generator γ^0 in (5.B.17) must be replaced by the expression

$$\gamma^0 = -\imath(a_0 + a_0^+) \tag{5.B.22}$$

with the related change in all subsequent formulas.

5. Irreducible representations: even and odd N

Let us classify the irreducible representations of Clifford algebras. The classification scheme is rather simple: they are almost unique with a difference between an odd and even values of dimension.

For *even dimensions*, $N = 2n$, the gamma matrix algebra is mapped to the fermionic oscillator algebra with n particles, so their irreducible representations also match. The irreducible representation of the fermionic algebra is unique and form 2^n-dimensional space [see proposition 10].

For *odd dimensions*, $N = 2n + 1$, as we have seen (5.B.11), the γ^{N+1} takes two values equal to $\sigma = \pm 1$. Using them, one can express one generator, say γ^{N-1} in terms of the others. For instance, for Euclidean algebra, we have:

$$\gamma^{N-1} = \sigma(-\imath)^N \gamma^{N-2} \dots \gamma^0. \tag{5.B.23}$$

Thus we arrive at the irreducible representation of the $N = 2n$ Clifford algebra described already. Thus, for $N = 2n+1$, there are two nonequivalent representations, each characterised by the value of $\sigma = \pm 1$ and having the dimension 2^n.

So, we have established the the *fundamental theorem of Dirac matrices*:

Theorem 4. *For even $N = 2n$ all irreducible representations of the Clifford algebra are equivalent. For odd $N = 2n+1$ there are two inequivalent irreducible representations, inherited from the $N = 2n$ case, each characterized by the value ± 1 of the central element γ^{N+1}. All representations are given by the Dirac matrices of dimension $d = 2^n$.*

Similarly, the Puli matrix representation in odd dimension, $N = 2n + 1$, is inherited from the even $N - 1$ case (5.B.13). For for definiteness, consider the Euclidean metrics and distinct the odd/even cases by denoting the Clifford algebra elements in odd dimension by capital letters. Then

$$\begin{aligned}
&\Gamma^{2i} = \gamma^{2i} = \sigma_1^{(i+1)} \prod_{j=i+2}^{n} \sigma_3^{(j)}, \qquad && \Gamma^{2i+1} = \gamma^{2i+1} = \sigma_2^{(i+1)} \prod_{j=i+2}^{n} \sigma_3^{(j)}, \qquad 0 \le i \le n-1, \\
&\Gamma^{2n} = \gamma^{2n} = \prod_{j=0}^{2n-1} \sigma_3^{(j)}, && \Gamma^{2n+2} = (\gamma^{2n})^2 = 1.
\end{aligned} \tag{5.B.24}$$

The last equation there defines a chirality matrix with square equal to the identity, $\sigma = 1$.

***Problem* 5.B.9:** Construct a Pauli matrix representation of the odd-dimensional Clifford algebra with $\sigma = -1$. *Hint:* Look at the next example.

***Example* 9:** In $N = 3$ dimension the first two gamma matrices are inherited from the two-dimensional case [see example 7] while the last one is determined by the chilality matrix:

$$\Gamma^0 = \sigma_1, \qquad \Gamma^1 = \sigma_2, \qquad \Gamma^2 = -\sigma_3. \tag{5.B.25}$$

The Γ^{N+1} is given by the identity matrix with $\sigma = 1$:

$$\Gamma^4 = \imath\Gamma^1\Gamma^2\Gamma^3 = (\sigma_3)^2 = I.$$

In order to get a second irreducible representation with $\sigma = -1$, it is enough to change a sign of one gamma matrix, say, of Γ^2. Thus, the two representations parameterized by $\sigma = \pm 1$ are given by

$$\Gamma^0 = \sigma_1, \qquad \Gamma^1 = \sigma_2, \qquad \Gamma^2 = -\sigma\sigma_3. \tag{5.B.26}$$

***Problem* 5.B.10:** Write explicitly the Pauli matrix representation of Euclidean five-dimensional Clifford algebra. *Hint:* use a Euclidean analog of the example 8.

6. Charge conjugation

In this subsection we will deal only with even dimensional Clifford algebras, $N = 2n$, which include also the usual four dimensional space-time with Minkowski metrics.

First we notice that if γ^μ are irreducible representation of the Clifford algebra, so are the representations defined by $-\gamma^\mu$, by the transposed matrices $\pm(\gamma^\mu)^\tau$, as well as complex conjugates $\pm(\gamma^\mu)^*$ and Hermitian conjugates $\pm(\gamma^\mu)^+$. Indeed, it is easy to see that all them satisfy the Clifford algebra defining relations (5.B.1).

***Problem* 5.B.11:** Why, for instance, $\pm(\gamma^\mu)^*$ are irreducible if γ^μ are so?

Due to theorem 4, all these representations are equivalent for even N, which is our case. In particular, there exists a matrix C which one may call the charge conjugation matrix, which intertwines between the representations γ^μ and $-(\gamma^\mu)^*$,

$$(\gamma^\mu)^* = -C\gamma^\mu C^{-1}. \tag{5.B.27}$$

Consider now the N-dimensional Dirac equation in electromagnetic field $A_\mu = A_\mu(x)$

$$\gamma^\mu(\imath\partial_\mu - eA_\mu)\psi = m\psi, \tag{5.B.28}$$

where $\psi = \psi(x)$ is the space-time dependent spinor field with mass m and electromagnetic charge e.

Using the charge conjugate matrix, one can construct another spinor

$$\psi^{(c)} = C^{-1}\psi^*. \tag{5.B.29}$$

which solves the Dirac equation with the opposite charge $-e$:

$$\gamma^\mu(\imath\partial_\mu + eA_\mu)\psi^{(c)} = m\psi^{(c)}. \tag{5.B.30}$$

***Problem* 5.B.12:** Verify the equation (5.B.30) for the wavefunction (5.B.29) using the definition of charge conjugate matrix (5.B.27).

Remember now that the Clifford algebra elements may be set to be (anti)Hermitian according to the relation (5.B.5). To exploit this property, first take a transpose of the equation (5.B.27) and get the first equation in the chain below,

$$(\gamma^\mu)^\tau = \epsilon_\mu C^\tau\gamma^\mu(C^{-1})^\tau, \qquad \gamma^\mu = \epsilon_\mu C^{-1}(\gamma^\mu)^\tau C, \qquad (\gamma^\mu)^\tau = \epsilon_\mu C\gamma^\mu C^{-1}, \tag{5.B.31}$$

where we have denoted

$$\epsilon_\mu = -\eta_{\mu\mu} = \pm 1.$$

The second equation in (5.B.31) is just a transposed version of the first one. The last identity s just the inverse of the second one. Comparing the first identity with the last one, we conclude that the matrix $C^{-1}C^\tau$ commuted with the Clifford algebra elements and therefore, according to the Schur lemma, must be a multiple of identity, which must be set to $\lambda = \pm 1$,

$$C^{-1}C^\tau\gamma^\mu = \gamma^\mu C^{-1}C^\tau, \qquad C^\tau = \pm C. \tag{5.B.32}$$

***Problem* 5.B.13:** Show that $C^\tau = \lambda C$ implies $\lambda = \pm 1$ for even dimensions N and $\lambda = 1$ for odd N. *Hint:* Take a double transpose of C.

So, the charge conjugate matrix is either symmetric or antisymmetric.

Now let us wort with the complex conjugate of the matrix C in a similar way Taking the complex conjugate of the equation (5.B.27) and applying it again, we get

$$\gamma^\mu = (C^* C)\gamma^\mu (C^* C)^{-1}. \tag{5.B.33}$$

We realise that the matrix C^*C commutes with the Dirac matrices and must therefore, according to the Schur lemma, be proportional to the identity matrix: $C^*C = \eta I$. So, one can rescale C such that $|\eta| = 1$. It can be shown that one can set it to be real and thus $\eta = \pm 1$. Its value is independent upon the representation of the Clifford algebra chosen.

7. Charge conjugation in Pauli matrix representation

In particular, consider the sigma matrix representation of the Euclidean Dirac matrices (5.B.13). It is easy to see that the gamma matrices with even indexes are real while they with odd indexes are pure imaginary due to a single σ_2 Pauli matrix in their product,

$$(\gamma^{2i})^* = \gamma^{2i}, \qquad (\gamma^{2i+1})^* = -\gamma^{2i+1}. \tag{5.B.34}$$

Consider the following two candidates for the charge conjugation matrix:

$$\begin{aligned} C_+ &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots, \\ C_- &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots, \end{aligned} \tag{5.B.35}$$

where we have $n = \frac{N}{2}$ terms in the product.

***Problem* 5.B.14:** Verify that both matrices $C_\pm$ satisfy the relations

$$C_\pm^+ = C_\pm = C_\pm^{-1}, \qquad C_\pm^\tau = (-1)^{\frac{1}{2}n(n\pm 1)} C_\pm. \tag{5.B.36}$$

These matrices obey the following relation with gamma matrices, which is easy to verify,

$$C_\pm \gamma^k C_\pm = \mp(-1)^{N+k}\gamma^k. \tag{5.B.37}$$

Indeed, take, for example, the C_+ case, which contains σ_2 at odd positions, and σ_1 – at even ones. Since both anticommute with σ_3, we acquire a sign factor $(-1)^{n-i-1}$ both for $k = 2i$ and $k = 2i+1$. The additional factors $(-1)^i$ and $(-1)^{i+1}$, respectively, for γ^{2i} and γ^{2i+1} appear due to σ_1 and σ_2 terms in the product (5.B.13) so that we have:

$$\begin{aligned} C_+\gamma^{2i}C_+ &= (-1)^{n-i-1}(-1)^i\gamma^{2i} = (-1)^{n-1}\gamma^{2i}, \\ C_+\gamma^{2i+1}C_+ &= (-1)^{n-i-1}(-1)^{i+1}\gamma^{2i+1} = (-1)^n\gamma^{2i+1}, \end{aligned} \tag{5.B.38}$$

which can be unified in a single relation (5.B.37).

***Problem* 5.B.15:** Check the relation (5.B.37) for C_-.

Using the relations (5.B.37), (5.B.34), we conclude that for odd values of n, the matrix $C = C_+$ satisfy the charge conjugation condition (5.B.27). For even n, one must take $C = C_-$ instead. Recall that the dimension of the Clifford algebra is even: $N = 2n$.

C. Spinors and (pseudo)orthogonal Lie algebras

1. Clifford algebra representation of $so(N)$ and $so(1, N-1)$

First recall the commutators among the standard $so(N)$ generators,

$$[L_{ij}, L_{kl}] = -\imath(\delta_{il}L_{jk} + \delta_{jk}L_{il} - \delta_{ik}L_{jl} - \delta_{jl}L_{ik}). \tag{5.C.1}$$

Now is is easy to verify that the above commutation relation is fulfilled, if we express the generators L_{ij} in terms of the γ matrices as follows:

$$L_{ij} = \frac{\imath}{4}[\gamma_i, \gamma_j]. \tag{5.C.2}$$

***Problem* 5.C.16:** Verify the commutations (5.C.1) for the generators (5.C.2). *Hint*: apply the defining anticommutation relations of the Clifford algebra.

The orthogonal generators are Hermitian: $L_{ij}^{+} = L_{ij}$.

Multiplying all but the first gamma matrices by the $\imath$, $\gamma_k \to \imath\gamma_k$ for $k = 1, \dots, N-1$, we get the Clifford algebra with generalized Minkowski metrics η_{ij}. The the relations (5.C.2) represent generators of pseudoorthogonal group $SO(1, N-1)$,

$$[L_{\mu\nu}, L_{\alpha\beta}] = -\imath(\eta_{\mu\beta}L_{\nu\alpha} +_{\nu\alpha} L_{\mu\beta} - \eta_{\mu\alpha}L_{\nu\beta} - \eta_{\nu\beta}L_{\mu\alpha}). \tag{5.C.3}$$

The generators are Hermitian or antiHermitian dependent on index values,

$$L_{\mu\nu}^{+} = \begin{cases} L_{\mu\nu} & \text{for} \quad 1 \le \mu, \nu \le N-1, \\ -L_{\mu\nu} & \text{for remaining } \mu, \nu. \end{cases} \tag{5.C.4}$$

The above equation can be verified using the (anti)unitary conditions on the gamma matrices (5.B.5)

$$\gamma_\mu^{+} = \eta_{\mu\mu}\gamma_\mu = \gamma_0\gamma_\mu\gamma_0.$$

Note that the indexes are lowered and upped with the help of the metrics tensor: $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$, etc.

The generators of the $SO(1, N-1)$ group split into two parts,

$$L_{0i} = \frac{\imath}{2}\gamma_0\gamma_i, \qquad L_{ij} = \frac{\imath}{2}\gamma_i\gamma_j, \qquad 1 \le i < j \le N-1. \tag{5.C.5}$$

The $\frac{(N-2)(N-1)}{2}$ matrices L_{ij} form $so(N-1)$ algebra, and the $N-1$ others, L_{0i}, describe Lorentz boosts.

2. Dirac and Weyl spinors in four dimensional Minkowski space

In the following we restrict ourself to the spinor representation in four dimensional Minkowski space ($N = 4$ and $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Consider the spin-$\frac{1}{2}$ relativistic fermions given by the $N = 4$ component spinor wave function $\psi = \psi(x)$ obeying the Dirac equation (5.B.28) which can be obtained from the variation of the relativistic invariant Lagrangian density

$$\mathcal{L} = \overline{\psi}(\imath\gamma^\mu\partial_\mu - m)\psi, \qquad \overline{\psi} = \psi^{+}\gamma^0. \tag{5.C.6}$$

The *Dirac basis* for the gamma matrices is given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{5.C.7}$$

where $k = 1, 2, 3$ is the space coordinate index.

Another common choice is the *Weyl* or *chiral representation* of the gamma matrices, in which γ^k remains the same but γ^0 is different, and so γ^5 is also different and diagonal,

$$\gamma^0 = \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^k = \beta\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{5.C.8}$$

Splitting the four component Weyl spinor into two parts

$$\psi = \psi_W = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{5.C.9}$$

we represent the Dirac equation as two coupled two component equations

$$\imath(\partial_t + \vec{\sigma}\cdot\vec{\partial})\psi_R = m\psi_L, \qquad \imath(\partial_t - \vec{\sigma}\cdot\vec{\partial})\psi_L = m\psi_R. \tag{5.C.10}$$

In Weyl representation, the generators of $SO(1, 3)$ group have block-diagonal form (5.C.5),

$$L_{0k} = \frac{\imath}{2}\begin{pmatrix} -\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \qquad L_k = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{5.C.11}$$

where $L_k = \epsilon_{ijk}L_{ij}$ is a three dimensional angular momentum pseudo-vector. We see that the $so(1, 3)$ representation on four-dimensional spinor usually referred as a bi-spinor, is reducible and splits into the two two-dimensional chiral components, right and left spinors. The right chiral component ψ_R transforms under the proper Lorentz group action as the complex conjugate to the left component, i.e. as ψ_L^*. Both representations are not equivalent.

***Problem* 5.C.17:** Calculate the values of the second order Casimir invariants of Lorentz group, $L_{\mu\nu}L^{\mu\nu}$ and $\epsilon_{\mu\nu\sigma\rho}L^{\mu\nu}L^{\sigma\rho}$ on the right and left chiral spinor representations. Do they coincide?

We are looking now for the change conjugation matrix in Weyl representation. Note that γ^2 is imaginary while the other γ^μ are real. One can take for that the matrix

$$C = \imath\gamma^2. \tag{5.C.12}$$

so that the charge conjugate spinor will be according to (5.B.29)

$$\psi^{(c)}(x) = \imath\gamma^2\psi(x)^*. \tag{5.C.13}$$

***Problem* 5.C.18:** Verify that the element (5.C.12) obey
1) the charge conjugate operator condition (5.B.27).
2) the condition $C^{-1} = C = C^*$.

3. Majorana fermions

There is also the Majorana basis, in which all of the Dirac matrices are imaginary and spinors are real. In terms of the Pauli matrices, it can be written as

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{5.C.14}$$

The charge conjugate spinor in Majorana representation take the simplest form. Indeed, singe all gamma matrices are imaginary, $(\gamma^\mu)^* = -\gamma^\mu$, we have for the complex conjugate Dirac equation (5.B.28) just

$$\gamma^\mu(\imath\partial_\mu + eA_\mu)\psi^* = m\psi^*$$

which allows to set the charge conjugate matrix to one, $C = 1$. So, an antiparticle is just the complex conjugate of a fermion:

$$\psi^{(c)} = \psi^*. \tag{5.C.15}$$

Now suppose the absence of the antiparticle for the Dirac spinor. More precisely, this means that a particle must coincide with its antiparticle. Since the transmission to the antiparticles implies the charge inversion, $e \to -e$, the charge conjugate operator must be applied to get the antiparticle wavefunction.This property is called a *Majorana condition*. In particular, Majorana representation we arrive at the real spinors,

$$\psi^{(c)}(x) = \psi(x) \qquad \text{so that} \qquad \psi(x)^* = \psi(x). \tag{5.C.16}$$

In Weyl representation, the above condition implies the dependence between the left and right chiral components of the spinor wavefunction,

$$\psi_L(x) = \imath\sigma_2\psi_R(x)^* \qquad \text{or} \qquad \psi_R(x) = \imath\sigma_2\psi_L(x)^*. \tag{5.C.17}$$

The wavefunction now is described actually by a singe two-component (left or right) complex spinor, instead of the four-component real spinor as in the Majorana representation.

***Problem* 5.C.19:** *1)* Verify the Majorana condition (5.C.17) in Weyl representation. *Hint*: apply the Weyl form of charge conjugation matrix (5.C.12), (5.C.8). *2)* Derive the Majorana condition in Dirac representation.

Substituting the constraint (5.C.17) into the Dirac equation (5.C.10), we get the similar equation for the Majorana fermions,

$$\imath(\partial_t + \vec{\sigma}\cdot\vec{\partial})\psi_R = \imath m\sigma_2\psi_R^* \qquad \text{or} \qquad \imath(\partial_t - \vec{\sigma}\cdot\vec{\partial})\psi_L = \imath m\sigma_2\psi_L^*. \tag{5.C.18}$$

4. Maps between Dirac, Weyl and Majorana spinors

The maps between three representations of the gamma matrices are given in the following way. A map between the Dirac ψ_D and Weyl ψ_W spinors is given by the same orthogonal matrix

$$\psi_W = S_{WD}\psi_D, \qquad S_{WD} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad S_{WD}^2 = 1. \tag{5.C.19}$$

Similarly, a map between the Majorana ψ_M and Dirac ψ_D spinors is provided by is given by the matrix

$$\psi_D = S_{DM}\psi_M, \qquad S_{DM} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{pmatrix}, \qquad S_{DM}^2 = 1. \tag{5.C.20}$$

As we see, both matrices are involutive and unitary.

Hence, the through map from the the Majorana spinor to the Weyl spinor is given by the product matrix, which also has to be unitary:

$$\psi_W = S_{WM}\psi_M, \qquad S_{WM} = S_{WD}S_{DM}, \qquad S_{MW}S_{MW}^{+} = 1. \tag{5.C.21}$$

So, more explicitly,

$$S_{WM} = \frac{1}{2}\begin{pmatrix} 1+\sigma_2 & \sigma_2 - 1 \\ 1-\sigma_2 & 1+\sigma_2 \end{pmatrix}, \qquad S_{MW} = S_{WM}^{+} = \frac{1}{2}\begin{pmatrix} 1+\sigma_2 & 1-\sigma_2 \\ \sigma_2 - 1 & 1+\sigma_2 \end{pmatrix}. \tag{5.C.22}$$

Of course, the same matrices intertwine between gamma matrices in Dirac γ_D^μ, Weyl γ_W^μ and Majorana γ_M^μ representations:

$$\gamma_W^\mu = S_{WD}\gamma_D^\mu S_{WD}, \qquad \gamma_D^\mu = S_{DM}\gamma_M^\mu S_{DM}, \qquad \gamma_W^\mu = S_{WM}\gamma_M S_{WM}^{+}. \tag{5.C.23}$$

***Problem* 5.C.20:** Verify the above mapping relations.

Literature

In four dimensional Minkowski space-time, the algebra of gamma matrices, spinor representation of Lorentz group, Dirac equation, etc. are described in detail in most books on the quantum field theory, see for example (Peskin, 1995). The higher dimensional Clifford algebra, its properties, spinor representations of orthogonal groups are described in (Georgi, 1999). The resent review on Majorana fermions and its applications in physics is provided in (Elliott, 2015).

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